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Spacelike Hypersurfaces with Free Boundary in the Minkowski Space under the Effect of a Timelike Potential

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Abstract: In this paper we consider a variational problem for spacelike hypersurfaces in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$, whose critical points are hypersurfaces supported in a spacelike hyperplane Π determined by two facts: the mean curvature is a linear function of the distance to Π and the hypersurface makes a constant angle with Π along its boundary. We prove that the hypersurface is rotational symmetric with respect to a straight-line orthogonal to Π and that each (non-empty) intersection with a parallel hyperplane to Π is a round $(n-1)$-sphere. A similar result is proved for hypersurfaces trapped between two parallel hyperplanes.

1. Introduction and Statement of Results

Consider the following variational problem: let Π be a spacelike hyperplane in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$ and denote by Π^+ one of the two halfspaces at which Π divides $\mathbb{L}^{n+1}$. Let M be a compact spacelike hypersurface whose boundary ∂M lies on Π and its interior, $\mathrm{int}(M)$, is included in Π^+. The hyperplane Π is called the *support hypersurface*. Let us denote Ω the bounded domain by ∂M on Π. In this setting, we consider all perturbations in such way that M is adhered to Π, that is, $\partial M \subset \Pi$, and $\mathrm{int}(M)$ remains in Π^+. We consider the following energy functional:

$$E = |M| - \cosh\beta|\Omega| + \int_M Y \, dM,$$

where $|M|$ and $|\Omega|$ denote the n-areas of M and Ω respectively. Here Y is a potential that, up constants, measures at each point the distance to Π. We say that Y is a *timelike potential* associated to Π (we shall drop the reference to Π if it is well understood in the context). We seek those configurations in a state of equilibrium, that is, when the energy is critical under any perturbation that does not change the volume enclosed by $M \cup \Omega$. According to the principle of virtual work, the equilibrium of the system is achieved if

1. the mean curvature of M is a linear function on the distance to Π and
2. the hyperbolic angle β with which M and Π intersect along ∂M is constant.

See Fig. 1. In such a situation, we shall say that M is a *stationary hypersurface*. In absence of a timelike potential Y, M is a hypersurface with constant mean curvature. Constant mean curvature hypersurfaces have interest in different problems in general relativity. We refer [14, 17, 20] and references therein. Alías and Pastor have proved the following result:

Theorem [4]. *Consider a compact spacelike surface M in $\mathbb{L}^3$ with constant mean curvature and supported in a plane Π. If the hyperbolic angle of contact between M and Π is constant along ∂M, then M must be an umbilical surface, that is, a planar disc or a hyperbolic cap.*

See also [18] for other results in Lorentzian space forms. The main argument used there is the holomorphicity of the Hopf differential in a surface with constant mean curvature. However, and just as they pointed out there, this method fails when the dimension ambient space is bigger than 3. In the present article, we extend the result of Alías and Pastor in two directions. First, we consider arbitrary dimension for the ambient space $\mathbb{L}^{n+1}$; and second, we assume the presence of a timelike potential corresponding to the support hyperplane Π. Our proof uses the *Alexandrov reflection method*. Such technique was firstly used by Alexandrov to prove that a closed embedded constant mean curvature surface in Euclidean 3-dimensional space must be a round sphere [1]. The proof uses the very hypersurface as barrier of comparison with itself and the Hopf maximum principle for elliptic equations.

Here, we prove a more general result:

Theorem 1. *Let $M \subset \mathbb{L}^{n+1}$ be a compact embedded spacelike hypersurface supported in a spacelike hyperplane Π. Assume that*

1. *M lies in one side of Π.*
2. *The mean curvature of M is a function that depends only on the distance to Π.*
3. *The hyperbolic angle that makes M with Π along ∂M is constant.*

Then there is a vertical straight-line L orthogonal to Π about which M is rotational symmetric. Moreover, M is topologically a n-ball and the intersection of M with a hyperplane orthogonal to L is a $(n-1)$-sphere whose center lies on the axis L. In the case that the mean curvature is constant, M is a piece of a hyperbolic hyperplane bounded by a round $(n-1)$-sphere or M is a domain of Π.

