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Graphs of Constant Mean Curvature in Hyperbolic Space

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Abstract. We study the problem of finding constant mean curvature graphs over a domain Ω of a totally geodesic hyperplane and an equidistant hypersurface Q of hyperbolic space. We find the existence of graphs of constant mean curvature H over mean convex domains $\Omega \subset Q$ and with boundary $\partial\Omega$ for $-H_{\partial\Omega} < H \leq |h|$, where $H_{\partial\Omega} > 0$ is the mean curvature of the boundary $\partial\Omega$. Here h is the mean curvature respectively of the geodesic hyperplane ($h = 0$) and of the equidistant hypersurface ($0 < |h| < 1$). The lower bound on H is optimal.

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1. Introduction and Statements of Results

One of the most important equations in differential geometry is the prescribed mean curvature equation. Physically, a constant mean curvature surface in Euclidean space describes a soap bubble trapping some air. If we think of a domain of a plane as a membrane, and if we blow up air, we construct constant mean curvature surfaces. Our intuition tells us that in a first moment, we will obtain graphs of functions that satisfy the constant mean curvature equation (in fact, this occurs in Euclidean 3-space if the enclosed volume is small enough [7]).

In this paper we study the Dirichlet problem for the mean curvature equation in hyperbolic space $\mathbf{H}^{n+1}$ of dimension $n+1$. Consider Q to be one of the following types of umbilical hypersurfaces in $\mathbf{H}^{n+1}$: a totally geodesic hyperplane, an equidistant hypersurface, and a horosphere. In Q we choose one orientation ξ. Let Ω be a smooth bounded domain of Q and f a smooth function in $\overline{\Omega}$. The graph on Ω determined by f is defined as follows. For each point $q \in \Omega$, there exists a unique unit speed geodesic $\gamma\colon \mathbf{R} \to \mathbf{H}^{n+1}$ with $\gamma(0) = q$ and $\xi(q)$ as the initial tangent vector. Let p be the point defined as $p = \gamma(f(q))$, that is, the point of the geodesic γ at distance $f(q)$ from Q. The graph Σ of f is defined as the set of all points

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$\gamma(f(q))$, $q \in \Omega$. Remark that each point p of Σ is determined by one point q of Ω and the number $f(q)$. This is so because the geodesics passing through the points of Ω and orthogonal to Ω do not intersect themselves (this is not the case if Q is a hyperbolic sphere).

If $f = 0$ along $\partial\Omega$, then the boundary of Σ is $\partial\Omega$.

The case where Q is a horosphere, has been extensively studied when Ω is a mean convex domain, that is, the minimum $H_{\partial\Omega}$ of the mean curvature of $\partial\Omega$ in Q is positive with respect to the inner orientation. Lin proved that a minimal graph ($H = 0$) exists with boundary $\partial\Omega$ [5]. In the interval $0 \leq H \leq 1$, the existence for such a graph was proved by Nelli and Spruck in [9]. Montiel and the author extended the range of existence to $-H_{\partial\Omega} < H \leq 1$ [8]. In this connection, we point out the considerable literature concerned with the existence of complete hypersurfaces of a constant mean curvature in hyperbolic space with asymptotic boundary at infinity (see [3] and references therein).

In this paper, we treat the Dirichlet problem for mean convex domains of totally geodesic hyperplanes and equidistant hypersurfaces. We prove the following theorem:

THEOREM 1.1. *Let Q be a totally geodesic hyperplane or an equidistant hypersurface and denote its mean curvature by h. Let Ω be a bounded domain in Q with smooth boundary $\partial\Omega$. Assume $\partial\Omega$ is mean convex, that is, the number*

$$H_{\partial\Omega} := \min_{q\in\partial\Omega} H^{\partial\Omega}(q)$$

is positive, where $H^{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$ in Q with respect to the inner orientation. Let H be a real number such that

$$-H_{\partial\Omega} < H \leq |h|.$$

Then there exists a graph on Ω with mean curvature H and bounded by $\partial\Omega$. Furthermore, if $H_{\partial\Omega} > 1$, then the interval of existence extends to $-H_{\partial\Omega} < H \leq 1$.

This result may be regarded as an analogue of the main theorem in [8] cited above. Remark that Theorem 1.1 cannot be improved, as the following case shows. Let $\partial\Omega$ be a round $(n-1)$-sphere in Q. Then the only graphs of constant mean curvature on Ω with boundary $\partial\Omega$ are pieces of umbilical hypersurfaces [6, prop. 2.1 and thm. 2.2]. Then it is not difficult to see that the mean curvature H satisfies $|H| < H_{\partial\Omega}$.

The technique for proving Theorem 1.1 is the continuity method applied to the equation for the mean curvature on Ω. This is an elliptic equation of the divergence type, and the difficult part of the proof is to obtain *a-priori* $C^{1,\alpha}$-bounds for solutions of the equation (see [2]). Starting from the trivial solution $f = 0$ on Ω (for $H = |h|$), that corresponds with the very domain Ω, appropriate C^0 and C^1-estimates will allow us to blow up this solution, thus obtaining solutions for each H until the bound $-H_{\partial\Omega}$.

The article is organized as follows. In Sections 2 and 3 we recall some basic facts of hyperbolic geometry and we derive the equation that satisfies the mean curvature. In Sections 4 and 5 we obtain, respectively, *a-priori* estimates for the C^0 and C^1 norms of the desired solutions. Then, in Section 6, Theorem 1.1 is proved using the continuity method.

