# Hypersurfaces with constant mean curvature in hyperbolic space[1]

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**Abstract**

Our aim is the study of constant mean curvature hypersurfaces ($H$-hypersurfaces) in hyperbolic space with non connected boundary with possible asymptotic boundary. We ask when the hypersurface inherits the symmetries of its boundary. Also, results of non-existence of $H$-hypersurfaces are obtained in relation with the value of $H$ and the distance between the boundary components. The methods by which we arrive at our conclusions are the tangency principle, the Alexandrov reflection method and the existence of a special family of $H$-hypersurfaces of revolution.

## 1 Introduction and preliminaries

In 1958 Alexandrov [1] showed that the round spheres are the only embedded closed hypersurfaces of constant mean curvature in $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$. The purpose of this paper is the study of smooth constant mean curvature hypersurfaces in $\mathbf{H}^{n+1}$ with non empty boundary. Recent progress have been obtained for several authors when the hypersurface is compact and the boundary is a codimension two sphere [3, 13,

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14, 15]. Denote by $H$ the mean curvature. When $|H| \leq 1$ the only compact *immersed* hypersurfaces of constant mean curvature $H$ bounded by a codimension two round sphere are the umbilical ones: domains of totally geodesic hyperplanes ($H = 0$), hyperspheres ($0 < |H| < 1$) and horospheres ($|H| = 1$) (see [3, 13]). However, for $|H| > 1$ it is still unknown if spherical caps are the only compact embedded hypersurfaces with spherical boundary. There is a qualitative difference with respect to the study of constant mean curvature hypersurfaces in hyperbolic and Euclidean spaces. The reason is that in $\mathbf{H}^{n+1}$ there are spheres which have mean curvature bounded away from zero with their radii tend to infinity. When $|H| = 1$, the behaviour of surfaces in $\mathbf{H}^3$ is looks as minimal surfaces in Euclidean space, and the existence of horospheres makes rich the treatment of the problems in $\mathbf{H}^{n+1}$. The similarity between 1-surfaces in $\mathbf{H}^3$ and minimal surfaces in $\mathbf{R}^3$ is made evident in [4], and it is the origin of a wide research on hypersurfaces of mean curvature 1 [16]. If $|H| > 1$, the properties are very different from the case $|H| < 1$ and some investigations have been done on the behaviour near the infinity [10].

Following in this direction, we deal what influence does the boundary have over the shape of a $H$-hypersurface in $\mathbf{H}^{n+1}$ and when, in the case that is embedded, the hypersurface inherits the symmetries of its boundary. In this sense and in order to set up definitions to be used later, we say that $M$ is a *hypersurface of revolution* if there exists a geodesic in $\mathbf{H}^{n+1}$ such that $M$ is invariant by the group of rotations of $\mathbf{H}^{n+1}$ that leaves this geodesic pointwise fixed.

We will work in the upper halfspace model of $\mathbf{H}^{n+1}$, that is,

$$\mathbf{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}; x_{n+1} > 0\}$$

equipped with the metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_{n+1}^2}{x_{n+1}^2}.$$

The hyperbolic space $\mathbf{H}^{n+1}$ has a natural compactification $\overline{\mathbf{H}^{n+1}} = \mathbf{H}^{n+1} \cup S^n(\infty)$, where $S^n(\infty)$ is identified with asymptotic classes of geodesics rays in $\mathbf{H}^{n+1}$. In the upper halfspace model of $\mathbf{H}^{n+1}$, the asymptotic boundary $S^n(\infty)$ of $\mathbf{H}^{n+1}$ is the one-point compactification of the hyperplane $\{x_{n+1} = 0\}$. Let $M$ be a subset of $\mathbf{H}^{n+1}$. We call the *asymptotic boundary* of $M$ the set $\partial_\infty M$ given by

$$\partial_\infty M = \overline{M} \cap S^n(\infty),$$

where $\overline{M}$ denotes the closure of $M$ in $\overline{\mathbf{H}^{n+1}}$. The concept of asymptotic boundary was introduced by Anderson [2] to prove that any closed submanifold in $S^n(\infty)$ is the asymptotic boundary of a minimal variety of $\mathbf{H}^{n+1}$. The notion of asymptotic boundary is important to understand the behaviour of noncompact constant mean curvature hypersurfaces (cf. [5, 6, 7, 9, 11]).

For future references, we shall refer any codimension two spheres in $\mathbf{H}^{n+1}$ or in $S^n(\infty)$ as *spheres*. Let $\phi : M \to \mathbf{H}^{n+1}$ be an isometric immersion of a smooth hypersurface $M$ with boundary $\partial M \neq \emptyset$, and let $C$ be a codimension two submanifold of $\mathbf{H}^{n+1}$. We say $C$ is the *boundary* of $\phi$ if $\phi$ maps diffeomorphically $\partial M$ onto $C$ and briefer, $M$ *with boundary* $C$ without particular references to the parametrization. Also, we call the *generalized boundary* of $M$ the set $\partial_g M$ given by $\phi(\partial M) \cup \partial_\infty \phi(M)$. Finally, we say that $M$ is an $H$*-hypersurface* if $\phi$ has constant mean curvature $H$, where $H \in \mathbf{R}$.

