# Constant mean curvature surfaces with planar boundary*

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## 1 Introduction

A constant mean curvature surface immersed in Euclidean three-space can be viewed as a surface where the exterior pressure and the surface tension forces are balanced. For this reason they are thought of as *soap bubbles or films* depending on the considered surface being either closed (that is, compact without boundary) or compact with non-empty boundary. With respect to the closed case, until 1986 the only known examples of constant mean curvature surfaces were the round spheres. In that year, Wente [**W4**] constructed genus one constant mean curvature surfaces which are non-embedded (see also [**Ab,Bo,PS**]). One year later, Kapouleas [**Ka**] did the same for genera bigger than two. These results activated in a remarkable way the research in this field and indicated the sharpness of the two principal theorems about closed constant mean curvature surfaces which were known at that time: the Hopf theorem, which asserts that the sphere is the only example with genus zero [**Ho**] and the Alexandrov theorem which says that the sphere is the only possible embedded example [**Al**].

With respect to the study of the space of compact constant mean curvature $H$ surfaces with prescribed non-empty boundary $\Gamma$, we do not know its structure even in the easiest case, when $\Gamma$ is a round circle with, for instance,

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unit radius. Heinz [**He2**] found that a necessary condition for existence in this situation is that $|H| \le 1$. The only known examples, excluding the trivial minimal case, are the two spherical caps with radius $1/|H|$, which are the only umbilical ones, and some non-embedded surfaces of genus bigger than two whose existence was proved by Kapouleas in [**Ka**].

In general, when $\Gamma$ is a Jordan curve in $\mathbb{R}^3$, the problem of existence of constant mean curvature $H$ surfaces $\Sigma$ with $\partial\Sigma = \Gamma$ has been studied by Heinz [**He1**], Hildebrant [**Hi**], Wente [**W1**], Werner [**We**] and Steffen [**St**] in the case of immersions from the two-dimensional disc. They solved the corresponding (disc-parametric) Plateau problem (when $H$ is small enough in terms of the geometry of the curve $\Gamma$) by showing existence of *small* (that is, contained in some sphere with radius less than $1/|H|$) solutions which are relative minimizers of the functional $A - 2HV$, $A$ being the area and $V$ the algebraic volume functionals, respectively. On the other hand, Serrin [**Se1**] proved, using continuity methods, existence of constant mean curvature graphs on some strictly convex planar domains. All these works have culminated in those of Brézis and Coron [**BrC**] and Struwe [**Str**], who showed that, with the same assumptions and $H \neq 0$, there exists a second non-minimizing *large* solution. This gave a solution to the Rellich conjecture.

From another point of view, Wente has solved in [**W2**] the (disc-parametric) Plateau problem when the volume is constrained to take a fixed value. Here, volume means the algebraic volume which will be defined in the following section, for any immersion $\phi$ from a compact oriented surface $\Sigma$ into $\mathbb{R}^3$. This algebraic volume depends on the choice of origin, but when $\partial\Sigma = \Gamma$ is planar, $\phi$ is an embedding above the plane and the origin is taken in that plane it coincides with the enclosed Lebesgue volume. Wente proved that, for any Jordan curve $\Gamma$ in $\mathbb{R}^3$ and any $V \in \mathbb{R}$, there exists a constant mean curvature immersion $\phi$ from the disc $D$ which minimizes the area among all the immersions from $D$ into $\mathbb{R}^3$ with $\phi(\partial D) = \Gamma$ and algebraic volume $V$. After this, Steffen and Wente have seen that the minimizers so obtained are regular [**SW**] and have studied its behaviour when the prescribed volume $V$ grows up to infinity [**W3**]. Without being precise, they converge to spheres outside a given compact set containing the boundary.

A third source of constant mean curvature surfaces with prescribed boundary is the isoperimetric problem. Suppose that the Jordan curve $\Gamma$ given as a prescribed boundary is the boundary $\partial\Omega$ of an embedded surface $\Omega$ (for instance, when $\Gamma$ is a planar curve and $\Omega$ is the corresponding planar domain). Given a positive number $V$, an isoperimetric region with respect to $(\Omega, V)$ is a region $M$ of volume $V$ in $\mathbb{R}^3$ such that $\partial M = \Omega \cup \Sigma$ and $\Sigma$ has least area among all the possible $M$. Existence of these isoperimetric regions is guaranteed in our case in the context of the geometric measure theory. Also, from the corresponding regularity theorems [**Alm**, p. 77, (6)] it turns out that their boundaries are smooth except in $\Gamma$ and so they give us embedded constant mean curvature surfaces, possibly of high genus. The problem is to understand what the isoperimetric regions look like when one varies the prescribed volume $V$ and what the behaviour of the corresponding mean curvature $H(V)$ is.

In this paper we will obtain certain results about these isoperimetric regions and constant mean curvature surfaces when the prescribed boundary $\Gamma$ is a convex planar Jordan curve. Precisely we will prove that

> *Given a bounded convex planar domain $\Omega$, there exists a positive constant $V_\Omega$ such that isoperimetric regions with respect to the pair $(\Omega, V)$ for $V \leq V_\Omega$ are bounded by $\Omega \cup \Sigma$ where $\Sigma$ is a constant mean curvature graph over $\Omega$.*

We will provide in this way a proof for what the G.A.N.G. at the University of Massachusetts, Amherst, had seen on their computers [**HR**]. More generally, we will able to show that

> *Given a convex closed planar curve $\Gamma$ there is a constant $V_\Gamma$ depending only on the curve such that any constant mean curvature compact surface with boundary $\Gamma$ and volume less than or equal to $V_\Gamma$ must be a graph.*

We will compute exactly the value of this critical volume in the case where the surface is a disc and the boundary curve is a circle:

*A constant mean curvature disc spanning a unit circle in Euclidean space is a small spherical cap provided that its volume is not bigger than* $2\pi/3$.

A corresponding uniqueness theorem for constant mean curvature surfaces which are small with respect to the area was already done in [**LMo**].

All these theorems come from a height estimate (in terms of the area, that is, a sort of monotonicity formula) for constant mean curvature compact surfaces with planar (not necessarily convex) boundary that we will get in Theorem 1, and from a suitable use of the E. Hopf maximum principle. This height estimate is more accurate than the corresponding estimate due to Serrin [**Se2**] when one deals with *small* surfaces, moreover, it has the advantage that it also works in the immersed case. It can be formulated in the following way (see Figure 1)

*Let* $\Sigma$ *be a constant mean curvature* $H$ *compact surface with planar boundary. Then the area of* $\Sigma$ *is greater than or equal to the area of the segment of a stack of spheres with the same mean curvature whose highest point and boundary plane are at the same heights as those of* $\Sigma$.

By the way, this estimate for the growth of the area will allow us to prove an existence theorem for graphs that, in some sense, improves, for planar boundary, the corresponding result by Serrin. In fact

*If* $\Gamma$ *is a convex closed planar curve with length* $L$ *and* $H$ *is a non-negative real number such that* $LH < \sqrt{3}\pi$, *then there exists a graph with constant mean curvature* $H$ *and boundary* $\Gamma$.

