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Constant Mean Curvature Graphs on Unbounded Convex Domains

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In this paper, we consider the Dirichlet problem for the constant mean curvature equation on an unbounded convex planar domain Ω. Let $H>0$. We prove that there exists a graph with constant mean curvature H and with boundary $\partial\Omega$ if and only if Ω is included in an infinite strip of width $\frac{1}{H}$. We also establish an existence result for convex bounded domains contained in a strip. © 2001 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a smooth surface M in three-dimensional Euclidean space, having a non-parametric representation $z=u(x, y)$ and consider the upwards orientation. Then M has mean curvature H provided that the function $u=u(x, y)$ satisfies the partial differential equation

$$L_H(u) \equiv \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - 2H = 0, \tag{1}$$

where div and ∇ are the Euclidean divergence and gradient operators. A solution of (1) is called an H-graph. The Dirichlet problem for Eq. (1) corresponding to a fixed (smooth) domain Ω in the (x, y)-plane and H a given constant consists of finding a solution of $L_H(u)=0$ in Ω taking on assigned continuous values on the boundary $\partial\Omega$ of Ω. Geometrically, this has the meaning that the resulting surface $z=u(x, y)$ is defined over Ω and spans a given space curve with single valued projection on the (x, y)-plane.

The first general existence theorem for H-graphs is due to Serrin. He showed in [9] that *when the curvature of $\partial\Omega$ satisfies*

$$\kappa \geqslant 2H > 0, \tag{2}$$

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then there exists a solution of (1) *for arbitrary prescribed boundary values.* His result requires, in particular, that Ω be bounded and convex. In this context, Serrin also showed that *the lower bound* (2) *is necessary for the existence of solutions with arbitrary boundary values.* Serrin's necessary and sufficient condition was a natural extension of that given by Finn in [3] who showed that *minimal graphs* ($H=0$) *exist over a bounded domain for arbitrary boundary values if and only if the domain is convex.*

Necessary Condition. In the same paper, Finn derived a necessary condition for the existence of a solution on the unbounded domain (strip) between two parallel lines: *If u is a solution of* (1) *on a strip, then necessarily the width h of the strip satisfies* $h \leqslant \frac{1}{H}$. Our first result generalizes this to unbounded convex domains and shows that a similarly strict condition is imposed on the geometry of Ω by the existence of a solution.

THEOREM 1.1. *Let* $H>0$ *and let* Ω *be an unbounded convex planar domain. If Eq.* (1) *has a solution in* Ω, *then* Ω *is included in a strip of width* $\frac{1}{H}$.

Sufficient Conditions. It is important to notice that the results of Finn and Serrin give sufficient conditions under which a domain admits solutions with *arbitrary boundary values.* We now consider the particular case of *zero boundary data.* In this particular case, there is a unique value $H_0>0$ depending on Ω for which solutions of (1) exist (as graphs over Ω) precisely for $|H| \leqslant H_0$ (a discussion of this point may be found in [7]). In [6] the author in collaboration with Montiel gives estimates on the value of H_0 in terms of the geometry of Ω. Further related results are also obtained by the author in [5]

THEOREM 1.2. *Let* $H>0$ *and let* Ω *be an unbounded convex planar domain included in a strip of width* $\frac{1}{H}$. *Then there exists an H-graph on* Ω *with boundary* $\partial\Omega$.

As a consequence of Theorems 1.1 and 1.2, we obtain:

COROLLARY 1.3. *Let* $H>0$ *and let* Ω *be an unbounded convex planar domain. Then there exists an H-graph in* Ω *with boundary* $\partial\Omega$ *if and only if* Ω *is included in a strip of width* $\frac{1}{H}$.

Finally, we establish an existence result of H-graphs for bounded convex domains included in a strip.

THEOREM 1.4. *Let* $H>0$ *and let* Ω *be a bounded convex domain included in a strip of width* $h>0$. *If* $H<\frac{1}{h}$, *there exists an H-graph on* Ω *whose boundary is* $\partial\Omega$.

