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Symmetry of stationary hypersurfaces in hyperbolic space*

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Abstract. We deposit a prescribed amount of liquid on an umbilical hypersurface Π of the hyperbolic space $\mathbb{H}^{n+1}$. Under the presence of a uniform gravity vector field directed towards Π, we seek the shape of such a liquid drop in a state of equilibrium of the mechanical system. The liquid-air interface is then modeled by a hypersurface under the condition that its mean curvature is a function of the distance from Π, together with the fact that the angle that makes with Π along its boundary is constant. We show that the hypersurface is rotational symmetric with respect to a geodesic orthogonal to Π. We extend this result to other configurations, for example, liquid bridges trapped between two umbilical hypersurfaces. Finally, we obtain a result which says that, under some assumptions on the mean curvature, an embedded hypersurface inherits a certain symmetry from its boundary.

Mathematics Subject Classifications. 53A10, 35Q35, 76B45, 35J65.

Key words. capillary problem, stationary hypersurface, hyperbolic space, tangency principle.

1. Introduction and Statement of Results

In the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, we consider the following mechanical system. We deposit a liquid drop of a prescribed volume into a solid hypersurface Π. We assume that no chemical reactions occur between the liquid and the solid substrate and that the materials are homogeneous. We assume the existence of a potential Y depending on Π, such as, a uniform gravity field directed towards Π. The energy of the physical system involves the area of the drop (the liquid-air interface) and the area of the region of contact of the drop with Π (the liquid-solid interface). Our interest are those configurations of such drops in a state of equilibrium, that is, when the energy of the physical system is critical under any perturbation of the system that do not change the amount of liquid of the drop.

From the mathematical viewpoint, the interior of the liquid drop is a bounded domain X of $\mathbb{H}^{n+1}$ whose boundary ∂X decomposes into $\partial X = \mathcal{S} \cup \Omega$, where $\mathcal{S}$ is

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the liquid-air interface and $\Omega = \overline{X} \cap \Pi$ is the region in Π occupied by the part of the drop that wets on Π. The hypersurface Π is called the *support hypersurface* and we shall assume that Π divides the ambient space $\mathbb{H}^{n+1}$ into two components. The fact that the liquid drop rests on Π means that $\mathcal{S}$ lies in one side of Π, that is, in one of the two components of $\mathbb{H}^{n+1} \backslash \Pi$. This has the significance that the liquid drop do not across Π beyond the liquid-solid interface; on the contrary case, the situation would be physically unrealizable. In equilibrium, we shall say then that $\mathcal{S}$ is a *stationary hypersurface*. According to the principle of virtual work, and when the equilibrium of the system is achieved, the possible shapes of a stationary hypersurface are given by two conditions, namely, the mean curvature of $\mathcal{S}$ is a function of its position in space, and $\mathcal{S}$ meets the support hypersurface in a prescribed angle. The angle β with which $\mathcal{S}$ and Π intersect along $\partial\mathcal{S} = \mathcal{S} \cap \Pi$ is determined as a physical constant depending only on the materials. Section 2 is devoted to the formulation of the variational problem of the physical system. We refer to the book of Finn [2] as an approximation to the interfacial phenomena.

In the present work the hypersurfaces to be considered as possible supports of our stationary hypersurfaces are given in the following definition:

DEFINITION 1.1. In hyperbolic space $\mathbb{H}^{n+1}$, we call a support hypersurface Π an umbilical non-bounded hypersurface. This means that Π is a totally geodesic hyperplane, an equidistant hypersurface or a horosphere.

Our proof uses the so-called Alexandrov reflection method. Such a technique was firstly used by Alexandrov to prove that a closed embedding of a constant mean curvature surface in Euclidean 3-dimensional space must be a round sphere [1]. The proof idea is to use the very hypersurface as a comparison hypersurface with itself and to apply the Hopf maximum principle for elliptic equations. In a more general setting, we show in Section 3:

THEOREM 1.2. *Let M be an embedded compact hypersurface in hyperbolic space $\mathbb{H}^{n+1}$. Assume that M rests on a support hypersurface Π and such that the following assumptions hold:*

(1) *M lies in one side of Π.*
(2) *The mean curvature of M is a function that depends only on the distance with respect to Π.*
(3) *The angle that makes M with Π along the boundary ∂M of M is constant.*

Then there exists a geodesic γ orthogonal to Π about which M is rotational symmetric.

In conclusion, and for stationary hypersurfaces, we obtain

> Consider a liquid drop in $\mathbb{H}^{n+1}$ resting on a support hypersurface Π. Let us assume the existence of a uniform gravity field directed towards to Π. In a state of mechanical equilibrium, the shape of the liquid drop is axially symmetric with respect to a geodesic orthogonal to Π.

