

## CYCLIC SURFACES OF CONSTANT GAUSS CURVATURE

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ABSTRACT. In this paper we prove that a surface in Euclidean three-space  $\mathbf{R}^3$  with nonzero constant Gauss curvature foliated by circles is a surface of revolution. In the case that the Gauss curvature vanishes on the surface, then the planes containing the circles must be parallel.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

A *cyclic surface* in  $\mathbf{R}^3$  is a surface foliated by pieces of circles, that is, it is generated by a smooth uniparametric family of pieces of circles. Surfaces of revolution are the best known examples of cyclic surfaces. In the eighteenth century, Euler proved that the catenoid is the only minimal surface of revolution. In 1860s Riemann found a family of embedded minimal surfaces foliated by circles in parallel planes. Each one of such surfaces is invariant by a group of translations and presents planar ends in a discrete set of heights ([7]). At the same time, Enneper proved that in a minimal cyclic surface, the foliating planes must be parallel ([3, 4]). As a consequence of Euler, Riemann and Enneper's works, we have that the catenoid and Riemann minimal examples are the only minimal cyclic surfaces in Euclidean space. A century later, Nitsche ([6]) studied in 1989 cyclic surfaces with nonzero constant mean curvature and he proved that the only such surfaces are the surfaces of revolution discovered by Delaunay in 1841 ([1]).

In this paper we study cyclic surfaces with constant Gauss curvature. As first examples, we have the family of surfaces of revolution (see [2] for a description of these surfaces). We shall prove that, except in the case that the Gauss curvature vanishes on the surface, the only cyclic surfaces with constant Gauss curvature are the surfaces of revolution. When the Gauss curvature is zero, the surface

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are not necessarily rotational, but we shall describe all them. We summarize the results of this paper in the next two theorems:

**Theorem 1.** *Let  $M$  be a surface in  $\mathbf{R}^3$  with constant Gauss curvature and foliated by pieces of circles. Then  $M$  is included in a sphere or the planes containing the circles of the foliation are parallel.*

**Theorem 2.** *Let  $M$  be a surface in  $\mathbf{R}^3$  with constant Gauss curvature  $K$  and foliated by pieces of circles in parallel planes.*

(1) *If  $K \neq 0$ , then  $M$  is a surface of revolution.*

(2) *If  $K = 0$ , then  $M$  can be parameterized, up a rigid motion of  $\mathbf{R}^3$ , as*

$$\mathbf{X}(u, v) = (a_1u + a_0, b_1u + b_0, u) + (r_1u + r_0)(\cos v, \sin v, 0), \quad a_0, a_1, b_0, b_1, r_0, r_1 \in \mathbf{R}.$$

As a corollary of both theorems, we obtain:

**Corollary 1.** *All cyclic surfaces in  $\mathbf{R}^3$  with nonzero constant Gauss curvature are surfaces of revolution.*

In this sense, this corollary may be viewed as the analogous result to the Nitsche's theorem of the case of nonvanishing constant mean curvature surfaces.

*Remark.* The author has extended these results of cyclic surfaces to other ambient spaces. See [5] as a survey on this subject and references therein.

## 2. PROOF OF RESULTS

Consider  $M$  a surface in  $\mathbf{R}^3$  with Gauss curvature  $K$ . Let  $\mathbf{X} = \mathbf{X}(u, v)$  be a local parametrization of  $M$  and let  $\nu$  denote the unit normal vector field on  $M$  given by

$$\nu = \frac{\mathbf{X}_u \wedge \mathbf{X}_v}{|\mathbf{X}_u \wedge \mathbf{X}_v|}, \quad \mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}, \quad \mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v},$$

where  $\wedge$  stands the cross product of  $\mathbf{R}^3$ . The metric  $\langle, \rangle$  in each tangent plane is determined by the first fundamental form

$$I = \langle d\mathbf{X}, d\mathbf{X} \rangle = Edu^2 + 2Fdudv + Gdv^2,$$

with differentiable coefficients

$$E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle, \quad F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle, \quad G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle.$$

The shape operator of the immersion is represented by the second fundamental form

$$II = -\langle d\nu, d\mathbf{X} \rangle = e du^2 + 2f dudv + g dv^2,$$

with differentiable coefficients

$$e = \langle \nu, \mathbf{X}_{uu} \rangle, \quad f = \langle \nu, \mathbf{X}_{uv} \rangle, \quad g = \langle \nu, \mathbf{X}_{vv} \rangle.$$

Under this parametrization  $\mathbf{X}$  the Gauss curvature  $K$  has the classical expression

$$(1) \quad K = \frac{eg - f^2}{EG - F^2}.$$

Let us denote by  $[\cdot, \cdot]$  the determinant in  $\mathbf{R}^3$  and put  $W = \det I = EG - F^2$ . With this notation (1) writes as

$$(2) \quad KW^2 = P \stackrel{\text{def}}{=} [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2.$$

Consider now that  $M$  is a cyclic surface with constant Gauss curvature. After a homothety, it may be assumed without loss of generality that the Gauss curvature is  $\varepsilon = -1, 0$  or  $1$ .