Moreover, we will give an optimal bound in order for a constant mean curvature compact *embedded* surface with convex planar boundary to be above the boundary plane. We will show that

*Let* $\Sigma$ *be a constant mean curvature* $H$ *compact embedded surface with convex planar boundary such that its area* $A$ *satisfies* $AH^2 \leq 2\pi$. *Then* $\Sigma$ *lies above the boundary plane. Moreover, if* $AH^2 \leq \pi$, *then* $\Sigma$ *is a graph.*

This answers a question posed by Rosenberg in [**HR**] with experimental evidences reported by Hoffman, at least in the case where the considered surfaces are *small*.

As a conclusion, we could summarize the results in this work by saying that constant mean curvature compact surfaces whose boundary is a convex planar curve are graphs (and so topologically discs), provided they are assumed to be small in some sense. It is interesting to point out that Ros and Rosenberg [**RR**] have obtained some nice related results for *large* constant mean curvature compact surfaces whose boundary is also a convex planar curve and which belong to one of the half-spaces determined by the boundary plane.

## 2 Preliminaries

Let $\Sigma$ be a compact (always supposed connected) surface whose boundary will be represented by $\partial\Sigma$. We will deal with immersions $\phi : \Sigma \to \mathbb{R}^3$ from $\Sigma$ into three-dimensional Euclidean space with constant mean curvature (abbreviated to *cmc* and to *cmc* $H$ when we emphasize the value $H$ of the mean curvature). We will represent by $ds^2$ the metric on $\Sigma$ induced from the Euclidean one in $\mathbb{R}^3$, by $dA$ and $A$ the associated measure and area respectively and, if we assume that the surface is oriented, by $V$ the algebraic volume of the immersion given by

$$V = -\frac{1}{3}\int_\Sigma \langle \phi, N\rangle \, dA,$$

where $N : \Sigma \to \mathbb{R}^3$ is the Gauss map corresponding to the given orientation. This is the signed volume of the cone constructed on the image $\phi(\Sigma)$ with vertex at the origin. Of course it depends on the choice of the origin. Notice that, when $H \neq 0$, our surface is necessarily orientable and, so, we may choose, and will do it from now on, the Gauss map for the immersion $\phi$ in such a way that $H \geq 0$.

A basic tool in this context is the often invoked maximum principle due to E. Hopf that can be stated as follows:

**Maximum principle** ([**HE**], see [**Ho**] or [**GT**]) *Let $\phi_1, \phi_2 : \Sigma \to \mathbb{R}^3$ two cmc immersions and $p \in \Sigma$ such that $\phi_1(p) = \phi_2(p)$ and $(d\phi_1)_p T_p\Sigma = (d\phi_2)_p T_p\Sigma$ (and $(d\phi_1)_p T_p\partial\Sigma = (d\phi_2)_p T_p\partial\Sigma$ if $p \in \partial\Sigma$). If the mean curvatures of $\phi_1$ and $\phi_2$ corresponding to a same choice of unit normal vector $N_p$ coincide and $\langle \phi_1, N_p\rangle \leq \langle \phi_2, N_p\rangle$ in some neighborhood of the point $p$, then $\phi_1 = \phi_2$ on $\Sigma$.*

That is, if two immersed cmc surfaces are tangent at some point, the mean curvatures agree for a common orientation and one of them is locally above the other one, then they coincide (recall: the surfaces are always connected). Together with this important principle, we have another useful tool that appeared the first time in [**K**] and later has been considered in a lot of works (see for example [**EBMR**], [**KKS**] and [**KK**]). In fact, suppose that the surface $\Sigma$ is orientable (for instance, in the non-minimal case) and let $N : \Sigma \to \mathbb{R}^3$ be a Gauss map for $\phi$ and $H$ the constant mean curvature corresponding to that choice of normal field. We may define a vector-valued one-form $\omega$ on $\Sigma$ by the equality

$$\omega = (H\phi + N) \wedge d\phi$$

where $\wedge$ stands for the vector product of $\mathbb{R}^3$. It can be easily checked that the fact that $H$ is constant is equivalent to the one-form $\omega$ is closed. Hence, we have

**Balancing formula** (see [**KKS**]) *Let $\phi : \Sigma \to \mathbb{R}^3$ a cmc $H$ immersion from an oriented compact surface into Euclidean space. Then*

$$\int_{\partial\Sigma} (H\phi + N) \wedge d\phi = 0.$$

*where $N$ is the corresponding Gauss map $N : \Sigma \to \mathbb{R}^3$.*

Roughly speaking this balancing formula says that a *cmc* immersed compact surface is in equilibrium with respect to the surface tension and the external pressure forces. Finally, we will also use below the so-called first Minkowski formula, more familiar in the boundaryless compact case (see [**MoR**] and references therein) and which is an immediate consequence of

$$\Delta|\phi|^2 = 4(1 + H\langle\phi, N\rangle),$$

where $\Delta$ is the Laplacian operator of the induced metric $ds^2$. Integrating, we obtain

**Minkowski formula** *Let $\phi : \Sigma \to \mathbb{R}^3$ be an immersion from an oriented compact cmc $H$ surface into Euclidean space. Then*

$$A - 3HV = -\frac{1}{2}\int_{\partial\Sigma}\langle\phi, \nu\rangle\, ds$$

*where $A$ is the area of the induced metric $ds^2$, $V$ denotes the algebraic volume of $\phi$ and $\nu$ is the inner conormal along the boundary $\partial\Sigma$.*

# 3 An a priori height estimate

We will begin to obtain an a priori estimate for the height of a compact *cmc* surface immersed in $\mathbb{R}^3$ measured from a plane when its boundary is contained in that plane. Our estimate has a different nature from the *height lemma* due to Serrin (see [**Se2**] and [**KKS**]). The Serrin result asserts that *a compact* cmc $H$ *graph,* $H > 0$*, with planar boundary has height at most* $1/H$ *above the boundary plane* and as a consequence of the Alexandrov reflection principle [**Al**] one has that *a compact* cmc $H$ *surface with planar boundary cannot extend more that* $2/H$ *above that plane*. Notice first that this height lemma is valid only for embedded surfaces and, second, the estimate is not sharp for the so-called *small cmc* surfaces because when $H$ tends to zero there are *cmc* $H$ surfaces with fixed boundary whose height tends to zero (small *cmc* surfaces) and others whose height grows up to infinity (large *cmc* surfaces). Our estimate will work in the immersed case and be accurate for small and large *cmc* surfaces. In the case where the surface $\Sigma$ is a disc it can be obtained from a paper by Wente [**W3**], although it is not explicitely written in that work.

An equivalent statement of the balancing formula which can be found in [**KKS**] is contained in the following result.