In Section 4, we consider liquid bridges between two supporting hypersurfaces, obtaining characterizations of the shapes of the possible stationary hypersurfaces. Finally, in Section 5, we give sufficient conditions so that a hypersurface inherits the symmetries of its boundary. In conclusion,

> Let $\mathbb{S}^{n-1}$ be a round $(n-1)$-dimensional sphere in a support hypersurface Π. If M is an embedded compact hypersurface with $\partial M = \mathbb{S}^{n-1}$ that lies in one side of Π and whose mean curvature depends on the distance to Π, then M must be axially symmetric with respect to a geodesic orthogonal to Π.

Theorem 1.2 extends a previous result due to Wente in Euclidean space [10]. Recently, the present author has studied the shape of an axial symmetric liquid drop in $\mathbb{H}^3$ supported in a horosphere [8].

2. Formulation of the Variational Problem

In this section we present the variational problem that we consider in hyperbolic space. Many of the results appear in the literature and we refer to them for more details [2]. Let $\mathbb{H}^{n+1}$ denote the $(n+1)$-dimensional hyperbolic space. We shall work in the upper halfspace model of $\mathbb{H}^{n+1}$, that is,

$$\mathbb{R}^{n+1}_{+} = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\}$$

equipped with the metric

$$\langle , \rangle = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}.$$

The hyperbolic space $\mathbb{H}^{n+1}$ has a natural compactification $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup \partial_\infty \mathbb{H}^{n+1}$, where $\partial_\infty \mathbb{H}^{n+1}$ can be identified with asymptotic classes of geodesics rays in $\mathbb{H}^{n+1}$. In the halfspace model of $\mathbb{H}^{n+1}$, $\partial_\infty \mathbb{H}^{n+1} = P_\infty \cup \{\infty\}$ is the one-point compactification of the hyperplane $P_\infty := \{x_{n+1} = 0\}$.

On the other hand, and refered to as the support hypersurfaces, in hyperbolic space $\mathbb{H}^{n+1}$ we have three cases of non-bounded umbilical hypersurfaces. The description in our model of $\mathbb{H}^{n+1}$ is the following:

(1) *Totally geodesic hyperplanes*. They are vertical Euclidean hyperplanes of $\mathbb{R}^{n+1}_{+}$ and Euclidean hemispheres of $\mathbb{R}^{n+1}_{+}$ that intersect P_∞ orthogonally.
(2) *Equidistant hypersurfaces*. They are tilted Euclidean planes transverse to P_∞ and Euclidean spherical caps, not hemispheres, tangent to P_∞.

(3) *Horospheres*. They are horizontal hyperplanes and Euclidean spheres of $\mathbb{R}^{n+1}_+$ tangent to P_∞.

Among the isometries of $\mathbb{H}^{n+1}$, we emphasize two of them. The first are the *hyperbolic translations*. In our model of $\mathbb{H}^{n+1}$, a hyperbolic translation is a Euclidean homothety centred at a point $p_0 \in P_\infty$ and a horizontal translation (parallel to P_∞). Another type of isometries are the *hyperbolic reflections* with respect to a geodesic hyperplane P. If $P \subset \mathbb{R}^{n+1}_+$ is a hemisphere centred at $p_0 \in P_\infty$, then it is an inversions with respect to p_0 that fix P and if P is a vertical hyperplane, the corresponding hyperbolic reflection is a Euclidean reflection with respect to P.

We now present the notion of a stationary hypersurface in $\mathbb{H}^{n+1}$. Consider a support hypersurface Π, which divides the space $\mathbb{H}^{n+1}$ into two non-bounded connected components. Let us orient Π by a unit vector field N_Π and consider $\mathbb{H}^{n+1}_+$ the component of $\mathbb{H}^{n+1}\setminus\Pi$ towards which N_Π is pointing. Let $x : M \to \mathbb{H}^{n+1}$ be a connected compact hypersurface with boundary ∂M, smooth even at ∂M such that $x(\mathrm{int}(M)) \subset \mathbb{H}^{n+1}_+$ and $x(\partial M) \subset \partial\mathbb{H}^{n+1} = \Pi$. A *variation* of x is a differentiable map $X : (-\epsilon, \epsilon) \times M \to \mathbb{H}^{n+1}$ such that $X_t : M \to \mathbb{H}^{n+1}$, $t \in (-\epsilon, \epsilon)$, defined by $X_t(p) = X(t, p)$, $p \in M$, is an immersion and $X_0 = x$. The functionals $A, S : (-\epsilon, \epsilon) \to \mathbb{R}$ defined by

$$A(t) = \int_M \mathrm{d}A_t, \quad S(t) = \int_{\Omega_t} \mathrm{d}\Pi,$$

measure, respectively, the area of M with the metric induced by X_t and the area of $\Omega_t \subset \Pi$, the region in Π bounded by $X_t(\partial M)$. Finally, the *volume function* $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_{[0,t]\times M} X^* \ \mathrm{d}V,$$

where $\mathrm{d}V$ is the canonical volume element of $\mathbb{H}^{n+1}$. The variation X is said to be *volume-preserving* if $V(t) = V(0)$ for all t. The *variational vector field* of X is defined on M by