We extend the concept of distance between two compact sets of $S^n(\infty)$ to subsets in $\mathbf{H}^{n+1}$ according to [5]. Let $P_1$ and $P_2$ be disjoint geodesic hyperplanes in $\mathbf{H}^{n+1}$. Then $P_1 \cup P_2$ divides $\mathbf{H}^{n+1}$ in three components. Let $D_1$ and $D_2$ be the two of them with boundary $P_1$ and $P_2$ respectively. Given two subsets $A_1$ and $A_2$ of $\mathbf{H}^{n+1}$, we say that $P_1$ and $P_2$ *separate* $A_1$ and $A_2$ if $A_i \subset D_i$, $i = 1, 2$. In this case, we call the *distance* between $A_1$ and $A_2$ the number

$$d(A_1, A_2) = \sup\{d(P_1, P_2);\ P_1 \text{ and } P_2 \text{ separate } A_1 \text{ and } A_2\},$$

where $d(P_1, P_2)$ denotes the distance between $P_1$ and $P_2$. In other case, we put $d(A_1, A_2) = 0$. Roughly speaking $d(A_1, A_2)$ is the largest distance between the 'parallel' geodesic hyperplanes that separate $A_1$ and $A_2$. When $A_1 \subset \mathbf{H}^{n+1}$ and $A_2 \subset S^n(\infty)$, we say that $P_1$ and $P_2$ separate $A_1$ and $A_2$ if $A_1 \subset D_1$ and $A_2 \subset \partial_\infty D_2$. Finally, if $A_1, A_2 \subset S^n(\infty)$, $P_1$ and $P_2$ separate $A_1$ and $A_2$ if $A_i \subset \partial_\infty D_i$, $i = 1, 2$. Let us observe that if the asymptotic boundary has an isolated point $p$, then $d(p, C) = \infty$ for each other component $C$ of $\partial_g M$.

Also, we need the concept of regularity at the asymptotic boundary. We say that $M$ is *regular at infinity* if $\overline{M} \subset \overline{\mathbf{H}^{n+1}}$ is a $C^2$-hypersurface (with boundary) of $\overline{\mathbf{H}^{n+1}}$, and $\partial_\infty M$ is a $C^2$-submanifold of $S^n(\infty)$.

Finally, we need the next definition. Two $(n-1)$-spheres $C_1$ and $C_2$ in $\mathbf{H}^{n+1}$ are called *coaxial* if there is a geodesic $\gamma$ such that $C_1 \cup C_2$ is invariant by the group of rotations that leaves pointwise fixed $\gamma$. The geodesic $\gamma$ will be called the rotation axis of $C_1 \cup C_2$. Since isometries in $\mathbf{H}^{n+1}$ induce conformal diffeomorphisms on $S^n(\infty)$, we can extend this definition when $C_1 \subset \mathbf{H}^{n+1}$ and $C_2 \subset \{x_{n+1} = 0\}$.

We are in position to give a brief summary of the main results. In Section 2 we use the Alexandrov reflection method to study the behaviour of a compact embedded $H$-hypersurface with non connected boundary in relation to the symmetries of its boundary. We establish (Corollaries 2.3 and 2.4):

*Let $C_1$ and $C_2$ be coaxial spheres of codimension two such that $C_1$ is included in a horosphere (or in a geodesic hyperplane) $Q_1$ and $C_2$ in $\{x_{n+1} = 0\} \subset S^n(\infty)$. Let $M \subset \mathbf{H}^{n+1}$ be an embedded $H$-hypersurface with $\partial_g M = C_1 \cup C_2$. If $M$ is* *included in the domain of $\mathbf{H}^{n+1} \setminus Q_1$ that contains $C_2$, then $M$ is a hypersurface of revolution.*

In Section 3 we consider the existence of compact $H$-hypersurfaces with non connected boundary in relation to the distance between their boundary components:

*Given $H \in (-1,1)$, there exists a constant $d_H$ depending only on $H$ such that if $C_1$ and $C_2$ are codimension two submanifolds with $d(C_1, C_2) > d_H$, then there exists no immersed connected compact $H$-hypersurfaces in $\mathbf{H}^{n+1}$ spanning $C_1 \cup C_2$.*

This result is a consequence of other one more general for hypersurfaces with non necessarily constant mean curvature function $H$ and $|H| < 1$ (Theorem 3.1). In this sense, if we recall that the only compact $H$-hypersurfaces in $\mathbf{H}^{n+1}$, $|H| \leq 1$, spanning a round $(n-1)$-sphere are the umbilical examples, we can observe that the behaviour of the $H$-hypersurfaces with $|H| \leq 1$ constitutes a new phenomena inside the general theory of constant mean curvature hypersurfaces in $\mathbf{H}^{n+1}$.

In Section 4 we study possible configurations of a $H$-hypersurface in $\mathbf{H}^{n+1}$ when the boundary components are included in geodesic hyperplanes of horospheres. In this direction, we show in Theorem 4.1:

*Let $C_1$ and $C_2$ be compact $(n-1)$-submanifolds included respectively in disjoint geodesic hyperplanes $P_1$ and $P_2$. There is a positive number $d_0$ such that if $d(C_1, C_2) \geq d_0$ and $H \neq 0$, any compact embedded connected $H$-hypersurface in $\mathbf{H}^{n+1}$ bounded by $C_1 \cup C_2$ and that intersects no the exterior of $C_i$ in $P_i$ must be included in the domain determined by $P_1$ and $P_2$.*

One may obtain analogous results when the boundary is included in disjoint horospheres (Theorem 4.3) and the case that one of the two boundary components lies in $S^n(\infty)$ (Corollary 4.5).

