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Some a priori bounds for solutions of the constant Gauss curvature equation

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Abstract

In this work, we give a priori height and gradient estimates for solutions of the prescribed constant Gauss curvature equation in Euclidean space. We shall consider convex radial graphs with positive constant mean curvature. The estimates are established by considering in such a graph, the Riemannian metric given by the second fundamental form of the immersion.
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1. Introduction and statement of results

Let Ω be a smooth domain (i.e. open and connected) on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. In this note, we consider the Dirichlet problem for the following equation of Monge–Ampère type:

$$\det(\rho^2 g_{ij} + 2\nabla_i\rho\nabla_j\rho - \rho\nabla_{ij}\rho) = Kg\rho^{2n-2}(\rho^2 + |\nabla\rho|)^{\frac{n+2}{2}} \quad \text{in } \Omega \tag{1}$$

with boundary data

$$\rho = \varphi \quad \text{on } \partial\Omega, \tag{2}$$

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where g_{ij} denotes the standard metric of $\mathbb{S}^n$, $g = \det(g_{ij})$, $\varphi \in C^\infty(\partial\Omega)$, $\varphi > 0$, and K is a positive constant.

A classical positive solution $\rho \in C^\infty(\bar{\Omega})$ of (1)–(2) realizes a smooth strictly convex hypersurface Σ of constant Gauss–Kronecker curvature (briefly Gauss curvature), which can be represented as

$$\Sigma = \{x(q) = \rho(q)q; q \in \bar{\Omega}\},$$

where x denotes the position vector of Σ in $\mathbb{R}^{n+1}$, and boundary values $x(q) = \varphi(q)q$ on $\partial\Omega$. See [4] for details. Moreover, the orientation N on Σ is given by

$$N(x(q)) = \frac{\nabla\rho(q) - \rho(q)q}{\sqrt{\rho^2(q) + |\nabla\rho|^2(q)}}. \tag{3}$$

We say that Σ is the radial graph of ρ, and in what follows, we call a K-hypersurface provided its Gauss curvature K is constant. The condition that Σ is a radial graph can be expressed by requiring that the supporting function $\langle N, x \rangle$ has sign on Σ and, in the particular case of the Dirichlet problem (1)–(2), this sign is negative.

The general technique employed in the solvability of the Dirichlet problem (1)–(2) is the method of continuity (see [1] in this context). We need that for the same boundary values φ on $\partial\Omega$, there exists $\rho^0 \in C^\infty(\bar{\Omega})$ whose graph is a K_0-hypersurface. For each t in $0 \leqslant t \leqslant 1$, we wish to find a solution $\rho^t \in C^{2,\alpha}(\bar{\Omega})$ of the family of Dirichlet problems:

$$\det(\rho^{t^2} g_{ij} + 2\nabla_i\rho^t\nabla_j\rho^t - \rho^t\nabla_{ij}\rho^t) = Ktg\rho^{t^{2n-2}}(\rho^{t^2} + |\nabla\rho^t|^2)^{\frac{n+2}{2}}$$
$$+ (1-t)K_0 g\rho^{0^{2n-2}}(\rho^{0^2} + |\nabla\rho^0|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega, \tag{4}$$

$$\rho^t = \varphi \quad \text{on } \partial\Omega.$$

Let the set A of $t \in [0,1]$ for which one can solve the equation for ρ^t. Because $0 \in A$, namely ρ^0, if one proves that A is open and closed, then $A = [0,1]$. The function ρ^1 is then our desired solution of (1)–(2).

In recent years, hypersurfaces of prescribed Gauss curvature have been subject to intensive studies. To mention a few examples, the Neumann boundary conditions is considered in [9], and for Dirichlet boundary conditions one can see [4–7,10,13], without claiming that this list of articles is complete.

The first main theorem of existence if due to Caffarelli et al. [1] and Krylov [8]. They proved that if $D \subset \mathbb{R}^n$ is a strictly convex planar domain, there exists a unique graph over D of constant Gauss curvature K, for K sufficiently small depending on the boundary data. Later, Guan and Spruck [4] proved that if Ω does not contain any hemisphere and it bounds a radial graph G over Ω with Gaussian curvature $K(G) > 0$, then for each $0 < K < \inf K(G)$, there exists a K-hypersurface on Ω.