Let us relate Theorem 1 with that of Alías and Pastor. First, we point out that a compact spacelike hypersurface in $\mathbb{L}^{n+1}$ is a graph on some domain of the support plane,

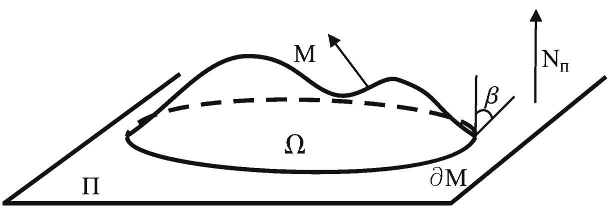

Fig. 1. A stationary hypersurface over a plane Π. The hyperbolic angle β is constant along ∂M

and so, it is embedded [4]. On the other hand, the fact that the mean curvature of M is constant corresponds with no assumption of a timelike potential in $\mathbb{L}^3$, as we will see in Sect. 2, Eq. (4). Moreover, an easy application of the maximum principle applied to the constant mean curvature equation implies that the hypersurface has no points in both sides of Π. Hence that M lies in one side of Π, unless that M is included in Π and in which case, M is a planar domain.

As conclusion of Theorem 1, we describe the shape of the critical points of the initial variational problem.

Corollary 1. *Let* Π *be a spacelike hyperplane in* $\mathbb{L}^{n+1}$. *Then any stationary embedded hypersurface* M *supported in* Π *is a hypersurface of revolution with respect to a straight-line orthogonal to* Π. *Moreover, each non-empty intersection of* M *with a parallel hyperplane to* Π *is a round* $(n-1)$*-sphere.*

A similar result was proved by Wente in Euclidean space [21]. Recently, the present author has detailed the size and shape of a stationary surface in $\mathbb{L}^3$ supported in a spacelike plane [16].

This paper is organized as follows. Section 2 is a preparatory introduction where we will formulate the variational problem. Next we present the background analysis of the maximum principle and in Sect. 4 we prove our main result, Theorem 1. In Sect. 5 we extend our result for stationary hypersurfaces trapped between two parallel hyperplanes and, finally, we summarize the results and the conclusions in Sect. 6.

2. Preliminaries

In this section we present the variational problem introduced in the above section. Much of these results appear in the literature and we refer to them for more details. See [4–8]. Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric $\langle , \rangle = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$, where $x = (x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{L}^{n+1}$. An immersion $x : M^n \to \mathbb{L}^{n+1}$ of a smooth n-manifold M is called *spacelike* if the induced metric on M is positive definite. Observe that $\boldsymbol{a} = (0, \ldots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on the space $\mathbb{L}^{n+1}$. This allows us the choice of a unit normal vector field N on M which is in the same time-orientation as $\boldsymbol{a}$, and hence that M is oriented by N. We will refer to N as the future-directed Gauss map of M.

The spacelike condition imposes topological restrictions to the immersion x. For example, there are not closed spacelike hypersurfaces and then, any compact spacelike hypersurface has non-empty boundary. If Γ is a $(n-1)$-submanifold in $\mathbb{L}^{n+1}$ and $x : M \to \mathbb{L}^{n+1}$ is a spacelike immersion of a compact hypersurface, we say that the boundary of M is Γ if the restriction $x : \partial M \to \Gamma$ is a diffeomorphism. For spacelike hypersurfaces, the projection $\pi : \mathbb{L}^{n+1} \to \{x_{n+1} = 0\}$, $\pi(x) = (x_1, \ldots, x_n)$, is a local diffeomorphism between $\mathrm{int}(M)$ and $\pi(\mathrm{int}(M))$. Thus, π is an open map and $\pi(\mathrm{int}(M))$ is a domain in Π. If M is compact, then $\pi : M \to \overline{\Omega}$ is a covering map. Thus, *any compact spacelike hypersurface whose boundary* Γ *is a graph over the boundary of an open region* $\Omega \subset \{x_{n+1} = 0\}$ *is a graph over* Ω. From now, we shall identify a point $p \in M$ with its image by x, namely $x(p)$.