**Lemma 1 (Balancing formula)** *Let $\phi : \Sigma \to \mathbb{R}^3$ be a* cmc $H$ *immersion from an oriented compact surface into Euclidean space such that $\phi(\partial\Sigma)$ is a planar curve. Then*

$$\int_{\partial\Sigma} \nu \, ds - 2H\bar{A}a = 0$$

*where $\bar{A}$ is the algebraic area of the curve $\phi_{|\partial\Sigma}$ and $a \in \mathbb{R}^3$ is a unit vector normal to the boundary plane.*

**Remark** Using $|\langle \nu, a\rangle| \leq 1$ in the equality of Lemma 1 we obtain

$$2H|\bar{A}| \leq L,$$

where $L$ is the length of the curve $\phi(\partial\Sigma)$. As a consequence we obtain a restriction for the range of possible values of the mean curvature $H$ of the immersion in terms of the geometry of the prescribed boundary, provided that this boundary has non-zero algebraic area (for instance, when it is a planar Jordan curve). In fact, we have

$$H \leq \frac{L}{2|\bar{A}|}. \tag{1}$$

**Theorem 1** *Let $\Sigma$ be a* cmc $H$ *compact surface immersed in Euclidean space with boundary belonging to a plane $P$. If $h$ denotes the height of $\Sigma$ with respect to $P$, we have that*

$$h \leq \frac{HA^+}{2\pi}$$

*where $A^+$ is the area of the region of $\Sigma$ above the plane $P$. The equality holds if and only if $\Sigma$ is a spherical cap.*

**Proof** Let $\phi : \Sigma \to \mathbb{R}^3$ be the immersion. The minimal case $H = 0$ is clear because, in this case, $\phi(\Sigma)$ is contained in the convex hull of $\phi(\partial\Sigma) \subset P$.

So, suppose that $H > 0$ and represent by $N$ the corresponding Gauss map. After a translation, we may assume that

$$P = \{x \in \mathbb{R}^3 \,|\, \langle x, a\rangle = 0\} \quad \text{for some } a \in \mathrm{R}^3, |a| = 1.$$

We consider the height function $f = \langle \phi, a\rangle : \Sigma \to \mathbb{R}$. Clearly we have that $f_{|\partial\Sigma} \equiv 0$. On the other hand, if $p \in \Sigma$ and $v \in T_p\Sigma$

$$\langle \nabla f_p, v\rangle = \langle (d\phi)_p(v), a\rangle \tag{2}$$

and, so, $p \in \Sigma$ is a critical point of $f$ if and only if the vector $a$ is perpendicular to $\phi$ at $p$, that is, if and only if $N_p = \pm a$. Then the set $C$ of the critical points of $f$ is contained in

$$\{p \in \Sigma \,|\, \langle N(p), b\rangle = \langle N(p), c\rangle = 0\},$$

where $a, b, c$ form an orthonormal basis of $\mathbb{R}^3$. In this way the set $C$ is contained in the intersection of nodal lines of two solutions $u_1 = \langle N, b\rangle$ and $u_2 = \langle N, c\rangle$ of the Schrödinger equation $\Delta u + |\sigma|^2 u = 0$, $\sigma$ being the second fundamental form of the immersion $\phi$. If these two solutions were identically zero, then we would have either $N = a$ or $N = -a$ everywhere on the surface and, so, the surface would be minimal, which is not the case. Thus, either $u_1$ or $u_2$ is not trivial and, so, its nodal line consists of a finite number of immersed circles (as we can see, for example, in [**Ch**]). Hence, this set $C$ of the critical points of $f$ has zero measure. In this situation, $A(t)$ is continuous and the coarea formula [**F**] gives us

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{|\nabla f|}\, ds_t \qquad t \in \mathbb{R},$$

where $A(t)$ is the area of $\Omega(t) = \{p \in \Sigma \,|\, f(p) \geq t\}$ and $\Gamma(t) = \{p \in \Sigma \,|\, f(p) = t\}$. If we denote by $L(t)$ the length of the planar curve $\Gamma(t)$, the Schwarz inequality yields

$$L(t)^2 \leq \int_{\Gamma(t)} |\nabla f|\, ds_t \int_{\Gamma(t)} \frac{1}{|\nabla f|}\, ds_t = -A'(t) \int_{\Gamma(t)} |\nabla f|\, ds_t. \tag{3}$$

But we have from (2) that, along the curve $\Gamma(t)$,

$$|\nabla f|^2 = 1 - \langle N, a\rangle^2 = \langle \nu^t, a\rangle^2$$

where $\nu^t : \Gamma(t) \to \mathbb{R}^3$ is the inner conormal of $\Omega(t)$. Since $\Omega(t)$ is above the plane $P_t = \{x \in \mathrm{R}^3 \,|\, \langle x, a\rangle = t\}$ we know that $\langle \nu^t, a\rangle \geq 0$. Hence

$$|\nabla f|_{|\Gamma(t)} = \langle \nu^t, a\rangle.$$

Then, (3) can be rewritten as follows

$$L(t)^2 \leq -A'(t) \int_{\Gamma(t)} \langle \nu^t, a\rangle \, ds_t \qquad t \in \mathrm{R}. \tag{4}$$

We know that $\Omega(t)$ is a compact surface with smooth boundary $\Gamma(t)$ for almost every $t \in \mathbb{R}$. Notice that, if $t < 0$, then $\partial\Sigma \subset \Omega(t)$ and so $\phi(\Gamma(t))$ has a component lying in $P$ and possibly others in $P_t$. But if $t \geq 0$, then $\phi(\Gamma(t)) \subset P_t$ and, in this case, we may use Lemma 1 above and obtain

$$\int_{\Gamma(t)} \langle \nu^t, a\rangle \, ds_t = 2H|\bar{A}(t)|$$

where $\bar{A}(t)$ is the algebraic area of the planar closed curve $\phi_{|\Gamma(t)}$. Thus if we substitute in (4) we have

$$L(t)^2 \leq -2HA'(t)|\bar{A}(t)| \quad \text{for every } t \geq 0. \tag{5}$$

Now, denote by $\Omega_1(t), \ldots, \Omega_{n_t}(t)$ the bounded domains which are determined in the plane $P_t$ by the closed curve $\phi_{|\Gamma(t)}$, and by $A_i(t)$ (with $i = 1, \ldots, n_t$) the Lebesgue area of the corresponding $\Omega_i(t)$. As one has that

$$\bar{A}(t) = \varepsilon_1 A_1(t) + \cdots + \varepsilon_{n_t} A_{n_t}(t)$$

where $\varepsilon_i \in \mathrm{Z}$ are the order numbers corresponding to the curves $\phi_{|\partial\Omega_i(t)}$, then

$$|\bar{A}(t)| \leq |\varepsilon_1| A_1(t) + \cdots + |\varepsilon_{n_t}| A_{n_t}(t).$$

On the other hand, if $L_i(t)$ is the length of the boundary of $\Omega_i(t)$, we have that $L(t) = |\varepsilon_1| L_1(t) + \cdots + |\varepsilon_{n_t}| L_{n_t}(t)$ and hence

$$L(t)^2 \geq \varepsilon_1^2 L_1(t)^2 + \cdots + \varepsilon_{n_t}^2 L_{n_t}(t)^2.$$