$$\xi(p) = \frac{\partial X}{\partial t}(p)\bigg|_{t=0}, \quad p \in M.$$

If we assume in the ambient space the existence of a potential energy $Y = Y(p)$, $p \in \mathbb{H}^{n+1}$, then resultant variation energy is

$$Y(t) = \int_M Y \ \mathrm{d}A_t.$$

The energy function $E : (-\epsilon, \epsilon) \to \mathbb{R}$ of the mechanical system is defined by

$$E(t) = A(t) + \cos(\beta) \ S(t) + Y(t),$$

where $\beta \in \mathbb{R}$ is an arbitrary real constant. The variation is called *admissible* if $X_t(\mathrm{int}(M)) \subset \mathbb{H}^{n+1}_+$ and $X_t(\partial M) \subset \Pi$ for all t. The immersion x is said to be *stationary* if $E'(0)=0$ for any volume preserving admissible variation of x. A standard variational argument shows that the first variation formula for the energy is

$$E'(0)=\int_M (-nH+Y+\lambda)\,\langle N,\xi\rangle\;\mathrm{d}M+\int_{\partial M}\langle \xi,\nu\rangle\,(\langle N,N_\Pi\rangle-\cos\beta)\;\mathrm{d}s.$$

Here N is a unit normal vector field along x, $\mathrm{d}s$ the volume element of ∂M induced by x, ν is the unit inward normal along ∂M, H is the mean curvature of x and λ is a Lagrange multiplier arising from the volume constraint. It follows that

PROPOSITION 2.1. *Let Π be a support hypersurface of $\mathbb{H}^{n+1}$ and let M be a compact hypersurface. Let us consider $x: M\to\mathbb{H}^{n+1}$ a smooth immersion such that $x(int(M))\subset\mathbb{H}^{n+1}_+$ and $x(\partial M)\subset\Pi$. Then x is stationary if and only if*

(1) *The mean curvature H satisfies the relation*

$$nH(p)=Y(p)+\lambda,\quad p\in M.$$

(2) *The hypersurface $\mathcal{S}=x(M)$ meets the support hypersurface Π in a constant angle β, that is, $\cos\beta=\langle N,N_\Pi\rangle$ along ∂M.*

On the other hand, when we talk of contact angle, it is implicitly assumed that the boundary regularity of M is enough to ensure that the idea of a normal to M at every boundary point makes sense. For this we will require M to be a sufficiently smooth hypersurface up to the boundary ∂M. When $Y=0$ on M, then x is an immersion of constant mean curvature. In this paper, our interest will center on the case for which the vector field $Y(p)$ depends on the distance to Π, as for example, a uniform gravitational potential directed towards Π, namely,

$$Y(p)=\kappa\;\mathrm{dist}(p,\Pi)+\lambda,$$

for constants κ and λ.

The key ingredient in our proofs is the *Tangency Principle* for the mean curvature equation. This is a consequence of the maximum principle of linear elliptic equations. Briefly, we recall in the hyperbolic context. In our model of the hyperbolic space $\mathbb{H}^{n+1}$, we must distinguish the case that M is a graph on a horizontal or vertical (Euclidean) hyperplane. Consider the first situation. Locally, M writes as $x_{n+1}=u(x_1,\ldots,x_n)$, where u is a smooth function in some domain Ω of $\mathbb{R}^n$. Considering this coordinate system and with respect to the unit upper normal vector, the mean curvature $H=H(x,u(x))$ of M satisfies

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=\frac{n}{u}\left(H-\frac{1}{\sqrt{1+|\nabla u|^2}}\right)\quad x=(x_1,\ldots,x_n) \tag{1}$$

on Ω. Assume that we have two smooth functions u_1 and u_2 defined on Ω whose mean curvature functions H_i of their graphs satisfy the inequality $H_1(x, u_1(x)) \leqslant H_2(x, u_2(x))$. A standard argument using the chain rule shows that the difference function $w = u_1 - u_2$ satisfies an inequality of type $Lw \leqslant 0$, where L is a locally uniformly elliptic operator. Then we are in position to apply Hopf's maximum principle to w, whether in its classical formulation [6] or its boundary point version [7]: see also [4, Ch. 3]. Consequently, we have the following result:

THEOREM 2.2 (Maximum Principle). *Let u_1 and u_2 be two smooth functions on a domain $\Omega \subset \mathbb{R}^n$ such that $H_1 \leqslant H_2$. Suppose that $u_1 \geqslant u_2$ on Ω and $u_1(x_0) = u_2(x_0)$, $x_0 \in \Omega$. Then $u_1(x) = u_2(x)$ on Ω. The same holds if $x_0 \in \partial\Omega$ with the extra hypothesis that $\partial u/\partial\nu = \partial v/\partial\nu$ at x_0, where ν is the outward unit normal to $\partial\Omega$.*

