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Rafael López

A note on radial graphs with constant mean curvature

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Abstract. Let Ω be a smooth domain on the unit sphere $\mathbb{S}^n$ whose closure is contained in an open hemisphere and denote by $\mathcal{H}$ the mean curvature of $\partial\Omega$ as a submanifold of Ω with respect to the inward unit normal. It is proved that for each real number H that satisfies $\inf \mathcal{H} > -H \geq 0$, there exists a unique radial graph on Ω bounded by $\partial\Omega$ with constant mean curvature H. The orientation on the graph is based on the normal that points on the opposite side as the radius vector.

1. Introduction and statement of results

Let Ω be a smooth domain (i.e. open and connected) of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the Euclidean $(n+1)$-space. We define the *radial graph* of a function $\rho \in C^2(\Omega) \cap C^0(\overline{\Omega})$ as the hypersurface

$$\Sigma = \{X(q) = \exp(\rho(q))q; q \in \Omega\},$$

where X denotes the position vector of Σ in $\mathbb{R}^{n+1}$. From the geometric viewpoint, a radial graph Σ is characterized as a hypersurface that is starshaped relative to the origin, that is, each ray emanating from the origin intersects Σ once at most. Consequently, if N is a unit normal vector field on Σ, the support function $\langle N(p), p\rangle$, $p \in \Omega$, is either positive or negative in the whole of Σ. Throughout this work, we shall assume the orientation opposite to the radius vector, that is, $\langle N(p), p\rangle < 0$ for all $p \in \Sigma$.

In the work we present here, we consider the problem of finding radial graphs with constant mean curvature $H \in \mathbb{R}$ (briefly radial H-graphs). The pioneering work on this subject is due to Radó [8]. He proved that for any Jordan space curve in $\mathbb{R}^3$ with single valued radial projection onto a convex curve of $\mathbb{S}^2$ bounds a minimal radial graph. More recently, Tausch showed that if $\Omega \subset \mathbb{S}^n$ is a convex set and Γ is a radial graph on $\partial\Omega$, then there exists a disc-type hypersurface of least area among all integral currents having boundary Γ. Moreover, this hypersurface can locally be represented as a radial graph [10].

Rafael López: Departamento de Geometría y Topología. Universidad de Granada. 18071 Granada, Spain. e-mail: rcamino@ugr.es

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On the other hand, Serrin [9, §23] studied the existence and uniqueness of radial graph with prescribed mean curvature. Taking into account the chosen orientation on a radial graph, he proved the following theorem.

Theorem 1 (Serrin). *Let Ω be a smooth domain in S^n whose closure is contained in an open hemisphere. Denote by $\mathcal{H}$ the mean curvature of $\partial\Omega$ as submanifold of Ω, computed with respect to the unit normal pointing to the interior of Ω. Let ψ a smooth function on $\partial\Omega$ and let H be a non-positive smooth function defined over the closure of Ω such that*

$$\mathcal{H}(q) \geq -\frac{n}{n-1} H(q)\, e^{\psi(q)} \geq 0 \tag{1}$$

for each $q \in \partial\Omega$. Then there exists a unique radial graph on Ω with mean curvature H and boundary data ψ.

In particular, if $\mathcal{H}(q) \geq 0$ at each point q of the boundary, there exists a minimal radial graph on Ω, for arbitrary value ψ on $\partial\Omega$. When $\psi \equiv 0$, that is, when the prescribed boundary is included in $\mathbb{S}^n$, the mean curvature condition (1) can be relaxed. In the present note, our main result may be stated as follows:

Theorem 2. *Let Ω be a smooth domain on the unit sphere $\mathbb{S}^n$ whose closure is contained in an open hemisphere and let $H \leq 0$. Denote by $\mathcal{H}$ the mean curvature of $\partial\Omega$ as above. If H satisfies*

$$\mathcal{H}(q) > -H$$

for each point $q \in \partial\Omega$, then there exists a unique radial H-graph on Ω with boundary $\partial\Omega$.

Although most of the computations presented in this work are well known for specialists, as far as I know the statement of Theorem 2 does not seem to have appeared previously in the literature.