By using all these inequalities and (5) we have for $t \geq 0$ that

$$\varepsilon_1^2 L_1(t)^2 + \cdots + \varepsilon_{n_t}^2 L_{n_t}(t)^2 \leq -2HA'(t)(|\varepsilon_1| A_1(t) + \cdots + |\varepsilon_{n_t}| A_{n_t}(t)).$$

If we take into account that $L_i(t)^2 \geq 4\pi A_i(t)$, from the isoperimetric inequality for the domain $\Omega_i(t)$, and that $\varepsilon_i^2 \geq |\varepsilon_i|$, we get

$$2\pi \leq -HA'(t) \qquad \text{for every } t \geq 0.$$

Integrating this inequality from 0 to $h = \max_{p\in\Sigma} f(p) \geq 0$ one gets

$$2\pi h \leq H(A(0) - A(h)) = HA^+$$

which is the inequality that we looked for.

If the equality holds with $H > 0$ then all the inequalities above become equalities. In particular, $|\nabla f|_{\Gamma(t)} = \langle \nu^t, a\rangle$ is a constant for any $t \geq 0$ and $\phi(\Gamma(t))$ is a circle. Thus, in a neighborhood of the point of $\Sigma$ where the greatest height is attained, the immersion $\phi$ is an embedding whose image is a *cmc* surface with a circle as boundary and lying above the boundary plane. The reflection principle [**Al**] of Alexandrov allows us to conclude that, in that neighborhood, $\phi$ is umbilic. By analyticity, $\Sigma$ must be a disc and $\phi$ an embedding whose image is a spherical cap.

**Remark** Theorem 1 above can be paraphrased in the following way: A *cmc* $H > 0$ compact surface immersed into Euclidean space with planar boundary has area bigger than the area of a right cylinder with its same height and radius $1/H$ unless it is a spherical cap of radius $1/H$ and, in this case, those two areas coincide. The latter is an old theorem by Archimedes, as one can see, for example, in [**Hea**, p. 293]. On the other hand, if we want to compare the area growth of our surface with that corresponding to a standard surface with the same constant mean curvature, we can easily see that Theorem 1 above says also that a *cmc* $H > 0$ compact immersed surface with planar boundary has area bigger than the area of a segment of a stack of spheres with radius $1/H$ whose highest point and boundary plane are the same as those of the surface (see the Figure 1).

Figure 1:

In the same hypothesis of Theorem 1 we have

$$h^- \geq -\frac{HA^-}{2\pi}$$

where $A^-$ means the area of the region of the surface $\Sigma$ which is below the boundary plane and $h^-$ is the lowest height of the surface under the plane containing its boundary. This yields the following

**Corollary 1** *If $\Sigma$ is a* cmc *$H$ compact surface immersed into Euclidean space with boundary contained in a plane $P$ and area $A$, then $\Sigma$ lies in a slab parallel to $P$ with height less than $HA/2\pi$, unless $\Sigma$ is a spherical cap (in which case the thinnest slab has height exactly $HA/2\pi$).*

The estimate which we have obtained for the height, together with the maximum principle, allow us to obtain bounds relative to other directions of space when we assume that the surface $\Sigma$ is *small* in certain sense.

**Corollary 2** *Let $\Sigma$ be a* cmc *$H$ compact surface immersed into Euclidean space with planar boundary. If the area $A$ of the surface satisfies $AH^2 \leq \pi$, then the surface $\Sigma$ lies inside the right cylinder determined by the convex hull of its boundary.*

**Proof** If $\Sigma$ is a spherical cap with mean curvature $H \geq 0$, one has that (see, for instance, [**LMo**])

$$\text{either } AH^2 = 2\pi(1 - \sqrt{1 - H^2}) \text{ or } AH^2 = 2\pi(1 + \sqrt{1 - H^2}),$$

depending on whether spherical cap is small or large. Since we are supposing that $AH^2 \leq \pi$ it must be small and the result is clear in this case.

In any other situation, from Corollary 1, the surface $\Sigma$ is contained in a slab parallel to the boundary plane $P$ with height less than $AH/2\pi \leq 1/2H$. Thus $\Sigma$ lies inside a slab parallel to $P$ and *symmetric* with respect to $P$ with height less than $1/H$. Suppose now that there is some point of $\Sigma$ projecting on a point $p \in P$ outside the convex hull $C$ of the boundary of $\Sigma$ and choose $q \in C$ minimizing the distance to $p$. Denote by $R$ the half-line of $P$ starting at $q$ and passing through $p$ and by $C_R$ a half-cylinder of radius $1/2H$ with axis belonging to $P$ and perpendicular to $R$. We move $C_R$ along $R$ far enough for it not to touch the surface $\Sigma$ and we place its concave side in front of $\Sigma$. Now we proceed to approach the half-cylinder $C_R$ to $\Sigma$ and in this way we get a first (and so tangential) contact point between the two surfaces. As the axis of $C_R$ lies inside $P$ and there is a point of $\Sigma$ projecting on the point $p$ outside the convex hull of the boundary, this contact point so obtained is non-boundary point of the surface $\Sigma$. It is also an interior point of the half-cylinder $C_R$ because $\Sigma$ is inside a slab with height less than the height $1/H$ of $C_R$. On the other hand, this half-cylinder has constant mean curvature $H$ with respect to the normal field pointing to its concave part. As we already know that $\Sigma$ is in that concave part, by elementary comparison, we have that this same choice of normal at the contact point gives mean curvature $H$ for $\Sigma$. The interior maximum principle would say that $C_R$ and $\Sigma$ agree. This is a contradiction. As a consequence, all the points of the surface $\Sigma$ must project on the convex hull of its boundary.

**Remark** The assumption $AH^2 \leq \pi$ in Corollary 2 above is not optimal. Probably one should obtain the same result under the hypothesis $AH^2 \leq 2\pi$. At least this occurs when the boundary of the surface $\Sigma$ is a circle.

In the proof of Corollary 2, a fundamental fact was that the slab of width

$1/H$ where the surface $\Sigma$ lies is symmetric with respect to the boundary plane $P$. This was the reason to suppose $AH^2 \leq \pi$. If we weaken this hypothesis to $AH^2 \leq 2\pi$, either $\Sigma$ is a small spherical cap or, using Corollary 1, $\Sigma$ is inside a slab $S$ parallel to $P$ with height less than $AH/2\pi \leq 1/H$. This slab $S$ is not necessarily symmetric with respect to $P$ as above. But we may utilize again half-cylinders of radius $1/2H$ and axis in the central plane of $S$ as barriers and obtain, by taking into account the maximum principle, that the surface $\Sigma$ is contained in the convex body of $\mathbb{R}^3$ delimited by the convex hull $C$ of $\partial\Sigma$ and the afore-mentioned half-cylinders which are tangent to $C$. As a consequence, if $\partial\Sigma$ is a convex curve then the surface $\Sigma$ does not intersect outside the corresponding planar convex domain. This will permit us to find a necessary condition in order that a *cmc* compact surface *embedded* into $\mathbb{R}^3$ with planar convex boundary stay in a half-space.

