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A characterization of hemispheres

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Abstract

We prove that for constant contact angle $\gamma = 0$, a capillary surface over a convex domain has no umbilical points unless that the surface is a hemisphere. The method involves the comparison of a lower hemisphere with the given surface at a second-ordered contact point and it is based on an argument of Alexandrov.

MSC: 53A10; 35J65

Keywords: Umbilical point; Capillary surface; Maximum principle

1. Introduction and statement of the result

We consider the family $\mathcal{F}$ of compact surfaces immersed in Euclidean 3-space $\mathbb{R}^3$, with constant mean curvature H. In the minimal case, that is, $H = 0$, the maximum principle assures that the surface must be contained in the convex hull of its boundary. In what follow, we suppose $H \neq 0$. In the family $\mathcal{F}$, the hemispheres have radius $\frac{1}{|H|}$ and their boundary is a circle of radius $1/|H|$. In the class $\mathcal{F}$, hemispheres have been characterized in several ways:

(1) They are the only embedded surfaces of $\mathcal{F}$ whose boundary is a convex curve and the surface is perpendicular to the plane along its boundary. This is a consequence of the reflection method of Alexandrov [1,4,17].
(2) Hemispheres are the only surfaces of $\mathcal{F}$ bounded by a circle of radius $1/|H|$ [3].
(3) In 1969, E. Heinz [13] showed that if Γ is a Jordan planar curve with length L bounding a domain of area A, any surface with constant mean curvature H having Γ as its boundary satisfies $|H| \leqslant L/(2A)$. In this context, hemispheres are the only surfaces of $\mathcal{F}$ of disc type and where the equality $|H| = L/2A$ occurs [16].
(4) Hemispheres are the only stable surfaces of $\mathcal{F}$ with free boundary in a plane [15].

In this note we consider capillary surfaces in the family $\mathcal{F}$. The physical interpretation of a capillary surface is the following: we put a liquid into a vertical cylindrical container with arbitrary cross section Ω and transport it to

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the space in absence of gravity. We consider smooth the shape of the surface $\mathcal{S}$ of the liquid and expressed non-parametrically as the graph of a function u defined over the cross section Ω. In an equilibrium state of the liquid, as a consequence of the least action principle of physics, the shape of the surface $\mathcal{S}$ is obtained when u minimizes the energy functional

$$E(u) = \sigma \int_{\Omega} \sqrt{1+|\nabla u|^2}\, \mathrm{d}x_1\, \mathrm{d}x_2 - \sigma\lambda \int_{\partial\Omega} u\, \mathrm{d}s,$$

under the volume constraint

$$\int_{\Omega} u\, \mathrm{d}x_1\, \mathrm{d}x_2 = \text{constant},$$

where $\mathrm{d}s$ is the arc length measure on $\partial\Omega$ and ∇u is the gradient of u. The constant σ is the surface tension and λ is a constant to be determined by the liquid and the (homogeneous) material of the wall. The equilibrium condition $\delta E(u) = 0$ under the volume constraint is given by the Euler condition

$$\operatorname{div} \mathsf{T}u = 2H \quad \text{in } \Omega, \tag{1}$$

where

$$\mathsf{T}u = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}},$$

H is a constant that coincides with the mean curvature of the capillary free surface $u = u(x_1, x_2)$ and together the physical boundary condition

$$\mathsf{T}u \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega, \tag{2}$$

which consists in prescribing the contact angle γ with which $\mathcal{S}$ meets the walls of the tube. Here ν is the exterior directed unit normal to $\partial\Omega$ and the angle γ is measured inside the fluid. For further physical and geometrical background information, see [9]. We may normalize so that H is positive and $0 \leqslant \gamma \leqslant \pi/2$: we suppose that done along this paper.

It should be noted here that the constant H cannot be prescribed; it is implicitly determined by the size of Ω, as it follows by applying the divergence theorem to (1)–(2):

$$H = \frac{|\partial\Omega|}{2|\Omega|} \cos\gamma,$$

where $|\Omega|$, $|\partial\Omega|$ denote the measure of Ω and $\partial\Omega$ respectively. It is well known that Eq. (1) is an elliptic equation of quasilinear type satisfying a maximum principle.