2. Preliminaries and Definitions

This section is devoted to fixing the notation, recalling a few properties of hyperbolic geometry, and introducing the concept of the graph. First, we recall the Minkowski model for hyperbolic space. Let $\mathbf{L}^{n+2}$ be the vector space $\mathbf{R}^{n+2} = \{x = (x_0, \ldots, x_{n+1}); x_i \in \mathbf{R}\}$ equipped with the Lorentzian metric

$$\langle , \rangle = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2.$$

Then the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$ can be viewed as

$$\mathbf{H}^{n+1} = \{x \in \mathbf{L}^{n+2}; \langle x, x\rangle = -1, x_0 > 0\}$$

with the induced metric from $\mathbf{L}^{n+2}$. In this model, the umbilical hypersurfaces are determined as the intersections between $\mathbf{H}^{n+1}$ with affine hyperplanes of $\mathbf{R}^{n+2}$. Depending on the causal character of the affine hyperplane, we have the following description of umbilical hypersurfaces. Let $a \in \mathbf{L}^{n+2}$ be a vector such that $\langle a, a\rangle = \epsilon$, with $\epsilon = -1, 0, 1$. Let $\tau \in \mathbf{R}$. Then the umbilical hypersurfaces of $\mathbf{H}^{n+1}$ are described as the sets

$$U_{a,\tau} = \{q \in \mathbf{H}^{n+1}; \langle q, a\rangle = \tau\}.$$

The kinds of umbilical hypersurfaces are the following: totally geodesic hyperplanes ($\epsilon = 1, \tau = 0$), equidistant hypersurfaces ($\epsilon = 1, \tau \neq 0$), horospheres ($\epsilon = 0, \tau \neq 0$), and hyperbolic spheres ($\epsilon = -1, |\tau| > 1$). A unit normal field on $U_{a,\tau}$ is defined as

$$\xi(q) = -\lambda(a + \tau q), \quad \lambda = \frac{1}{\sqrt{\tau^2 + \epsilon}}. \tag{1}$$

The shape operator D on $U_{a,\tau}$ is given by $D\xi = \lambda\tau\mathrm{Id}$. Thus, the mean curvature is constant on $U_{a,\tau}$ and is given by $h = \lambda\tau$: $h = 0$ for totally geodesic hyperplanes, $h \in (0, 1)$ for equidistant hypersurfaces, $h = 1$ for horospheres, and $h > 1$ for hyperbolic spheres. In Euclidean space, the ambient space is foliated by totally geodesic (parallel) hyperplanes. One of the differences between $\mathbf{H}^{n+1}$ and Euclidean space is that, besides the totally geodesic hyperplanes, $\mathbf{H}^{n+1}$ can be foliated by equidistant hypersurfaces and horospheres. This fact will be applied in our proofs.

Now we give the definition of *graph* in hyperbolic space that generalizes what occurs in Euclidean space. Recall that a geodesic in the Minkowski model for

$\mathbf{H}^{n+1}$ through a point x and with initial velocity v is given by $\gamma(s) = \cosh(s)x + \sinh(s)v$. According to what was stated in the Introduction, we have the following definition:

DEFINITION 2.1. Consider the Minkowski model for hyperbolic space $\mathbf{H}^{n+1}$ and let $U_{a,\tau}$ be a totally geodesic hyperplane, an equidistant hypersurfaces or a horosphere where an orientation ξ has been fixed. Consider Ω a domain in $U_{a,\tau}$ and let f be a smooth function on Ω. The graph Σ of the function f is defined as

$$\Sigma = \{\cosh f(q)q + \sinh f(q)\xi(q);\, q \in \Omega\}.$$

Each point $p \in \Sigma$ is determined by the pair $(q, f(q)) \in \Omega \times \mathbf{R}$, where p is the point of the geodesic starting from q with velocity $\xi(q)$ at distance $f(q)$ from q. In particular, if the choice of the orientation in $U_{a,\tau}$ is as in (1), the graph Σ is explicitly given by

$$\Sigma = \{(\cosh f(q) - h \sinh f(q))q - \lambda \sinh f(q)a;\, q \in \Omega\}.$$

It is not difficult to see that the fact that Σ is a graph is equivalent to $\langle N, a\rangle \neq 0$ on Σ, where N is a unit normal field on Σ.

Remark 1. The definition of the graph on a domain of a totally geodesic hyperplane that appears in [1] does not correspond to our case. Exactly, the function u that describes the graph in [1] associates for each $q \in \Omega$ one point in one of the two horocycles passing by q and orthogonal to the hyperplane. Therefore, the existence result in [1] does not correspond to Theorem 1.1 in the present paper.

Convention. Throughout this paper, if $U_{a,\tau}$ is a totally geodesic hyperplane or an equidistant hypersurface, we will choose τ to be nonnegative (it suffices to replace $-a$ by a if it should not be so). The orientation on $U_{a,\tau}$ will be ξ as in (1) and then the mean curvature h is nonnegative. Also, the orientation N on the graph Σ will be so that $\langle N, a\rangle < 0$.

For the establishment of $C^{1,\alpha}$-bounds for our solution, we need two elliptic equations that enclose important information about the geometry of a constant mean curvature hypersurface M in $\mathbf{H}^{n+1}$. Consider the Minkowski model of $\mathbf{H}^{n+1}$ and let $x\colon M \to \mathbf{H}^{n+1}$ be an immersion of constant mean curvature H. If $a \in \mathbf{L}^{n+2}$, a brief calculation shows that (see [4]):

$$\Delta\langle x, a\rangle = n\langle x, a\rangle + nH\langle N, a\rangle, \tag{2}$$

$$\Delta\langle N, a\rangle = -nH\langle x, a\rangle - |\sigma|^2\langle N, a\rangle, \tag{3}$$

where Δ stands for the Beltrami–Laplacian operator of Σ and σ is the second fundamental form of x.

A characteristic of our graphs with constant mean curvature is that they lie in one of the two sides that determine the umbilical hypersurface that contains its boundary. The following result is a particular case of theorem 2.2 in [6]. For the sake of completeness, we prove this result adapted to our situation.