# 2 Symmetries of embedded $H$-hypersurfaces

Let us consider an embedded $H$-hypersurface $M \subset \mathbf{H}^{n+1}$ with boundary. If one studies whether $M$ inherits the symmetries of its boundary, an important tool is the so-called Alexandrov reflection method [1], which is based in the classical maximum principle for elliptic equations. The following version of the maximum principle for hypersurfaces can be stated (see [6] for details and definitions):

**Proposition 2.1 (Tangency principle)** *Let $M_1$ and $M_2$ be two oriented constant mean curvature hypersurfaces in $\mathbf{H}^{n+1}$ of mean curvature $H_1 \leq H_2$ respectively. If $M_1$ and $M_2$ have a point $p$ of common tangency, either in the interior or in the (analytic) boundary, and $M_1$ lies above $M_2$ near $p$, then $M_1 = M_2$ in a neighbourhood of $p$*

With the aid of the Alexandrov method, it is proved that a compact embedded $H$-hypersurface in $\mathbf{H}^{n+1}$ spanning a sphere and contained in one of the two halfspaces determined by the geodesic hyperplane that contains the boundary must be a hypersurface of revolution. Others applications can be viewed in [5, 7, 11, 15]. This section is devoted to derive some consequences of this technique in the case that the boundary of the hypersurface is not connected. We start by considering the following result that is similar as for $H$-hypersurfaces in $\mathbf{R}^{n+1}$ (for example, see [12, Theorem 2.1]).

**Proposition 2.2** *Let $Q_1$ and $Q_2$ be two horospheres with the same asymptotic boundary. Let $C_1$ and $C_2$ be coaxial spheres of codimension two included in $Q_1$ and $Q_2$ respectively. Let $M$ be a compact embedded $H$-hypersurface in $\mathbf{H}^{n+1}$ with boundary $C_1 \cup C_2$. If $M$ is included in the domain determined by $Q_1$ and $Q_2$, then $M$ is a hypersurface of revolution.*

*Proof:* The proof is a standard application of the Alexandrov reflection technique. We make hyperbolic reflections with respect to a family of totally geodesic hyperplanes orthogonal to $Q_1 \cup Q_2$. The fact that $M$ is included in the domain defined by $Q_1$ and $Q_2$ leads that the possible contact points between $M$ and its successive reflections occurs at interior or boundary points where the tangency principle can be applied. As consequence, $M$ inherits the symmetries of its boundary. In our case, $M$ is a hypersurface of revolution.

$\square$

The same reasoning holds when one of the two boundary components lies in $S^n(\infty)$:

**Corollary 2.3** *Let $C_1$ and $C_2$ be coaxial spheres of codimension two such that $C_1$ is included in a horosphere $Q_1$ and $C_2 \subset S^n(\infty)$. Let $M$ be an embedded $H$-hypersurface in $\mathbf{H}^{n+1}$ with $\partial_g M = C_1 \cup C_2$. If $M$ is included in the domain of $\mathbf{H}^{n+1} \setminus Q_1$ that contains $C_2$, then $M$ is a hypersurface of revolution.*

Remark 1: We observe that the same argument applies even if $C_2$ is a single point $p \in S^n(\infty)$ provided the geodesic joining the centre of $C_1$ and $p$ defines a group of rotations in $\overline{\mathbf{H}^{n+1}}$ that leaves invariant $C_1 \cup \{p\}$.

It is remarkable that the same process in Proposition 2.2 does not work when the hypersurface is included in the domain determined by two disjoint geodesic hyperplanes.

This anomaly is partly explained by the fact that there exists no a one-parameter family of geodesic hyperplanes orthogonal to both hyperplanes. However, we have the analogous result to Corollary 2.3 when one of the boundary components lies at infinity.

**Corollary 2.4** *Let $C_1$ and $C_2$ be coaxial spheres of codimension two where $C_1$ is included in a geodesic hyperplane $P_1$, $C_2 \subset S^n(\infty)$ and $C_2 \cap \partial_\infty P_1 = \emptyset$. Let $M$ be an embedded $H$-hypersurface in $\mathbf{H}^{n+1}$ with $\partial_g M = C_1 \cup C_2$. If $M$ is included in the component of $\mathbf{H}^{n+1} \setminus P_1$ that contains $C_2$, then $M$ is a hypersurface of revolution.*

*Proof:* Without loss of generality, we assume that the rotation axis $\gamma$ of $C_1 \cup C_2$ is the $x_{n+1}$-axis. Consider $\Omega_1 \subset P_1$ and $\Omega_2 \subset \{x_{n+1} = 0\}$ the bounded domains determined by $C_1$ and $C_2$ respectively. We construct the embedded hypersurface $M \cup \Omega_1 \cup \Omega_2$ and let $W$ denote the bounded domain that determines in $\{x_{n+1} > 0\}$ (Figure 1).

Let $P$ be a geodesic hyperplane containing $\gamma$ and we shall prove that $P$ is a hyperplane of symmetry of $M$. Let $\alpha$ be an infinite geodesic in $P_1$ that intersects $P$ orthogonally at $\gamma \cap P_1$. Let $\{P(t); t \in \mathbf{R}\}$ be the one-parameter family of geodesic hyperplanes of $\mathbf{H}^{n+1}$ such that for each $t$, $P(t)$ intersects $\alpha$ orthogonally at the point $\alpha(t)$. We use the Alexandrov method with the family $P(t)$. The key fact is that the domain determined by $P_1$ and $S^n(\infty)$ that contains $M$ is invariant by the hyperbolic reflections with respect to $P(t)$ (see Figure 1). Now, the Alexandrov reflection method finishes with the proof.