Consider a compact spacelike hypersurface $M \subset \mathbb{L}^{n+1}$ whose boundary ∂M is on a spacelike hyperplane Π, which must be of spacelike-type. Without loss of generality and after an isometry of the ambient, we assume that $\Pi = \{x_{n+1} = 0\}$ and that M is the graph of a function u on a domain of Π. Although the boundary ∂M is possibly non-connected, the causal character on M implies the existence of a component of ∂M, named Γ_0, such that $\pi(\mathrm{int}(M))$ is contained in the bounded domain determined by Γ_0 in Π. Therefore, M defines an "interior" domain, that is, there exists a bounded region $\Omega \subset \Pi$ such that $M \cup \Omega$ determines in $\mathbb{R}^{n+1}$ a bounded domain B, called the "interior" of M.

For spacelike hypersurfaces of $\mathbb{L}^{n+1}$, the notions of the first and the second fundamental form are defined in the same way as in the Euclidean space. In a classical notation, they are given by

$$\mathrm{I} = \sum_{i,j}^{n} g_{ij} dx_i\, dx_j, \qquad \mathrm{II} = \sum_{i,j}^{n} h_{ij} dx_i\, dx_j,$$

where $g_{ij} = \langle \partial_i x, \partial_j x\rangle$ is the induced metric on M by x and $h_{ij} = \langle \partial_i N, \partial_j x\rangle$. Then the mean curvature H of x is

$$H = \frac{1}{n}\mathrm{trace}\, [(g_{ij})^{-1}(h_{ij})].$$

Assume that M is the graph of a smooth function $u = u(x_1, \ldots, x_n)$ defined over a domain $\Omega \subset \Pi$. The spacelike condition implies $|\nabla u| < 1$, where ∇ is the gradient operator in $\mathbb{R}^n$ and the Gauss map is

$$N = \frac{(\nabla u, 1)}{\sqrt{1 - |\nabla u|^2}}.$$

According to this orientation, the mean curvature H at the point $(x, u(x))$, $x \in \Omega$, satisfies the equation

$$(1 - |\nabla u|^2)\Delta u - \sum_{i,j}^{n} u_i u_j u_{ij} = nH(1 - |\nabla u|^2)^{3/2}. \tag{1}$$

This equation can alternatively be written in divergence form

$$\mathrm{div}(Tu) = nH, \qquad Tu = \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}. \tag{2}$$

We present now the notion of stationary hypersurface in $\mathbb{L}^{n+1}$. Consider a spacelike hyperplane Π, that divides $\mathbb{L}^{n+1}$ into two halfspaces. Let us orient Π by the future-directed unit timelike vector field N_Π and consider $\mathbb{L}^{n+1}_+$ the component of $\mathbb{L}^{n+1} \setminus \Pi$ towards where N_Π points to. Let $x : M \to \mathbb{L}^{n+1}$ be a connected compact hypersurface with boundary ∂M, smooth even at ∂M such that $x(\mathrm{int}(M)) \subset \mathbb{L}^{n+1}_+$ and $x(\partial M) \subset \Pi$. A *variation* of M is a differentiable map $X : (-\epsilon, \epsilon) \times M \to \mathbb{L}^{n+1}$ such that $X_t : M \to \mathbb{L}^{n+1}$ defined by $X_t(p) = X(t, p)$, $p \in M$, is an immersion and $X_0 = x$. The variation is called *admissible* if $X_t(\mathrm{int}(M)) \subset \mathbb{L}^{n+1}_+$ and $X_t(\partial M) \subset \Pi$ for all t. The functionals $A, S : (-\epsilon, \epsilon) \to \mathbb{R}$ defined by

$$A(t) = \int_M dA_t, \qquad S(t) = \int_{\Omega_t} d\Pi,$$

measure, respectively, the n-area of M with the metric induced by X_t and the n-area of $\Omega_t \subset \Pi$, the region in Π bounded by $X_t(\partial M)$. Finally, the *volume function* $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_{[0,t]\times M} X^* \, dV,$$

where dV is the canonical volume element of $\mathbb{L}^{n+1}$. The variation X is said to be *volume-preserving* if $V(t) = V(0)$ for all t. The *variational vector field* of X is

$$\xi(p) = \frac{\partial X}{\partial t}(p)\bigg|_{t=0}.$$

If we assume the existence of a potential Y, then resultant variation energy is

$$Y(t) = \int_M Y \, dA_t.$$

The energy function $E : (-\epsilon, \epsilon) \to \mathbb{R}$ of the mechanical system is defined by