THEOREM 2.2. *Let Q be a totally geodesic hyperplane or an equidistant hypersurface of $\mathbf{H}^{n+1}$ and let Ω be a bounded domain of Q. Let Σ be a graph on Ω with boundary $\partial\Omega$. If Σ has constant mean curvature, then either $\Sigma = \Omega$ or Σ is included in one of the two components of $\mathbf{H}^{n+1} \setminus Q$.*

Proof. We use the upper-halfspace model for $\mathbf{H}^{n+1}$, i.e.,

$$\mathbf{R}^{n+1}_{+} = \{(x_1, \cdots, x_{n+1}) \in \mathbf{R}^{n+1}; x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$\frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2}{x_{n+1}^2}.$$

Up to an isometry of the ambient, we suppose that

$$Q = \{(x_1, \ldots, x_{n+1} \in \mathbf{R}^{n+1}_{+}; x_n = \cot(\theta) x_{n+1}\},$$

where $0 < \theta \leq \pi/2$. Remark that Q is a vector hyperplane that makes an angle θ with $x_{n+1} = 0$. If $\theta = \pi/2$, Q is a totally geodesic hyperplane; in another case, Q is an equidistant hypersurface.

Denote $w = (0, \ldots, 0, -\sin\theta, \cos\theta)$. Choose in Q the orientation N_Q given by $N_Q = x_{n+1} w$. According to this orientation, the mean curvature of Q is $h = \cos\theta \geq 0$. Let N be the unit normal vector field in Σ such that $\langle N, N_Q\rangle$ is positive.

Assume that Σ does not agree with Ω. By contradiction, suppose that Σ contains points in both sides of Q. Consider the foliation of $\mathbf{H}^{n+1}$ defined by $\{Q(t); t \in \mathbf{R}\}$, where $Q(t)$ is a horizontal translation of Q in the direction $t(0, \ldots, 1, 0)$:

$$Q(t) = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}_{+}; x_n = t + \cot(\theta) x_{n+1}\}.$$

Consider in $Q(t)$ the unit normal vector field $N_{Q(t)} = N_Q$. Since Σ is compact, for t being small enough, $\Sigma \cap Q(t) = \emptyset$. Letting $t \to 0$, set $t_0 < 0$ the first moment such that $Q(t_0)$ touches Σ. Since the boundary of Σ is $\partial\Omega$, the intersection between Σ and $Q(t_0)$ occurs at some interior point. Let $x_0 \in \Sigma \cap Q(t_0)$. According to the chosen orientations on Σ and $Q(t_0)$, we have $N(x_0) = N_Q(x_0)$. By comparing the mean curvature of both hypersurfaces, the maximum principle yields $h > H$.

A similar reasoning with positive numbers t implies there exists $t_1 > 0$ such that $\Sigma \cap Q(t_1) \neq \emptyset$ and $\Sigma \cap Q(t) = \emptyset$ for $t > t_1$. The maximum principle implies $H > h$, which is a contradiction. □

Examining the above proof, we know on what side of Q lies Σ. With the orientations chosen on Σ and Q, we have that if $H = h$, then $\Sigma = \Omega$; if $H < h$, then Σ is included in the component W of $\mathbf{H}^{n+1} \setminus Q$ where N_Q points to, and if $H > h$, $\Sigma \subset \mathbf{H}^{n+1} \setminus \overline{W}$. Remark that if Q is a totally geodesic hyperplane, then the equation that satisfies Q is $x_n = 0$. In this situation, if H is negative, then $\Sigma \subset \{x_n < 0\}$. For our convenience, we need to establish these facts in the Minkowski model of $\mathbf{H}^{n+1}$. We easily obtain the following corollary:

COROLLARY 2.3. *Let $U_{a,\tau}$ be a totally geodesic hyperplane or an equidistant hypersurface and let Ω be a bounded domain of Q. Let f be a smooth function on Ω, $f = 0$ in $\partial\Omega$, whose graph Σ has constant mean curvature H.*

(1) *If $H = h$, then $\Sigma = \Omega$, i.e. $f = 0$ on Ω.*
(2) *If $H < h$, then $\langle x, a\rangle < \tau$, for $x \in \Sigma \setminus \partial\Sigma$, i.e. $f > 0$ on $\Omega \setminus \partial\Omega$.*
(3) *If $H > h$, then $\langle x, a\rangle > \tau$, for $x \in \Sigma \setminus \partial\Sigma$, i.e. $f < 0$ on $\Omega \setminus \partial\Omega$.*

Moreover, if $H \geq 0$, then $\langle x, a\rangle \geq 0$, for $x \in \Sigma$.

Proof. It remains to consider the case $H \geq 0$. By contradiction, suppose that $\langle x, a\rangle$ assumes negative values on Σ. Let $x_0 \in \Sigma$ be a relative minimum of this function where $\langle x_0, a\rangle < 0$. Then $\Delta\langle x, a\rangle(x_0) \geq 0$, this is a contradiction of Equation (2). □

When Q is a totally geodesic hyperplane, we have the following property that is analogous to what happens with constant mean curvature graphs in Euclidean space:

PROPOSITION 2.4 (Symmetry). *Let Ω be a domain of a totally geodesic hyperplane Q and let f be a smooth function on Ω whose graph has constant mean curvature H. Then* graph$(-f)$ *has constant mean curvature $-H$.*

Proof. Consider the upper-halfspace model for $\mathbf{H}^{n+1}$. After an isometry of $\mathbf{H}^{n+1}$, let Q be the hyperplane defined as $x_n = 0$. Then the (Euclidean) reflection across to Q, i.e.

$$(x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, -x_n, x_{n+1})$$

is an isometry of $\mathbf{H}^{n+1}$ that leaves pointwise fixed Q. The surface obtained by reflecting the graph (f) across to Q is the graph $(-f)$ and this concludes the proof. □

Corollary 2.3 and the symmetry property allow us to consider, without loss of generality, that the mean curvature satisfies $H \leq 0$ in the proof of Theorem 1.1, that is, $f > 0$ on Ω.