Theorems 1.2 and 1.4 assert us that the Dirichlet problem with zero boundary data associated to the constant mean curvature equation (1) can be solved for convex domains included in a strip of width $\frac{1}{H}$. When Ω is not bounded, the result is optimum. However, when Ω is a bounded convex domain included in a strip, it is to be wished that the assumption $H < \frac{1}{h}$ on the width of the strip can be relaxed in some sense, as it occurs when Ω is a disk.

2. PROOF OF RESULTS

Proof of Theorem 1.1. Let u be a solution of (1) in Ω and let G denote the graph of u. Next result follows the same ideas as in [3].

LEMMA 2.1. *For each $p \in \Omega$, the disk $D(p, \frac{1}{H})$ centred at p and with radius $\frac{1}{H}$ is not included in Ω.*

Proof. We proceed by contradiction. Let Σ be a sphere of radius $\frac{1}{H}$ whose center lies on the straight-line through p that is orthogonal to the (x, y)-plane. Lift Σ vertically upwards until Σ is completely above G. Then, let us descend Σ until to find a first point of contact of Σ and G. Since $D(p, \frac{1}{H}) \subset \Omega$, the contact point is interior and the maximum principle gives us a contradiction. ∎

We continue with the proof of Theorem 1.1. From Lemma 2.1 and the convexity of Ω, it follows that Ω is included in a strip B' of width $\frac{2}{H}$. By means of a rigid motion, we can assume that the strip is given by

$$B' = \left\{(x, y);\ -\frac{1}{H} < y < \frac{1}{H}\right\}. \tag{3}$$

Since Ω is a convex domain, then we have two possibilities: if the x-coordinate is not bounded from above or below in Ω, then Ω is a strip included in B'; in other case, Ω is asymptotic to a strip included in B'. In the first case, it is known that the width of Ω is less than $\frac{1}{H}$. Consider the second one. Without loss of generality, we suppose that $\Omega \subset \{x > 0\}$ and $\bar{\Omega} \cap \{x = 0\} \neq \emptyset$. Let $B'_h \subset B'$ be the strip of width $h > 0$ such that Ω is asymptotic to the boundary of B'_h. For each $n > 0$, let $\{y_1(n), y_2(n)\} = \Omega \cap \{x = n\}$, with $y_1(n) < y_2(n)$. For each $m > n > 0$, consider $\Omega_{n,m} = \Omega \cap \{(x, y);\ n < x < m,\ y_1(n) < y < y_2(n)\}$. An integration of (1) over $\Omega_{n,m}$ yields

$$
\begin{aligned}
2H(m-n)(y_2(n)-y_1(n)) &= 2H\,\mathrm{area}(\Omega_{n,m}) \\
&= \left|\int_{\partial\Omega_{n,m}} \frac{\langle \nabla u, \nu\rangle}{\sqrt{1+|\nabla u|^2}}\right| \leqslant \mathrm{length}(\partial\Omega_{n,m}) \\
&= 2(m-n+y_2(n)-y_1(n)) \leqslant 2\left(m-n+\frac{2}{H}\right),
\end{aligned}
$$

where ν denotes the exterior directed unit normal along $\partial\Omega_{n,m}$. Letting $m\to\infty$, we have $y_2(n)-y_1(n)\leqslant\frac{1}{H}$. Since the two branches of $\partial\Omega$ are asymptotic to $\partial B'_h$, in the limit case ($n\to\infty$), we conclude $h\leqslant\frac{1}{H}$, and the theorem is proved.

Proof of Theorem 1.2. Without loss of generality, we suppose that the strip B in the statement of the theorem is given by $B=\{(x,y);\ -\frac{1}{2H}< y<\frac{1}{2H}\}$. Solutions of Eq. (1) on unbounded convex planar domains and with zero boundary data on the boundary satisfy a priori height estimates. Let u be a solution of (1) such that $u=0$ on $\partial\Omega$, being Ω an unbounded domain. First, since u is zero along $\partial\Omega$, u is bounded in Ω (see [8, Lemma 2.4]). Suppose now that Ω is convex. Then a standard barrier argument [1] by using halfcylinders of radius $\frac{1}{2H}$ of type $z(x,y)=-\sqrt{1/4H^2-y^2}$ and the plane $z=0$ gives us

$$
z(x,y)\leqslant u(x,y)\leqslant 0, \qquad (x,y)\in\Omega. \tag{4}
$$

Now we are in position to prove Theorem 2. There exists two types of unbounded convex domains in B: either an infinite strip parallel to the x-axis, or a domain with the x-coordinate is lower (or upper) bounded and asymptotic to a strip included in B.