The problem of existence of radial graphs of prescribed constant mean curvature leads to a quasilinear elliptic equation (see formula (4) below). In providing the setting of our result, it would be necessary to point out some differences with the Dirichlet problem for the mean curvature equation in a planar domain $D \subset \mathbb{R}^n$ (also so-called vertical graphs):

$$\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = nH \quad \text{ in } D \tag{2}$$

where H is a constant. Investigations of such equation have historically attracted the interest for geometers. We refer the reader to e.g. [1] for a modern treatment in the theory of existence and uniqueness of equation (2). Different kinds of boundary conditions have been imposed to equation (2) in order to obtain existence of solution. For example, if D is a bounded domain with $\mathcal{H} \geq n|H| > 0$, Serrin [9] proved that for any arbitrary given $(n-1)$-submanifold Γ with single valued projection onto ∂D, there exists a unique solution of (2) whose graph has boundary Γ [9]. In the case of zero boundary data, it suffices in assuming $\mathcal{H} > |H| \geq 0$ (see e.g. [4]). The techniques employed can be generalized in other ambient spaces. This occurs in hyperbolic space, where the same hypothesis $\mathcal{H} > |H| > 0$ assures the existence of constant mean curvature graphs on domains of a totally geodesic hyperplane or a horosphere ([5],[6]).

However differences appear between the Dirichlet problems for radial and vertical graphs. This is the case of the uniqueness (see Example 1 below in Section 2), where no uniqueness holds if H is positive. In fact, the sign of H plays a fundamental role for existence and uniqueness of radial H-graphs (recall that a symmetry property of (2) for zero boundary data shows that the sign on H is irrelevant). Exactly, there exist three intervals in the range of H where the behavior of the Dirichlet problem for radial graphs presents important differences, namely, $(-\infty, 0]$, $[0, 1]$ and $(1, +\infty)$. Definitively, this shows that the Dirichlet problem for radial H-graphs has a richness that deserves well of its own interest. In this sense, it has been some activity in the existence and uniqueness of a starshaped closed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature (see e.g. [11] and references therein).

This paper is organized as follows. In Section 2, we introduce the Dirichlet problem and discuss some properties of uniqueness. In Section 3 we shall derive C^0 and C^1 estimates for the desired solutions. In Section 4 we then prove Theorem 2 using the continuity method.

2. Some facts about the Dirichlet problem

In proving Theorem 2, it is convenient to establish the associated Dirichlet problem. Consider a smooth radial graph $\Sigma = \text{graph}(\rho)$ over a domain Ω on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, where $\rho \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Let $e_1, \ldots, e_n$ be a smooth local frame field on $\mathbb{S}^n$ and let ∇ denote the covariant differentiation on $\mathbb{S}^n$. We use the notation $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j$. Let $\theta_{ij} = \langle e_i, e_j \rangle$ denote the metric on $\mathbb{S}^n$. The metric of $X(q) = e^{\rho(q)} q$ is then given in terms of ρ by

$$g_{ij} = \langle \nabla_i X, \nabla_j X \rangle = e^{2\rho}(\theta_{ij} + \nabla_i \rho \nabla_j \rho),$$

where $\langle, \rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. The inward unit normal to X is

$$N(X(q)) = \frac{\nabla \rho(q) - q}{\sqrt{1 + |\nabla \rho|^2(q)}}, \tag{3}$$

where $\nabla \rho = \text{grad}\, \rho$. This the Gauss map N on Σ satisfies the inequality $\langle N(X(q)), X(q) \rangle < 0$, and thus, it agrees with our choice of the orientation on radial graphs. The second fundamental form of X is

$$\sigma_{ij} = \langle \nabla_{ij} X, N \rangle = \frac{e^{\rho}(\theta_{ij} + 2\nabla_i \rho \nabla_j \rho - \nabla_{ij} \rho)}{\sqrt{1 + |\nabla \rho|^2}}.$$