3. The Mean Curvature Equation

In this section we establish the Dirichlet problem that we need to solve and we study the nature of the prescribed mean curvature equation associated to this problem. Consider the upper-halfspace for $\mathbf{H}^{n+1}$. Let Q be a totally geodesic hyperplane or an equidistant hypersurface. After an isometry of $\mathbf{H}^{n+1}$, Q is given by $x_n = \cot(\theta) x_{n+1}$. Let Ω be a bounded domain in Q. Parametrize Ω in rectangular coordinates $(x_1, \ldots, x_{n+1})$. The geodesic through a point of Q is given by

$$s \to \left(x_1, \ldots, x_{n-1}, x_n \frac{\cos(s+\theta)}{\cos\theta}, x_n \frac{\sin(s+\theta)}{\cos\theta}\right).$$

Then the graph Σ is locally given by the parametrization

$$(x_1, \dots, x_n) \to \left(x_1, \dots, x_{n-1}, x_n \frac{\cos(u+\theta)}{\cos\theta}, x_n \frac{\sin(u+\theta)}{\cos\theta}\right),$$

where $u = u(x_1, \dots, x_n)$ is a smooth function on the variables x_i. Let $y_i = x_i$, $1 \leq i \leq n-1$ and $y_n = x_n(\cos(u+\theta))/\cos\theta)$ as new independent variables. Then the new parametrization of Σ becomes

$$(y_1, \dots, y_n) \to (y_1, \dots, y_n, y_n \tan(u+\theta)),$$

or, in other words, Σ satisfies the equation $y_{n+1} = g(y_1, \dots, y_n)$, where

$$g(y_1, \dots, y_n) = y_n \tan(u+\theta).$$

Now recall the relation between H and the mean curvature H' of Σ with respect to the Euclidean metric:

$$H = x_{n+1} H' + x_{n+1} \circ N,$$

where N is an (Euclidean) unit normal vector field on Σ (see, for example, [8]). From the classical formula of H' in terms of g, we have

$$E_H(u) =: \operatorname{div} \frac{\nabla g}{\sqrt{1+|\nabla g|^2}} - \frac{n}{g}\left(H - \frac{1}{\sqrt{1+|\nabla g|^2}}\right) = 0.$$

Consequently, our problem of finding constant mean curvature graphs is equivalent to solving the boundary-value problem

$$(P_H) = \begin{cases} E_H(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

One can easily check that the operator E_H is quasilinear elliptic (uniformly elliptic if $|\nabla u|$ remains bounded) and that there exists the trivial solution $u = 0$ for the value $H = \cos\theta$. This solution corresponds to the domain Ω.

The Schauder theory requires C^1-uniform bounds of the function u in (P_H) to establish $C^{1,\alpha}$ uniform bounds. Hence, we would need C^0 and C^1 estimates for the solution u of $E_H(u) = 0$. First, we obtain C^0 and C^1 estimates for u provided that we have the corresponding bounds for the function f that define the graph (see Definition 2.1), since the relation between the functions u and f is

$$\cos(u+\theta) = \frac{A\,\mathrm{e}^{2f} - 1}{A\,\mathrm{e}^{2f} + 1}, \quad A = \frac{1-\cos\theta}{1+\cos\theta}.$$

However, it is more convenient to work with other related functions that help us for searching *a-priori* estimates. First, we begin with C^0-bounds. With the above notation, if $x = (q, f(q)) \in \Sigma$, we have

$$\begin{aligned} \langle x, a\rangle &= (\cosh f(q) - h \sinh f(q))\tau - \lambda \sinh f(q) \\ &= \tau \cosh f(q) - (\lambda + h\tau) \sinh f(q) = \tau \cosh f(q) - \frac{1}{\lambda} \sinh f(q). \end{aligned}$$

Hence, we obtain

$$f = \log \frac{\lambda}{(1-h)} \frac{1}{\langle x, a\rangle + \sqrt{1 + \langle x, a\rangle^2}}. \tag{4}$$

With the objective of obtaining C^1-bounds of a solution f of our Dirichlet problem, we have to estimate $|\nabla f|$ on Ω. The gradient of f is related with the function $\langle N, a\rangle$, where N is a unit normal vector field on $\Sigma = \text{graph}(f)$. More precisely, a computation of N yields the following: if $x = (q, f(q)) \in \Sigma$,

$$N(x) = -\frac{1}{A(h, f(q), \nabla f(q))}(\cosh f(q) - h \sinh f(q))(h \cosh f(q) - \\ - \sinh f(q))q + \nabla f(q) + \lambda \cosh f(q)(\cosh f(q) - h \sinh f(q))a),$$

where

$$A(h, f(q), \nabla f(q)) = \sqrt{(\cosh f(q) - h \sinh f(q))^2 + |\nabla f|^2(q)}.$$

With the choice of N, we obtain

$$\langle N(q), a\rangle = -\frac{(\cosh f(q) - h \sinh f(q))^2}{\lambda\sqrt{(\cosh f(q) - h \sinh f(q))^2 + |\nabla f|^2(q)}}.$$

Thus

$$|\nabla f|(q) = (\cosh f(q) - h \sinh f(q))\sqrt{\frac{(\cosh f(q) - h \sinh f(q))^2}{\lambda^2\langle N(q), a\rangle^2} - 1}. \tag{5}$$

Collecting this work, identities (4) and (5) get

PROPOSITION 3.1. *Let $U_{a,\tau}$ be either a totally geodesic hyperplane or an equidistant hypersurface in $\mathbf{H}^{n+1}$ and let Ω be a domain in $U_{a,\tau}$. Let f be a smooth function on Ω.*

(1) *To obtain a-priori C^0-bounds for f, it suffices to have a-priori C^0-bounds of the function $\langle x, a\rangle$, $x \in \Sigma$.*
(2) *To obtain a-priori C^1-bounds for f, it suffices to have a-priori C^0-bounds of the function $\langle x, a\rangle$, $x \in \Sigma$ and a-priori bound $C > 0$ such that $\langle N, a\rangle \leq -C$ on Σ.*

4. *A Priori* C^0-Bounds

In order to find a solution of our Dirichlet problem (P_H), the program of the continuity method requires us to find *a-priori* $C^{1,\alpha}$-estimates for functions whose graphs have constant mean curvature. In this section, we first seek C^0-estimates for our solutions. We start by considering the case that the mean curvature satisfies $|H| \leq 1$.