In the case that M is a graph on a vertical Euclidean hyperplane, for instance, $x_1 = u(x_2, \ldots, x_{n+1})$, the mean curvature $H(x, u(x))$ in this coordinate system satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{n}{x_{n+1}}\left(H + \frac{\partial u/\partial x_{n+1}}{\sqrt{1+|\nabla u|^2}}\right).$$

A similar reasoning leads to the corresponding Maximum Principle throughout the x_1-direction. We translate geometrically the maximum principle in terms of hypersurfaces. Let M_1 and M_2 be two orientable hypersurfaces in $\mathbb{H}^{n+1}$ which are tangent at a point p and whose unit normal vectors at p agree. Locally around p, let us write M_i, $i = 1, 2$ as graphs of smooth functions u_i (either in the horizontal or vertical coordinate system). If $u_1 \geqslant u_2$ in a neighbourhood of p, then we say that M_1 lies above M_2 in a neighbourhood of p. Then the Maximum Principle implies:

THEOREM 2.3 (Tangency Principle). *Let M_1 and M_2 be two orientable hypersurfaces in $\mathbb{H}^{n+1}$ with mean curvature H_1 and H_2 respectively. Suppose that $H_1 \leqslant H_2$. If M_1 and M_2 have a common tangent (interior or boundary) point p and M_1 lies above M_2 in a neighbourhood of p, then M_1 and M_2 agree in an open set that involves p in its interior.*

3. Proof of Theorem 1.2

The method of proof used in this work is the Alexandrov reflection method. Such a technique has been used in a variety of situations in differential geometry. See also the called "method of moving planes" in the context of the theory of partial differential equations, beginning with the classical works [3, 9]. In our situation, we first show

Claim. For each tangent vector v to Π, there exists a totally geodesic hyperplane (briefly hyperplane) P_v orthogonal to v and Π, such that M is invariant by the hyperbolic reflection through to P_v.

Once showed the Claim, the proof of Theorem 1.2 continues as follows. Let us consider $p_0 \in \Pi$ and tangent vectors $v_1, \ldots, v_n$ to Π at p_0 and orthogonal themselves. Let $P_{v_1}, \ldots, P_{v_n}$ be the totally geodesic hyperplanes given by the Claim. Since M is a bounded set, one concludes that these hyperplanes define an orthogonal geodesic $\gamma = P_{v_1} \cap \cdots \cap P_{v_n}$ to Π, $p_0 \in \gamma$, such that the hyperbolic reflection of M across P_{v_i} keeps invariant M. As a consequence, M is axially symmetric with respect to the axis γ. Moreover, the proof will show that P_{v_i} divides M into two sets, namely, M_i^+, M_i^-, such that each one of them is a graph over the domain $M \cap P_{v_i}$, $1 \leqslant i \leqslant n$.

We proceed to show the Claim. We distinguish each one of the types of support hypersurfaces. We do the details of proof in the first case, and briefly, the corresponding modifications in the other ones.

Case 1: Π is a totally geodesic hyperplane. After an ambient isometry, we can assume that Π is a hemisphere in $\mathbb{R}^{n+1}_+$ whose boundary lies in P_∞. The boundary ∂M intersects Π in a finite number of disjoint, connected, compact embedded submanifolds Γ_i of Π. Then it is possible to attach to M a finite number of bounded domains $D_j \subset \Pi$ such that $D := \cup_j D_j = \partial M$ and $G = D \cup M$ is an n-dimensional compact connected topological submanifold without boundary of $\mathbb{R}^{n+1}_+$. Hence G is orientable and Alexander duality [5] implies that G separates $\mathbb{R}^{n+1}_+$ into two closed components with common boundary G. Let W be the bounded component, called the "inside" of G. By hypothesis, G is included in the domain $\mathbb{H}^{n+1}_+$.

Let $p_0 \in D$ and let v be a unit tangent vector to Π at p_0. Let us denote $\gamma = \gamma(t)$ the unit speed geodesic included in Π with $\gamma(0) = p_0$ and velocity v. In particular, γ intersects ∂M for positive and negative values of t. This geodesic γ determines a one-parameter family of hyperplanes $P(t)$, $t \in \mathbb{R}$, such that $P(t)$ intersects γ orthogonally at $\gamma(t)$. See Figure 3. Moreover, the set of all $P(t)$ is a foliation of $\mathbb{H}^{n+1}$. For each t, we say the right side of $P(t)$ the component of $\mathbb{H}^{n+1} \backslash P(t)$ that contains $\gamma(s)$, for $s > t$. Since M is compact, for t near ∞, $P(t)$ does not intersect M. Letting $t \searrow -\infty$, one gets a first hyperplane $P(t_1)$ that reaches M, that is, $P(t_1) \cap M \neq \emptyset$ but $P(t) \cap M = \emptyset$ if $t > t_1$. Now, when we decrease t and for values close to t_1, the (closed) part of M on the right of $P(t)$, which we denote by $M(t)^+$, is a graph over a domain of $P(t)$.