$\square$

Remark 2: From the proof, we observe that the assumption '$C_2 \cap \partial_\infty P_1 = \emptyset$' in Corollary 2.4 can be replaced by either of the two following situations:

1. $C_2 = \partial_\infty P_1$ and $M \subset \mathbf{H}^{n+1} \setminus P_1$, or
2. $C_2$ is a component of $\partial_\infty \gamma$.

To end this section, we consider other kind of symmetry for $H$-hypersurfaces with boundary. Let $P$ be a geodesic hyperplane and $\gamma$ a geodesic orthogonal to $P$. If $\Omega$ is a domain in $P$, we call the *solid cylinder* $K(\Omega, \gamma)$ determined by $\Omega$ with respect to $\gamma$, the set of all hyperbolic translations of $\Omega$ along $\gamma$, i.e., if $q \in \Omega$ and $l_q$ is the integral curve of the Killing vector field associated to the hyperbolic translation along $\gamma$, $K(\Omega, \gamma) = \cup_{q \in \Omega} l_q$.

**Proposition 2.5** *Let $P_1$ and $P_2$ be two geodesic hyperplanes such that $P_2$ is a hyperbolic translation with respect to a geodesic $\gamma$ orthogonal to $P_1$. Let $C_1$ be a closed codimension two submanifold in $P_1$ and $C_2$ the corresponding translation in $P_2$. Let $\Omega_1 \subset P_1$ be the*

Figure 1:

*bounded domain determined by $C_1$. Assume that $M \subset \mathbf{H}^{n+1}$ is a compact embedded connected $H$-hypersurface spanning $C_1 \cup C_2$ such that $M \cap \partial K(\Omega_1, \gamma) = C_1 \cup C_2$ and either*

$$M \subset K(\Omega_1, \gamma) \quad \textit{or} \quad M \subset \mathbf{H}^{n+1} \setminus \overline{K(\Omega_1, \gamma)}.$$

*Then $M$ is symmetric with respect to the hyperplane $P_3$ that is equidistant from $P_1$ and $P_2$. Moreover, $P_3$ divides $M$ in two graphs on $P_3$.*

*Proof:* After an isometry in $\mathbf{H}^{n+1}$, we can suppose that $\gamma$ is the $x_{n+1}$-axis and thus, $P_1$ and $P_2$ are two Euclidean concentric hemispheres in $x_{n+1} \geq 0$ centred at the origin. Consider the family of geodesic hyperplanes $\{P(t); t \geq 0\}$, where $P(0) = P_1$ and $P(t)$ is the hyperbolic translation along $\gamma$: the parameter $t$ denotes the distance between $P(t)$ and $P(0)$. Assume that $P_2 = P(d)$, that is, $d$ is the distance between $P_1$ and $P_2$. Let $W$ denote the bounded domain determined by

$$B = M \cup (\partial K(\Omega_1, \gamma) \cap (\cup_{0 \leq t \leq d} P(t)) .$$

The hypothesis $M \cap \partial K(\Omega_1, \gamma) = C_1 \cup C_2$ assures that $B$ is an embedded closed hypersurface (non smooth along $C_1 \cup C_2$). By the argument of Corollary 2.3, we reflect $M$ with respect to hyperplanes $P(t)$ coming $t = \infty$ until the first time so that the reflected of $M$ touches $M$ again. By the tangency principle, it cannot exist a tangent point.

Thus the reflection process continues until the intermediate position between $P_1$ and $P_2$. Changing the roles of $P_1$ and $P_2$, it concludes that $P_3$ is a hyperplane of symmetry of $M$. Furthermore the Alexandrov technique implies that

$$M_1 = M \cap (\cup_{t \leq d/2} P(t)) \quad \text{and} \quad M_2 = M \cap (\cup_{t \geq d/2} P(t))$$

are graphs on $P_3$ (by graph we mean that for each point $q$ of $P_3$, $l_q$ meets $M_i$ just at a single point at most).

□

# 3 Non-existence of $H$-hypersurfaces with boundary

This section is devoted to analyse how the distance between two boundary components of a $H$-hypersurface $M$ determines some aspects related with the shape of $M$. A classic result in the minimal surfaces theory states that a minimal surface $M$ in $\mathbf{R}^3$ bounded by a pair of sufficiently distant curves cannot be connected. For this, we include the boundary curves in a solid cylinder. One proves the result by 'pinching' the surface with a family of catenoids having the same rotation axis as the cylinder, but smaller and smaller 'neck-size', and applying the maximum principle at the moment of the first contact. As in [5], we use the same argument with an analogous family of constant mean curvature surfaces of revolution studied by J. Gomes in [8, 9]. With the aid of these surfaces, we will prove that hypersurfaces of small mean curvature in hyperbolic space are disconnected if their boundary components are sufficiently distant.

Next we describe this kind of hypersurfaces (see [9] and [5]). Let $h \in [0, 1)$ and a geodesic $\gamma$ in $\mathbf{H}^{n+1}$. Then there exists a uniparametric family $\{M^h_\lambda; \lambda \geq 0\}$ of (non-umbilical) embedded $h$-hypersurfaces of revolution with the same axis of rotation. This axis is orthogonal to $\gamma$. Moreover

1. Each $M^h_\lambda$ is symmetric with respect to $\gamma$.
2. The asymptotic boundary is formed by two disjoint spheres: $\partial_\infty M^h_\lambda = S^h_{1,\lambda} \cup S^h_{2,\lambda}$.
3. If $P^h_{1,\lambda}, P^h_{2,\lambda}$ are the geodesic hyperplanes of $\mathbf{H}^{n+1}$ such that $\partial_\infty P^h_{i,\lambda} = S^h_{i,\lambda}$, $i = 1, 2$, then $M^h_\lambda$ is included in the component of $\overline{\mathbf{H}^{n+1}} \setminus (P^h_{1,\lambda} \cup P^h_{2,\lambda})$ whose boundary is $S^h_{1,\lambda} \cup S^h_{2,\lambda}$.
4. Each hypersurface $M^h_\lambda$ divides $\mathbf{H}^{n+1}$ in two components, one of them contains the rotation axis. The mean curvature vector points towards this component.