**Corollary 3** *Let $\Sigma$ be a* cmc *$H$ compact surface embedded into Euclidean space whose boundary is a convex planar curve contained in a plane $P$. If $AH^2 \leq 2\pi$, where $A$ is the area of the surface, then $\Sigma$ stays in a half-space determined by $P$ and is transverse to $P$ along the boundary. By Alexandrov reflection, $\Sigma$ inherits the symmetries of its boundary.*

**Proof** We already know that, if $\Omega$ is a convex domain in $P$ with $\partial\Omega = \partial\Sigma$, then $\Sigma \cup \operatorname{ext}\Omega = \emptyset$. Then one can consider a hemi-sphere under the plane $P$ whose boundary disc $D$ is contained in $P$ and is large enough that $\Omega \subset \operatorname{int} D$. Thus $\Sigma \cup (D - \Omega) \cup (S - D)$ is a compact surface embedded into $\mathbb{R}^3$ and so determines an interior domain, say $W$. Choose a Gauss map $N$ for $\Sigma$ in such a way that $N$ points into $W$ at each point. It turns out that, if there are points of the surface $\Sigma$ in both half-spaces determined by $P$, then $N$ takes the same value at the points where the height function attains its maximum and minimum respectively. Reversing $N$ if necessary, we can conclude that the Gauss map of $\Sigma$ (for which $H > 0$) takes the same value at the highest and at the lowest points of the surface. Lower a sphere of radius $1/H$ to the highest point or pushing it up to the lowest one we obtain a contradiction using the (interior) maximum principle. Thus the surface lies in one of the half-spaces determined by the plane $P$ and rises on it less than or equal to

$1/H$. Using again half-cylinders of radius $1/2H$ and axis in a plane parallel to $P$ at height $1/2H$ as barriers, the (boundary) maximum principle shows us that the surface is transversal along its boundary.

**Remark** Combining the two results obtained in Corollary 3 above and in Corollary 2 one can conclude that a *cmc H* compact surface embedded into Euclidean space with convex planar boundary and $AH^2 \leq \pi$ must be contained in one of the half-spaces determined by the boundary plane and, moreover, inside the right cylinder with that boundary as cross-section. Using Alexandrov reflection we can assert that *if* $\Sigma$ *is a* cmc $H$ *compact surface embedded into* $\mathbb{R}^3$ *with planar convex boundary and area* $A$ *satisfying* $AH^2 \leq \pi$*, then* $\Sigma$ *is a graph.*

**Remark** From (1), we have that $H^2 \leq L^2/4\bar{A}^2$ where $L$ is the length of $\partial\Sigma$ and $\bar{A}$ is the area of the planar convex domain $\Omega$ which determines $\partial\Sigma$. Hence *if the* cmc $H$ *compact surface* $\Sigma$ *embedded in Euclidean space with planar convex boundary has area* $A$ *satisfying*

$$A \leq \frac{8\pi\bar{A}^2}{L^2},$$

*then* $\Sigma$ *lies in one of the half-spaces determined by the boundary plane and is transversal to this plane along* $\partial\Sigma$. Notice that, when the convex curve $\partial\Sigma$ is close to a circle, then the isoperimetric quotient $4\pi\bar{A}/L^2$ approximates to 1 and so the upper bound above is near to $2\bar{A}$.

Now we are going to see that our height estimate for *cmc* compact surfaces immersed into $\mathbb{R}^3$ allows us to obtain sometimes gradient estimates which will serve to find an existence theorem for graphs, improving the one obtained by Serrin in [**Se1**] when the prescribed boundary is planar.

**Corollary 4** *Let* $\Gamma$ *be a convex closed planar curve with length* $L$*. If* $H$ *is a non-negative real number such that* $LH < \sqrt{3}\pi$*, then there exists a* cmc $H$ *graph whose boundary is* $\Gamma$*.*

**Proof** We denote by $\Omega$ the planar convex domain enclosed by the curve $\Gamma$. To prove the existence of a *cmc* $H$ graph with boundary $\Gamma$ is equivalent to finding a solution $u$ of the Dirichlet problem corresponding to the equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 2H. \tag{6}$$

This can be achieved using a continuity method (see [**GT**, Theorem 13.8] and [**KK**] for an example in the same context) provided we are able to get a priori $C^1(\bar{\Omega})$ estimates independent of $H$ for each solution of (6) with $0 \leq H \leq c$ and $Lc < \sqrt{3}\pi$. For $H = 0$ we have the trivial solution $u \equiv 0$, and the implicit function theorem can be invoked in order to guarantee existence for $0 < H \leq c_0$, where $c_0$ is a positive small number. So, we will work in the range $c_0 \leq H \leq c$. For every solution $u$ of (6) with these restrictions on $H$ we know that the height lemma for graphs due to Serrin [**Se2**] implies

$$u \leq \frac{1}{H} \leq \frac{1}{c_0}.$$

So it remains to look for global gradient estimates of our solutions on $\bar{\Omega}$. But general considerations about quasilinear equations like (6) [**GT**, 11.3] or a detailed analysis of our concrete case show that it suffices to obtain these gradient estimates on the boundary $\partial\Omega = \Gamma$. Let $u_H$ be a solution of (6) and $\Sigma_H$ the corresponding *cmc* $H$ surface with $c_0 \leq H \leq c$. Then we have existence of solutions $u_h$ for all $h$ with $0 \leq h \leq H$. All the corresponding surfaces $\Sigma_h$ are topologically discs whose corresponding Gauss curvature functions $K_h$ satisfy $K_h \leq h^2$. So we may apply them the following isoperimetric inequality due to Barbosa and do Carmo [**BdC**]

$$L^2 \geq 4\pi A_h \left(1 - \frac{A_h h^2}{4\pi}\right).$$

That is, the area $A_h$ of $\Sigma_h$ satisfies the following second degree inequality

$$h^2 A_h^2 - 4\pi A_h + L^2 \geq 0,$$

which has discriminant $16\pi^2 - 4L^2h^2 > 16\pi^2 - L^2c^2 > \pi^2 > 0$. Then

$$\text{either} \quad A_h \leq \frac{2\pi - \sqrt{4\pi^2 - L^2h^2}}{h^2} \quad \text{or} \quad A_h \geq \frac{2\pi + \sqrt{4\pi^2 - L^2h^2}}{h^2}.$$