Let $M(t)^*$ be the hyperbolic reflection of $M(t)^+$ through $P(t)$, which is contained in W. Since each $P(t)$ is orthogonal to Π, the hyperbolic reflection about $P(t)$ leaves Π and $\mathbb{H}^{n+1}_+$ invariant. Because the hyperbolic reflection is an isometry of $\mathbb{H}^{n+1}$, the mean curvature remains invariant by the reflection. Further, the mean curvature vector of $M(t)^*$ is the reflection of the mean curvature vector of $M(t)^+$. By the fact that the mean curvature of M depends only on the distance to Π, the value of the mean curvature agrees in those points at the same height. We now continue $t \searrow -\infty$, and reflecting $M(t)^+$ about $P(t)$, successively until one

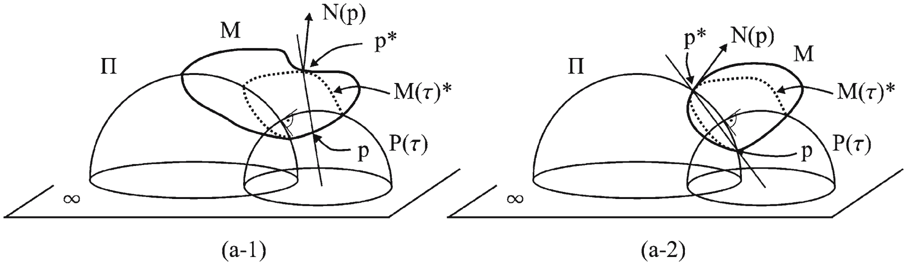

Figure 1. Case (a-1): p^* is an interior point; Case (a-2): p^* is a boundary point.

again reaches the first point of contact between $M(t)^*$ with the left side of M in respect to $P(t)$, which will be denoted by $M(t)^-$. This occurs because M is bounded, $P(0)\cap M\neq\emptyset$ and for t near $-\infty$, $P(t)\cap M=\emptyset$. Consider the first hyperplane $P(\tau)$ where one of the following conditions fails to hold:

(a) $\mathrm{int}(M(\tau)^*)\subset W$.
(b) $M(\tau)^+$ is a graph over a part of $P(\tau)$ and no point of $M(\tau)^+$ has a tangent hyperplane orthogonal to Π.

If (a) fails first, we have that M and $M(\tau)^*$ touch at some interior point p^*: see Figure 1 (a-1), or at a boundary point p^*, with $p^*\in\partial M\cap\partial M(\tau)^*$: see Figure 1(a-2). Here p^* is the reflection of a point p of $M(\tau)^+$. Either if p^* is an interior or boundary point, the tangent hyperplanes of $M(\tau)^-$ and $M(\tau)^*$ agree at p. In the second possibility, this fact is due to that the angle β of contact is constant along ∂M, condition (2) of Proposition 2.1. Because $P(\tau)$ keeps invariant Π, we have dist $(p,\Pi)=\mathrm{dist}(p^*,\Pi)$ and, by the hypothesis on the mean curvature of M, $H(p)=H(p^*)$. Moreover, the Gauss maps of both $M(\tau)^*$ and $M(\tau)^-$ at such point p^* are the same. Then one applies the Tangency Principle to $M(\tau)^-$ and $M(\tau)^*$ at the point where they touch to conclude that $P(\tau)$ is a hyperplane of symmetry of M.

If (b) fails first, then we have that there exists a point $p\in\partial M(\tau)^+\subset P(\tau)$ where: either the tangent hyperplane of $M(\tau)^+$ becomes orthogonal to $P(\tau)$, Figure 2(b-1); or $p\in\partial M\cap P(\tau)$, Figure 2 (b-2). In both cases, the reflection p^* agrees with the point p. In the former possibility, one can apply the Boundary Tangency Principle to $M(\tau)^*$ and $M(\tau)^-$ to conclude that $P(\tau)$ is a hyperplane of the symmetry of M; in the second one, the corresponding tangent hyperplanes of $M(\tau)^-$ and $M(\tau)^*$ are identical because the angle of contact between M and Π is constant along ∂M. Then one applies the Tangency Principle at a corner point (see details in [9]) obtaining that $P(\tau)$ is a hyperplane of the symmetry of M.

The hyperplane $P(\tau)$ is the hyperplane P_v that we are looking for in the Claim. This completes the proof of the Claim. In the rest of the cases, we only present the necessary modifications to do.