The difficult part in the method of continuity is the proof that A is closed. To see this, one has to find a priori estimates up to the second derivatives for solutions ρ^t of the family of equations given in (4). Here, some kind of existence of a strictly convex subsolution taking the same boundary value is assumed to Eq. (1) in order to derive the necessary a priori estimates for the prospective solutions ρ^t. Established these estimates, the $C^{2,\alpha}$ and higher order estimates follow from the classical elliptic theory [1].

In this paper, we shall obtain C^0 and C^1 bounds of solutions of (1)–(2). We first give optimal a priori C^0 estimates for such solutions that depend only on K and the boundary values of φ. More precisely:

Theorem 1. *Let Ω be a smooth domain of $\mathbb{S}^n$ and let $\varphi \in C^\infty(\partial\Omega)$, $\varphi > 0$. Denote*

$$M = \sup_{q \in \partial\Omega} \varphi(q).$$

If $\rho \in C^\infty(\bar{\Omega})$ is a positive solution of the Dirichlet problem (1)–(2), *then we have*

$$\rho \leqslant \frac{1 + \sqrt{1 + M^2 K^{\frac{2}{n}}}}{K^{\frac{1}{n}}}. \tag{5}$$

It is worthwhile to point out that Rosenberg proved an height estimate of K-graphs over planar domains of $\mathbb{R}^{n+1}$ and $K > 0$ [12]. Exactly, if Σ is a K-graph over $D \subset \mathbb{R}^n$ and $\partial\Sigma = \partial D$, then the maximum height h that Σ can rise above the plane containing $\partial\Sigma$ satisfies the inequality

$$h \leqslant \frac{1}{K^{\frac{1}{n}}}. \tag{6}$$

A second result refers to the C^1 norm of the solutions of (1). Our motivation is the following. Let us consider the Dirichlet problem (1)–(2) for the boundary values $\varphi \equiv 1$, that is, we seek K-hypersurfaces with boundary $\partial\Omega$. According to the result given in [4], if Ω is included in a hemisphere of $\mathbb{S}^n$, for each K, $0 < K < 1$ there exists a K-hypersurface bounded by $\partial\Omega$. However, it is natural to think that thanks to the method of continuity, we can obtain solutions for $K \geqslant 1$. To do this, we start with the solution $\rho^0 \equiv 1$, what corresponds with the very domain Ω. Then one could blow up from the domain Ω to get K-radial graphs with fixed boundary $\partial\Omega$ and $K \geqslant 1$, provided that we can control the C^2 norms for all solutions ρ^t of the auxiliary problems (4). The next result gives us C^1 estimates of solutions of (1) assuming a *convexity condition* on the boundary $\partial\Omega$:

Theorem 2. *Let Ω be a smooth domain of $\mathbb{S}^n$ whose closure is included in a hemisphere and denote by $\mathscr{K}$ the Gauss curvature of $\partial\Omega$ as submanifold Ω with respect to the* *inward unit normal. Let K be a positive number such that*

$$1 \leqslant K^{\frac{n-1}{n}} < \inf_{q \in \partial\Omega} \mathscr{K}(q). \tag{7}$$

Then there exists a positive constant $C(K, \mathscr{K})$ *depending only on K and* $\mathscr{K}$, *such that if* ρ *is a positive solution of* (1)–(2) *with* $\varphi \equiv 1$, *the following inequality holds*:

$$\sup_{\Omega} |\nabla\rho| \leqslant C(K, \mathscr{K}). \tag{8}$$

The proofs of Theorems 1 and 2 are inspired by ideas of Calabi [2] and Pogorelov [11] by considering the Riemannian metric induced by the second fundamental form. We compute the Laplacian of the modulus and supporting functions defined on the K-hypersurface. We then apply the same techniques as in [12]. Using the fact that the Gauss curvature is constant, our results are, essentially, a consequence of the maximum principle for elliptic equations. Immediately following Theorem 2, we can ask if hypothesis (7) on this Theorem suffices to assure the solvability of the Dirichlet problem (1)–(2).