$$E(t) = A(t) - \cosh\beta \; S(t) + Y(t), \tag{3}$$

where $\beta \in \mathbb{R}$ is a constant. We say that the immersion x is *stationary* if $E'(0) = 0$ for any volume preserving admissible variation of x. One can show that the first variation formula for the energy is:

$$E'(0) = \int_M (-nH + Y + \lambda)\,\langle N, \xi\rangle \, dM + \int_{\partial M} \langle \xi, \nu_\Pi\rangle \,(\cosh\beta + \langle N, N_\Pi\rangle)\, ds,$$

where ν_Π is the inner unitary conormal to Ω along ∂M. Thus, we have

Proposition 1. *Let Π be a spacelike hyperplane in $\mathbb{L}^{n+1}$ and let M be a compact hypersurface. Let us consider $x : M \to \mathbb{L}^{n+1}$ a smooth spacelike immersion such that $x(int(M)) \subset \mathbb{L}^{n+1}_+$ and $x(\partial M) \subset \Pi$. Then x is stationary if and only if*

1. *The mean curvature H of x satisfies the relation*

$$nH(p) = Y(p) + \lambda, \qquad p \in M, \tag{4}$$

   where Y is a potential and λ is a Lagrange parameter determined by an eventual volume constraint;
2. *The hypersurface $M = x(M)$ meets the support hyperplane Π with a constant hyperbolic angle β, and* $\cosh\beta = -\langle N, N_\Pi\rangle$ *along ∂M.*

Our interest in this article lies in the case for which Y is a timelike potential associated to Π. As we have supposed that $\Pi = \{x_{n+1} = 0\}$,

$$Y(p) = \kappa x_{n+1}(p), \tag{5}$$

for a constant κ. When $\kappa = 0$, $Y = 0$ and the mean curvature H of the hypersurface M is constant, with $H = \lambda/n$.

On the other hand, when we talk of contact angle, it is implicitly assumed that the boundary regularity of M is enough to ensure that the idea of a normal vector to M

at every boundary point makes sense. For this, we will require M to be a sufficiently smooth hypersurface up to the boundary ∂M. The contact angle β is given by

$$\cosh \beta = -\langle N, N_\Pi \rangle = \frac{1}{\sqrt{1-|\nabla u|^2}} \tag{6}$$

along ∂M. The constancy of the angle β implies that $|\nabla u|$ is constant along ∂M, and consequently, the Euclidean angle between M and Π along ∂M is also constant.

The constant λ is a Lagrange multiplier arising from the volume constraint: since M is a graph on Ω, the volume V enclosed by $M \cup \Omega$ is

$$V = \int_\Omega u \, d\Omega.$$

By combining (4), (5), (6) and the divergence theorem in (2), we obtain $\kappa V + \lambda|\Omega| = \cosh \beta |\partial\Omega|$ or

$$\lambda = \frac{\cosh \beta |\partial \Omega| - \kappa V}{|\Omega|}.$$

3. The Maximum Principle

We consider M_1 and M_2 two spacelike graphs in $\mathbb{L}^{n+1}$ defined respectively by two functions u^i, $i = 1, 2$. We suppose that both functions are defined in the same domain $\Omega \subset \mathbb{R}^n = \{x_{n+1} = 0\}$. We know that the mean curvature H_i of M_i satisfies

$$\mathrm{div}(Tu^i) = nH_i(x), \qquad |\nabla u^i| < 1$$

in Ω. Assume that for each $x \in \Omega$, we have the inequality $H_1(x) \leq H_2(x)$. The operator $\mathrm{div}(Tu)$ may be written in the form

$$\mathrm{div}(Tu) = \frac{1}{W}\sum_i^n u_{ii} + \frac{1}{W^3}\sum_{i,j}^n u_i u_j u_{ij}, \qquad W = \sqrt{1-|\nabla u|^2},$$

where the subscript i indicates the differentiation with respect to the variable x_i. We can write $\mathrm{div}(Tu) = \sum_{i,j}^n a_{ij}(x, u, p)u_{ij}$ with $p \in \mathbb{R}^n$, $p_i = u_i$, and where

$$\sum_{i,j}^n a_{ij}(x, u, p)\xi_i\xi_j = \frac{(1-|p|^2)|\xi|^2 + \langle \xi, p\rangle^2}{W^3}, \qquad \xi \in \mathbb{R}^n.$$