5. The distance function $d^h(\lambda) = d(S^h_{1,\lambda}, S^h_{2,\lambda}) = d(P^h_{1,\lambda}, P^h_{2,\lambda})$ satisfies $d^h(0) = 0$, is increasing near $\lambda = 0$, reaches a maximum $d_h > 0$ and decreases monotonically to zero as $\lambda \to \infty$. The value of $d^h(\lambda)$ is

$$d^h(\lambda) = \int_\lambda^\infty \frac{\sinh\lambda\cosh\lambda - h\,\sinh^2\lambda + h\,\sinh^2 x}{\cosh x\sqrt{\sinh^4 x\,\cosh^4 x - \left(\sinh\lambda\,\cosh\lambda - h\,\sinh^2\lambda + h\,\sinh^2 x\right)^2}}dx$$

and the number $d_h$ depends only on $h$.

The first result is an extension of Theorem 1 in [5] in the case that the hypersurface has non empty boundary. In fact, it is possible to consider immersed hypersurfaces with non necessarily constant mean curvature.

**Theorem 3.1** *Let $h \in (0,1)$. Then there exists a positive constant $d_h$ depending only on $h$ with the following property: let $M$ be a connected hypersurface immersed in $\mathbf{H}^{n+1}$ with mean curvature function $H$ and $H \leq h$. Then it holds that $d(C, \partial_g M \setminus C) \leq d_h$ for any component $C$ of $\partial_g M$. The equality holds if and only if $M$ is a hypersurface of revolution.*

*Proof:* The value of $d_h$ is given by the maximum of the function $d^h$ defined previously. The proof is by contradiction. Let $C_2$ be a component of $\partial_g M \setminus C_1$. Since $d(C_1, C_2) > d_h$, there exist two disjoint geodesic hyperplanes $P_1$ and $P_2$ separating $C_1$ and $C_2$ and $d(P_1, P_2) > d_h$. Let $D_1$ and $D_2$ be the two components of $\mathbf{H}^{n+1} \setminus (P_1 \cup P_2)$ such that $\partial D_i = P_i$ and $C_i \subset D_i$. Call $D_3$ the other component with $\partial D_3 = P_1 \cup P_2$. There exists a geodesic $\gamma$ in $D_3$ such that the corresponding family $\{M^h_\lambda; \lambda > 0\}$ is included in $D_3$.

For small number $\lambda > 0$, $M \cap M^h_\lambda = \emptyset$ because in the domain $D_3$ there are no components of $\partial_g M$. Let $\lambda \to \infty$. Since $M$ is connected, there exists a first time $\lambda_0$ such that $M^h_{\lambda_0}$ has an intersection point $p$ with $M$. Because $M^h_{\lambda_0} \subset D_3$, this point $p$ must be interior in $M$. Thus the tangent spaces of $M$ and $M^h_{\lambda_0}$ agree at $p$. Choose an orientation of $M^h_{\lambda_0}$ to have positive mean curvature. Then the hypersurface $M$ is over $M^h_{\lambda_0}$ (Figure 2). Since $H \leq h$, the tangency principle assures that $M^h_{\lambda_0}$ agrees with $M$ in an open set. This is a contradicition with the fact that $D_3 \cap \partial_\infty M = \emptyset$. Consequently, $d(C_1, C_2) \leq d_h$.

Finally, if $d(C_1, C_2) = d_h$, the tangency principle yields that $\partial_\infty M^h_{\lambda_0} = \partial_\infty(P_1 \cup P_2)$. In this case, $M^h_{\lambda_0}$ has a common tangent point with $M$ and the tangency principle gets again that $M^h_{\lambda_0} = M$, $H = h$ and $M$ is a hypersurface of revolution.

□

As a particular case of Theorem 3.1 we have:

Figure 2:

**Corollary 3.2** *Given $H \in (-1, 1)$, there exists a constant $d_H$ depending only on $H$ such that if $C_1$ and $C_2$ are codimension two submanifolds with $d(C_1, C_2) > d_H$, then there exists no immersed connected compact $H$-hypersurfaces in $\mathbf{H}^{n+1}$ spanning $C_1 \cup C_2$.*

This result says us that for $|H| < 1$ there are no connected $H$-hypersurfaces in $\mathbf{H}^{n+1}$ with boundary components sufficiently distant. In this sense, the behaviour of the $H$-hypersurfaces in $\mathbf{H}^{n+1}$ where $H \in (-1, 1)$ is similar as minimal surfaces in $\mathbf{R}^3$. In the Euclidean ambient this result is not true for nonzero constant mean curvature surfaces: given $H \in \mathbf{R}$, a piece of a right cylinder of radius $\frac{1}{2|H|}$ can be bounded by two circles with arbitrary distance.

We have as consequence of Theorem 3.1:

**Corollary 3.3** *Let $M$ be a connected $H$-hypersurface with boundary in $\mathbf{H}^{n+1}$ with $|H| < 1$. Then the asymptotic boundary of $M$ has not isolated points.*

The following result is the analogous one to Theorem 2 in [5].

