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Constant mean curvature hypersurfaces foliated by spheres*

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Abstract: We ask when a constant mean curvature n-submanifold foliated by spheres in one of the Euclidean, hyperbolic and Lorentz–Minkowski spaces ($\mathbf{E}^{n+1}$, $\mathbf{H}^{n+1}$ or $\mathbf{L}^{n+1}$), is a hypersurface of revolution. In $\mathbf{E}^{n+1}$ and $\mathbf{L}^{n+1}$ we will assume that the spheres lie in parallel hyperplanes and in the case of hyperbolic space $\mathbf{H}^{n+1}$, the spheres will be contained in parallel horospheres. Finally, Riemann examples in $\mathbf{L}^3$ are constructed, that is, non-rotational spacelike surfaces foliated by circles in parallel planes.

Keywords: Foliation, mean curvature, Aleksandrov reflection method.

MS classification: 53A10, 53C42.

1. Introduction and statements of results

In Euclidean 3-space $\mathbf{E}^3$, the only complete minimal surface of revolution is the catenoid. In particular, the catenoid is fibred by circles in parallel planes. There exist other minimal surfaces in $\mathbf{E}^3$ foliated by circles in parallel planes which were discovered by Riemann [14]. A Riemann surface is a simply periodic embedded minimal surface that is described in terms of elliptic functions on a twice punctured rectangular torus. Its two ends are flat. Enneper [3] proved that catenoids and Riemann examples are the only minimal surfaces foliated by circles.

Nitsche found all surfaces with non-zero constant mean curvature in $\mathbf{E}^3$ generated by a one-parameter family of circles. In [13], he proved that the surface must be a sphere or, in the non-spherical case, the circles must lie in parallel planes. In the latter case, the only possibilities are the surfaces of revolution determined in 1841 by Delaunay [2].

In the arbitrary dimension, Jagy studied minimal submanifolds in $\mathbf{E}^{n+1}$, $n \geqslant 3$, generated by a one-parameter family of hyperspheres. He showed that the hyperplanes containing the hyperspheres are parallel again, but, in contrast to what happens in $\mathbf{E}^3$, the hypersurface must be rotationally symmetric with respect to an axis. In this situation, the hypersurface obtained is the higher catenoid.

In this article, we deal with (connected) n-dimensional submanifolds in three different ambients: Euclidean, hyperbolic and Lorentz–Minkowski $(n+1)$-dimensional spaces. We shall

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consider that the submanifolds are foliated by $(n-1)$-hyperspheres in parallel hyperplanes (throughout this paper, hyperspheres will be called spheres for simplicity). More precisely:

1. Constant mean curvature n-submanifolds in Euclidean space $\mathbf{E}^{n+1}$ foliated by spheres in parallel hyperplanes.

2. Constant mean curvature n-submanifolds in hyperbolic space $\mathbf{H}^{n+1}$ foliated by spheres in parallel horospheres or parallel hyperplanes.

3. Constant mean curvature spacelike n-submanifolds in Lorentz–Minkowski space $\mathbf{L}^{n+1}$ foliated by spheres in parallel spacelike hyperplanes.

The methods that we apply in our proofs are based on the following fact. Consider the Euclidean case. A smooth hypersurface M^n in $\mathbf{E}^{n+1}$ can be written locally as the level set for a function. We orient M by the unit normal field $N=-\nabla f/|\nabla f|$. Then the mean curvature H of M is given by

$$nH=-\operatorname{div}\frac{\nabla f}{|\nabla f|},$$

where div denotes the divergence of the unit normal field N. An easy computation gives us

$$nH\,|\nabla f|^3=\Delta f\,|\nabla f|^2-\operatorname{Hess} f(\nabla f,\nabla f), \tag{1}$$

where ∇, Δ and Hess denote the gradient, laplacian and hessian operators respectively computed with the Euclidean metric. The idea is to express in terms of f the property that M is 'foliated by spheres in parallel hyperplanes.' The explicit computation of (1) and the fact that H is constant will impose restrictions on f that will conclude our results. In hyperbolic and Lorentz–Minkowski spaces, the reasoning is similar.