PROOF OF THEOREM 1. We follow the same ideas as in [6]. Let  $\Gamma = \Gamma(u)$  be an orthogonal smooth curve to each  $u$ -plane of the foliation and denote by  $u$  its arc-length parameter. We assume that the planes of the foliation are not parallel and we shall arrive that  $M$  is included in a sphere. Let  $\mathbf{t}$  be the unit tangent vector to  $\Gamma$ . Consider the Frenet frame of the curve  $\Gamma$ ,  $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ , where  $\mathbf{n}$  and  $\mathbf{b}$  denote the normal and binormal vectors respectively. Locally we parameterize  $M$  by

$$(3) \quad \mathbf{X}(u, v) = \mathbf{c}(u) + r(u)(\cos v \mathbf{n}(u) + \sin v \mathbf{b}(u)),$$

where  $r = r(u) > 0$  and  $\mathbf{c} = \mathbf{c}(u)$  denote the radius and centre of each  $u$ -circle of the foliation. Consider the Frenet equations of the curve  $\Gamma$ :

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} + \sigma \mathbf{b} \\ \mathbf{b}' &= -\sigma \mathbf{n} \end{aligned}$$

where the prime  $'$  denotes the derivative with respect to the  $u$ -parameter and  $\kappa$  and  $\sigma$  are the curvature and torsion of  $\Gamma$ , respectively. Observe that  $\kappa \neq 0$  because  $\Gamma$  is not a straight-line. Also, set

$$(4) \quad \mathbf{c}' = \alpha \mathbf{t} + \beta \mathbf{n} + \gamma \mathbf{b},$$

where  $\alpha, \beta, \gamma$  are smooth functions on  $u$ .

By using the Frenet equations and (4), a straightforward computation of  $W^2$  and  $P$  shows that both functions can be expressed by trigonometric polynomial

on  $\cos nv, \sin nv$ . Exactly, there exist smooth functions on  $u$ , namely  $A_n$  and  $B_n$ , such that (2) writes as

$$(5) \quad 0 = \varepsilon W^2 - P = A_0 + \sum_{n=1}^4 (A_n \cos nv + B_n \sin nv), \quad \varepsilon = -1, 0, \text{ or } 1.$$

Since this is an expression on the independent trigonometric terms  $\cos nv$  and  $\sin nv$ , all coefficients  $A_i, B_i$  vanish. We discuss two cases:

*Case  $\varepsilon = 0$ .* Equation (5) reduces simply as  $P = 0$ . A straightforward computation shows that

$$A_4 = \frac{\kappa^2 r^4 (\beta^2 + \kappa^2 r^2 - \gamma^2)}{8} = 0$$

$$B_4 = \frac{\kappa^2 r^4 \beta \gamma}{4} = 0.$$

According the expression of  $B_4$ , let us distinguish two possibilities:

- (1)  $\beta = 0$ . From  $A_4 = 0$ , it implies

$$\frac{\kappa^2 r^4 (-\gamma^2 + \kappa^2 r^2)}{8} = 0,$$

and then  $\gamma = \pm \kappa r$ . The computation of the coefficient  $A_3$  leads

$$A_3 = \frac{-3\kappa^3 r^5 \alpha}{4} = 0$$

and thus  $\alpha = 0$ . Then  $B_3 = 3\kappa^3 r^5 r' / 4 = 0$ , that is,  $r' = 0$ . In this situation,  $A_2 = \kappa^4 r^6 = 0$ , contradiction.

- (2)  $\gamma = 0$ . From the coefficient of  $A_4$ , we have

$$A_4 = \frac{r^4 (\beta^2 + \kappa^2 r^2)^2}{8} = 0,$$

which is a contradiction again.

*Case  $\varepsilon = \pm 1$ .* In this situation, the coefficient  $B_4$  is

$$B_4 = \frac{\beta \gamma r^4 (2\beta^2 - 2\gamma^2 - \varepsilon \kappa^2 + 2\kappa^2 r^2)}{4} = 0.$$

Consider the following cases:

- (1)  $\beta = 0$ . Now we have

$$A_4 = \frac{r^4 (-\gamma^2 + \kappa^2 r^2) (-\gamma^2 - \varepsilon \kappa^2 + \kappa^2 r^2)}{8} = 0$$

$$A_3 = \frac{\alpha \kappa r^5 (4\gamma^2 + 3\varepsilon \kappa^2 - 4\kappa^2 r^2)}{4} = 0.$$

In view of the expression of  $A_3$ , we distinguish two possibilities according to the value of  $\alpha$ .