**Theorem 3.4** *Let $M \subset \mathbf{H}^{n+1}$ be an embedded $H$-hypersurface with boundary and regular at infinity such that $\partial_\infty M \neq \emptyset$. Then $|H| \leq 1$ and*

1. *If* $|H| < 1$, *then* $\partial_\infty \overline{M} = \partial_\infty M$ *and* $M$ *is nowhere tangent to* $S^n(\infty)$.

2. *If* $|H| = 1$, *then* $\overline{M}$ *is everywhere tangent to* $S^n(\infty)$ *along* $\partial\overline{M} \cap S^n(\infty)$.

*Proof:* After a rigid motion in $\mathbf{H}^{n+1}$, consider a horosphere $Q = \{x_{n+1} = a\}$, $a > 0$, such that the boundary $\partial M$ of $M$ lies in $\{x_{n+1} > a\}$, $M$ is transverse to $Q$ and $\partial_\infty M \subset \{x_{n+1} = 0\}$. Set $M^* = M \cap \{x_{n+1} \leq a\}$. Then $M^*$ spans a set of closed submanifods $C_1, \ldots, C_k$ included on $Q$. We decompose $M^*$ in the following way. Let $\varepsilon$ be a small positive number. For each $i = 1, \ldots, k$, let $C_i^-(\varepsilon)$ be the bounded $(n-1)$-submanifold on $M^*$ near $C_i$ obtained by intersecting $M^*$ with the horosphere $\{x_{n+1} = a - \varepsilon\}$. Remove from $M^*$ the annuli bounded by $C_i \cup C_i^-(\varepsilon)$ and let us attach the domains $D_i^-(\varepsilon)$ bounded by $C_i^-(\varepsilon)$ in $\{x_{n+1} = a - \epsilon\}$ obtaining an embedded hypersurface $B$. We use different values of $\varepsilon$ when several $C_i$ are concentric. Let $M^*$ be any component of $B$ such that $\partial_\infty M^* \neq \emptyset$. Then we have an embedded hypersurface (not smooth along $C_i^-(\varepsilon)$ for some values $i$) that divides $\mathbf{H}^{n+1}$ in two components, denoted by $I$ and $O$. Without loss of generality, assume that for $H \geq 0$, the unit normal vector field $N$ of $M^*$ points towards $I$.

Firstly we prove that $H \leq 1$. We assume on the contrary, $H > 1$. Since $\partial_\infty M^* \neq \emptyset$, let $\Sigma$ be a sphere of constant mean curvature $H$, with small Euclidean radius and sufficiently close to $S^n(\infty)$ so that $\Sigma$ is inside $I$. This is possible because a sphere with centre in $p = (p_1, \ldots, p_{n+1})$ and Euclidean radius $s > 0$, $s < p_{n+1}$ has $H = \frac{p_{n+1}}{s}$ as mean curvature. Move $\Sigma$ by horizontal translations towards $M^*$, i.e., according to a parallel direction to the $x_{n+1}$-hyperplane. Then the first intersection point $q$ between $M^*$ and $\Sigma$ is interior. In this case, the normal vector field of $M^*$ and $\Sigma$ agree at $q$ because both vectors point towards $I$. This is a contradiction with the tangency principle because $\partial_\infty \Sigma = \emptyset$. Thus $H \leq 1$.

If $H < 1$, Corollary 3.3 assures that $\partial_\infty M^*$ does not contain isolated points. Thus $\overline{M^*}$ is not tangent to $S^n(\infty)$ and so, $\partial_\infty \overline{M^*}$ agrees with $\partial_\infty M^*$.

If $H = 1$ and $\partial_\infty \overline{M^*} \neq \emptyset$, the domain $\partial_\infty I$ is not empty. Let $p \in O$ and $\Sigma_p$ be a horosphere with $\partial_\infty \Sigma_p = \{p\}$ and with Euclidean radius sufficiently small so that $\Sigma_p - \{p\}$ is included in $I$. This is possible by the regularity at infinity. Let us move $\Sigma_p$ towards $M^*$ by horizontal translations. In view of the tangency principle and since $\partial \Sigma_p = \emptyset$, the first intersection point lies at the boundary of $\partial_\infty I$. Moreover $\overline{M^*}$ is tangent to $S^n(\infty)$ at this point. By moving $\Sigma_p$ in each horizontal direction, we concluded that $M^*$ is tangent to $S^n(\infty)$ at any point.

$\square$

# 4 Certain configurations of $H$-hypersurfaces with boundary

We consider embedded hypersurfaces with nonzero mean curvature such that the boundary components are included in geodesic hyperplanes or horospheres. The method of proof introduced in Theorem 3.1 allows us to obtain information about the geometry of the hypersurface in relation with these hyperplanes or horospheres.