2. Preliminaries

Let Σ be a smooth hypersurface and $x : \Sigma \to \mathbb{R}^{n+1}$ a convex immersion with positive Gauss curvature. This means that the eigenvalues of the second fundamental form σ are positive anywhere. Then Σ is an orientable hypersurface. This occurs as follows. Locally, for each $p \in \Sigma$, there exists a neighborhood V of p that lies to one side of the tangent space $T_p\Sigma$ to Σ at p. This allows us to orient Σ by a unit normal vector field $N : \Sigma \to \mathbb{S}^n$: the choice of $N(p)$ is that points to V and

$$\sigma_p(u, v) = -\langle dN_p(u), v \rangle, \quad u, v \in T_p\Sigma,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^{n+1}$. With this orientation N, σ is a Riemannian metric on Σ. Throughout this work, we shall assume this orientation on the hypersurfaces.

Denote by $\lambda_1, \ldots, \lambda_n$ the principal curvature of x. The Gauss and mean curvature K and H curvature of x are defined as

$$K = \lambda_1 \ldots \lambda_n, \quad H = \frac{\lambda_1 + \cdots + \lambda_n}{n}.$$

Choose a point $p \in \Sigma$ and an orthonormal basis $e_1, \ldots, e_n$ for the metric σ in the tangent space $T_p\Sigma$ of Σ at p. Extend this basis to a frame, in a suitable neighborhood $V \subset \Sigma$ of p, by parallel transporting each e_i, $i = 1, \ldots, n$ with the connection ∇^σ along geodesics issuing from p. This frame and its extensions to a neighborhood of p in

$\mathbb{R}^{n+1}$ will again be denoted by $e_1, \ldots, e_n$. Notice that $\nabla^{\sigma}_{e_i} e_j(p) = 0$ and $[e_i, e_j](p) = 0$, for all $i, j = 1, \ldots, n$. Denote by $\overline{\nabla}$ the connection on $\mathbb{R}^{n+1}$.

Lemma 3. *Assume that the Gauss curvature K is constant. With the above notation, the following identities hold*:

$$\sum_{i=1}^{n} \langle \overline{\nabla}_i \overline{\nabla}_i N, e_j \rangle (p) = 0, \quad \textit{for all } 1 \leqslant j \leqslant n, \tag{9}$$

$$\sum_{i=1}^{n} \langle \overline{\nabla}_i \overline{\nabla}_i N, N \rangle (p) = -nH(p). \tag{10}$$

Proof. (See also [3]). Denote by g_{ij} the metric of Σ, that is, $g_{ij} = \langle e_i, e_j \rangle$, $g = \det(g_{ij})$ and (g^{ij}) the inverse of (g_{ij}). Then

$$\overline{\nabla}_i N = -\sum_{k=1}^{n} g^{ik} e_k. \tag{11}$$

Because $\{e_i\}$ is an orthonormal frame for the metric σ,

$$K = \frac{1}{g}, \quad nH = \mathrm{trace}(g^{ij}) = \sum_{i=1}^{n} g^{ii}.$$

Since $\langle N, e_j \rangle = 0$, we obtain,

$$\langle \overline{\nabla}_i N, e_j \rangle = -\langle N, \overline{\nabla}_i e_j \rangle = -\sigma(e_i, e_j) = -\delta_{ij}.$$

Hence

$$\langle \overline{\nabla}_i \overline{\nabla}_i N, e_j \rangle + \langle \overline{\nabla}_i N, \overline{\nabla}_i e_j \rangle = 0.$$

Using $[e_i, e_j](p) = 0$ and (11), we have

$$\sum_{i=1}^{n} \langle \overline{\nabla}_i \overline{\nabla}_i N, e_j \rangle (p) = -\sum_{i=1}^{n} \langle \overline{\nabla}_i N, \overline{\nabla}_j e_i \rangle (p) = \sum_{k,i=1}^{n} g^{ik} \langle e_k, \overline{\nabla}_j e_i \rangle.$$