Case 2: Π is an equidistant hypersurface. We change γ by an equidistant curve $\sigma=\sigma(t)$ included in Π through p_0 at velocity v. Consider the 1-parameter family

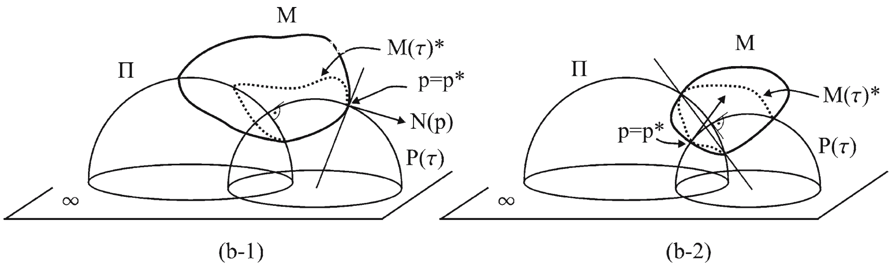

Figure 2. Case (b-1): $p^* = p \in \partial M(\tau)^+$ is an interior point; Case (b-2): $p^* = p \in \partial M \cap P(\tau)$.

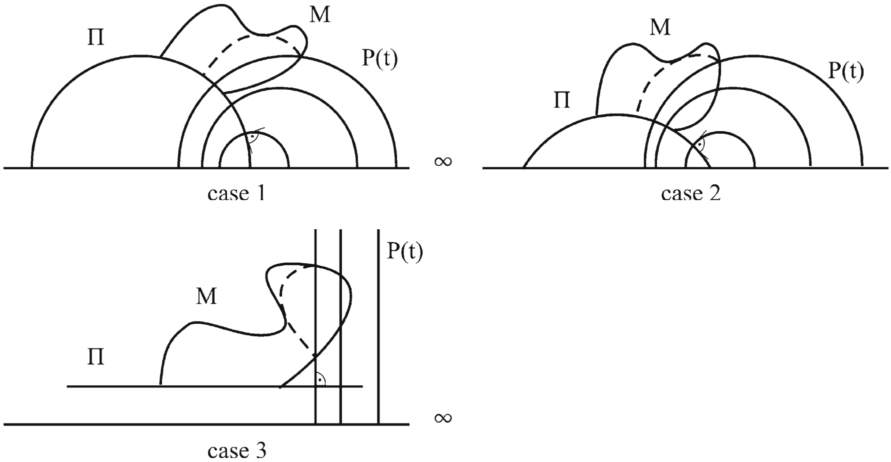

Figure 3. The Alexandrov reflection method in the cases that Π is: (1) a geodesic hyperplane; (2) an equidistant hypersurface and; (3) a horosphere.

of totally geodesic hyperplanes with $p_0 \in P(0)$ and $P(t)$ cutting Π orthogonally at $\sigma(t)$. See Figure 3.

Case 3: Π is a horosphere. Without loss of generality, we assume that Π is a horizontal hyperplane. We choose the horocycle $\varphi = \varphi(t)$ contained in Π with $\varphi(0) = p_0$ and $\varphi'(0) = v$. See Figure 3.

In both cases, the key of the proof is that the hyperbolic reflections across the hyperplanes $P(t)$ keep Π and $\mathbb{H}^{n+1}_+$ invariant. Thus, in the Alexandrov method, the reflection of $M(t)^+$, $M(t)^*$, remains in the very component $\mathbb{H}^{n+1}_+$, as it occurs with $M(t)^-$. In this way, one can do reflections until to reach M again at the first contact point.

This completes the proof of Theorem 1.2

4. Liquid Bridges between two Parallel Supports

The same reflection technique can be used in other configurations. For example, liquid bridges interconnecting a set of support hypersurfaces. The physical

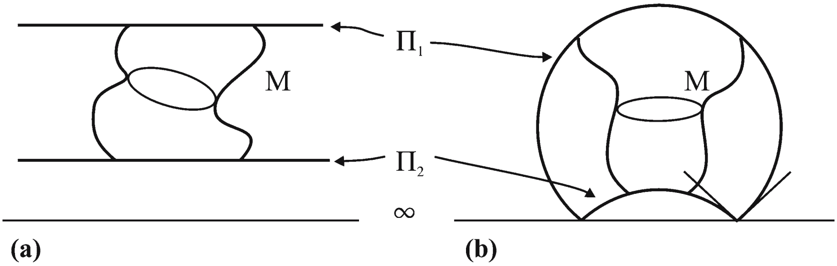

Figure 4. Liquid bridges trapped between: (a) two parallel horospheres; (b) two parallel equidistant hypersurfaces.

situation that we will consider is an amount of liquid drop trapped between two homogeneous 'parallel' support hypersurfaces Π_1 and Π_2. By two parallel support hypersurfaces we mean that Π_1 and Π_2 are two umbilical hypersurfaces with mean curvatures $|H_1| = |H_2|$ and the same, non-empty boundary at infinity, $\partial_\infty \Pi_1 = \partial_\infty \Pi_2 \neq \emptyset$. The cases that appear are (see Figure 4):

(1) Two horospheres with the same point at the infinity. In such a case, and after an isometry of the ambient, Π_1 and Π_2 are two horizontal hyperplanes of $\mathbb{R}^{n+1}_+$.
(2) Two equidistant hypersurfaces. In the upper halfspace model, and up an isometry of $\mathbb{H}^{n+1}$, Π_1 and Π_2 are two (different) spherical caps in $\mathbb{R}^{n+1}_+$, with the same Euclidean radius and such that $\partial_\infty \Pi_1 = \partial_\infty \Pi_2$ is a $(n-1)$-sphere of P_∞.