As a consequence

$$0 < \lambda(x, u, p)|\xi|^2 \leq \sum_{i,j}^n a_{ij}(x, u, p)\xi_i\xi_j \leq \Lambda(x, u, p)|\xi|^2,$$

where

$$\lambda(x, u, p) = \frac{1}{W} \qquad \Lambda(x, u, p) = \frac{1}{W^3}.$$

Then the operator is elliptic for $|p| < 1$ and uniformly elliptic for compact domains. Let

$$\phi(x, p, r) = \mathrm{div}(Tu) = nH(x), \tag{7}$$

where $r = (r_{ij})$, $r_{ij} = u_{ij}$. Then ϕ is a smooth function defined in $\Omega \times D \times \mathbb{R}^{n^2}$ given explicitly by

$$\phi(x, p, r) = \frac{1}{\sqrt{1-|p|^2}} \sum_{i,j}^{n} \left(\delta_{ij} + \frac{p_i p_j}{1-|p|^2}\right) r_{ij},$$

where D is the unit open disc of $\mathbb{R}^n$. For each $u = u^i$, $i = 1, 2$, we will use the notation p^i, r^i and H_i for each i . Since $H_1 \leq H_2$, a standard argument by using the chain rule shows then

$$\begin{aligned} 0 &\leq \phi(x, p^2, r^2) - \phi(x, p^1, r^1) \\ &= \sum_{i,j}^{n} \int_0^1 \frac{\partial \phi}{\partial r_{ij}}(\theta(t))\, dt\, w_{ij} + \sum_{j}^{n} \int_0^1 \frac{\partial \phi}{\partial p_j}(\theta(t))\, dt\, w_j := Lw, \end{aligned}$$

where $w = u^1 - u^2$, $w_i = \partial w/\partial x_i$, $w_{ij} = \partial^2 w/\partial x_i \partial x_j$ and $\theta = \theta(t) = (x, tp^2 + (1-t)p^1, tr^2 + (1-t)r^1)$. The right hand side of the above equation defines an elliptic operator L because

$$|\xi|^2 \leq \int_0^1 \frac{1}{W(\theta(t))}\, dt |\xi|^2 \leq Lw \leq \max\left\{\frac{1}{W_1^3}, \frac{1}{W_2^3}\right\} |\xi|^2,$$

and $W_i = \sqrt{1-|\nabla u^i|^2}$. Since the coefficients a_{ij} are locally bounded, L is locally uniformly elliptic and we are in position to apply the Hopf maximum principle to the difference function w, whether in its classical formulation ([11]) or its boundary point version ([12]): see also [10, Ch. 3]. Consequently, we have proved the following result:

Theorem 2 (The touching principle). *Let u, v be two smooth solutions to the same prescribed mean curvature equation (7) on a domain $\Omega \subset \mathbb{R}^n$. Suppose that $u \leq v$ on Ω and $u(x_0) = v(r_0)$, $x_0 \in \Omega$. Then $u(x) = v(x)$ on Ω. The same holds if $p \in \partial\Omega$ with the extra hypothesis that $\partial u/\partial \nu = \partial v/\partial \nu$ at x_0, where ν is the outward unit normal to $\partial\Omega$.*

4. Proof of Theorem 1

The method of proof used in this work is the Alexandrov reflection method and it may be adapted to the present situation. For expository reasons, we will describe it briefly. Such techniques have been used in a variety of situations in differential geometry. See also the so-called "method of moving plane" in the context of the theory of partial differential equations (for example [9, 19]).

The proof consists in showing that for each hyperplane P orthogonal to Π, there exists some hyperplane P^* parallel to P such that M is invariant by the symmetry with respect to P^*. In showing this fact for such hyperplane, then M is a hypersurface of revolution whose axis L is the intersection of all hyperplanes P^*. In addition, the proof shows that M is a graph over some domain of P^* in each side. Consequently, each intersection of M with a hyperplane parallel to Π is a round $(n-1)$-sphere.