(1) In the case that Ω is an infinite strip, by means of using a rigid motion, we can suppose that $\Omega=\{(x,y);\ -h<y<h\}$ with $h\leqslant\frac{1}{2H}$. Then the piece of the cylinder of radius $\frac{1}{2H}$ given by

$$
u(x,y)=\sqrt{\frac{1}{4H^2}-\frac{1}{h^2}}-\sqrt{\frac{1}{4H^2}-y^2}, \qquad (x,y)\in\Omega,
$$

is a solution of (1) with $u=0$ on $\partial\Omega$.

(2) Now consider that Ω is not a strip. In order to find a solution u of the corresponding Dirichlet problem, we shall use of the classical Perron method of superfunctions for (1) (see [4]; also [1] for an example in the same context). This will be possible provided that for sufficiently small disks there exist solutions of (1) with arbitrary continuous boundary values. Let $v\in C^0(\overline{\Omega})$ be a continuous function in $\overline{\Omega}$ and let D be a closed disk in Ω. We denote by $\bar{v}$ the unique solution of the Dirichlet problem

$L_H(\bar{v})=0$ in D satisfying the condition $\bar{v}=v$ on ∂D. The existence of the function $\bar{v}$ is assured by the Serrin result stated in Introduction: as $D\subset B$, the radius of the disk D is less than $\frac{1}{2H}$ and so, the curvature of ∂D is greater than $2H$. We define the uniquely continuous function $M_D(v)$ in Ω as

$$M_D(v)(p)=\begin{cases}\bar{v}(p), & p\in D\\ v(p), & p\in \Omega-D.\end{cases}$$

Following [4], we call a function $v\in C^0(\bar{\Omega})$ supersolution in Ω if it satisfies $v\geqslant M_D(v)$ for every disk D in Ω. On the other hand, a continuous function in $\bar{\Omega}$ will be called a superfunction relative to 0 if v is a supersolution in Ω and if, on ∂D, $v\geqslant 0$.

Let F denote the class all superfunctions relative to 0. The set F is not empty since $0\in F$: for any disk $D\subset\Omega$, $0\geqslant M_D(0)$ in D by the maximum principle of Eq. (1). The next lemma gives some properties of F that are necessary in the Perron technique.

LEMMA 2.2. (1) *If* $\{v_1, ..., v_n\}\subset F$, *then* $\min\{v_1, ..., v_n\}$ *also belongs to* F.

(2) *If* $v\in F$ *and if* D *is a disk in* Ω, *then* $M_D(v)\in F$.

Proof. (1) Let D be a disk in Ω and denote $v=\min\{v_1, ..., v_n\}$. On ∂D, we have $v_i\geqslant v$ and the maximum principle gives $M_D(v_i)\geqslant M_D(v)$ for each i. Then $\min\{M_D(v_1), ..., M_D(v_n)\}\geqslant M_D(v)$. Furthermore, as the functions v_i are supersolutions, $v_i\geqslant M_D(v_i)$ on D and then $v\geqslant \min\{M_D(v_1), ..., M_D(v_n)\}$. As a consequence, we obtain the desired inequality $v\geqslant M_D(v)$. This proves that v is a supersolution. On the other hand, $v_i\geqslant 0$ on $\partial\Omega$. Thus $v\geqslant 0$ on $\partial\Omega$, and we arrive that v is a superfunction in Ω relative to 0.

(2) First, we must show that

$$M_D(v)\geqslant M_{D'}(M_D(v))$$

for any disk $D'\subset\Omega$. This clearly holds if D' lies either completely inside or completely outside of D. Thus, we only consider the case where D' is partly inside and partly outside of D. As $v\in F$, it follows that $v\geqslant M_D(v)$ and the maximum principle yields $\bar{v}\geqslant \overline{M_D(v)}$ on D'. The fact that $v\geqslant M_{D'}(v)$ and the maximum principle leads again

$$v\geqslant M_{D'}(v)\geqslant M_{D'}(M_D(v)). \tag{5}$$

We now observe that by (5), we have $M_D(v)=v\geqslant M_{D'}(M_D(v))$ in $D'-D$. We study the situation in $D'\cap D$.