Since

$$\langle e_k, \overline{\nabla}_j e_i \rangle = \tfrac{1}{2} (e_j \langle e_i, e_k \rangle + e_i \langle e_j, e_k \rangle - e_k \langle e_j, e_i \rangle),$$

we have

$$\begin{aligned}\sum_{i=1}^{n}\langle\overline{\nabla}_i\overline{\nabla}_i N, e_j\rangle(p) =& \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}e_j(g_{ik})(p)+\frac{1}{2}\sum_{i=1}^{n} e_i\left(\sum_{k=1}^{n} g^{ik}g_{kj}\right)(p)\\ &-\frac{1}{2}\sum_{k=1}^{n} e_k\left(\sum_{i=1}^{n} g^{ki}g_{ij}\right)(p)=\frac{1}{2}\sum_{i,k=1}^{n} g^{ik}e_j(g_{ik})(p)\\ =&\frac{1}{2}\frac{e_j(g)}{g}(p)=\frac{1}{2}e_j(\log g)(p)=\frac{1}{2}e_j\log\left(\frac{1}{K}\right)(p)=0,\end{aligned}$$

where in the last identity, we use the fact of the constancy of K. This proves (9).

On the other hand, since $\langle N,N\rangle=1$, $\langle\overline{\nabla}_i N,N\rangle=0$. Thus, and from (11),

$$\begin{aligned}\sum_{i=1}^{n}\langle\overline{\nabla}_i\overline{\nabla}_i N,N\rangle(p) &= -\sum_{i=1}^{n}\langle\overline{\nabla}_i N,\overline{\nabla}_i N\rangle(p)=\sum_{i,j=1}^{n} g^{ij}\langle e_j,\overline{\nabla}_i N\rangle(p)\\ &= -\sum_{i,j=1}^{n} g^{ij}\delta_{ij}(p)=-\sum_{i=1}^{n} g^{ii}(p)=-nH(p).\end{aligned}$$

This completes the proof. $\square$

Proposition 4. *Let $x:\Sigma\to\mathbb{R}^{n+1}$ be a convex immersion with positive Gauss curvature K. Let a be a fixed vector in $\mathbb{R}^{n+1}$. Denote by Δ^σ the Laplacian operator in Σ with the metric σ. Then the function $\langle x,a\rangle$ satisfies*

$$\Delta^\sigma\langle x,a\rangle=n\langle N,a\rangle. \tag{12}$$

Moreover, if K is constant, the function $\langle N,a\rangle$ satisfies

$$\Delta^\sigma\langle N,a\rangle+nH\langle N,a\rangle=0. \tag{13}$$

Proof. Let $p\in\Sigma$ and consider a geodesic moving frame $e_1,\dots,e_n$ in a neighborhood of p for the metric σ as in Lemma 3. Then

$$\begin{aligned}\Delta^\sigma\langle x,a\rangle &= \sum_{i=1}^{n}\sigma(\nabla_i^\sigma\nabla_i^\sigma\langle x,a\rangle,e_i)=\sum_{i=1}^{n} e_ie_i\langle x,a\rangle=\sum_{i=1}^{n} e_i\langle e_i,a\rangle\\ &= \sum_{i=1}^{n}\langle\sigma(e_i,e_i)N,a\rangle=n\langle N,a\rangle.\end{aligned}$$

This proves (12).

Now, let us assume that K is constant. Using (9) and (10), we obtain

$$\begin{aligned}\Delta^\sigma \langle N, a\rangle &= \sum_{i=1}^{n} \sigma(\nabla_i^\sigma \nabla^\sigma \langle N, a\rangle, e_i) = \sum_{i=1}^{n} e_i e_i \langle N, a\rangle \\ &= \sum_{i=1}^{n} \langle \overline{\nabla}_i \overline{\nabla}_i N, a\rangle = \sum_{i,j=1}^{n} \langle \overline{\nabla}_i \overline{\nabla}_i N, N\rangle \langle N, a\rangle \\ &= -nH\langle N, a\rangle,\end{aligned}$$

and identity (13) follows. □

Corollary 5. *Let Σ be a smooth hypersurface and let $x : \Sigma \to \mathbb{R}^{n+1}$ be a convex immersion of positive constant Gauss curvature K. Then modulus function $|x|^2$ satisfies*

$$\Delta^\sigma |x|^2 = 2n\langle N, x\rangle + 2\frac{S_n}{K}, \tag{14}$$

where $S_n = \sum_{j=1}^{n} (\lambda_1 \dots \hat{\lambda}_j \dots \lambda_n)$ and, as usual, $\hat{\lambda}_j$ means that λ_j is missing.