**Theorem 4.1** *Let $C_1$ and $C_2$ be compact $(n-1)$-submanifolds in disjoint geodesic hyperplanes $P_1$ and $P_2$ of $\mathbf{H}^{n+1}$ and let $\Omega_i \subset P_i$ be the domains bounded by $C_i$, $i=1,2$. There exists a positive number $d_0$ with the following property: if $d(C_1, C_2) \geq d_0$ any connected compact embedded hypersurface $M$ of mean curvature function $H \neq 0$ spanning $C_1 \cup C_2$ and such that $M \cap (P_i \setminus \overline{\Omega_i}) = \emptyset$, $i=1,2$ must be included in the domain $S$ determined by $P_1$ and $P_2$.*

*Proof:* Let us take $d_0$ the maximum of the function $d^0(\lambda)$ given in Section 3. By standard hyperbolic geometry and after an isometry in $\mathbf{H}^{n+1}$, the asymptotic boundaries of $P_1$ and $P_2$ are coaxial spheres in $\{x_{n+1}=0\}$ (see Figure 3). Without loss of generality, we suppose that $P_1$ lies above $P_2$ respect to the positive direction of the $x_{n+1}$-axis. Since $H \neq 0$, we can choose an orientation of $M$ by the unit normal vector field $N$ such that the mean curvature function $H$ is positive.

Let us attach to $M$ two great geodesic balls $O_i \subset P_i$, with $\overline{\Omega_i} \subset O_i$. We denote by $P_1^+$ the upper halfspace determined by $P_1$, that is, the component of $\mathbf{H}^{n+1} \setminus P_1$ that does not contain $P_2$. Also, let $P_2^-$ be the halspace below $P_2$. We consider appropriate compact hypersurfaces $S_1$ and $S_2$ such that $\partial S_i = \partial O_i$, $S_1 \subset P_1^+$, $S_2 \subset P_2^-$ and $M$ does not intersect $S_1 \cup S_2$ (for example, spherical geodesic domains. See Figure 3). This is possible by the compactness of $M$. Hence, we obtain a closed embedded hypersurface

$$M \cup (S_1 \cup (O_1 \setminus \overline{\Omega_i})) \cup (S_2 \cup (O_2 \setminus \overline{\Omega_2}))$$

enclosing a domain $W$ non smooth on $C_i \cup \partial O_i$, $i=1,2$.

By contradiction, suppose that $M$ contains points outside $S$. If some part of $M$ lies above $P_1$, let $p \in M$ be the highest point with respect to $P_1$. Then $p \notin \partial M$. Consider an Euclidean hemisphere $P_3$ in $x_{n+1} \geq 0$ such that $\partial P_3 \subset \{x_{n+1}=0\}$ is a concentric sphere with $\partial P_1$ and such that $p \in P_3$. In hyperbolic geometry, $P_3$ is a geodesic hyperplanes and satisfies that $M$ lies below $P_3$. Since $P_3$ is a minimal hypersurface, by comparying $P_3$ and $M$ at $p$, the tangency principle leads that $N(p)$ points towards $\mathbf{H}^{n+1} \setminus \overline{W}$ (Figure 3). In the case that $M$ contains points below $P_2$, we consider geodesic hyperplanes that

Figure 3:

come from $S^n(\infty)$ below $M$ touching $M$ at the lowest (interior) point with respect to $P_2$. Again, $N$ points towards $\mathbf{H}^{n+1} \setminus \overline{W}$.

Consider the corresponding family of minimal hypersurfaces $\{M_\lambda^0; \lambda > 0\}$. Since $d(C_1, C_2) \geq d_0$, each hypersurface $M_\lambda^0$ lies in the domain $S$. For small number $\lambda > 0$ and since $M$ is compact, $M_\lambda^0$ is included in $\mathbf{H}^{n+1} \setminus \overline{W}$. Now we increase $\lambda \to \infty$ until the first intersection point $q$ with $M$ for the value $\lambda = \lambda_0$. This point is not at the boundary of $M$, because $M_\lambda^0 \subset S$ for each $\lambda > 0$. Since $N(q)$ points outside $W$, $M_{\lambda_0}^0$ lies above $M$ in a neighbourhood of $q$: this is a contradiction by the tangency principle, because the mean curvature $H$ on $M$ is positive. Hence, we conclude that $M \subset S$ and this completes the proof of the theorem.

$\square$

When the boundary is formed by two round spheres, we obtain as a consequence:

**Corollary 4.2** *Let $C_1$ and $C_2$ be coaxial spheres of the same radii $r_0$ included in hyperplanes $P_1$ and $P_2$ respectively such that $d(C_1, C_2) \geq d_0$. Let $M \subset \mathbf{H}^{n+1}$ be a compact embedded hypersurface of mean curvature function $H$, $0 < |H| \leq 1$, and with boundary $C_1 \cup C_2$. Then $M$ is included in the domain determined by $P_1$ and $P_2$.*

*Proof:* After an isometry of $\mathbf{H}^{n+1}$, we suppose that the $x_{n+1}$-axis is the rotation axis of $C_1 \cup C_2$. Let us denote $C(r)$ the cylinder of radius $r > 0$ around $\gamma$, that is, the set of points at distance $r$ from $\gamma$:

$$C(r) = \{(x_1, \ldots, x_{n+1}) \in \mathbf{H}^{n+1}; \sum_{i=1}^{n} x_i^2 = (\sinh^2 r) x_{n+1}^2\}.$$

Let us call $B(r)$ the domain in $\mathbf{H}^{n+1} \setminus C(r)$ that contains $\gamma$. Let us choose the orientation of $C(r)$ such that its mean curvature $H(r)$ is positive. Then the unit normal vector field points towards $B(r)$ and its value is

$$H(r) = \frac{1}{2}(\tanh r + \coth r) > 1.$$

Since $M$ is compact, consider a sufficiently large number $r$ such that $M$ is included in the domain $B(r)$. Let us decrease $r$ until the first cylinder $C(r)$ that touches $M$. Because $H(r) > 1 \geq |H|$, the tangency principle implies $r = r_0$, $M \subset \overline{B(r_0)}$ and $M \cap \partial C(r_0) = C_1 \cup C_2$. In particular, $M$ does not intersect the exterior of $C_i$ in $P_i$. Now, we apply Theorem 4.1.