THEOREM 4.1. *Let Q be either a totally geodesic hyperplane or an equidistant hypersurface and let Ω be a bounded domain in Q. Then there exists a constant $C_1 = C_1(\Omega) > 0$ such that if f is a smooth function on Ω, $f = 0$ in $\partial\Omega$, whose graph Σ has constant mean curvature H, $|H| \leq 1$, then*

$$-C_1 \leq f \leq C_1.$$

Moreover, if $-1 \leq H \leq h$ (resp. $h \leq H \leq 1$), then $0 \leq f \leq C_1$ (resp. $-C_1 \leq f \leq 0$).

Proof. Without loss of generality, we can assume that $a = (0, \ldots, 0, 1)$. Then the equation that satisfies Q is $x_{n+1} = \tau$ and $\langle x, a\rangle = x_{n+1}$. Following Equation (9) in [8], $0 < x_0 \leq \max_{\partial\Omega} x_0 = -B_1$, where $B_1 = B_1(\Omega)$ is a negative constant. For each $x \in \Sigma$, we have

$$-1 = \langle x, x\rangle = -x_0^2 + \sum_{i=1}^{n+1} x_i^2 \geq -x_0^2 + x_{n+1}^2.$$

Then

$$-\sqrt{B_1^2 - 1} \leq \langle x, a\rangle \leq \sqrt{B_1^2 - 1}.$$

The constant $\sqrt{B_1^2 - 1}$ provides us with the correspondent constant C_1 according to Proposition 3.1. The final part is a consequence of Corollary 2.3. $\square$

We also need C^0-bounds in the range $|H| > 1$. The next theorem follows ideas due to Serrin (see also [4] and [8]).

THEOREM 4.2. *Let Q be either a totally geodesic hyperplane or an equidistant hypersurface and let Ω be a bounded domain in Q. Consider H as a real number with $|H| > 1$. Then there exist constants $C_2 = C_2(Q, H)$ and $C_3 = C_3(Q, H)$, such that if f is a smooth function on Ω, $f = 0$ in $\partial\Omega$ whose graph Σ has constant mean curvature H, then $C_2 \leq f \leq C_3$.*

Proof. From Equations (2) and (3), we have

$$\Delta(H\langle x, a\rangle + \langle N, a\rangle) = (nH^2 - |\sigma|^2)\langle N, a\rangle \geq 0.$$

The maximum principle yields

$$H\langle x, a\rangle + \langle N, a\rangle \leq \max_{q\in\partial\Omega}(H\langle x, a\rangle + \langle N, a\rangle) \leq H\tau. \tag{6}$$

For each $x \in \Sigma$,

$$\langle a, a\rangle = 1 = \langle a^T, a^T\rangle^2 + \langle N, a\rangle^2 - \langle x, a\rangle^2 \geq \langle N, a\rangle^2 - \langle x, a\rangle^2,$$

where a^T denotes the tangent part of the vector a on Σ. Then

$$\langle N, a\rangle^2 \leq 1 + \langle x, a\rangle^2.$$

Inequality (6) then implies

$$H(\langle x, a\rangle - \tau) \le -\langle N, a\rangle \le \sqrt{1 + \langle x, a\rangle^2}. \tag{7}$$

From Corollary 2.3, if $H > 1$, $\langle x, a\rangle \ge \tau$ and if $H < -1$, $\langle x, a\rangle \le \tau$. Squaring (7), we obtain

$$H^2(\langle x, a\rangle - \tau)^2 \le 1 + \langle x, a\rangle^2$$

or, equivalently,

$$(H^2 - 1)\langle x, a\rangle^2 - 2\tau H^2\langle x, a\rangle + H^2\tau^2 - 1 \le 0.$$

Consequently,

$$\frac{\tau H^2 - \sqrt{H^2\tau^2 + H^2 - 1}}{H^2 - 1} \le \langle x, a\rangle \le \frac{\tau H^2 + \sqrt{H^2\tau^2 + H^2 - 1}}{H^2 - 1}. \tag{8}$$

Hence, we have the lower and upper bounds for the function $\langle x, a\rangle$. The C^0 estimate for f follows from Proposition 3.1 and this completes the proof. □

Remark 2. Inequality (8) can be compared with the corresponding height estimate that appears in [4] for a graph on a domain of totally geodesic hyperplanes. In this case, [4] gives

$$f(q)| \le \text{arc sinh}\frac{1}{\sqrt{H^2 - 1}}.$$

By identity (4), this implies

$$-\frac{1}{\sqrt{H^2 - 1}} \le \langle x, a\rangle \le \frac{1}{\sqrt{H^2 - 1}}.$$

This is given in (8) for $\tau = 0$. Thus, (8) extends the height estimates to graphs on equidistant hypersurfaces.

5. *A-Priori* C^1-Bounds

Continuing our treatment of the analysis of $C^{1,\alpha}$-bounds, in this section we derive the C^1-estimates for the solution of the Dirichlet problem (P_H).

THEOREM 5.1. *Let Q be either a totally geodesic hyperplane or an equidistant hypersurface and let Ω be a bounded domain in Q such that $\partial\Omega$ is mean convex. Let H be a real number such that $-H_{\partial\Omega} < H < H_{\partial\Omega}$. Then there exists a constant $C_4 = C_4(Q, \Omega, H)$ such that, if f is a smooth function on f, $f = 0$ in $\partial\Omega$ whose graph Σ has with constant mean curvature H, we have* $\sup_\Omega |\nabla f| \le C_4$.