(a)  $\alpha = 0$ . Then  $A_1 = -\kappa^2 r^5 \gamma \sigma = 0$ . Thus  $\sigma = 0$  or  $\gamma = 0$ .

(i):  $\sigma = 0$ . Now  $A_4 = 0$  yields two cases depending on the value of  $\gamma$ . If  $\gamma = \pm \kappa r$ , then  $B_3 = 0$  gives  $\kappa^3 r^5 r' = 0$  and so  $r' = 0$ . In this case,  $A_2 = 0$  yields  $\kappa^4 r^6 = 0$ , contradiction. Now consider the case  $\gamma = \pm \kappa \sqrt{r^2 - \varepsilon}$ . Suppose  $\varepsilon = 1$ . If  $r^2 = 1$ , then  $\gamma = 0$  and as a consequence the line  $\mathbf{c}$  reduces at one point  $\mathbf{c}_0$ . In this case,  $M$  is included in the sphere of radius 1 centered at  $\mathbf{c}_0$ . If  $r^2 - 1 \neq 0$ , and according of equation  $B_3 = 0$ , we have  $\kappa^2 r^6 r' = 0$  and thus  $r' = 0$  again. In this case, the coefficient  $A_0$  is

$$A_0 = \kappa^4 r^6 (r^2 - 1) = 0,$$

contradiction. When  $\varepsilon = -1$ , identity  $B_3 = 0$  implies  $\kappa^3 r^6 r' = 0$ , and then  $r' = 0$ . With these conditions,  $A_0 = \kappa^4 r^6 (1 + r^2) = 0$ : contradiction.

(ii):  $\gamma = 0$ . In this case,

$$A_4 = \frac{\kappa^4 r^6 (r^2 - \varepsilon)}{8} = 0.$$

If  $\varepsilon = 1$  then  $r^2 = 1$ . Since  $\alpha = \beta = \gamma = 0$ , identity (4) implies that  $\mathbf{c}$  is a constant  $\mathbf{c}_0 \in \mathbf{R}^3$ . Thus the surface  $M$  is included in the sphere of radius 1 centered at  $\mathbf{c}_0$ . In the case  $\varepsilon = -1$ , we get a contradiction.

(b)  $\alpha \neq 0$ . From  $A_3 = 0$  we have

$$(6) \quad 4\gamma^2 + 3\varepsilon\kappa^2 - 4\kappa^2 r^2 = 0.$$

On the other hand,  $A_4 = 0$  implies  $\gamma = \pm \kappa r$  or  $\gamma = \pm \kappa \sqrt{r^2 - \varepsilon}$ .

Both cases are in contradiction with the identity (6).

(2)  $\gamma = 0$ . Now

$$A_4 = \frac{r^4 (\beta^2 + \kappa^2 r^2) (\beta^2 - \varepsilon \kappa^2 + \kappa^2 r^2)}{8} = 0.$$

If  $\varepsilon = -1$ , we obtain a contradiction. Suppose now  $\varepsilon = 1$ . Then  $\beta = \pm \kappa \sqrt{1 - r^2}$ . If  $r^2 - 1 = 0$ , then  $\beta = 0$  and this case has been discussed previously. So, let us assume  $r^2 - 1 \neq 0$ . Substituting in the corresponding expression of  $A_3$ , we obtain

$$(7) \quad \alpha + \frac{r r'}{\sqrt{1 - r^2}} = 0.$$

Now the coefficient  $A_2$  implies  $\kappa^2 r^6 r'^2 = 0$  and thus  $r' = 0$ . Equation (7) leads  $\alpha = 0$ . Then  $r$  is constant and equation (4) writes

$$\mathbf{c}' = \pm \kappa \sqrt{1-r^2} \mathbf{n} = \pm \sqrt{1-r^2} \mathbf{t}'.$$

By integrating, we conclude that there exists  $\mathbf{c}_0 \in \mathbf{R}^3$  such that  $\mathbf{c}(u) = \mathbf{c}_0 + \sqrt{1-r^2} \mathbf{t}(u)$ . It follows that the parametrization (3) of  $M$  is given by

$$\mathbf{X}(u, v) = \mathbf{c}_0 + \sqrt{1-r^2} \mathbf{t}(u) + r(\cos v \mathbf{n}(u) + \sin v \mathbf{b}(u)).$$

Therefore  $|\mathbf{X}(u, v) - \mathbf{c}_0|^2 = 1$  and  $M$  is included in a sphere centered at  $\mathbf{c}_0$  of radius 1.