For this, we work as follows. Fix a hyperplane P orthogonal to Π and consider the foliation of all translated copies of P along a straight-line orthogonal to P. Then coming from the infinity towards M doing such translations, one makes successive symmetries about these hyperplanes and looks to the possible first point of tangent touching contact with M again. Then we use the very hypersurface M as comparison hypersurface with itself and the touching principle concludes that in that new position P^*, the hypersurface M is invariant by the symmetry with respect to P^*.

After an ambient isometry, we can suppose that the support hyperplane Π is $\Pi = \{x \in \mathbb{R}^{n+1}; x_{n+1} = 0\}$ and that the mean curvature H of M depends only on the x_{n+1}-coordinate, that is, $H(x) = H(x_{n+1}(x))$ for any $x \in M$. Without loss of generality, we assume that M lies over the hyperplane Π. Let Ω be the bounded region in Π bounded by ∂M such that $M \cup \Omega$ is a closed embedded hypersurface. Therefore, $M \cup \Omega$ determines two domains in $\mathbb{R}^{n+1}$, namely A and B, where we denote, respectively, the non-bounded and the interior domain determined by M in $\mathbb{R}^{n+1}$. Recall that in our situation, both hyperbolic and Euclidean angles between M and Π are constant along ∂M.

Let P be a fixed vertical hyperplane far away from M so $P \subset A$. Let $P(t)$ be the 1-parameter family of translated copy of P, where we choose the parameter t such that $P(t)$, $t > 0$, is included in the connected component determined by P which contains M. Here $t = \mathrm{dist}(P(t), P)$, hence $P(0) = P$. Translating P towards M parallel to itself (say, to the right) one gets a first plane $P(t_1)$ that reaches M, that is, $P(t_1) \cap M \neq \emptyset$ but if $t < t_1$ then $P(t) \cap M = \emptyset$. Furthermore, the spacelike character of M implies that $P(t_1)$ touches M only at boundary points. Now, when we move P a little more to the right from $t = t_1$, until a hyperplane $P(t)$, the (closed) part of M on the left of $P(t)$, which we denote by $M(t)^-$, is a graph (with respect to the horizontal) over a domain in $P(t)$ and no point of $M(t)^-$ has a horizontal tangent hyperplane. We denote $M(t)^+$ the part of on the right of $P(t)$.

Let $M(t)^*$ be the symmetry of $M(t)^-$ through $P(t)$. We know then that for $\epsilon > 0$ sufficiently small, $M(t)^* \subset B, t \in (t_1, t_1+\epsilon)$. Recall that the symmetry with respect to a vertical hyperplane is an isometry of $\mathbb{L}^{n+1}$, and so, the mean curvature remains invariant by the symmetry. Because the mean curvature of M depends only on the height with respect to Π, the mean curvature is the same for all points of $M(t)^+$ and $M(t)^*$ at the same height. We continue now moving $P(t)$ to the right, and reflecting $M(t)^-$ about $P(t)$, successively until one reaches a first point of contact of the reflection of M with $M(t)^+$.

Consider the first parallel hyperplane $P(\tau)$ where one of the following conditions fails to hold (see Fig. 2 and 3):

1. $int(M(\tau)^*) \subset int(B)$.
2. $M(\tau)^-$ is a graph over a part of $P(\tau)$ and no point of $M(\tau)^-$ has a horizontal tangent hyperplane.

If 1) fails first, $M(\tau)^+$ and $M(\tau)^*$ touch at some interior point p (Fig. 2, (a)), or at a boundary point p, with $p \in \partial M \cap \partial M(\tau)^*$ (Fig. 2 (b)). The fact that M lies over Π prohibits the possibility that $p \in \partial M$ and p is a reflection of an interior point of $M(\tau)^-$. Thus the tangent hyperplanes of $M(\tau)^+$ and $M(\tau)^*$ agree at p (in the latter case, we use that the hyperbolic angle between M and Π along ∂M is constant). In addition, the reflections invert normal vectors and the Gauss maps N of $M(\tau)^*$ and $M(\tau)^+$ at such point p are the same. Then one applies the touching principle to $M(\tau)^+$ and $M(\tau)^*$ at