On the other hand, the supporting function $\langle N, x\rangle$ satisfies

$$\Delta^\sigma \langle N, x\rangle = -n - nH\langle N, x\rangle. \tag{15}$$

Proof. Let $a_1, \dots, a_{n+1}$ be is an orthonormal basis of $\mathbb{R}^{n+1}$, and denote by $x_i = \langle x, a_i\rangle$, $N_i = \langle N, a_i\rangle$, $1 \leqslant i \leqslant n+1$, the coordinate functions of x and N, respectively. Consider again $e_1, \dots, e_n$, a geodesic moving frame around p as in Lemma 3. We know from (11) that

$$\begin{aligned}\nabla^\sigma x_i &= \sum_{k=1}^{n} \langle a_i, e_k\rangle e_k, \\ \nabla^\sigma N_i &= -\sum_{j,k=1}^{n} g^{kj} \langle a_i, e_j\rangle e_k.\end{aligned}$$

Then Proposition 4 and the Green's identity imply

$$\begin{aligned}\Delta^\sigma |x|^2 &= 2\sum_{i=1}^{n+1} (x_i \Delta^\sigma x_i + \sigma(\nabla^\sigma x_i, \nabla^\sigma x_i)) \\ &= 2\sum_{i=1}^{n+1} \left(n x_i N_i + \sum_{j=1}^{n} \langle a_i, e_j\rangle^2 \right) = 2n\langle N, x\rangle + 2\sum_{j=1}^{n} g_{jj}.\end{aligned}$$

Since $\{e_1, \ldots, e_n\}$ is an orthonormal basis for σ,

$$\sum_{j=1}^{n} g_{jj} = \mathrm{trace}(g_{ij}) = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \frac{S_n}{K}.$$

On the other hand, and again by Proposition 4, we obtain for the supporting function $\langle N, x \rangle$,

$$\begin{aligned}\Delta^{\sigma}\langle N, x\rangle &= \sum_{i=1}^{n+1}(N_i \Delta^{\sigma} x_i + x_i \Delta^{\sigma} N_i + 2\sigma(\nabla^{\sigma} x_i, \nabla^{\sigma} N_i))\\ &= \sum_{i=1}^{n+1}\left(nN_i^2 - nHx_iN_i - 2\sum_{j,k=1}^{n}(\langle a_i, e_j\rangle \langle a_i, e_k\rangle g^{kj})\right)\\ &= n - nH\langle N, x\rangle - 2\sum_{j,k=1}^{n} g_{jk}g^{kj} = -n - nH\langle N, x\rangle,\end{aligned}$$

and this completes the proof of (15). $\square$

3. Proof of Theorems 1 and 2

Let $x : \Sigma \to \mathbb{R}^{n+1}$ be a convex radial graph over a domain $\Omega \subset \mathbb{S}^n$ and with positive constant Gauss curvature K. We consider on Σ the orientation N so that the second fundamental form σ is a positive definite quadratic form (and thus, H is a positive function). Since Σ is a radial graph, the function supporting $\langle N, x \rangle$ either positive or negative in the whole hypersurface Σ.

For clarity in this section, we first prove the following lemma:

Lemma 6. *With the above assumptions, the following holds*:

(1) *If the sign of* $\langle N, x \rangle$ *is positive, then*

$$|p| \leqslant \sup_{p \in \partial\Sigma} |p|, \quad \textit{for all } p \in \Sigma.$$

(2) *Assume that* Ω *is strictly included in a hemisphere and* $\partial\Sigma = \partial\Omega$. *If* $K \geqslant 1$ *and* $\langle N, x \rangle$ *is negative on* Σ, *then* $|p| \geqslant 1$, *for all* $p \in \Sigma$.