It should be pointed out that up until now, the only known example of a maximal surface ($H=0$) in the Lorentz–Minkowski 3-space $\mathbf{L}^3$ foliated by circles in parallel spacelike planes is the Lorentzian catenoid (see [7]). A major goal of this paper is the construction in $\mathbf{L}^3$ of a family of non-rotational maximal surfaces foliated by circles in parallel spacelike planes (Theorem 4.1). Together with the Lorentzian catenoid, these new surfaces comprise all the maximal surfaces in $\mathbf{L}^3$ foliated by circles in parallel spacelike planes. In this sense, these surfaces play the same role as Riemann minimal surfaces in Euclidean space $\mathbf{E}^3$. A different approach to the maximal spacelike surfaces in $\mathbf{L}^3$ *via* the Weierstrass–Enneper representation has been studied in [8]. Finally, the case of spacelike surface with nonzero constant mean curvature and foliated by circles has been fully studied by the author [10, 11].

We can summarize the results obtained in the following way:

Let X be one of the following $(n+1)$-dimensional spaces: Euclidean space $\mathbf{E}^{n+1}$, hyperbolic space $\mathbf{H}^{n+1}$ and Lorentz–Minkowski space $\mathbf{L}^{n+1}$. Let M^n be an n-dimensional submanifold in X of constant mean curvature H.

1. *If $X=\mathbf{E}^{n+1}$, $n\geqslant 3$, $H\neq 0$ and M is foliated by spheres in parallel hyperplanes, then M is a hypersurface of revolution* (*Theorem* 2.1).

2. *If $X=\mathbf{H}^{n+1}$ and M is foliated by spheres in parallel horospheres, then M is a hypersurface of revolution* (*Theorem* 3.3).

3. *If* $X = \mathbf{L}^{n+1}$ *and* M *is foliated by spheres in parallel spacelike hyperplanes then* (*Theorem* 4.1):

(a) M *is a hypersurface of revolution if* $H \neq 0$ *or* $H = 0$ *and* $n \geqslant 3$;

(b) *if* $H = 0$ *and* $n = 2$, M *is the Lorentzian catenoid or* M *belongs to a non-rotational one-parametric family of maximal surfaces.*

Added in proof. We have just known the existence of reference [6] where part of our results are also studied.

2. Hypersurfaces in Euclidean space

Nitsche proved that a surface M of constant mean curvature $H \neq 0$ in $\mathbf{E}^3$ and foliated by circles in parallel planes must be a Delaunay surface [13]. We obtain this result in $\mathbf{E}^{n+1}$.

Theorem 2.1. *Let* M^n *be an n-dimensional submanifold of* $\mathbf{E}^{n+1}$ *of constant mean curvature* $H \neq 0$ *and foliated by spheres in parallel hyperplanes. Then* M^n *is a hypersurface of revolution.*

Proof. Without loss of generality, assume that each hyperplane of the foliation is parallel to $x_{n+1} = 0$. Let $P_1 = \{x_{n+1} = t_1\}$ and $P_2 = \{x_{n+1} = t_2\}$ be two hyperplanes of the foliation with $t_1 < t_2$. Consider M^* as the piece of M between P_1 and P_2. We use the Aleksandrov reflection method in Euclidean space [1]. This method is based on the classical Hopf maximum principle [4], which stated that if two hypersurfaces with the same mean curvature are tangent at a common point p and one hypersurface (locally) lies by the side of the other one, then they agree in a neighbourhood of p.

The Aleksandrov method involves successive reflections across each family of parallel hyperplanes. A standard application of this technique with hyperplanes orthogonal to the foliation hyperplanes, shows that M^* inherits the symmetries of its boundary $\partial M^* = (M^* \cap P_1) \cup (M^* \cap P_2)$ (see [9] for the three-dimensional case and [15] for the minimal case). Therefore, for each $t_1 \leqslant t \leqslant t_2$, the centers of each level $M \cap \{x_{n+1} = t\}$ lie in the same 2-plane. After a translation, we can assume that this 2-plane is defined by $x_2 = \cdots = x_n = 0$. Let us parametrize the centers of the spheres by $t \longmapsto (c(t), 0, \ldots, 0, t)$, $t \in [t_1, t_2]$. Furthermore, M^* is the level set of a smooth function f given by

$$f = (x_1 - c(t))^2 + \sum_{i=2}^{n} x_i^2 - r(t)^2, \tag{2}$$

where $r(t) > 0$ denotes the sphere radius at the level $x_{n+1} = t$. We shall prove that the line of the centers is a straight-line orthogonal to the hyperplane $x_{n+1} = 0$, that is, c is a constant map. This should show that M^* is a hypersurface of revolution.