It was observed by Bernstein [2] that Ω cannot strictly contain a disc of radius $1/H$. In fact, an integration of (1) over a closed disc D_r of radius r and included in Ω yields $r < 1/H$. Even more, Finn [8] found a new characterization of hemispheres as follows

Theorem 1. *If Ω contains the open disc $D_{1/H}$ and $u = u(x_1, x_2)$ is a solution of (1), then Ω coincides with this disc and $u(x_1, x_2)$ describes a hemisphere of radius $1/H$.*

A point p in a surface is called an umbilical point if the two principal curvatures at p are equal. Using isothermic parameters and the Codazzi equations, one can show that the umbilical points of a surface are the set of zeroes of a differential 2-form known as Hopf differential. But in a surface with constant mean curvature in Euclidean space, this differential form is holomorphic [14, p. 137]. Thus, umbilical points are isolated unless that the surface is a subset of a totally umbilical immersion. Our aim is to prove that the number of umbilical points of a capillary surface with contact angle $\gamma = 0$ in a convex cylinder is zero, unless the surface is a hemisphere. More precisely, we have:

Theorem 2. *Let $u = u(x_1, x_2)$ be a solution of* $\operatorname{div} \mathsf{T}u = 2H$ *in Ω with the boundary condition*

$$\mathsf{T}u \cdot \nu = 1 \quad \textit{on } \partial\Omega, \tag{3}$$

where Ω is a bounded convex domain of the plane, ν denotes the unit outer normal vector on the boundary $\partial\Omega$. Then the solution surface $u = u(x_1, x_2)$ cannot have umbilical points unless that u describes a lower hemisphere $u(x) = u_0 - \sqrt{\frac{1}{H^2} - |x|^2}$, $x = (x_1, x_2)$.

The proof of Theorem 2 follows from the comparison of a hemisphere with the given surface at a second-ordered contact point. This procedure is an argument due to Alexandrov [1] and that it has been used in various contexts by a number of authors. We cite three of them: a priori gradient bounds for the solutions of the constant mean curvature equation are obtained by Finn and Giusti comparing with moon surfaces [12]; by using half-cylinders, Chen and Huang proved that for $\gamma = 0$, capillary surfaces in absence of gravity over convex domains are necessarily convex [5]; Finn used nodoids to obtain an estimate for the Gaussian curvature of a nonparametric surface of constant mean curvature [10].

2. The proof of the result

Our proof is inspired by [5]. As we noticed in the Introduction, Chen and Huang use half-cylinders as comparison surfaces, while we use half-spheres. Without loss of generality, we may identify the (x_1, x_2)-coordinates with the plane $\Pi = \{(x_1, x_2, x_3); x_3 = 0\}$ in Euclidean 3-space $\mathbb{R}^3$. Denote

$$\mathcal{S} = \{(x_1, x_2, u(x_1, x_2)); (x_1, x_2) \in \Omega\}$$

the graph of a solution $u = u(x_1, x_2)$ of (1)–(3). With our assumptions, the mean curvature H is computed with respect to the upwards unit normal vector $N = (-\nabla u, 1)/\sqrt{1 + |\nabla u|^2}$.

Assume that the surface $\mathcal{S}$ has umbilical points and let $p_0 = (x_0, u(x_0))$ be an umbilical point of the graph $\mathcal{S}$. We assume without loss of generality that $x_0 = (0, 0)$. We construct the comparison surface

$$v = v(x_1, x_2) = -\sqrt{\frac{1}{H^2} - x_1^2 - x_2^2}, \qquad x_1^2 + x_2^2 < \frac{1}{H^2},$$

that is, a lower hemisphere $\mathcal{H}$ of the same mean curvature H as $\mathcal{S}$. Thus, it is possible to move $\mathcal{H}$ by horizontal and vertical displacements until to put it in such a way that $\mathcal{H}$ and $\mathcal{S}$ are tangent at the point p_0. In particular, both surfaces $\mathcal{S}$ and $\mathcal{H}$ have at the point p_0 equal mean and Gaussian curvature. Thus the two surfaces have a contact of (at least) second order at p_0.