Given two parallel support hypersurfaces, we call the slab determined by Π_1 and Π_2 the component of $\mathbb{H}^{n+1} \backslash (\Pi_1 \cup \Pi_2)$ whose boundary is $\Pi_1 \cup \Pi_2$. In this setting, we consider embedded compact hypersurfaces M included in the slab defined by Π_1 and Π_2 and whose boundary satisfies $\partial M \cap \Pi_i \neq \emptyset$, $i = 1, 2$. We say then that M is a *liquid bridge* interconnecting Π_1 and Π_2. In such case, the term S in the energy functional E is the area of the domains that the bridge wets in each one of the support hypersurfaces. Again, in a state of equilibrium, the angle between the normal to the liquid bridge and the normal to Π_i along their contact is constant (and possibly different in each support Π_i). The Alexandrov reflection method again yields the following

THEOREM 4.1 *Let Π_1 and Π_2 be two parallel support hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$. Consider M an embedded compact hypersurface in $\mathbb{H}^{n+1}$ included in the slab determined by Π_1 and Π_2. Assume that the mean curvature of M depends only on the distance to Π_i and that the angle of contact between M and Π_i is constant along ∂M in each one of the two support hypersurfaces. Then M is rotationally symmetric with respect to a geodesic orthogonal to Π_i.*

Proof. The argument is similar as in Theorem 1.2. Only it suffices to point out some remarks. Assume, for instance, that Π_i is two parallel horospheres. After an isometry of the ambient, we see Π_i as two horizontal hyperplanes. First, we can attach to M a finite number of bounded domains $D_j \subset \Pi_1 \cup \Pi_2$ such that the boundary of $D := \cup_j D_j$ satisfies $\partial D = \partial M$ and $G = D \cup M$ is a n-dimensional compact connected topological submanifold without boundary of $\mathbb{R}^{n+1}_+$. Again G divides $\mathbb{R}^{n+1}_+$ into two components defining an "inside" W (necessarily included in the slab determined by Π_1 and Π_2).

On the other hand, we fix a point $p_0 \in D$ with $p_0 \in \Pi_1$. For each tangent vector v to D at p_0, we consider the horocycle φ that through p_0 at $t = 0$ with velocity v. The one-parameter family of geodesic hyperplanes $P(t)$ orthogonal to φ at t are vertical parallel hyperplanes. The fact that Π_1 and Π_2 are parallel supports implies that each $P(t)$ is also orthogonal to Π_2. Moreover, the hyperbolic reflections across $P(t)$ leaves invariant both Π_i as well as the slab. Here, and in the upper halfspace model for $\mathbb{H}^{n+1}$, a hyperbolic reflection is a Euclidean reflection across $P(t)$. Now the proof follows the same steps as in Theorem 1.2.

When Π_1 and Π_2 are two parallel equidistant hypersurfaces, we take an equidistant curve $\sigma \subset \Pi_1$ instead of φ. Again, if $P(t)$ is a totally geodesic hyperplane orthogonal to Π_1, then it is also orthogonal to Π_2. Moreover, the hyperbolic reflection through $P(t)$ leaves Π_i and the slab invariant. $\square$

As the conclusion of Theorem 4.1,

Consider a liquid bridge in $\mathbb{H}^{n+1}$ trapped between two parallel support hypersurfaces. Assume the existence of a uniform gravity field directed towards the supports. If the liquid bridge reaches a state of mechanical equilibrium, then it is rotationally symmetric with respect to a geodesic orthogonal to both support hypersurfaces.

5. Other Configurations of Stationary Hypersurfaces

The Alexandrow technique allows us to obtain other results of symmetry. The next theorem says that, under certain conditions, an embedded compact hypersurface of $\mathbb{H}^{n+1}$ inherits the symmetry of its boundary.

THEOREM 5.1. *Let Π be a support hypersurface in $\mathbb{H}^{n+1}$ and let us consider Γ an $(n-1)$-dimensional submanifold of $\mathbb{H}^{n+1}$ included in Π. Assume the following two conditions:*

(1) *There exists a totally geodesic hyperplane P orthogonal to Π such that Γ decomposes into $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^+ \subset P^+$, $\Gamma^- \subset P^-$, being P^+ and P^- the two components of $\mathbb{H}^{n+1} \backslash P$ and such that Γ^- is the hyperbolic reflection of Γ^+ about P.*
(2) *There exists a bounded domain Ω in $P \cap \Pi$ and a nonnegative smooth function f defined on Ω such that f is positive in Ω, identically zero on $\partial\Omega$ and Γ^+ is the graph of f.*

Let M be an embedded compact hypersurface in $\mathbb{H}^{n+1}_+$ whose boundary is Γ. If the mean curvature of M depends only on the distance to Π, then M is invariant by the hyperbolic reflection through P.