We assume, by contradiction, that there is a sub-interval of $[t_1, t_2]$ where c is not constant and so, $c' \neq 0$. Without loss of generality, we suppose this interval is $[t_1, t_2]$. Now, we use identity (1). Computations are the same as [5] and, for the sake of completeness, we repeat them:

$$\begin{aligned}
\nabla f &= 2\big(x_1 - c, x_2, \dots, x_n, -(x_1 - c)c' - rr'\big),\\
|\nabla f|^2 &= 4\big(r^2 + [(x_1 - c)c' + rr']^2\big),\\
\Delta f &= 2\big(n + c'^2 - r'^2 - rr'' - (x_1 - c)c''\big),\\
\operatorname{Hess} f &= 2\begin{pmatrix} 1 & 0 & \cdots & -c' \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -c' & 0 & \cdots & c'^2 - r'^2 - rr'' - (x_1 - c)c'' \end{pmatrix},\\
\operatorname{Hess} f(\nabla f, \nabla f) &= 8\big(r^2 + 2c'(x_1 - c)[(x_1 - c)c' + rr']\\
&\qquad + [(x_1 - c)c' + rr']^2[c'^2 - rr'' - r'^2 - (x_1 - c)c'']\big).
\end{aligned}$$

On the other hand, the left-hand side of (1) is

$$8nH\big(r^2 + [(x_1 - c)c' + rr']^2\big)^{3/2}.$$

Let us fix a section t. Since x_1 is varied, we introduce the variable λ by

$$\lambda = \frac{(x_1 - c)c' + rr'}{r}. \tag{3}$$

Since $c' \neq 0$ for each sphere of the foliation of M^*, λ takes values in an interval of the line $\mathbb{R}$. By using (3), we regard identity (1) as a polynomial on λ where the coefficients are functions of the independent variable t. The right-hand side of (1) is a 2-degree polynomial $a_0 + a_1\lambda + a_2\lambda^2$:

$$8nHr(1 + \lambda^2)^{3/2} = a_0 + a_1\lambda + a_2\lambda^2. \tag{4}$$

Squaring (4) and examining the leader coefficients, we have $n^2H^2r^2 = 0$, which is a contradiction because $H \neq 0$. Therefore $c'(t) = 0$. Since t is arbitrary, then c is constant and so, M^* is a hypersurface of revolution. Since M^* is an arbitrary piece of M, then M is a hypersurface of revolution. □

3. Hypersurfaces in hyperbolic space

Let us consider the upper halfspace model of hyperbolic space

$$\mathbf{H}^{n+1} =: \mathbb{R}^{n+1}_+ = \big\{x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\big\}$$

equipped with the metric

$$ds^2 = \frac{(dx_1)^2 + \cdots + (dx_{n+1})^2}{x_{n+1}^2}.$$

Hyperbolic space $\mathbf{H}^{n+1}$ has a natural compactification $\overline{\mathbf{H}^{n+1}} = \mathbf{H}^{n+1} \cup \partial_\infty \mathbf{H}^{n+1}$, where $\partial_\infty \mathbf{H}^{n+1}$ can be identified with asymptotic classes of geodesic rays in $\mathbf{H}^{n+1}$. In the upper halfspace model, $\partial_\infty \mathbf{H}^{n+1} = \{x_{n+1} = 0\} \cup \{\infty\}$ is the one-point compactification of the hyperplane $x_{n+1} = 0$.

We will deal with hypersurfaces foliated by spheres included either in horospheres or hyperplanes in two natural situations, which will be merely called *parallel horospheres* or *parallel hyperplanes*. For our own convenience, we give our definition.

Definition 3.1. A one-parameter family of horospheres or geodesic hyperplanes are called *parallel* if their asymptotic boundaries agree at exactly one point.

Since the asymptotic boundary of a horosphere is exactly one point, 'parallel horospheres' means that they have the same asymptotic boundary. By means of using an isometry of $\mathbf{H}^{n+1}$, one can describe a family of parallel horospheres as Euclidean hyperplanes in $\{x_{n+1} > 0\}$ parallel to the hyperplane $x_{n+1} = 0$ in the Euclidean sense. In the same way, a family of geodesic parallel hyperplanes can be viewed as Euclidean hyperplanes parallel to the hyperplane $x_n = 0$. Also, in our model for $\mathbf{H}^{n+1}$, $(n-1)$-spheres are simply Euclidean $(n-1)$-spheres included in $\mathbb{R}^{n+1}_+$

In the proofs, we will write a hypersurface M in hyperbolic space locally as the level set of a smooth function f. So, we need the analogous formula (1) to describe the mean curvature H of M in terms of f. In our model of $\mathbf{H}^{n+1}$, the hyperbolic metric is conformal with the Euclidean metric supported by $\mathbb{R}^{n+1}_+$. A straightforward computation gives us the relation between the mean curvatures of M with the two induced metrics.