(a) If $p \in \partial(D' \cap D) \cap \partial D$, $M_D(v)(p) = v(p)$ and by (5), $M_D(v)(p) \geqslant M_{D'}(M_D(v))(p)$.

(b) If $p \in \partial(D' \cap D) \cap \partial D'$, $M_{D'}(M_D(v))(p) = M_D(v)(p)$.

As a consequence, it follows

$$M_D(v) \geqslant M_{D'}(M_D(v)) \qquad \text{on } \partial(D' \cap D)$$

and the maximum principles yields the desired inequality $M_D(v) \geqslant M_{D'}(M_D(v))$ in $D' \cap D$. Finally, since $\partial\Omega \subset \bar{\Omega} - D$, $M_D(v) = v \geqslant 0$ on $\partial\Omega$. Hence, we conclude that $M_D(v)$ is a superfunction in Ω relative to 0. ▮

Let z be the function defined in B given by $z(x, y) = -\sqrt{1/4H^2 - y^2}$. Let $Z = z_{|\Omega}$ be the restriction of z to the domain Ω. Consider

$$F^* = \{v \in F;\, Z \leqslant v \leqslant 0 \text{ in } \Omega\}.$$

Remark that the functions of F^* are uniformly bounded. Also, the functions of F^* satisfy the properties (1) and (2) of Lemma 2.2. For the property (1), $Z \leqslant v_i \leqslant 0$ and the same occurs with the function $\min\{v_1, ..., v_n\}$. In relation to the second property, let $v \in F^*$. As $Z \leqslant v \leqslant 0$, the maximum principle implies $M_D(Z) \leqslant M_D(v) \leqslant M_D(0)$ on D. But $M_D(Z) = Z$ and $M_D(0) \leqslant 0$.

Define

$$u = \inf\{v;\, v \in F^*\} = \inf\{M_D(v);\, v \in F^*, D \subset \Omega\}.$$

Perron's method shows that the function u is a solution of equation $L_H(u) = 0$ in Ω. We want to prove that u is continuous in $\bar{\Omega}$ and assumes 0 as boundary values. The fact that u is continuous in Ω is a consequence of Harnack's principle: if D is a disk in Ω, the functions $M_D(v)$ are uniformly bounded and this implies uniform boundedness of their first derivatives (see [9]). Thus the family $\{M_D(v);\, v \in F^*\}$ is uniformly continuous in D and the same occurs for the function u.

It remains to show that the Perron solution actually takes the value 0 on the boundary. The study of the boundary behaviour of the Perron solution is determined by the geometric properties of the boundary of Ω. This can be accomplished by the technique of barriers. Indeed, let $p \in \partial\Omega$. Consider an (infinite) quarter-cylinder C_p of radius $\frac{1}{2H}$ whose axis is parallel to the tangent line l to $\partial\Omega$ at the point p. Now let us place C_p in such a way that the axis and one component of C_p lie in the (x, y)-plane. Also, assume that $C_p \subset \{z \leqslant 0\}$, the concave side of C_p is looking towards p and $\partial C_p \cap \{z = 0\}$ lies in the component of $\{z = 0\} - l$ that does not contain Ω. By the convexity of $\partial\Omega$, we get $C_p \cap G = \emptyset$, where G is the graph of the function u. Move C_p towards Ω by horizontal translations until to touch G. By taking account of maximum principle, the convexity of Ω and the height estimate (4), the first point of contact between C_p and G occurs at p. Denote by v_p the function whose graph is C_p. Consequently, the functions v_p and $w=0$ are a modulus of continuity in a neighbourhood of p. Since $v_p(p)=w(p)=0$, we obtain $u(p)=0$. This completes the proof of Theorem 2.

Remark 1. The solution obtained in Theorem 1.2 is unique at infinity [2]. In the paper [3] mentioned above, Finn also conjectured that the half-cylinder of radius $1/2H$ was the only H-graph in a strip of width $1/H$. A counterexample can be obtained from the 1990 paper of Collin [1] who succeeded in proving the existence of H-graphs over a strip of width $1/H$ with boundary formed by two convex functions on $\partial\Omega$.