The mean curvature H of Σ is defined by

$$nH = \sum_{ij} g^{ij} \sigma_{ij},$$

where (g^{ij}) is the inverse of (g_{ij}). After a straightforward computation, the mean curvature H satisfies the following equation:

$$\operatorname{div}\frac{\nabla\rho}{\sqrt{1+|\nabla\rho|^2}} = n\left(-He^{\rho}+\frac{1}{\sqrt{1+|\nabla\rho|^2}}\right), \tag{4}$$

where div denotes the divergence operator with the standard metric of $\mathbb{S}^n$. Equation (4) is a quasilinear elliptic equation in $\Omega \subset \mathbb{S}^n$ of divergence form ([1], Chapter 10) and the machinery of Schauder theory can be used in the problem of existence. Thus Theorem 2 is equivalent to find a unique solution $\rho \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet problem

$$Q[\rho] \equiv \operatorname{div}\frac{\nabla\rho}{\sqrt{1+|\nabla\rho|^2}} + n\left(He^{\rho}-\frac{1}{\sqrt{1+|\nabla\rho|^2}}\right) = 0 \qquad \text{in } \Omega \tag{5}$$

with boundary data,

$$\rho = 0 \qquad \text{on } \partial\Omega. \tag{6}$$

Following the method of continuity, the solvability of the Dirichlet problem (5)–(6) can be established provided we can introduce a real parameter $\tau \in [0, 1]$ into the boundary-value problems

$$(D_\tau) \qquad Q_\tau[\rho_\tau] = 0 \text{ in } \Omega \qquad \rho_\tau = 0 \text{ on } \partial\Omega,$$

where Q_τ is equal that Q in (5), except that we replace H by τH, and such that the following holds: for $\tau = 0$ the problem has a solution of class $C^{2,\alpha}$; for $\tau = 1$ we obtain the given problem (5)–(6). Finally, for all $\tau \in [0, 1]$ we have to find uniform *a priori* estimates in the $C^{2,\alpha}$ norm for all solutions $C^{2,\alpha}(\overline{\Omega})$ of the auxiliary problems (D_τ). Following the usual Schauder approach [1, Th. 13.8], $C^1(\overline{\Omega})$ uniform bounds imply $C^{2,\alpha}(\overline{\Omega})$ uniform bounds for $\alpha \in (0, 1)$. Therefore, one has to seek a positive constant M, independently of τ, such that the estimate

$$\| \rho_\tau \|_{C^1(\overline{\Omega})} = \sup_\Omega \rho_\tau + \sup_\Omega |\nabla\rho_\tau| \leq M$$

holds for any $C^{2,\alpha}(\overline{\Omega})$-solution ρ_τ of (D_τ)

It is convenient to point out that a standard argument by using the Hopf maximum principle proves uniqueness of solutions in the Dirichlet problem for the vertical graphs. In general, solutions to equation (5)–(6) are not unique, as it shows the following example for $H > 0$:

Example 1. Let

$$\frac{1}{\sqrt{2}} \leq r < 1, \qquad a \in [0, \frac{2r^2-1}{\sqrt{1-r^2}}], \qquad R_a = \sqrt{1+a^2+2a\sqrt{1-r^2}}.$$

Consider $x = (x_1, \ldots, x_{n+1})$ and

$$S(a) = \{x \in \mathbb{R}^{n+1}; \sum_{i=1}^n x_i^2 + (x_{n+1}+a)^2 = R_a^2\}.$$

This hypersurface $S(a)$ is a sphere centered at $(0, \ldots, -a)$ with radius R_a. Let $\Sigma_1(a)$ be the part of $S(a)$ above the hyperplane $P : x_{n+1} = \sqrt{1-r^2}$ and let $\Sigma_2(a)$ be the reflection of $S(a)\backslash\overline{\Sigma_1(a)}$ with respect to P. The condition on a assures that both hypersurfaces are radial $1/R_a$-graphs in the domain Ω defined by

$$\Omega = \{x \in \mathbb{S}^n; x_{n+1} > \sqrt{1-r^2}\}.$$

Moreover, the mean curvature $H_a = 1/R_a$, satisfies $0 < H_a \leq 1$. Let us observe that the case $a = 0$ leads $S(0) = \mathbb{S}^n$. Hence $\Sigma_1(0) = \Omega$ and $\Sigma_2(0)$ is the reflection of $\mathbb{S}^n\backslash\overline{\Omega}$ with respect to the hyperplane P.

However uniqueness of the solution holds if $H \leq 0$.