Proof. We distinguish two cases:

First case. Suppose $-H_{\partial\Omega} < H \leq h$. From Corollary 2.3 we have $\langle x, a\rangle \leq \tau$. Thus $\langle \nu, a\rangle \leq 0$ in $\partial\Omega$ where ν denotes the inner conormal along $\partial\Omega$. In view of (6), there exists a boundary point $q_0 \in \partial\Omega$ such that the function $H\langle x, a\rangle + \langle N, a\rangle$ attains its maximum, that is,

$$H\langle x, a\rangle + \langle N, a\rangle \leq H\tau + \langle N(q_0), a\rangle. \tag{9}$$

Furthermore, (6) implies

$$H\langle \nu(q_0), a\rangle + \langle dN_{q_0}\nu(q_0), a\rangle \leq 0, \tag{10}$$

where N is a unit normal field on Σ such that $\langle N, a\rangle \leq 0$. A computation of (10) leads $(H - \sigma(\nu, \nu)(q_0))\langle \nu(q_0), a\rangle \leq 0$, where σ is the second fundamental form of Σ. Since $\langle \nu, a\rangle$ is nonpositive along $\partial\Omega$, we obtain

$$H - \sigma(\nu, \nu)(q_0) \geq 0. \tag{11}$$

On the other hand, if $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal frame of $\partial\Omega$ and η is the unit normal field to $\partial\Omega$ in Q pointing to Ω, we have

$$\begin{aligned} \sigma(v_i, v_i) &= \langle N, \alpha_i''(0)\rangle \\ &= -\lambda h\langle N(q_0), a\rangle + \sigma^{\partial\Omega}(v_i, v_i)\langle N(q_0), \eta(q_0)\rangle, \end{aligned} \tag{12}$$

where $\alpha_i\colon I \to Q$ is a smooth curve with $\alpha_i(0) = q_0$, $\alpha_i'(0) = v_i$, $1 \leq i \leq n-1$ and $\sigma^{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ in Q. Since, along $\partial\Omega$,

$$1 = \langle N, N\rangle = \langle N, \eta\rangle^2 + \langle N, \xi\rangle^2 = \langle N, \eta\rangle^2 + \lambda^2\langle N, a\rangle^2,$$

then

$$\langle N, \eta\rangle = -\sqrt{1 - \lambda^2\langle N, a\rangle^2}. \tag{13}$$

By identity (13), and summing (12) from $i = 1$ to $n - 1$, we obtain

$$\sum_{i=1}^{n-1} \sigma(v_i, v_i) = -\lambda h(n-1)\langle N(q_0), a\rangle - (n-1)H^{\partial\Omega}(q_0)\sqrt{1 - \lambda^2\langle N(q_0), a\rangle^2}.$$

Then (11) yields

$$-\lambda h\langle N(q_0), a\rangle - H^{\partial\Omega}(q_0)\sqrt{1 - \lambda^2\langle N(q_0), a\rangle^2} \geq H \tag{14}$$

or, equivalently,

$$H^{\partial\Omega}(q_0)\sqrt{1 - \lambda^2\langle N(q_0), a\rangle^2} \leq -\lambda h\langle N(q_0), a\rangle - H.$$

Squaring this inequality, we obtain

$$\begin{aligned}&\lambda^2(h^2+H^{\partial\Omega}(q_0)^2)\langle N(q_0),a\rangle^2\\&\quad+2\lambda hH\langle N(q_0),a\rangle+H^2-H^{\partial\Omega}(q_0)^2\geq 0.\end{aligned}\tag{15}$$

Consider $\alpha_1(H,q_0)$, $\alpha_2(H,q_0)$ to be the two roots of the left-hand side in (15), with $\alpha_1(H,q_0)\leq\alpha_2(H,q_0)$. Then

$$\alpha_2(H,q_0)=\frac{-hH+H^{\partial\Omega}(q_0)\sqrt{h^2+H^{\partial\Omega}(q_0)^2-H^2}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}.$$

Inequality (15) implies that $\langle N(q_0),a\rangle\leq\alpha_1(H,q_0)$ or $\langle N(q_0),a\rangle\geq\alpha_2(H,q_0)$. We distinguish three cases:

(1) Assume $H\leq 0$. In this case, $\alpha_2(H,q_0)>0$ because $H\leq 0$. Since $\langle N(q_0),a\rangle<0$, then $\langle N(q_0),a\rangle\leq\alpha_1(H,q_0)$, that is

$$\langle N(q_0),a\rangle\leq\frac{-hH-H^{\partial\Omega}(q_0)\sqrt{h^2+H^{\partial\Omega}(q_0)^2-H^2}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}.$$

Since, by hypothesis, $H^{\partial\Omega}(q_0)\geq H_{\partial\Omega}>-H$, $\alpha_1(H,q_0)$ is a negative number. Using (9), we can estimate

$$\begin{aligned}\langle N,a\rangle&\leq H(\tau-\langle x,a\rangle)+N(q_0),a\rangle\leq\langle N(q_0),a\rangle\\&\leq\frac{-hH-H^{\partial\Omega}(q_0)\sqrt{h^2+H^{\partial\Omega}(q_0)^2-H^2}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}\end{aligned}\tag{16}$$

Denote

$$\overline{H}_{\partial\Omega}=\max_{q\in\partial\Omega}H^{\partial\Omega}(q)>0.$$

By (16), one directly verifies that

$$\begin{aligned}\langle N,a\rangle&\leq\frac{-hH-H_{\partial\Omega}\sqrt{h^2+H_{\partial\Omega}^2-H^2}}{\lambda(h^2+\overline{H}_{\partial\Omega}^2)}\\&\leq\frac{H_{\partial\Omega}}{\lambda(h^2+\overline{H}_{\partial\Omega}^2)}\Big(h-\sqrt{h^2+H_{\partial\Omega}^2-H^2}\Big).\end{aligned}\tag{17}$$

Remark that this case comprises of the case where Q is a totally geodesic hypersurface, since $h=0$. Therefore, the rest of the proof only assumes that Q is an equidistant hypersurface.