Proof. The first assertion is a consequence of the maximum principle applied to Eq. (14). For the second statement and without loss of generality, let $D = \{p \in \mathbb{S}^n ; p_{n+1} > 0\}$ be the hemisphere containing Ω. Assume to the contrary, that is, there are interior points of Σ in the open ball $B = \{p \in \mathbb{R}^{n+1} ; |p| < 1\}$. Take the family of spherical caps C_t, $0 \leqslant t < 1$ such that $\partial C_t = \partial D$, $\mathrm{int}(C_t) \subset B \cap \{p \in \mathbb{R}^{n+1}; p_{n+1} > 0\}$ and where t denotes the Gauss curvature of C_t. Starting from the disc C_0, we blow up until the first point of contact between some C_{t_0} and Σ, $t_0 < 1$. This occurs at some common interior point of both hypersurfaces. Then the comparison principle for Σ and C_{t_0} yields a contradiction. $\square$

Now we are in position to prove Theorem 1. Denote by Σ the graph of the function ρ in the statement of theorem. We need to estimate the modulus function of Σ, since $|p| = |x(q)| = \rho(q)$, $q \in \bar{\Omega}$. If the supporting function $\langle N, x \rangle$ is positive, then Lemma 6 gives $|p| \leqslant M$, what proves (5).

We then assume that $\langle N, x \rangle$ is negative in Σ. Combining (14) and (15),

$$\Delta^\sigma \left(\frac{K^{\frac{1}{n}}}{2} |x|^2 + \langle N, x \rangle \right) = n(K^{\frac{1}{n}} - H) \langle N, x \rangle + \frac{S_n - nK^{\frac{n-1}{n}}}{K^{\frac{n-1}{n}}}. \tag{16}$$

Inequality between the geometric and arithmetic average gives $K^{1/n} - H \leqslant 0$. Using again this inequality for each one of the summands of S_n, we have

$$\frac{S_n}{n} = \frac{1}{n} \sum_{j=1}^{n} (\lambda_1 \ldots \hat{\lambda}_j \ldots \lambda_n) \geqslant \left(\prod_{j=1}^{n} (\lambda_1 \ldots \hat{\lambda}_j \ldots \lambda_n) \right)^{\frac{1}{n}} = K^{\frac{n-1}{n}}.$$

In view of (16), we then obtain

$$\Delta^\sigma \left(\frac{K^{\frac{1}{n}}}{2} |x|^2 + \langle N, x \rangle \right) \geqslant 0.$$

The maximum principle for elliptic equations gives us

$$\frac{K^{\frac{1}{n}}}{2} |x|^2 + \langle N, x \rangle \leqslant \max_{p \in \partial \Sigma} \left(\frac{K^{\frac{1}{n}}}{2} |p|^2 + \langle N(p), p \rangle \right) \leqslant \frac{K^{\frac{1}{n}}}{2} M^2. \tag{17}$$

Using the fact that $\langle N, x \rangle \geqslant -|x|$, one has

$$K^{\frac{1}{n}} |x|^2 - 2|x| - K^{\frac{1}{n}} M^2 \leqslant 0.$$

This inequality is quadratic in $|x|$, and we easily obtain from it estimate (5). This completes Theorem 1.

Remark 7. In the case that the boundary values φ is constant, namely $\varphi \equiv R > 0$, inequality (5) written as

$$\rho \leqslant \frac{1 + \sqrt{1 + R^2 K^{\frac{2}{n}}}}{K^{\frac{1}{n}}}. \tag{18}$$

In this situation, we get equality in (18) when Σ is a spherical cap that meets the sphere $\mathbb{S}^n(R)$ of radius R orthogonally along its boundary. Exactly, let $r \in (0,1)$ and $\Omega = \{x \in \mathbb{S}^n; x_{n+1} > \sqrt{1-r^2}\}$. Let

$$S = \left\{ x \in \mathbb{R}^{n+1}; \sum_{i=1}^{n} x_i^2 + \left(x_{n+1} - \frac{R}{\sqrt{1-r^2}} \right)^2 = \frac{R^2 r^2}{1-r^2} \right\}.$$

Then the spherical cap $S \cap \{x_{n+1} \geqslant R\sqrt{1-r^2}\}$ attains estimate (18).

Remark 8. We point out that Rosenberg's estimate (6) is obtained from (18) by subtracting R from each side and letting R tend to $+\infty$.