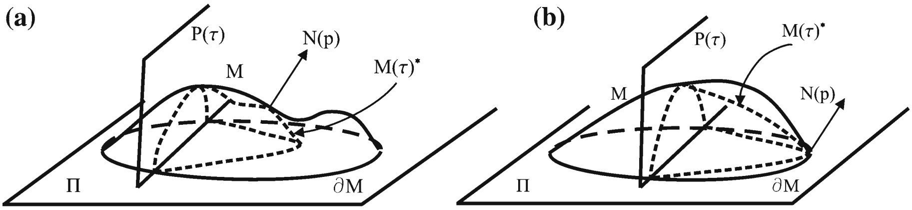

Fig. 2. The Alexandrov reflection method: (Case 1)

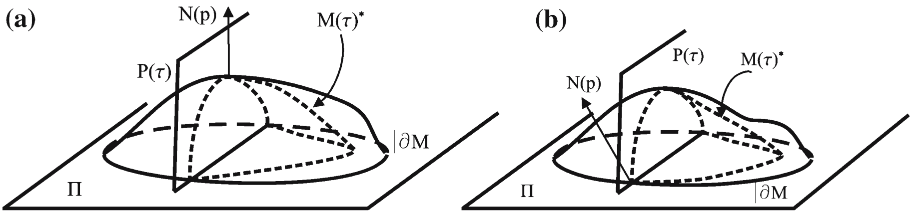

Fig. 3. The Alexandrov reflection method: (Case 2)

the point where they touch to conclude that $M(\tau)^+ = M(\tau)^*$. This means that $P(\tau)$ is a hyperplane of symmetry of M.

If 2) fails first, then there exists a point p where the tangent hyperplane of $M(\tau)^-$ becomes horizontal is on $\partial(M(\tau)^-) \subset P(\tau)$ (Fig. 3 (a)) or $p \in \partial M \cap P(\tau)$ (Fig. 3 (b)). In the former possibility one can apply the boundary touching principle to $M(\tau)^*$ and $M(\tau)^+$ to conclude that $P(\tau)$ is a hyperplane of symmetry of M; in the second one, the corresponding tangent hyperplanes of M and $M(\tau)^*$ are identical because the hyperbolic angle with the $\boldsymbol{a}$ direction is the same at p. Then one applies the maximum principle at a corner point (see details in [19]).

Thus, for each vertical hyperplane P, some parallel translate of P, namely $P^* = P(\tau)$, is a hyperplane of symmetry of M and this proves that M is a hypersurface of revolution.

To finish with the proof, we consider the situation of absence of the timelike potential. We know that a stationary hypersurface M with free boundary supported in a spacelike hyperplane Π must be a hypersurface of revolution. Set $|x| = r$, $x \in \mathbb{R}^n$. After an isometry of the ambient, we assume that the rotation axis is the x_{n+1}-line. Then M is obtained by the rotation of the profile of a function $u : [0, R] \to \mathbb{R}$ with boundary conditions

$$u(0) = u_0, \qquad u'(0) = 0.$$

Equation (2) becomes an ordinary differential equation and it converts into

$$\frac{1}{r^{n-1}} \frac{d}{dr} \left(\frac{r^{n-1} u'(r)}{\sqrt{1 - u'(r)^2}} \right) = \kappa\, u(r) + \lambda, \qquad 0 \leq r < R. \tag{8}$$

In the case that we are treating, $\kappa = 0$, the solution corresponds with a constant mean curvature hypersurface, with $H = \lambda/n$. A direct integration of (8) leads to (up to constants)

$$u(r) = \sqrt{\frac{n^2}{\lambda^2} + r^2} \qquad \text{if } \lambda \neq 0$$

$$u(r) = 0, \qquad \text{if } \lambda = 0.$$

In the first case, u describes a hyperbolic hyperplane of mean curvature λ/n; in the second one, we obtain that M is a domain of Π. This completes the proof of Theorem 1.

The Alexandrov reflection method applies in a similar situation as in Theorem 1, where the condition on the angle is replaced by certain symmetry of the boundary. The next result generalizes those obtained in [2, 3] and its proof is omitted.