Proof. The proof follows the same steps as in Theorem 1.2. We use the same notation. Denote again D the bounded domain of Π enclosed by Γ, and let W the bounded domain in $\mathbb{H}^{n+1}$ defined by $M \cup D$. For simplicity, we only consider the case that Π is a geodesic hyperplane. Let us take γ a geodesic in Π that across orthogonally the hyperplane P. Let $P(t)$ be the one-parameter family of geodesic hyperplanes orthogonal to γ at t and parametrized so that $P(0)=P$.

Again and for large t, $P(s)\cap M=\emptyset$ for $s>t$. Now decrease t, $t\searrow 0$, and we arrive at the first time t_1 such that $P(t_1)$ touches M. From here, we continue decreasing t and doing the reflection through $P(t)$ of $M(t)^+$. The embeddness property of M implies the existence of $\epsilon>0$ such that for $t\in(t_1-\epsilon,t_1)$, the hyperbolic reflection of $M(t)^+$ lies in W. We continue with $t\searrow 0$ until the value τ so that $M(\tau)^*$ reaches again with the very hypersurface M, that is, with $M(\tau)^-$. Recall that the hyperbolic reflection keeps Π invariant. As in Theorem 1.2, either (a) or (b) holds. By hypothesis and because $P(0)=P$ is a hyperplane of symmetry of Γ, the number τ satisfies $\tau\geqslant 0$.

Claim. $\tau=0$.

By contradiction, we suppose that $\tau>0$. When (a) fails, the case (a-2) that p^* is a boundary point cannot occur since $\Gamma^-\subset\Pi$ is the reflection of Γ^+ through $P(0)$. Thus p^* must be an interior point. Then the Tangency Principle says that $P(\tau)$ is a hyperplane of symmetry, which is false by our assumptions about Γ and the fact that $\tau>0$. If (b) holds, the case (b-2) $p\in\partial M\cap P(\tau)$ is impossible again. Then $p\in\partial M(\tau)^+\subset P(\tau)$ where the tangent hyperplane of $M(\tau)^+$ at p becomes orthogonal to $P(\tau)$. The Tangency Principle, in its boundary version, implies that $P(\tau)$ is a hyperplane of symmetry of M, in particular, of Γ. Since $\tau>0$, this gives a contradiction. Hence, τ cannot be positive and must be zero. This shows the Claim.

Therefore the hyperbolic reflection $M(0)^*$ of $M(0)^+$ through $P(0)=P$ lies in the left side of $P(0)$ and is included in the domain W. Now, we make a similar reasoning but with hyperplanes coming from $-\infty$. The same argument says that the reflection of $M(0)^-$ with respect to the hyperplane P lies in W. This proves that P is a hyperplane of the symmetry of M. □

As a consequence of Theorem 5.1, we consider the case that the boundary of M is a round sphere.

COROLLARY 5.2. *Let $\mathbb{S}^{n-1}$ be a round sphere in a support hypersurface Π. Let M be an embedded compact hypersurface with $M\subset\mathbb{H}^{n+1}_+$ and whose boundary is $\mathbb{S}^{n-1}$. If the mean curvature of M depends only on the distance to Π, then M is axially symmetric with respect to a geodesic orthogonal to Π.*

For liquid bridges, we have a similar results as in Theorem 4.1. Without proof, we state

THEOREM 5.3. *Consider* Π_1 *and* Π_2 *two parallel support hypersurfaces and let* $\Gamma = \Gamma_1 \cup \Gamma_2$ *be an* $(n-1)$*-dimensional submanifold of* $\mathbb{H}^{n+1}$*, with* $\Gamma_i \subset \Pi_i$*,* $i = 1, 2$*. Suppose that*

(1) *There exists a totally geodesic hyperplane* P *orthogonal to* Π_i *such that* Γ *decomposes into* $\Gamma = \Gamma^+ \cup \Gamma^-$ *where* $\Gamma^+ \subset P^+$*,* $\Gamma^- \subset P^-$*, being* P^+ *and* P^- *the two components of* $\mathbb{H}^{n+1} \backslash P$ *and such that* Γ^- *is the reflection of* Γ^+ *about* P*.*
(2) *There exists two bounded domains* $\Omega_i \subset P \cap \Pi_i$ *and two nonnegative smooth function* f_i *defined on* Ω_i *such that* f_i *is positive in* Ω_i*, identically zero on* $\partial\Omega_i$ *and* Γ_i^+ *is the graph of* f_i*.*

Let M *be an embedded compact hypersurface in the slab determined by* Π_1 *and* Π_2 *whose boundary is* Γ*. If the mean curvature depends only on the distance to* Π*, then* M *is symmetric with respect to* P*. In the particular case that* Γ *is the union of two coaxial* $(n-1)$*-spheres, the hypersurface* M *is rotationally symmetric with respect to the geodesic acrossing the centers of* Γ_i*.*

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