Proposition 1 (Uniqueness for H ≤ 0). *Let Σ_1 and Σ_2 be two radial graphs of two functions ρ_1 and ρ_2 respectively defined on $\Omega \subset \mathbb{S}^n$. Represent by H_1 and H_2 their (not necessarily constant) mean curvatures, respectively. If $\rho_1 \leq \rho_2$ on $\partial\Omega$ and if $H_1 \circ \rho_1 \leq H_2 \circ \rho_2 \leq 0$, then $\rho_1 \leq \rho_2$ in Ω. In particular, there is uniqueness of solutions for the Dirichlet problem (5)–(6).*

Proof. Assume that $\rho_1 > \rho_2$ at some point of Ω. By homotheties, we lift Σ_2 outwards until it lies entirely above the hypersurface Σ_1 (with respect to the radius vector). Then lower it by homotheties again until an initial point p of contact occurs. Let $\Sigma_2' = t_0\Sigma_2$ be the position of Σ_2 at this time ($t_0 > 1$). Since Σ_2' is tangent to Σ_1 at p the normal vectors to Σ_2' and Σ_1 coincide. Then Σ_2' lies locally above Σ_1 and an application of the maximum principle implies that the mean curvature H_2' at that point cannot exceed the mean curvature H_1 of Σ_1. However, and because $H_2 \leq 0$, we have

$$H_2' = \frac{1}{t_0} H_2 > H_2 \geq H_1,$$

obtaining a contradiction. □

Remark 1. One cannot expect solvability of the Dirichlet problem (5)–(6) without any assumptions on H and the size of the domain Ω. In fact, if Ω is a domain *containing an open hemisphere of* $\mathbb{S}^n$, the Dirichlet problem for $H \leq 0$ is not solvable: this is an application of the maximum principle between radial H-graphs and hyperplanes.

Remark 2. As a direct consequence of the maximum principle, a vertical H-graph with planar boundary lies completely in one side of the hyperplane containing the boundary. An interesting question is whether this occurs in our setting, that is, if a radial H-graph on $\Omega \subset \mathbb{S}^n$ and with boundary $\partial\Omega$ lies in one side of Ω (this holds when Ω is included in a hemisphere and $|H| > 1$, see [3]).

Remark 3. In the simplest case of domain Ω, that is, when Ω is a totally geodesic disc of $\mathbb{S}^n$, appropriate spherical caps with boundary $\partial\Omega$ are radial H-graphs on Ω. In this case, $\partial\Omega$ is a $(n-1)$-sphere. But no more examples are known. In the case that $\overline{\Omega}$ is included in a open hemisphere, an argument comparing our radial H-graph with hyperplanes together the Alexandrov reflection method shows that the graph must be a spherical cap. However it is still opened the following

Conjecture. Spherical caps and planar n-discs are the only radial graphs in $\mathbb{R}^{n+1}$ with constant mean curvature and with boundary a $(n-1)$-sphere.

If the Conjecture is true, it would be the *boundary version* of a beautiful theorem in classical differential geometry due to Jellet in 1853, that asserts that a starshaped closed surface with constant mean curvature is a round sphere [2].

3. A priori C^0 and C^1 estimates

In this section we shall derive *a priori* estimates that we need to establish for the existence of a solution of (5)–(6). The modulus function together the support function defined on the radial graph shall allow to obtain these estimates.
(a) *Bounds for ρ on Ω.* If $p = X(q) = e^{\rho(q)}q$ denotes the position vector of a radial graph Σ, $|p| = e^{\rho}$. Then C^0 estimates of ρ corresponds with bounds in the modulus of each point of the graph. Denote

$$m = \min_{p\in\partial\Sigma} |p|^2, \qquad M = \max_{p\in\partial\Sigma} |p|^2.$$

Theorem 3. *Let Ω be a domain in $\mathbb{S}^n$ and let Σ be a radial H-graph. Then:*

1. *If $H < 0$,*
$$\frac{-1+\sqrt{1+mH^2}}{-H} \leq |p| \leq \sqrt{M}, \qquad p \in \Sigma \tag{7}$$
2. *If $H > 0$,*
$$|p| \leq \frac{1+\sqrt{1+MH^2}}{H}, \qquad p \in \Sigma \tag{8}$$
3. *If $H = 0$,*
$$|p| \leq \sqrt{M}, \ \ p \in \Sigma \tag{9}$$