Since $A_h$ depends continuously on $h$ and the first inequality occurs when $h = 0$, we have that this first inequality holds for all $h$ with $0 \leq h \leq H$, and so, in particular, for $h = H$. This gives

$$\frac{A_H H}{2\pi} \leq \frac{2\pi - \sqrt{4\pi^2 - L^2 H^2}}{2\pi H}.$$

From here it is not difficult to see, using our assumption $c < \sqrt{3}\pi/L$ , that there exists a real number $\varepsilon(c) > 0$ depending only on the constant $c$ such that

$$\frac{A_H H}{2\pi} \leq \frac{1}{2H} - \varepsilon(c).$$

Now our Theorem 1 says that the graph $\Sigma_H$ rises above the plane $P$ containing the curve $\Gamma = \partial\Sigma_H$ less than or equal to $1/2H - \varepsilon(c)$. Consider any slab $S$ with height $1/2H$ parallel to $P$ and move the graph $\Sigma_H$ in such a way that the boundary plane $P$ is at height $\varepsilon(c)/2$ above the lowest limiting plane of $S$. For each point $p \in \Gamma$ we denote by $C_p$ the piece belonging to $S$ of a right cylinder of radius $1/2H$ whose axis lies in the plane $R$ and is parallel to the tangent line to $\Gamma$ at $p$. This surface $C_p$ clearly has constant mean curvature $H$. We separate $C_p$ and $\Sigma_H$ for them not to intersect with the concave side of $C_p$ in front of the surface $\Sigma_H$. After this, we slide $C_p$ toward the surface until they touch each other for the first time. The maximum principle and the fact that $\Gamma$ is convex imply that this contact point is $p$. As this holds for every $p \in \Gamma$, the surface $\Sigma_H$ is inside the convex body delimited by the curve $\Gamma$ and all the cylinder pieces $C_p$ with $p \in \Gamma$. Then the slope of our graph at each point $p$ of the boundary $\Gamma$ is less than the slope of the corresponding $C_p$. But this cylinder piece cuts the plane $P$ with an angle which does not depend on $p$ and which is less than $\pi/2$. Hence, if $a \in \mathbb{R}^3$ is a unit vector orthogonal to the plane $P$, we have obtained that there exists a positive number $\delta(c)$ such that

$$\langle \nu_p, a \rangle \leq 1 - \delta(c) \quad \text{for each } p \in \Gamma,$$

where $\nu_p$ is the inner conormal to $\Sigma_H$ at $p$. It suffices to observe that

$$\langle \nu, a \rangle = \frac{1}{\sqrt{1 + |\nabla u_H|^2}}$$

in order to conclude that we have found the global gradient estimate that we were looking for.

**Remark** The existence theorem due to Serrin [**Se1**] asserts that, for a convex closed planar curve $\Gamma$ with curvature function $k$ such that $k \geq 2H > 0$, there exist *cmc H* graphs *with arbitrary boundary Dirichlet data*. In this sense, his theorem and our Corollary 4 are not comparable, but the Serrin result has the drawback, when applied to our situation of vanishing boundary data, that it requires the curve $\Gamma$ to be *strictly convex*. This is not the case in our hypothesis. On the other hand, an easy consequence of Corollary 4 is the fact that *if* $\Gamma$ *is a planar closed convex curve whose curvature function* $k$ *satisfies* $\sqrt{3}k > 2H$, *then there exists a* cmc $H$ *graph whose boundary is* $\Gamma$.

# 4 Small volume *cmc* compact surfaces

Our aim at the beginning this work was to understand how the solutions of Plateau's problem with constrained (algebraic) volume look and how they behave when the prescribed volume is modified. More generally we wanted to know how the shape of a *cmc* compact surface (with prescribed boundary) and some geometric quantities (such as the area and the mean curvature) change as functions of the algebraic volume. When this volume is very large this has been done by Wente in [**W4**], at least in the case of immersions from the disc which minimize the area with fixed volume. A new approach for embedded surfaces with convex planar boundary can be seen in a forthcoming paper by Ros and Rosenberg [**RR**]. However nothing is known about the shape, the area or the behaviour of the mean curvature for *cmc* compact surfaces with assigned boundary when the corresponding algebraic volume is small, aside from the *experimental* evidence obtained on the computers of the Amherst G.A.N.G. pointed out by Hoffman in [**HR**].

Before starting with the results, we are going to comment on some relevant facts concerning the algebraic volume of oriented compact immersed surfaces. For an immersion $\phi : \Sigma \to \mathbb{R}^3$ from an oriented compact surface $\Sigma$ with non-empty boundary $\partial\Sigma$ into Euclidean space, we had recalled in Section 2 that its algebraic volume $V$ is defined as

$$V = -\frac{1}{3}\int_\Sigma \langle \phi, N\rangle \, dA,$$

where $N$ represents the Gauss map for the immersion $\phi$ compatible with the orientation of $\Sigma$. Of course, this $V$ depends on the choice of the origin in $\mathbb{R}^3$. But *in the case of* cmc *immersions with planar boundary the algebraic volume is independent of the choosen origin provided it is taken in the boundary plane*. This can be easily shown from the fact that, in this case, $\Delta\phi = 2HN$, by combining the divergence theorem and balancing formula. So, we will make such a choice from now.

On the other hand, assume that the boundary of our immersed surface $\phi(\partial\Sigma)$ is a Jordan curve $\Gamma$ in the plane $P$ and let $\Omega$ be the corresponding bounded domain in $P$. In this situation, if $\phi$ is an embedding with image $M$ such that $M \cup \Omega$ bounds a region $W$ of $\mathbb{R}^3$, then it turns out from the divergence theorem applied to the position vector field $\phi$ on $W$ that

$$\operatorname{vol} W = -\frac{1}{3}\int_{M\cup\Omega} h,$$

where $\operatorname{vol} W$ is the Lebesgue volume in $\mathbb{R}^3$ and $h = \langle \phi, N\rangle$ is the support function of the surface $M \cup \Omega$ corresponding to the inner normal field of $W$. In the last equality the choice of origin is not significant, but if it is taken in $P$ the integral on the right side becomes

$$\operatorname{vol} W = -\frac{1}{3}\int_M h = -\frac{1}{3}\int_\Sigma \langle \phi, N\rangle \, dA = V,$$

that is, *the algebraic volume of the embedding $\phi$ coincides with the Lebesgue measure of the region enclosed by its image $M$ together with the convex planar domain $\Omega$ determined by its boundary $\Gamma$.*

Another remarkable peculiarity concerning the algebraic volume of *cmc* $H > 0$ immersions from compact orientable surfaces with planar boundary is that it is always positive (recall: we consider the orientation given by the Gauss map for which $H > 0$). This assertion follows from the equality

$$2HV = \int_\Sigma |\nabla \langle \phi, a\rangle|^2 \, dA,$$

which can be easily derived by using the divergence theorem. *This proves that $V \geq 0$ and $V = 0$ only in the case that the image of the immersion lies into the plane $P$, that is, in the minimal case $H = 0$.*

After these observations we start to study *cmc* $H > 0$ immersions $\phi$ from compact surfaces $\Sigma$ into Euclidean three-space spanning a *convex* planar curve $\Gamma$ (that is, $\phi(\partial\Sigma) = \Gamma$) which are *small* from the volume point of view.