Denote Ω' the disc of radius $1/H$ obtained by the orthogonal projection of $\mathcal{H}$ onto Π. Theorem 1 stated in Section 1 says us that if Ω contains the open disc Ω', then $\Omega = \Omega'$ and $u = v$. In this case, Theorem 2 is proved. Assume now $\partial\Omega' \cap (\Pi \setminus \overline{\Omega}) \neq \emptyset$. Let D be the component of $\Omega \cap \Omega'$ that contains the origin. In particular, the convexity of Ω implies that the boundary of D contains at most two pieces of circular arcs of $\partial\Omega'$.

In the domain D we define the difference function $w = u - v$. The function w satisfies a quasilinear elliptic second order partial differential equation without zero-order term and that it has a specified type of contact at a given point where w vanishes. For this, let $\mathcal{Z}$ be the zero set of w. We need to control the local behavior of $\mathcal{Z}$ around the origin $(0, 0)$. The special properties of w were used by Alexandrov [1] to prove that the spheres are the only embedded closed surfaces with constant mean curvature (see also, [14, Chapter VII]). In our context, since $\mathcal{S}$ and $\mathcal{H}$ have second order contact at p_0, the set $\mathcal{Z}$ is a union of piecewise smooth arcs intersecting at x_0 and it divides a neighborhood $\mathcal{U}$ of x_0 into at least six components sharing x_0 as common boundary point, in which the signs of w alternate. For a proof, see e.g. [5,7,10].

A component of $\overline{D} \setminus \mathcal{Z}$ that contains a component of $\mathcal{U} \cap (D \setminus \mathcal{Z})$ must intersect ∂D because, in otherwise we have two functions, namely w and 0, defined in some domain, both satisfy the Dirichlet problem (1)–(3), violating the uniqueness of such problem. In the same way, two distinct components of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ cannot be included in the same component of $\overline{D} \setminus \mathcal{Z}$.

Thus there are at least six arcs of ∂D where w alternates sign. See Fig. 1. If $\Gamma' \subset \partial D$ is an open circular arc of $\partial\Omega'$, then

$$1 = \mathsf{T}v \cdot \nu \geqslant \mathsf{T}u \cdot \nu. \tag{4}$$

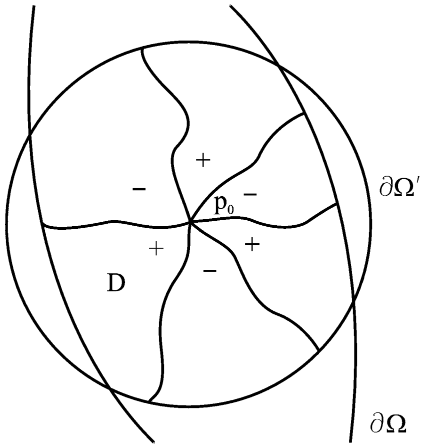

Fig. 1. Proof of Theorem 2; the function w alternates the sign in at least six domains emanating from the contact point x_0.

Claim 1. *Let $\mathcal{K}$ be a component of $\overline{D} \setminus \mathcal{Z}$ where $w > 0$. Then we have that $\mathcal{K}$ must intersect $\partial D \cap \partial \Omega$.*