Proof. (Compare with Proposition 2.1 in [7] for vertical H-graphs). A direct computation leads to obtain the following formulae for hypersurfaces with constant mean curvature:

$$\Delta|p|^2 = 2n + 2nH\langle N(p), p\rangle \tag{10}$$

$$\Delta\langle N(p), p\rangle = -nH - |\sigma|^2(p)\langle N(p), p\rangle, \tag{11}$$

where Δ is the Laplace-Beltrami operator of the induced metric on the hypersurface Σ and $|\sigma|$ is the length of the second fundamental form σ. Then

$$\Delta\left(\frac{H}{2}|p|^2 + \langle N(p), p\rangle\right) = (nH^2 - |\sigma|^2(p))\langle N(p), p\rangle \geq 0.$$

The maximum principle for elliptic equations gives us

$$\frac{H}{2}|p|^2 + \langle N(p), p\rangle \leq \max_{p\in\partial\Sigma}\left(\frac{H}{2}|p|^2 + \langle N(p), p\rangle\right) \tag{12}$$

and this concludes, for the case $H \neq 0$, the proof of Theorem in the usual way. Concerning to the case $H = 0$, the maximum principle for equation (10) yields directly (9). $\square$

In the case that the boundary $\partial\Sigma$ lies in $\mathbb{S}^n$, the statement (7) asserts

$$\frac{-1+\sqrt{1+H^2}}{-H} \leq |p| \leq 1 \tag{13}$$

for each $p \in \Sigma$.

It is interesting to observe that estimates (7), (8) and (9) are the best ones possible as shows the following example:

Example 2. Let $r \in (0,1)$ and $\Omega = \{x \in \mathbb{S}^n; x_{n+1} > \sqrt{1-r^2}\}$.
Let

$$S(r) = \{x \in \mathbb{R}^{n+1}; \sum_{i=1}^n x_i + (x_{n+1} - \frac{1}{\sqrt{1-r^2}})^2 = \frac{r^2}{1-r^2}\}.$$

The hypersurface $S(r)$ is a sphere of radius $R = r/\sqrt{1-r^2}$ centered at the point $(0,\ldots,1/\sqrt{1-r^2})$. Then $\partial\Omega$ determines two spherical caps on $\mathbb{S}^n$, namely Σ_1 and Σ_2, that are radial graphs on Ω. It is easy to see that Σ_1 and Σ_2 attain the bounds (7) and (8) of Theorem 3, respectively. We also note that both graphs are tangent to the cone determined by $\partial\Omega$. Finally, $\{x_{n+1} = \sqrt{1-r^2}\} \cap \{|x| \leq 1\}$ is a radial minimal graph on Ω and $|p| \leq 1$.

Remark 4. In the case that $H > 0$, it is possible to have *a priori* lower estimates for ρ, provided *the closure of* Ω *is included in an open hemisphere*. This improves the estimate (8). Indeed, we assume without loss of generality that Ω is included in the upper hemisphere of $\mathbb{S}^n$. Denote $\mu = \min_{q\in\partial\Omega} x_{n+1}(q) > 0$. Then we compare Σ with parallel hyperplanes to $x_{n+1} = 0$ that come from the halfspace $x_{n+1} < 0$. Recall that the orientation on Σ points in the opposite direction of the vector position. Then the maximum principle for hypersurfaces with constant mean curvature assures that the first contact point between the hyperplanes and Σ occurs at some boundary point. In particular, $|p| \geq x_{n+1}(p) \geq \mu$, for all $p \in \Sigma$, and thus, $\log\mu \leq \rho$ on Ω.

(b) *Bounds for* $|\nabla\rho|$ *on* Ω. We now proceed to estimate the first derivatives of ρ when $H \leq 0$ and $\rho = 0$ on $\partial\Omega$. In order to see this, it will be convenient to express $\nabla\rho$ in terms of the support function of Σ. The expression of N on Σ given in (3) yields

$$\langle N(p), p\rangle = \langle N(X(q)), X(q)\rangle = -\frac{e^{\rho(q)}}{\sqrt{1+|\nabla\rho|^2(q)}}. \tag{14}$$

Thus C^1 bounds of ρ are obtained provided that we have bounds for the support function together the C^0 estimates obtained for ρ in Theorem 3.