**Lemma 2** *Let $\phi_k : \Sigma \to \mathbb{R}^3$, $k \in N$, be a sequence of* cmc $H_k$ *immersions from a compact surface $\Sigma$ into Euclidean space with $\phi_k(\partial\Sigma) = \Gamma_k$, where each $\Gamma_k$ is a star-shaped planar Jordan curve. Suppose that* $\lim_{k\to\infty} \Gamma_k = \Gamma$*, where $\Gamma$ is another star-shaped planar Jordan curve, and that the sequence of the corresponding algebraic volumes satisfies* $\lim_{k\to\infty} V_k = 0$*. Then*

$$\lim_{k\to\infty} A_k = \bar{A} \quad \textit{and} \quad \lim_{k\to\infty} H_k = 0,$$

*$A_k$ being the area of $\phi_k$ and $\bar{A}$ the area of the planar domain determined by the limiting curve $\Gamma$.*

**Proof** For each $k \in \mathrm{N}$, let $\Omega_k$ be the planar domain determined by the respective Jordan curve $\Gamma_k$, $\bar{A}_k$ the corresponding (algebraic or Lebesgue) area and $L_k$ the length of $\Gamma_k$. Since $\Gamma_k$ tends to the curve $\Gamma$ when $k$ goes to infinity, we have

$$\lim_{k\to\infty} \bar{A}_k = \bar{A} \qquad \lim_{k\to\infty} L_k = L,$$

where $L$ is the length of $\Gamma$. Using the inequality remarked in (1) and as $\bar{A} > 0$ we observe that

$$0 \leq \lim_{k\to\infty} H_k \leq \lim_{k\to\infty} \frac{L_k}{2\bar{A}_k} = \frac{L}{2\bar{A}},$$

and so the sequence of the mean curvatures $H_k$ is bounded. Then, from our hypothesis about the volume sequence $V_k$, one sees that

$$\lim_{k\to\infty} H_k V_k = 0. \tag{7}$$

Consider now the Minkowski formula for each *cmc* immersion $\phi_k$ and, as $\Gamma_k$ is planar, we may choose an inner unit normal $n_k$ for $\partial\Omega_k$ in the corresponding plane $P_k$ to obtain

$$A_k - 3H_kV_k = -\frac{1}{2}\int_{\partial\Sigma}\langle\phi_k,\nu_k\rangle\,ds_k = -\frac{1}{2}\int_{\partial\Sigma}\langle\phi_k,n_k\rangle\langle n_k,\nu_k\rangle\,ds_k,$$

where $\nu_k$ is the upward conormal to $\phi_k$. As the curve $\Gamma_k$ is star-shaped, after a suitable choice of $n_k$, its support function $\langle\phi_k,n_k\rangle$ is negative on $\partial\Sigma$. Also $\langle\nu_k,n_k\rangle\leq 1$ and so

$$A_k - 3H_kV_k \leq -\frac{1}{2}\int_{\partial\Sigma}\langle\phi_k,n_k\rangle\,ds_k = \bar{A}_k.$$

On the other hand, the surface $\phi_k(\Sigma)$ projects on the correponding plane $P_k$ onto a region including $\Omega_k$ because, if there were a point in $\Omega_k$ such that the straight line passing through it perpendicularly to the plane $P_k$ did not touch the surface $\phi_k(\Sigma)$, then the curve $\phi_k(\partial\Sigma)=\Gamma_k$ would not be homologous to zero in the shadow of $\phi_k(\Sigma)$ on $P_k$. However the boundary $\partial\Sigma$ is null-homologous in the surface $\Sigma$ and this is a contradiction. Hence

$$A_k \geq \bar{A}_k.$$

Taking limits in the two last inequalities, we may conclude from (7) that

$$\lim_{k\to\infty} A_k = \bar{A},$$

which was the first conclusion that we looked for. By the way, from these very inequalities, it follows

$$\bar{A} = \lim_{k\to\infty} -\frac{1}{2}\int_{\partial\Sigma}\langle\phi_k,\nu_k\rangle\,ds_k = \lim_{k\to\infty} -\frac{1}{2}\int_{\partial\Sigma}\langle\phi_k,n_k\rangle\,ds_k$$

and so

$$\lim_{k\to\infty}\int_{\partial\Sigma} -\langle\phi_k,n_k\rangle\,(1-\langle n_k,\nu_k\rangle)\,\frac{ds_k}{ds}\,ds = 0,$$

where $ds$ is the arc-length element for $\Gamma$. Since our boundary curves are star-shaped the integrands above are non-negative and so Fatou's Lemma implies

$$\liminf_{k\to\infty} -\langle\phi_k,n_k\rangle\,(1-\langle n_k,\nu_k\rangle)\,\frac{ds_k}{ds} = 0.$$

But $\lim_{k\to\infty}\langle\phi_k, n_k\rangle(ds_k/ds) < 0$ because the sequence $\Gamma_k$ converges to the star-shaped curve $\Gamma$. Then, there is a subsequence (which will be suitably relabelled) of the original sequence of immersions such that $\lim_{k\to\infty}\langle\nu_k, n_k\rangle = 1$. Let $a_k$ be a unit vector of $\mathbb{R}^3$ orthogonal to the plane $P_k$ where the curve lies. Then

$$\lim_{k\to\infty}\langle\nu_k, a_k\rangle = 0.$$

Hence, our Lemma 1 and the Lebesgue bounded convergence theorem show us that $\lim_{k\to\infty} H_k\bar{A}_k = 0$. But we already knew that the sequence $\bar{A}_k$ tends to $\bar{A} > 0$ when $k$ goes to infinity. So, there exists a subsequence of our sequence of *cmc* immersions with $\lim_{k\to\infty} H_k = 0$. As this holds for every sequence, the second conclusion that we wanted to obtain follows.

Combining the two assertions of Lemma 2 above and Corollary 2 one sees that, given a star-shaped planar Jordan curve $\Gamma$, any *cmc* compact surface $\Sigma$ immersed into Euclidean space with boundary $\Gamma$ and algebraic volume small enough lies inside of the right cylinder whose base is the convex hull of its $\Gamma$. On the other hand, if the volume of $\Sigma$ is sufficiently small, from Lemma 2, its mean curvature will be small and, using Corollary 4, provided that we require $\Gamma$ to be convex, there will exist a *cmc* graph $G$ with boundary $\Gamma$ and with the same mean curvature $H$ as $\Sigma$. Hence, if $\Gamma$ is a convex closed planar curve, there is a positive number $V_\Gamma$ such that any *cmc* compact surface $\Sigma$ immersed into $\mathrm{R}^3$ with $\partial\Sigma = \Gamma$ and $V \leq V_\Gamma$ lies inside the right cylinder determined by the domain $\Omega$ which $\Gamma$ bounds and such that there exists a *cmc* graph $G$ on $\Omega$ with the same curvature as $\Sigma$ and $\partial G = \partial\Sigma = \Gamma$. The following result will permit us to compare $G$ and $\Sigma$ (cf. the *positive flux lemma* in [**KK**]).