(2) Assume $0 < H \leq h$. Since $\partial\Omega$ is mean convex, we have that $\alpha_1(0, q) < 0 < \alpha_2(0, q)$ for any $q \in \partial\Omega$. By continuity and because $\partial\Omega$ is a compact set, there exists a (small) constant $h_0 = h_0(\Omega, Q) < h$, such that $\alpha_1(H, q) < 0 < \alpha_2(H, q)$ for all $H \in [0, h_0]$, $q \in \partial\Omega$. Hence, $\langle N(q_0), a\rangle \leq \alpha_1(H, q_0)$. Then, for each $H \in [0, h_0]$ and by virtue of (9),

$$\begin{aligned}\langle N, a\rangle &\leq H\tau - H\langle x, a\rangle + \langle N(q_0), a\rangle \leq H\tau + \langle N(q_0), a\rangle \\ &\leq \frac{Hh(h^2 + H^{\partial\Omega}(q_0)^2) - hH - H^{\partial\Omega}(q_0)\sqrt{h^2 + H^{\partial\Omega}(q_0)^2 - H^2}}{\lambda(h^2 + H^{\partial\Omega}(q_0)^2)} \\ &\leq \frac{Hh(h^2 + \overline{H}^2_{\partial\Omega}) - H^2_{\partial\Omega}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})} \qquad (18)\end{aligned}$$

where, the second inequality we apply Corollary 2.3 to assure that $\langle x, a\rangle \geq 0$. Choose h_0 small so that $h_0 h(h^2 + \overline{H}^2_{\partial\Omega}) - H^2_{\partial\Omega} < 0$. Then (18) becomes

$$\langle N, a\rangle \leq \frac{h_0 h(h^2 + \overline{H}^2_{\partial\Omega}) - H^2_{\partial\Omega}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})} < 0.$$

(3) Assume $H \in [h_0, h]$. By (14), we have $-\lambda h\langle N(q_0), a\rangle \geq H$ or, equivalently, $\langle N(q_0), a\rangle \leq -(H/\lambda h)$. Corollary 2.3 and inequality (9) yield

$$\begin{aligned}\langle N, a\rangle &\leq H\left(\tau - \langle x, a\rangle - \frac{1}{\lambda h}\right) = H\left(-\langle x, a\rangle + \frac{h^2 - 1}{\lambda h}\right) \\ &\leq H\frac{h^2 - 1}{\lambda h} \leq h_0\frac{h^2 - 1}{\lambda h} = -\frac{h_0}{\tau} < 0 \qquad (19)\end{aligned}$$

Second case. Suppose $h \leq H < H_{\partial\Omega}$. The proof is similar and we only outline it. In this case, $\langle x, a\rangle \geq \tau$. With appropriate changes in (11), (13) and (14), we obtain the same inequality (15). Since $\alpha_2(H, q_0)$ is positive again (because $H_{\partial\Omega} > H$), we see that $\langle N(q_0), a\rangle \leq \alpha_1(H, q_0)$. As in (17), we have

$$\begin{aligned}\langle N, a\rangle &\leq H\tau - H\langle x, a\rangle + \langle N(q_0), a\rangle \leq H\tau + \langle N(q_0), a\rangle \\ &\leq \frac{-hH - H_{\partial\Omega}\sqrt{h^2 + H^2_{\partial\Omega} - H^2}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})} \\ &\leq -\frac{hH_{\partial\Omega}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})} = -\tau\frac{H_{\partial\Omega}}{h^2 + \overline{H}^2_{\partial\Omega}}. \qquad (20)\end{aligned}$$

Set

$$-C = \max\left\{\frac{H_{\partial\Omega}}{\lambda(h^2+\overline{H}^2_{\partial\Omega})}\left(h-\sqrt{h^2+H^2_{\partial\Omega}-H^2}\right),\right.$$
$$\left.\frac{h_0 h(h^2+\overline{H}^2_{\partial\Omega})-H^2_{\partial\Omega}}{\lambda(h^2+\overline{H}^2_{\partial\Omega})}, -\frac{h_0}{\tau}, -\tau\frac{H_{\partial\Omega}}{h^2+\overline{H}^2_{\partial\Omega})}\right\}.$$

This constant is negative and only depends on Q, Ω and H. Recall that if Q is a totally geodesic hyperplane, then let

$$C = -\frac{H_{\partial\Omega}\sqrt{H^2_{\partial\Omega}-H^2}}{\overline{H}^2_{\partial\Omega}}.$$

Now we obtain that $\langle N, a\rangle \leq -C$. It follows from Proposition 3.1, Theorems 4.1 and 4.2, the desired C^1-estimates for the function f. □

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1 using the method of continuity. As usual, the proof is based on the establishment of global $C^{1,\alpha}$ *a-priori* estimates for prospective solutions. Consider the set $\mathcal{S}$ defined as

$$\mathcal{S} = \{H \in (-H_{\partial\Omega}, h];\ \text{there exists a smooth function on } \Omega,\ f=0$$
$$\text{in } \partial\Omega \text{ whose graph } \Sigma \text{ has constant mean curvature } H\}.$$

Since $0 \in \mathcal{S}$ (for $f = 0$), then $\mathcal{S}$ is not an empty set.