Proof of Theorem 1.4. Let Ω be a convex bounded domain included in a strip of width h. If $H<\frac{1}{h}$, the solvability of the Dirichlet problem

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 2H & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

can be achieved using the continuity method. Following the usual Leray–Schauder approach (see [4, Theorem 13.8]), solvability of (6) follows if there is a constant $M>0$ such that an apriori bound

$$|u|_{C^1(\Omega)} = \sup_\Omega |u| + \sup_\Omega |\nabla u| < M$$

holds for any solution u of (6) with $0\leqslant H\leqslant c$, where c is a fixed real positive number such that $c<\frac{1}{h}$. The constant M is required to be independent of u and H. Standard theory of quasilinear equations assures that the estimate of $|\nabla u|$ in Ω is obtained if we have bounds of $|\nabla u|$ along $\partial\Omega$.

For $H=0$, we have the trivial solution $u=0$ and from the implicit function theorem applied to Eq. (1), there exists a range of solutions for (6) with $0\leqslant H\leqslant c_0$, for a small number c_0. In the interval $[0, c_0]$, there exists uniform C^0-bounds depending only on Ω. For every solution u of (6) with $c_0\leqslant H\leqslant c$ the height estimate for graphs due to Serrin implies that $|u|\leqslant 1/|H|\leqslant 1/c_0$ ([9]; see also [8]). Thus we obtain an apriori estimation of $\sup_\Omega |u|$.

On the other hand, C^1-estimates of u along the boundary $\partial\Omega$ are obtained provided that we are able to establish apriori estimates of the slope of the H-graph G of u along its boundary $\partial G=\partial\Omega$. This estimate will be accomplished with quarter-cylinders as geometrical barrier surfaces as in

[6, Corollary 4]. This is summarized as follows. Without loss of generality, consider that Ω is contained in the strip

$$B^* = \left\{ (x, y);\ -\frac{c_1}{2} < y < \frac{c_1}{2} \right\},$$

with $c_1 < h/2$. We need first a slight improvement on the height estimates for u. By comparing with halfcylinders of radius $\frac{h}{2}$ as in Theorem 1.2, the maximum principle gets immediately $-\frac{h}{2} < u \leqslant 0$ (remark that $H \leqslant c < \frac{1}{h}$). Let K be the halfcylinder

$$K = \left\{ (x, y, z);\ -\frac{h}{2} < y < \frac{h}{2},\ z = -\frac{1}{2}\sqrt{h^2 - 4y^2} \right\},$$

whose mean curvature is $\frac{1}{h}$. Consider K' the piece of K that projects orthogonally onto B^*. Then the maximum principle asserts that we can move upwards K' until to touch the first time the plane $z = 0$ and without contacting with G. Therefore,

$$c_2 =:\ -\tfrac{1}{2}\left(h - \sqrt{h^2 - 4c_1^2}\right) < u(x, y) \leqslant 0, \qquad (x, y) \in \Omega. \tag{7}$$

In order to estimate the slope of the graph G, let $p \in \partial G = \partial\Omega$ be an arbitrary boundary point. Consider K_p a quarter-cylinder of radius $\frac{h}{2}$ whose axis is parallel to the tangent line l to $\partial\Omega$ at the point p and with the same characteristics as C_p in Theorem 1.2. Applying the estimate (7), we move up K_p slightly without touching G until to obtain a piece K_p' of K_p such that its height is c_2 with respect to the plane $z = 0$. After a rigid motion, the surface K_p' can be viewed as the half portion of K'. By moving K_p' towards p and by the convexity of Ω, the same reasoning as in Theorem 1.2 proves that the slope of G at the point p is less than the slope of K' along its boundary. This estimate depends only on h and c_2, or in others words, it is independent on H. This completes the proof of the theorem.

Remark 2. It is worth pointing out that the bounded domain Ω in Theorem 1.4 is not necessarily *strictly* convex, in contrast to the Serrin result in [9]. As a consequence, the boundary of our domain Ω can contain pieces of straight-lines or, in other situation, the size of Ω can be too big inside the strip.

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