**Lemma 3** *Let $\Gamma$ be a planar closed Jordan curve and $\Omega$ the corresponding bounded planar domain. Suppose that there exists a* cmc *$H$ graph on $\Omega$ and denote by $C$ the right cylinder with cross-section $\Omega$. Then, any compact* cmc *$H$ surface $\Sigma$ immersed into Euclidean space with $\partial\Sigma = \Gamma$ and lying in the inside of $C$ either coincide with $G$ or with its reflection about the boundary plane.*

**Proof** We first show that $\Sigma$ must be under the graph $G$. Suppose in fact that this were not the case. Then we translate $G$ upwards so it does not touch $\Sigma$ and then we drop it until it reaches a contact point $p$ with $\Sigma$ for the first time and, so, a final position tangent to $\Sigma$. Denote by $G'$ the translated graph. We have that $p \notin \partial G'$ because $\Sigma \subset C$ and, since $\Sigma$ rised on the boundary plane more than $G$, we know that $p \notin \partial\Sigma$. On the other hand, as $G'$ is a graph, the unit vector $N_p$ normal to $G'$ at the point $p$ for which the mean curvature $H$ is positive points downwards. Moreover, $\Sigma$ is now under $G'$ and so its mean curvature at $p$ must be greater than or equal to that of $G'$. As a conclusion, the mean curvatures of $G'$ and $\Sigma$ agree for the same choice of normal vector at $p$. Then, from the maximum principle, the surfaces $G'$ and $\Sigma$ should be coincide and, so, we would reach a patent contradiction because $\partial G'$ and $\partial\Sigma$ are at different heights. Thus we see that $\Sigma$ is under $G$ as we had claimed. The same reasoning is showing that $\Sigma$ is above $G^*$, $G^*$ being the reflection of $G$ about the plane $P$ containing $\Gamma$. Then, if $a$ is a unit vector perpendicular to $P$ and $\nu_\Sigma$, $\nu_G$ represent the inner conormals of $\Sigma$ and $G$ respectively along their common boundary $\Gamma$, we have that

$$|\langle \nu_\Sigma, a\rangle| \leq \langle \nu_G, a\rangle.$$

If the equality held at some point of $\Gamma$, the (boundary) maximum principle would say to us that either $\Sigma = G$ or $\Sigma = G^*$ as we claimed. In other case we would obtain that

$$|\langle \nu_\Sigma, a\rangle| < \langle \nu_G, a\rangle$$

and, integrating this strict inequality on $\Gamma$ and using balancing formula (in Lemma 1), we would attain a contradiction because the mean curvatures of $\Sigma$ and $G$ are the same.

**Remark** It is convenient to point out that this proof is similar to the proof of the positive flux lemma in the paper [**KK**] by Korevaar and Kusner.

After the uniqueness result stated in Lemma 3 above, and from the consequences that we have mentioned from Lemma 2 and Corollaries 2 and 4, we may state the following theorem about *cmc* compact surfaces with convex boundary and small volume.

**Theorem 2** *Let $\Gamma$ be a convex closed planar curve. There exists a positive number $V_\Gamma$ depending only on $\Gamma$ such that, for any $0 \leq V \leq V_\Gamma$, there is a* cmc *graph $G_V$ with boundary $\Gamma$ and enclosed volume $V$. Moreover this graph $G_V$ is the only* cmc *compact surface immersed into Euclidean space whose boundary is $\Gamma$ and whose algebraic volume is $V$.*

Since the smooth solutions to the isoperimetric problem are *cmc* surfaces (because they are minimizers and so critical points for the area when the volume is constrained to take a fixed value) and using the existence and regularity results arising from geometric measure theory [**Alm**] remarked in the Introduction of this paper, it follows that

**Theorem 3** *Let $\Omega$ be a bounded convex planar domain. Then there is a positive number $V_\Omega$ such that, for any real number $0 \leq V \leq V_\Omega$, the isoperimetric region corresponding to the pair $(\Omega, V)$ is bounded by $\Omega \cup G$ where $G$ is a* cmc *graph on $\Omega$ with boundary $\partial\Omega$.*

In order to conclude this work and obtain all the results that we discussed in the Introduction, it remains to show that, in case of the surfaces of disc-type and when the prescribed boundary is a circle, the best constants whose existence are asserted in the last two theorems are computable. In fact, we have

**Theorem 4** *An immersed* cmc *disc spanning a unit circle in Euclidean space is a small spherical cap provided that its algebraic volume is less than or equal to the volume of a unit radius half-sphere.*

**Proof** We will let $D$ be the two-dimensional disc and $\phi : D \to \mathbb{R}^3$ a *cmc $H$* immersion with area $A$ and algebraic volume $V$. Clearly we can exclude from consideration the minimal case $H = 0$. So, we assume that $H > 0$ and use the corresponding orientation. From our hypothesis, $\phi(\partial D)$ is a unit circle, say

$$\{p \in \mathbb{R}^3 \,|\, |p| = 1,\ \langle p, a\rangle = 0\},$$

where $a$ is a unit vector. If $\nu$ is the inner conormal of $\phi$ along the boundary $\partial D$, the Schwarz inequality yields

$$\left(\int_{\partial D} \langle \nu, \phi \rangle \, ds\right)^2 \leq 2\pi \int_{\partial D} \langle \nu, \phi \rangle^2 \, ds = 2\pi \int_{\partial D} \left(1 - \langle \nu, a \rangle^2\right) \, ds. \tag{8}$$

Using again the Schwarz inequality and balancing formula, we have

$$\int_{\partial D} \langle \nu, a \rangle^2 \, ds \geq \frac{1}{2\pi} \left(\int_{\partial D} \langle \nu, a \rangle \, ds\right)^2 = 2\pi H^2.$$

Substituting in (8), we obtain

$$\left|\int_{\partial D} \langle \nu, \phi \rangle \, ds\right| \leq 2\pi \sqrt{1 - H^2}.$$

This inequality provides an upper bound for the right member in the Minkowski formula for the immersion $\phi$. Then

$$A - 3HV \leq \pi \sqrt{1 - H^2}. \tag{9}$$

On the other hand, we have proved in [**LMo**] that, if $\phi(D)$ is not the small spherical cap with mean curvature $H$, then the area $A$ must be greater than that of the corresponding big spherical cap. That is

$$A > \frac{2\pi}{H^2} \left(1 + \sqrt{1 - H^2}\right).$$

Hence, from (9) and this last inequality, we get (unless the surface is a small spherical cap) that

$$V > f(H) = \frac{2\pi}{3H^3} \left(1 + \sqrt{1 - H^2}\right) - \frac{\pi}{3H} \sqrt{1 - H^2}.$$

As the mean curvature $H$ moves in the range $0 < H \leq 1$ (see [**LMo**] or [**He2**]), we can look for the minimum of the increasing function $f$ above in this range and conclude that $V > 2\pi/3$, as we had asserted.

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