Let $\Omega \subset \mathbb{S}^n$ be a smooth domain whose boundary $\partial\Omega$ has mean curvature $\mathcal{H}$ measured by the inward unit normal (as submanifold of Ω).

Theorem 4. *Let* $\Omega \subset \mathbb{S}^n$ *be a smooth domain of* $\mathbb{S}^n$ *and let* $H \leq 0$. *Assume that* $\inf\mathcal{H} > -H$. *Then there exists a positive constant* $C = C(\Omega, H)$ *depending only on* Ω *and* H, *such that if* ρ *is a solution of (5)–(6), we have*

$$\sup_\Omega |\nabla\rho| \leq C.$$

Proof. Denote by Σ the graph of ρ. From (7), $|p| \leq 1$ on Σ and one has from the maximum principle $\langle \nu(p), p\rangle \leq 0$ along $\partial\Omega$, where ν represents the inner unit conormal vector along $\partial\Omega$. Inequality (12) assures that the function $H|p|^2/2 + \langle N(p), p\rangle$ attains its maximum at some point $q \in \partial\Omega$:

$$\frac{H}{2}|p|^2 + \langle N(p), p\rangle \leq \frac{H}{2} + \langle N(q), q\rangle \quad \text{for all } p \in \Sigma. \tag{15}$$

Thus the maximum principle gives us

$$H\langle \nu(q), q\rangle + \langle dN_q \nu(q), q\rangle \leq 0.$$

Consequently,

$$(H - \sigma(\nu(q), \nu(q)))\langle \nu(q), q\rangle \leq 0.$$

The above inequality and the fact of $\langle \nu(q), q\rangle \leq 0$ yields $H \geq \sigma(\nu(q), \nu(q))$. As $nH = -\text{trace}(dN)$, we have

$$(n-1)H \leq \sum_{i=1}^{n-1} \sigma(e_i, e_i), \tag{16}$$

where $e_1, \ldots, e_{n-1}$ is an orthonormal basis of $T_q\partial\Sigma$. Now, we may decompose the second fundamental form σ of the immersion $\partial\Omega \to S^n$ as the sum of the second fundamental form σ^Ω of $\partial\Omega$ into Ω and the second fundamental form of Ω into S^n. Thus

$$\sigma(e_i, e_i) = \sigma^\Omega(e_i, e_i)\langle \nu(q), q\rangle - \langle N(q), q\rangle.$$

Summing from $i = 1$ to $i = n-1$ and using inequality (16), we have

$$\langle N(q), q)\rangle + H \leq -\mathcal{H}(q)\sqrt{1 - \langle N(q), q\rangle^2}.$$

Since $\langle N(q), q\rangle \leq 0$, we have

$$\langle N(q), q\rangle \leq \frac{-H - \mathcal{H}(q)\sqrt{\mathcal{H}^2(q) + 1 - H^2}}{1 + \mathcal{H}^2(q)}.$$

Introduce

$$\kappa_0 = \min_{q\in\partial\Omega} \mathcal{H}(q) \qquad \kappa_1 = \max_{q\in\partial\Omega} \mathcal{H}(q).$$

Taking into account the above inequality and (15), it follows that

$$\begin{aligned}\langle N(p), p\rangle &\leq \frac{H}{2}(1 - |p|^2) - \frac{H + \mathcal{H}(q)\sqrt{\mathcal{H}^2(q) + 1 - H^2}}{1 + \mathcal{H}^2(q)} \\ &\leq -\frac{H + \mathcal{H}(q)\sqrt{\mathcal{H}^2(q) + 1 - H^2}}{1 + \mathcal{H}^2(q)} \\ &\leq -\frac{H + \kappa_0\sqrt{\kappa_0^2 + 1 - H^2}}{1 + \kappa_1^2} =: b_1, \end{aligned} \tag{17}$$

for each $p \in \Sigma$. Let us remark that $b_1 < 0$ ($\mathcal{H} \geq \kappa_0 > -H$) and that this constant $b_1 = b_1(\Omega, H)$ depends only on Ω and H. According (14) and (7), the number b_1 determines a constant $C(\Omega, H)$ such that $|\nabla\rho| \leq C$. $\square$

Remark 5. Let us note that if $H \leq 0$ and by using (11), we have $\Delta\langle N(p), p\rangle \geq 0$. Thus the maximum principle says us that the function $\langle N(p), p\rangle$ attains its maximum at some point of $\partial\Sigma$. In this case (no assumption on $\mathcal{H}$), $\sup_\Omega |\nabla\rho| = \sup_{\partial\Omega} |\nabla\rho|$.