Lemma 3.2. *Let M be an oriented hypersurface immersed in $\mathbb{R}^{n+1}_+$ and let ds_0^2 and ds^2 be respectively the Euclidean and hyperbolic metrics on $\mathbb{R}^{n+1}_+$. Let N be a Gauss map for the immersion $M \to (\mathbb{R}^{n+1}_+, ds_0^2)$ and consider the orientation on $M \to (\mathbb{R}^{n+1}_+, ds^2)$ given by $x_{n+1}N$. Denote by h and H the mean curvatures of M for the immersion of M in $(\mathbb{R}^{n+1}_+, ds_0^2)$ and $(\mathbb{R}^{n+1}_+, ds^2)$ respectively. Then, for each $p \in M$,*

$$H(p) = x_{n+1}(p)h(p) + N_{n+1}(p), \tag{5}$$

where $N_{n+1}(p)$ denotes the x_{n+1}-coordinate of $N(p)$

Equation (1) and relation (5) tells us that if M is a hypersurface in $\mathbf{H}^{n+1}$ of constant mean curvature H given by the level set of $f = 0$, then

$$nH|\nabla f|^3 = nN_{n+1}|\nabla f|^3 + x_{n+1}\big(\Delta f|\nabla f|^2 - \operatorname{Hess} f(\nabla f, \nabla f)\big), \tag{6}$$

where ∇, Δ and Hess denote as (1).

We are in a position to study constant mean curvature submanifolds in $\mathbf{H}^{n+1}$ foliated by spheres in parallel horospheres. In contrast with the Euclidean case (when $H = 0$), the only possibility will be that the hypersurface is a rotational hypersurface with a geodesic as the axis of revolution.

Theorem 3.3. *Let M^n be an n-dimensional submanifold in $\mathbf{H}^{n+1}$ of constant mean curvature and foliated by spheres in parallel horospheres. Then M is a hypersurface of revolution, that is, there exists a geodesic γ such that M is invariant by the group of isometries that leaves γ fixed pointwise.*

Proof. As we have pointed out, we can assume that the horospheres are Euclidean hyperplanes of $\mathbb{R}^{n+1}_+$ parallel to the hyperplane $x_{n+1} = 0$. Consider M^* as a piece of M between two levels

$P_1 = \{x_{n+1} = t_1\}$ and $P_2 = \{x_{n+1} = t_2\}$, $t_1 < t_2$. The Aleksandrov reflection can be applied in our case as in Theorem 2.1, where by reflections we mean *hyperbolic reflections* across totally geodesic hyperplanes. In our model for $\mathbf{H}^{n+1}$, these hyperbolic reflections are regarded as Euclidean reflections across vertical hyperplanes and Euclidean inversions with respect to spheres meeting orthogonally $\partial_\infty \mathbf{H}^{n+1}$. Therefore we can parametrize the Euclidean centers of spheres $M^* \cap \{x_{n+1} = t\}$ by $t \longmapsto (c(t), 0, \ldots, 0, t)$, where $r(t) > 0$ denotes the Euclidean radius for each t. Then the surface M^* is the level set for the same function f defined in (2).

We proceed by contradiction. So, we suppose that $c' \neq 0$ in the interval $[t_1, t_2]$. The x_{n+1}-coordinate of the Gauss map N of $M^* \subset \mathbf{E}^{n+1}$ is given by

$$N_{n+1} = -\frac{(x_1 - c)c' + rr'}{\sqrt{r^2 + [(x_1 - c)c' + rr']^2}}.$$

Let us fix the level $x_{n+1} = t$. By using (3), equation (6) can written in the following way:

$$nrH(1 + \lambda^2)^{3/2} = nr\lambda(1 + \lambda^2) + x_{n+1}(a_0 + a_1\lambda + a_2\lambda^2), \tag{7}$$

where a_i are coefficients that do not depend on λ. The right-hand side in (7) is a 3-degree polynomial: $b_0 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3$. Squaring both sides in (7), the identity of the leader coefficients gives