We show that the set $\mathcal{S}$ is open. This is accomplished by using the implicit function theorem for Banach spaces. Let $x\colon M \to \mathbf{H}^{n+1}$ be an isometric immersion, where M is a compact manifold with $\partial M \neq \emptyset$. For each $u \in C_0^{1,\alpha}(M)$, we define $x_t\colon M \to \mathbf{H}^{n+1}$ the map $x_t(p) = \exp_{x(p)}(tu(p)N(p))$, where N denotes a unit normal vector field along M in $\mathbf{H}^{n+1}$. For t near zero, x_t is an immersion. Let H be the mean curvature function and define $J\colon C_0^{1,\alpha}(M) \to C^\alpha(M)$ by

$$J(u)(p) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} H(x_t(p))).$$

The operator J is the linearization of H at x associated to normal variations given by N. Then the Jacobi operator J is given by $J = \Delta + |\sigma|^2 - n$.

Assume that $H \in \mathcal{S}$ and Σ is the associated graph for H. Consider the function $\psi = \langle N, a\rangle$. By virtue of Equations (2) and (3), we have

$$J\psi = -n(H\langle x, a\rangle + \psi).$$

If $H \leq 0$, inequality (6) implies $J\psi \geq 0$. If $H \in (0, h]$, (18) and (19) imply, respectively,

$$H\langle x, a\rangle + \langle N, a\rangle \leq H\tau + \langle N(q_0), a\rangle \leq H\tau + \alpha_1(H, q_0) \leq 0,$$
$$H\langle x, a\rangle + \langle N, a\rangle \leq -\frac{H}{\tau} \leq 0.$$

In conclusion $J\psi \geq 0$. Therefore J is a Fredholm operator of index zero. By the implicit function theorem, there exists a neighbourhood of H, namely $(H-\delta, H+\delta)$, $\delta > 0$, such that for each H' belonging to this interval, there exists a graph on Ω of constant mean curvature H'.

Finally, the proof ends by using the continuity method, provided *a-priori* $C^{1,\alpha}$-bounds independent of H are established for each solution with $c \leq H \leq h$ and $-H_{\partial\Omega} < c$. We now turn our attention to the upper bound for $\langle N, a\rangle$ in Theorem 5.1. Consider

(1) *Case* $-1 \leq c$. Theorem 4.1 gives C^0-estimates depending only on Q and Ω. On the other hand, in (17) and when $H \leq 0$, we get

$$\langle N, a\rangle \leq \frac{H_{\partial\Omega}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})}(h - \sqrt{h^2 + H^2_{\partial\Omega} - H^2}).$$

Call

$$C' = \max\left\{\frac{-H_{\partial\Omega}}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})}(h - \sqrt{h^2 + H^2_{\partial\Omega} - H^2}), \frac{h_0}{\tau}, \frac{H^2_{\partial\Omega} - h_0 h(h^2 + \overline{H}^2_{\partial\Omega})}{\lambda(h^2 + \overline{H}^2_{\partial\Omega})}\right\};$$

when Q is a totally geodesic hyperplane, take

$$C' = \frac{H_{\partial\Omega}\sqrt{H^2_{\partial\Omega} - c^2}}{\overline{H}^2_{\partial\Omega}}.$$

This number C' depends only on Q and Ω and verifies $\langle N, a\rangle \leq -C'$. In view of Proposition 3.1, there are C^1-estimates. The continuity method and the fact that $\mathcal{S}$ is open imply that, for each H, $c \leq H \leq h$, there exists a function f on Ω vanishing on its boundary and whose graph has constant mean curvature H.

(2) *Case* $c < -1$. The above construction for $-1 \leq c$ assures the existence of graphs with constant mean curvature until the value $H = -1$. Because $\mathcal{S}$ is open, in a neighbourhood of this value there is an interval of solutions, namely $[-1-\delta, -1]$, where $\delta = \delta(Q, \Omega) > 0$. Let $c \leq -1-\delta$. The bound

C' for $\langle N, a\rangle$ holds as above. Then, for each graph on Ω whose constant mean curvature H satisfies $c \leq H \leq -1-\delta$, Corollary 2.3 and the estimate (8) imply

$$\begin{aligned}\tau \ \geq\ \langle x, a\rangle &\geq \frac{\tau-\sqrt{c^2\tau^2+c^2-1}}{H^2-1}\\ &\geq \frac{\tau-\sqrt{c^2\tau^2+c^2-1}}{\delta^2+2\delta}.\end{aligned}$$

Thus we have C^0-bounds (independent on H) in the interval $[c, -1-\delta]$ and this finishes the proof of Theorem 1.1.

Finally, it remains to examine the final statement of Theorem 1.1. Let $\mathcal{S}$ be as in the definition but changing the interval of H by $(-H_{\partial\Omega}, 1]$. Assume $H_{\partial\Omega} > 1$. The C^0-bounds are derived in Theorem 4.1 and the C^1-estimates hold as above, since in the interval $[h, 1]$ we have

$$\langle N, a\rangle \leq -\tau\frac{H_{\partial\Omega}}{h^2+\overline{H}^2_{\partial\Omega}}.$$

Let us prove that $[h, 1]$ is open in $\mathcal{S}$. We only have to verify that $J\psi \geq 0$. Since $H_{\partial\Omega} > 1$, inequality (20) implies

$$\begin{aligned}H\langle x, a\rangle \ +\ \langle N, a\rangle &\leq H\tau - \frac{Hh+H^{\partial\Omega}(q_0)\sqrt{h^2+H^{\partial\Omega}(q_0)^2-H^2}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}\\ &\leq \frac{Hh(h^2+H^{\partial\Omega}(q_0)^2-1)-H^{\partial\Omega}(q_0)\sqrt{h^2+H^{\partial\Omega}(q_0)^2-1}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}\\ &\leq \frac{\sqrt{h^2+H^{\partial\Omega}(q_0)^2-1}}{\lambda(h^2+H^{\partial\Omega}(q_0)^2)}\left(\sqrt{h^2+H^{\partial\Omega}(q_0)^2-1}-H^{\partial\Omega}(q_0)\right) \leq 0\end{aligned}$$

and J follows being a Fredholm operator of zero index. This completes the proof of Theorem 1.1.

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