4. Proof of the Theorem 2

In Section 3, we have obtained *a priori* C^0 and C^1 estimates for prospective solutions of (5)–(6). Once such estimates have been established, we shall prove existence of the Dirichlet problem by means of the continuity method ([1]). We briefly explain this technique with some detail. Assume the hypothesis of Theorem 2. Define the set

$$J = \{\tau \in [0, 1]; \exists\rho_\tau \text{ solution of } (D_\tau)\}.$$

In this situation, one has to show that J is a non-void, open and closed subset of $[0, 1]$, and hence, $J = [0, 1]$. In particular, $1 \in J$, proving Theorem 2. We shall show by steps.

First, we prove that $0 \in J$. As we observed in the introduction, this is a consequence if one applies Serrin's theorem. Let us note that this moment is the only place we need hypothesis that Ω is included in a hemisphere and it allows us to start the existence procedure.

In a second step, we prove that J is open in $[0, 1]$. This is accomplished by using the implicit function theorem for Banach spaces. Consider $\tau \in J$ and let us see that the Dirichlet problem (D_β) can be solved for each β in a certain interval around τ. Denote by Σ_τ the graph hypersurface corresponding to ρ_τ. For the invertibility, one just needs to make sure that the null space of the linearisation of the mean curvature of Σ_τ is trivial. Define a map

$$h : C_0^{2,\alpha}(\Sigma_\tau) \rightarrow C_0^{\alpha}(\Sigma_\tau)$$

taking each u onto the mean curvature function of the normal graph on Σ_τ corresponding to the function u. This map h between both Banach spaces has as the linearisation the Jacobi operator of the hypersurface Σ_τ, that is,

$$L(u)(p) = (dh)_0 = \Delta + |\sigma|^2,$$

where Δ is the Laplace-Beltrami operator in Σ_τ and σ is its the second fundamental form. Here $L(u)$ is a self-adjoint linear elliptic operator. The kernel of L is trivial since

$$L\langle N(p), p\rangle = -nH,$$

(recall that $H \leq 0$). Thus, L is a operator of index zero. Using the Riesz spectral theory of compact operators, the Fredholm alternative applies and the invertibility of (5)–(6) is assured around τ. The implicit function theorem in Banach spaces guarantees then an interval of solutions of the auxiliary problems (D_β) in a neighborhood of τ.

It remains to see that J is closed in [0, 1]. This is equivalent if we have C^0 and C^1 global estimates of each solution ρ_τ of (D_τ) independent on τ. As a consequence of Theorem 3 and Proposition 1, for each solution ρ_τ, $\tau \in [0, 1]$, we have

$$\frac{-1+\sqrt{1+H^2}}{-H} \leq exp(\rho_\tau) \leq 1.$$

Thus

$$\log \frac{-1+\sqrt{1+H^2}}{|H|} \leq \rho_\tau \leq 0,$$

obtaining the desired C^0 *a priori* bounds for ρ_τ independent on τ.

Turning to the estimates for the first derivatives of ρ, we use Theorem 4. We need only estimate b_1 in (17) from above by a bound independent on τ. But

$$b_1 = -\frac{\tau H + \kappa_0\sqrt{\kappa_0^2 + 1 - \tau^2 H^2}}{1+\kappa_1^2} \leq -\frac{H + \kappa_0\sqrt{\kappa_0^2+1-H^2}}{1+\kappa_1^2} < 0,$$

and thus giving the bounds for $\| \rho \|_{C^{1,\alpha}(\overline{\Omega})}$, as requires for the Schauder procedure. Thus, there is a $C^{2,\alpha}(\overline{\Omega})$ solution of the Dirichlet problem as claimed. The elliptic regularity theory implies that $\rho \in C^\infty(\overline{\Omega})$. Uniqueness of solutions follows from Proposition 1. This completes the proof of Theorem 2.

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