$$n^2r^2H^2 = n^2r^2.$$

Thus $H^2 = 1$. Since the square of the left-hand side in (7) is a polynomial with non odd terms in λ, the coefficients of λ^5 and λ^3 vanish on the right-hand side. The 5-degree coefficient yields $b_2b_3 = 0$. Since $b_3 = n^2r^2 \neq 0$, then $b_2 = 0$. Now, the λ^3-term gives $b_0b_3 = 0$ and then, $b_0 = 0$. However $H^2 = 1$ and the λ^0-term on the left-hand side of (7) is $n^2r^2H^2 = n^2r^2 \neq 0$. This contradiction leads to $c' = 0$ on $[t_1, t_2]$, that is, c is constant. Therefore M is a hypersurface of revolution with the geodesic $\gamma(t) = (c, 0, \ldots, 0, t)$ being the rotation axis. □

The second part of this section is concerned with submanifolds in $\mathbf{H}^{n+1}$ foliated by spheres in parallel geodesic hyperplanes. Let M be a n-submanifold of constant mean curvature in $\mathbf{H}^{n+1}$. By means of using an isometry of the ambient, we suppose that the foliation of M is given by hyperplanes parallel to $x_n = 0$. As in Theorem 3.3, we pick a piece of M denoted as M^* between two hyperplanes P_1, P_2 of the foliation. In this situation, it is not possible to use Aleksandrov technique to show that the centers of the spheres of the foliation lie in a 2-plane: there does not exist a family of parallel geodesic hyperplanes orthogonal to *both* hyperplanes P_1 and P_2. One case where the Aleksandrov technique works is when for each $2 \leqslant i \leqslant n$, $(M^* \cap P_1) \cup (M^* \cap P_2)$ is invariant under some hyperbolic reflection across a geodesic hyperplane parallel to $x_i = 0$. In this case, Aleksandrov method proves that the line of the sphere centers of the foliation lies in a 2-plane of $\mathbf{H}^{n+1}$.

Theorem 3.4. *Let M^n be an n-dimensional submanifold in $\mathbf{H}^{n+1}$ of constant mean curvature and foliated by spheres in parallel geodesic hyperplanes. Assume there exist two geodesic hyperplanes P_1 and P_2 of the foliation such that $(M \cap P_1) \cup (M \cap P_2)$ is invariant under hyperbolic reflections across $n - 1$ orthogonal geodesic hyperplanes and all them orthogonal to $P_1 \cup P_2$ as well. Then M is a totally umbilical hypersurface.*

Proof. We take M^* the piece of M between the two geodesic hyperplanes containing the two spheres of the hypothesis. By means of an isometry, we assume the centers of the spheres that foliate M^* can be parametrized by

$$\gamma(t) = (0, \ldots, 0, t, c(t)).$$

Then M^* is the level set of

$$f = \sum_{i=1}^{n-1} x_i^2 + [x_{n+1} - c(t)]^2 - r(t)^2 = 0,$$

where $c(t)$ and $r(t)$ denote as in the proof of Theorem 2.1. By contradiction, let us assume that $c' \neq 0$ on $[t_1, t_2]$. Again let us fix a level t of the foliation and let

$$\lambda = \frac{(x_{n+1} - c)c' + rr'}{r}. \tag{8}$$

A computation of identity (6) becomes the polynomial equation on λ:

$$\begin{aligned} nrH(1+\lambda^2)^{3/2} &= -\frac{nr(\lambda - r')}{c'}(1+\lambda^2) + x_{n+1}(d_0 + d_1\lambda + d_2\lambda^2) \\ &= e_0 + e_1\lambda + e_2\lambda^2 + e_3\lambda^3. \end{aligned} \tag{9}$$

It is easy to check that

$$d_0 = n - 1 + c'^2 - r'^2 - rr'' - rc'', \qquad d_1 = 2r' - \frac{rc''}{c'}, \qquad d_2 = n - 2.$$

Notice that from (8)

$$x_{n+1} = \frac{r}{c'}(\lambda - r') + c.$$

Squaring (9), the equality of the λ^6-terms gives

$$n^2H^2 = \frac{4}{c'^2}. \tag{10}$$

In particular, $e_3 \neq 0$. As in Theorem 2.1, all odd terms of the polynomial $(e_0+e_1\lambda+e_2\lambda^2+e_3\lambda^3)^2$ are zero. Thus $e_0 = e_2 = 0$. If we regard the square of the left-hand side in (9) and by considering the independent term, we obtain $n^2r^2H^2 = 0$, in contradiction with (10).