We now proceed to show Theorem 2. Let the setting be as in Theorem 2 and consider ρ a solution of (1)–(2). Denote by Σ the corresponding K-hypersurface defined by ρ. In deriving the gradient estimate, it is much more convenient to express $\nabla\rho$ in terms of the supporting function $\langle N, x \rangle$ of Σ. Recall that the orientation N is given by (3), and then

$$\langle N, x \rangle = -\frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}} < 0. \tag{19}$$

Moreover, Lemma 6 implies $|p| \geqslant 1$ for each $p \in \Sigma$.

Thus a priori C^1 estimates of ρ are reduced to control the supporting function $\langle N, x \rangle$, together C^0 estimates of the function ρ.

Let η (resp. v) be the inward unit normal vector of $\partial\Omega$ (resp. $\partial\Sigma$) regarded as submanifold of Ω (resp. Σ). Because $|p| \geqslant 1$ on Σ, we know $\langle v_p, p \rangle \geqslant 0$, for all $p \in \partial\Omega$.

Now $M = 1$ in (5). Inequality (17) assures that the maximum is achieved at some boundary point $q \in \partial\Omega$:

$$\frac{K^{\frac{1}{n}}}{2}|p|^2 + \langle N(p), p \rangle \leqslant \frac{K^{\frac{1}{n}}}{2} + \langle N(q), q \rangle, \quad \text{for all } p \in \Sigma. \tag{20}$$

Thus the maximum principle gives us

$$K^{\frac{1}{n}} \langle v_q, q \rangle + \langle dN_q v_q, q \rangle \leqslant 0,$$

and consequently,

$$(K^{\frac{1}{n}} - \sigma_q(v_q, v_q))\langle v_q, q\rangle \leqslant 0. \tag{21}$$

If $\langle v_q, q\rangle = 0$, the (boundary version) maximum principle between Σ and $\bar{\Omega} \subset \mathbb{S}^n$ concludes that $\Sigma = \bar{\Omega}$. Thus, we can assume that $\langle v_q, q\rangle > 0$. Then (21) yields

$$K^{\frac{1}{n}} \leqslant \sigma_q(v_q, v_q). \tag{22}$$

Denote by σ^{Ω} the second fundamental form of the immersion $\partial\Omega \hookrightarrow \Omega$, and by $\mu_1, \ldots, \mu_{n-1}$ their principal curvatures. Consider $e_1, \ldots, e_{n-1}$ an orthonormal basis of $T_q\partial\Sigma$, such that $\sigma_q^{\partial\Omega}(e_i, e_j) = \delta_{ij}\mu_i$. Then we have

$$\begin{aligned}\sigma_q(e_i, e_i) &= \sigma_q^{\Omega}(e_i, e_i)\langle N(q), \eta_q\rangle - \langle N(q), q\rangle \\ &= \mu_i(q)\sqrt{1 - \langle N(q), q\rangle^2} - \langle N(q), q\rangle, \quad 1 \leqslant i \leqslant n-1,\end{aligned}$$

$$\sigma_q(e_i, e_j) = -\langle e_i, e_j\rangle\langle N(q), q\rangle = 0, \quad i \neq j.$$

We claim that

$$\sigma_q(e_i, v_q) = 0, \quad \text{for all } i = 1, \ldots, n-1.$$

Combining (18) and (20), the restriction of function $\langle N, x\rangle$ into $\partial\Omega$ has a maximum at q. Because $\langle N, x\rangle^2 + \langle N, \eta\rangle^2 = 1$ along $\partial\Omega$, and $\langle N, x\rangle$, $\langle N, \eta\rangle$ has sign along $\partial\Omega$, q is also a critical point of $\langle N, \eta\rangle$ along $\partial\Omega$. Hence, for all $1 \leqslant i \leqslant n-1$,

$$e_i\langle N, x\rangle = 0, \quad e_i\langle N, \eta\rangle = 0.$$

Thus, $\nabla_{e_i}N$ is a tangent vector to $\mathbb{S}^n$ at the point q. In view of this,

$$\sigma_q(e_i, v_q) = -\langle \nabla_{e_i}N, v_q\rangle = -\langle \nabla_{e_i}N, \eta_q\rangle\langle v_q, \eta_q\rangle.$$