□

REMARK 3: The map $H \longmapsto d_H = \max d^H$ is a monotone increasing function on $H$. This allows us compare Theorem 3.1 with Theorem 4.1 in the case that the boundary of an embedded $H$-hypersurface is formed by two compact submanifolds $C_1, C_2$ in geodesic hyperplanes $P_1$ and $P_2$ respectively: if $0 < H < 1$, the distance between $P_1$ and $P_2$ is $d_H$ at most; moreover, if $d_0 < d(P_1, P_2) \leq d_H$ and $M$ does not intersect the exterior of the bounded geodesic domains determined by the boundary, then the hypersurface is included in the domain determined by $P_1$ and $P_2$.

When the boundary is included in horospheres, we have the following result for $|H| \geq 1$:

**Theorem 4.3** *Let $Q_1$ and $Q_2$ be two horospheres with the same asymptotic boundary. Let $C_1$ and $C_2$ be compact $(n-1)$-submanifolds included in $Q_1$ and $Q_2$ respectively and let $\Omega_1$ and $\Omega_2$ be the two domains that bound in $Q_1$ and $Q_2$. Then there is a positive number $d_0$ with the following property: if $d(C_1, C_2) \geq d_0$, any connected compact embedded hypersurface $M \subset \mathbf{H}^{n+1}$ of mean curvature function $H$, $|H| \geq 1$, bounded by $C_1 \cup C_2$ and $M \cap (Q_i \setminus \overline{\Omega_i}) = \emptyset$, $i = 1, 2$ is included in the domain $S$ defined by $Q_1$ and $Q_2$. In particular, if $C_1$ and $C_2$ are coaxial spheres and $H$ is constant, $M$ is a hypersurface of revolution.*

*Proof:* The reasoning is similar to Theorem 4.1 and we indicate it briefly. Choose an orientation on $M$ such the mean curvature is positive. Suppose, contrary to the assertion that $M$ contains points outside $S$. Construct a domain $W$ by attaching two 'caps' $S_1$ and $S_2$ in a similar way as the proof of Theorem 4.1. After an isometry of $\mathbf{H}^{n+1}$, we consider $Q_1 = \{x_{n+1} = a_1\}$ and $Q_2 = \{x_{n+1} = a_2\}$ with $0 < a_2 < a_1$. We have two possibilities: if $M$ contains points below $Q_2$, we come from $S^n(\infty)$ by geodesic hyperplanes until to touch the lowest point of $M$ (with respect to $Q_2$). The tangency principle assures that the unit normal vector field $N$ of $M$ points outside $W$. In the case that $M$ contains points above $Q_1$, we place a horosphere with asymptotic boundary $\infty \in S^n(\infty)$ in the highest point of $M$ with respect to $Q_1$. The tangency principle assures again that $N$ points outside $W$. Now a similar reasoning as Theorem 4.1 works in the same way. In the case that the boundary is spherical, we apply Proposition 2.2.

$\square$

As consequence of Corollary 4.2 and Theorem 4.3, we have the following result proved in [7]:

**Corollary 4.4** *Let $C_1$ and $C_2$ be codimension two spheres with the same radii. Then there exists $d_0 > 0$ such that if $d(C_1, C_2) \geq d_0$ any connected compact embedded $H$-hypersurface $M$ in $\mathbf{H}^{n+1}$ with $|H| = 1$ and bounded by $C_1 \cup C_2$ is a hypersurface of revolution.*

With appropriate modifications in the statements, Theorems 4.1 and 4.3 generalize to the case that one of the two boundary components is included in $S^n(\infty)$.

**Corollary 4.5** *Let $C_1$ and $C_2$ be codimension two compact submanifolds of $\mathbf{H}^{n+1}$, where $C_2 \subset S^n(\infty)$ and assume that $d(C_1, C_2) \geq d_0$ ($d_0$ is the number given in Theorem 4.1).*

1. *Suppose that $C_1$ and $C_2$ are included in disjoint geodesic hyperplanes $P_1$ and $P_2$ respectively, where $C_2 = \partial_\infty P_2$. Then any connected embedded hypersurface $M$ of mean curvature function $H \neq 0$ with $\partial_g M = C_1 \cup C_2$ and $M \cap (P_1 \setminus \overline{\Omega_1}) = \emptyset$ is included in the domain determined by $P_1$ and $P_2$.*

2. *Suppose that $C_1$ is included in a geodesic hyperplane $P_1$. Let $\Omega_2$ be one of the two domains in $S^n(\infty)$ determined by $\partial_\infty P_1$. Assume that $C_2$ is a subset of $\Omega_2$. Then any connected embedded hypersurface of mean curvature function $H \neq 0$ with generalized boundary $C_1 \cup C_2$ that does not intersects $P_1 \setminus \overline{\Omega_1}$ is included in the domain determined by $P_1$ and $\Omega_2$.*

3. *Suppose that* $C_1$ *is included in a horosphere* $Q_1$ *and* $\partial_\infty Q_1 \notin C_2$. *Then any connected embedded hypersurface of mean curvature function* $|H| \geq 1$ *with* $\partial_g M = C_1 \cup C_2$ *that does not intersect* $Q_1 \setminus \overline{\Omega_1}$ *is included in the domain of* $\mathbf{H}^{n+1} \setminus Q_1$ *that contains* $C_2$.

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