For the proof of the claim, we assume, by contradiction that $\mathcal{K}$ meets ∂D only in $\partial\Omega'$. We decompose $\partial\mathcal{K} = \Gamma \cup \Gamma'$, where $\Gamma' \subset \partial\Omega'$ and $\Gamma \subset D$. Then the divergence theorem yields

$$0 = \int_{\mathcal{K}} (\operatorname{div} \mathsf{T}u - \operatorname{div} \mathsf{T}v) = \oint_{\partial\mathcal{K}} (\mathsf{T}u - \mathsf{T}v) \cdot \nu$$
$$= \int_{\Gamma} (\mathsf{T}u - \mathsf{T}v) \cdot \nu + \int_{\Gamma'} (\mathsf{T}u - \mathsf{T}v) \cdot \nu, \tag{5}$$

where ν is the outer normal on the boundary $\partial\mathcal{K}$. By (4), the second summand in (5) is nonpositive. Moreover using the Schwarz inequality, we always have

$$(\nabla u - \nabla v) \cdot (\mathsf{T}u - \mathsf{T}v) \geqslant 0 \tag{6}$$

and equality holds if and only if $\nabla u = \nabla v$. As $u > v$ on $\mathcal{K}$, we infer $(\nabla u - \nabla v) \cdot \nu \leqslant 0$ along Γ. Thus, from (6) we obtain $(\mathsf{T}u - \mathsf{T}v) \cdot \nu \leqslant 0$ along Γ. In virtue of (5), this implies $(\mathsf{T}u - \mathsf{T}v) \cdot \nu = 0$ on $\partial\mathcal{K}$, and so, $\nabla u = \nabla v$ in $\mathcal{K}$. The uniqueness of the Neumann problem of (1) in $\mathcal{K}$ asserts $u = v$ in $\mathcal{K}$: a contradiction. Then, the claim is proved.

In Claim 1 we have used that $\mathsf{T}v \cdot \nu = 1$ on $\partial\Omega'$. In the same way, the boundary condition (3) for the solution u in arcs of $\partial\Omega$ allows one to show that

Claim 2. *Any component $\mathcal{K}$ of $\overline{D} \setminus \mathcal{Z}$ where $w < 0$ must meet $\partial D \cap \partial\Omega'$.*

We are now in position to end with the proof of Theorem 2. As conclusion of the above reasonings, we have at least six arcs of ∂D where w alternates sign. Moreover, they must alternatively distribute along the (at most) four arcs of ∂D. Hence, there are at least two consecutive arcs of $\partial(\overline{D} \setminus \mathcal{Z}) \cap \partial D$ lying only in $\partial\Omega$ or in $\partial\Omega'$. This is a contradiction with the statements of the two claims (for example, in Fig. 1, one of the (−) regions leads to a contradiction). Hence there is no umbilical point on the surface.

Remark 1. One can also argue using the maximum principle proved by Concus and Finn [6]: we have u and v two functions in $C^2(\Omega)$ such that $\operatorname{div} \mathsf{T}u = \operatorname{div} \mathsf{T}v$ in Ω and $\mathsf{T}u \cdot \nu \geqslant \mathsf{T}v \cdot \nu$ on $\partial\Omega$. Then $u = v + \text{constant}$ in Ω.

Remark 2. There are capillary surfaces for $\gamma \neq 0$ in the boundary data (3) and without umbilical points, provided the domain is not convex. In the class of Delaunay surfaces i.e. rotational, constant mean curvature surfaces, we consider the nodoids: there exist embedded pieces of nodoids that are graphs on non-convex domains, such that the value of $\mathsf{T}u \cdot \nu$ is ± 1 along the boundary of the domain (see [10,11]). The Gaussian curvature of such pieces of nodoids is negative and so, there are no umbilical points. See Fig. 2.

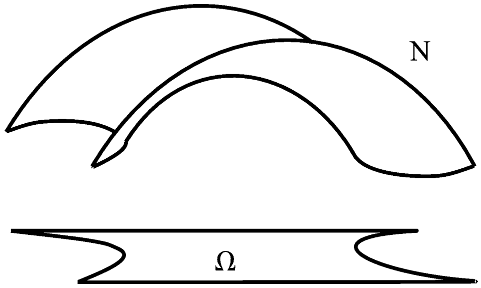

Fig. 2. Piece of a nodoid N that is a graph on a non-convex domain Ω with $\mathsf{T}u \cdot \nu = \pm 1$ along $\partial\Omega$.

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