Finally

$$\langle \nabla_{e_i}N, \eta_q\rangle = e_i\langle N, \eta\rangle = 0.$$

As the Gauss curvature is given by the determinant of the second fundamental form in an orthonormal basis, from (22) and together with that fact that $\langle N, x\rangle$ is negative, we have

$$
\begin{aligned}
K &= \sigma_q(v_q, v_q)\det(\sigma_q(e_i, e_j))\\
&= \sigma_q(v_q, v_q)\prod_{i=1}^{n-1}\left(\mu_i(q)\sqrt{1-\langle N(q), q\rangle^2} - \langle N(q), q\rangle\right)\\
&\geqslant K^{\frac{1}{n}}\mathscr{K}(q)(1-\langle N(q), q\rangle^2)^{\frac{n-1}{2}}.
\end{aligned}
$$

Then

$$
K^{\frac{n-1}{n}} \geqslant \mathscr{K}(q)(1-\langle N(q), q\rangle^2)^{\frac{n-1}{2}}. \tag{23}
$$

Set

$$
\mathscr{K}_0 = \inf_{p\in\partial\Omega} \mathscr{K}(p).
$$

Hypothesis (7) leads a constant $C_1 = C_1(K, \mathscr{K})$ such that $K^{\frac{n-1}{n}} < C_1 < \mathscr{K}_0$. In virtue of (23), one has

$$
(1-\langle N(q), q\rangle^2)^{\frac{n-1}{2}} < C_2 := \frac{C_1}{\mathscr{K}_0} < 1.
$$

Thus

$$
\langle N(q), q\rangle < -\sqrt{1-C_2^{\frac{2}{n-1}}} < 0.
$$

Taking into account the above inequality and that $|p| \geqslant 1$ (see Lemma 6), it follows from (20) that

$$
\langle N(p), p\rangle \leqslant \frac{K^{\frac{1}{n}}}{2}(1-|p|^2) + \langle N(q), q\rangle \leqslant C_3 := -\sqrt{1-C_2^{\frac{2}{n-1}}}. \tag{24}
$$

Remark that the negative number C_3 depends only on $\mathscr{K}$ and K.

Let $C_4 = C_4(K)$ be the right-hand of estimate (18). Then from (19) and (24), we obtain finally

$$
|\nabla\rho| \leqslant C(K, \mathscr{K}) =: -\frac{C_4(K)^2}{C_3(K, \mathscr{K})},
$$

that proves (8). This ends the proof of Theorem 2. $\square$

References

[1] L.A. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, I. Monge–Ampère equations, Comm. Pure Appl. Math. 37 (1984) 369–402.

[2] E. Calabi, Improper affine hyperspheres of convex type and a generalization of theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.

[3] J. Gálvez, A. Martínez, The Gauss map and second fundamental form of surfaces in $\mathbb{R}^3$, Geom. Dedicata 81 (2000) 181–192.

[4] B. Guan, J. Spruck, Boundary-value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature, Ann. of Math. 138 (1993) 601–624.

[5] B. Guan, J. Spruck, The existence of hypersurfaces of constant Gauss curvature with prescribed boundary, preprint.

[6] D. Hoffman, H. Rosenberg, J. Spruck, Boundary value problems for surfaces of constant Gauss curvature, Comm. Pure Appl. Math. 45 (1992) 1051–1062.

[7] N.M. Ivochkina, F. Tomi, Locally convex hypersurfaces of prescribed curvature and boundary, Cal. Var. Partial Differential Equations 7 (1998) 293–314.

[8] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, Izv. Math. 47 (1983) 95–108.

[9] P.L. Lions, N.S. Trudinger, J.I. Urbas, The Neumann problem for equations of Monge–Ampère type, Comm. Pure Appl. Math. 39 (1986) 539–563.

[10] T. Nehring, Hypersurfaces of prescribed Gauss curvature and boundary in Riemannian manifolds, J. Reine Angew. Math. 501 (1998) 143–170.

[11] A.V. Pogorelov, On a regular solution of the n-dimensional analogue of the Monge–Ampère, Soviet Math. Dokl. 12 (1971) 1436–1440.

[12] H. Rosenberg, Hypersurfaces of constant curvature in space forms, Bull. Sci. Math. 117 (1993) 211–239.

[13] N.S. Trudinger, X. Wang, On locally convex hypersurfaces with boundary, J. Reine Angew. Math. 551 (2002) 11–32.