(3)  $\beta\gamma \neq 0$ . From the value of  $B_4$  we have

$$(8) \quad \gamma^2 = \beta^2 - \varepsilon \frac{\kappa^2}{2} + \kappa^2 r^2.$$

Substituting in  $A_4$  we conclude

$$(9) \quad 16\beta^4 - 8\varepsilon\beta^2\kappa^2 + \kappa^4 + 16\beta^2\kappa^2r^2 = 0.$$

If  $\varepsilon = -1$ , we get a contradiction. Assume  $\varepsilon = 1$ . Identity (8) yields  $\beta^2 - \frac{\kappa^2}{2} + \kappa^2 r^2 \geq 0$ . Substituting in (9), we conclude

$$0 = 16\beta^4 - 8\beta^2\kappa^2 + \kappa^4 + 16\beta^2\kappa^2r^2 \geq \kappa^4,$$

contradiction.

PROOF OF THEOREM 2. Without loss of generality, we assume that the planes of the foliation are parallel to the  $(x_1, x_2)$ -plane. Let

$$\mathbf{X}(u, v) = (a(u) + r(u) \cos v, b(u) + r(u) \sin v, u), u \in I, v \in J,$$

be a local parametrization of  $M$ . A computation of  $W$  and  $P$  yields

$$W = r^2 \left( 1 + \frac{a'^2 + b'^2}{2} + r'^2 \right) + (2a'r') \cos v + (2b'r') \sin v + \frac{a'^2 - b'^2}{2} \cos 2v + (a'b') \sin 2v.$$

$$P = -r^3 r'' - r^3 a'' \cos v - r^3 b'' \sin v.$$

(1) *Case*  $\varepsilon = \pm 1$ . Again the identity (5) writes as

$$A_0(u) + \sum_{n=1}^4 A_n(u) \cos nv + \sum_{n=1}^4 B_n(u) \sin nv = 0.$$

A computation yields

$$A_4 = \varepsilon \frac{r^4}{8} (a'^4 - 6a'^2 b'^2 + b'^4)$$

$$B_4 = \varepsilon \frac{r^4}{2} a' b' (a'^2 - b'^2)$$

Assume  $a' \neq 0$  and  $b' \neq 0$  at some  $u$ -interval. Since  $B_4 = 0$ ,  $a' = \pm b'$ . Taking in account  $A_4 = 0$ , it follows that  $r^4 b'^4 = 0$ , which is a contradiction. Hence  $a' = 0$  or  $b' = 0$ . Firstly assume  $a' = 0$  and  $b' \neq 0$  at some  $u$ -interval. The equation  $A_4 = 0$  yields  $r^4 b'^4 = 0$  and we obtain a contradiction again. The case  $a' \neq 0$  and  $b' = 0$  at some  $u$ -interval is analogous and gives also a contradiction. As a consequence,  $a' = b' = 0$ . This means that the circles generating  $M$  are coaxial and  $M$  must be a surface of revolution.

- (2) *Case*  $\varepsilon = 0$ . Identity (5) writes as  $P = 0$ . In view of the above expression of  $P$ , it follows that  $r'' = a'' = b'' = 0$ . As consequence, there are constants  $r_0, r_1, a_0, a_1, b_0, b_1$ , such that

$$r(u) = r_1 u + r_0$$

$$a(u) = a_1 u + a_0$$

$$b(u) = b_1 u + b_0$$

and we have the second statement of Theorem 2.

#### REFERENCES

- [1] C. Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante*, J. Math. Pure Appl. **6** (1841) 309–320.
- [2] L. Eisenhart, *Differential Geometry*, Gin and Co. Boston, 1909.
- [3] A. Enneper, *Ueber die cyclischen Flächen*, Nach. Königl. Ges. d. Wissensch. Göttingen, Math. Phys. Kl., (1866) 243–249.
- [4] A. Enneper, *Die cyklichen Flächen*, Z. Math. Phys., **14** (1869) 393–421.
- [5] R. López, *Cyclic hypersurfaces of constant curvature*, to appear in *Advances Studies in Mathematics*.
- [6] J.C.C. Nitsche, *Cyclic surfaces of constant mean curvature*, Nachr. Akad. Wiss. Göttingen Math. Phys. II **1** (1989) 1–5.
- [7] B. Riemann, *Über die Flächen vom kleinsten Inhalt bei gegebener Begrenzung*, Abh. Königl. Ges. d. Wissensch. Göttingen, Mathema. Cl. **13** (1868) 329–333.

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