Corollary 2. *Let $\Gamma \subset \mathbb{L}^{n+1}$ be a closed $(n-1)$-submanifold included in a spacelike plane Π and symmetric with respect to a straight-line $L \subset \Pi$. Let M be a spacelike embedded hypersurface spanning Γ. Assume*

1. *Each component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L.*
2. *M lies in one side of Π.*
3. *The mean curvature of M is a function that depends only on the distance with respect to Π.*

Then the plane P orthogonal to Π with $L \subset P$ is a hyperplane of symmetry of M. Moreover, each component of $M \setminus (M \cap P)$ is a graph on P. In the particular case that Γ is a round $(n-1)$-sphere, M is a hypersurface of revolution and the intersection of M with a hyperplane parallel to Π is a round $(n-1)$-sphere.

5. Bridges Between Two Parallel Hyperplanes

The Alexandrov reflection technique can be used in other possible configurations. For example, hypersurfaces interconnecting a set of spacelike hyperplanes. The setting that we will consider is that a stationary hypersurface is trapped between two parallel spacelike hyperplanes Π_1 and Π_2. Usually the hypersurface is called a bridge. In such case, the term S in (3) is the n-area of the domains that ∂M bounds in each one of the hyperplanes. Again, in a state of equilibrium, the angle between the normal vector to the bridge and Π_i along their lines of contact is constant (and possibly with different values in each hyperplane Π_i). The Alexandrov reflection method yields again the following

Theorem 3. *Let Π_1 and Π_2 be two parallel spacelike hyperplanes in $\mathbb{L}^{n+1}$. Consider M a spacelike embedded compact hypersurface included in the slab determined by $\Pi_1 \cup \Pi_2$ and whose boundary ∂M intersects both Π_1 and Π_2. Assume that the mean curvature of M depends only on the distance to Π_i and the hyperbolic contact angle between M and Π_i is constant along ∂M in each one of the two hyperplanes. Then M is rotational symmetric with respect to a straight-line orthogonal to Π_i. Moreover, each (non-empty) intersection of M with a parallel hyperplane to Π_i is a round $(n-1)$-sphere.*

Corollary 3. *Let M be a stationary embedded hypersurface in $\mathbb{L}^{n+1}$ trapped between two parallel hyperplanes $\Pi_1 \cup \Pi_2$. Then M is a hypersurface of revolution with respect to a straight-line orthogonal to the support hyperplanes Π_i. Moreover, each (non-empty) intersection of M with a parallel hyperplane to Π_i is a round $(n-1)$-sphere.*

Remark 1. In the case that M has constant mean curvature, M is not necessarily a piece of a hyperbolic hyperplane. The family of constant mean curvature spacelike hypersurfaces bounded by two axial $(n-1)$-spheres in parallel hyperplanes is richer and according to Eq. (8), the function u is determined by elliptic integrals. See [15].

6. Final Discussions and Conclusions

In the Lorentz-Minkowski space $\mathbb{L}^{n+1}$, we have considered the variational problem of an embedded compact spacelike hypersurface M resting on a spacelike hyperplane Π. The forces involved in the system are to the n-areas of the hypersurface and the domain that M bounds in Π. Furthermore, we assume the existence of a timelike potential determined by Π. Our interest was the possible shapes of the hypersurface when it reaches an equilibrium: the energy of the system is critical under any perturbation of the hypersurface such that we maintain its adherence to the plate Π and the enclosed volume.

The so-called Alexandrov reflection method allows to prove that the hypersurface is rotational symmetric with respect to a line orthogonal to the support hyperplane. Moreover the intersection with a parallel hyperplane to Π is a round $(n-1)$-sphere. This extends the result proved in [4] both for arbitrary dimension and for a more general mean curvature function of the hypersurface. A similar result has been obtained for bridges between parallel hyperplanes.

Another interesting support hypersurface occurs when Π is a hyperbolic hyperplane. One can believe that the only stationary hypersurface resting on a hyperbolic hyperplane are pieces of hyperbolic hyperplanes. This is true in $\mathbb{L}^3$ under the assumption of the constancy of the mean curvature [4]. However, we do not know if the same remains true under the effect of a timelike potential. We remark that a hyperbolic plane is also a hypersurface of revolution, which it makes one think that our conclusions can extend to this situation.

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