As a conclusion, c is constant, that is, γ is a horizontal Euclidean straight-line. Returning to (6) and putting $c' = 0$, we have

$$nHr(1 + r'^2)^{3/2} = (-rr'' - 1 - r'^2)x_{n+1} + nc(1 + r'^2).$$

Consider that x_{n+1} varies in this identity. Then the radius $r = r(t)$ satisfies the next two differential equations:

$$rr'' + 1 + r'^2 = 0, \tag{11}$$

$$Hr\sqrt{1 + r'^2} = c. \tag{12}$$

Each solution of (12) verifies (11) and the solutions of (11) are circles. Therefore, if we look at M^* as a subset of $\mathbb{R}^{n+1}_+$, then M^* is an open set of an n-dimensional Euclidean sphere. From the hyperbolic viewpoint, this n-sphere is an umbilical hypersurface of $\mathbf{H}^{n+1}$ and hence, M is a totally umbilical hypersurface. $\square$

4. Hypersurfaces in Lorentz–Minkowski space

Let $\mathbf{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle\cdot,\cdot\rangle = (dx_1)^2 + \cdots + (dx_n)^2 - (dx_{n+1})^2,$$

where $(x_1 \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. An M hypersurface immersed in $\mathbf{L}^{n+1}$ is *spacelike* if the induced metric is a Riemannian metric on M. When the hypersurface is (locally) the level set of $f = 0$, and the fact M is spacelike means that ∇f is a vector orthogonal to M of timelike character:

$$\langle\nabla f, \nabla f\rangle < 0. \tag{13}$$

Let us orient M by the unit normal field $N = -\nabla f/|\nabla f|$, where

$$|\nabla f| = \sqrt{-\langle\nabla f, \nabla f\rangle} = \sqrt{-\sum_{i=1}^{n} f_i^2 + f_{n+1}^2}.$$

Here f_j denotes the partial derivative of the function $f = f(x_1, \ldots, x_{n+1})$ with respect to the x_j-coordinate. Now, if H is the mean curvature calculated with this orientation, then

$$nH = -\operatorname{Div}\frac{\nabla f}{|\nabla f|},$$

where Div denotes the divergence with the Lorentzian metric. A straightforward computation gives

$$nH|\nabla f|^3 = \langle\nabla f, \nabla f\rangle\Delta f - \operatorname{Hess} f(\nabla f, \nabla f), \tag{14}$$

with

$$\nabla f = (f_1, \ldots, f_n, -f_{n+1}),$$

$$\Delta f = \sum_{i=1}^{n} f_{i,i} - f_{n+1,n+1},$$

$$\operatorname{Hess} f(\nabla f, \nabla f) = \sum_{i,j=1}^{n} f_i f_j f_{i,j}.$$

In the present section we study constant mean curvature spacelike hypersurfaces in Lorentz–Minkowski space $\mathbf{L}^{n+1}$ foliated by spheres in parallel spacelike hyperplanes. After using a Lorentz transformation, we can assume that these hyperplanes are parallel to the hyperplane $x_{n+1} = 0$. In this case, these spheres can be viewed as Euclidean spheres in horizontal hyperplanes.

Theorem 4.1. *Let M^n be a spacelike n-dimensional submanifold in $\mathbf{L}^{n+1}$ of constant mean curvature H and foliated by spheres in parallel spacelike hyperplanes.*

1. *If $H \neq 0$, then M is a hypersurface of revolution.*
2. *If $H = 0$ and*

(a) $n \geqslant 3$, *then M is a hypersurface of revolution.*

(b) $n = 2$, *then M is a surface of revolution or M belongs to a one-parameter family of non-rotational maximal surfaces.*

Proof. A spacelike hypersurface of constant mean curvature in $\mathbf{L}^{n+1}$ satisfies (locally) an elliptic equation to which we can apply the classical maximum principle (see, for instance, [16]). So, the Aleksandrov technique works in our situation. Let M^* be an arbitrary piece of M between two hyperplanes of the foliation: $x_{n+1} = t_1$, $x_{n+1} = t_2$, $t_1 < t_2$. Again, we apply the Aleksandrov reflection method by hyperplanes orthogonal to $x_{n+1} = 0$ as in Theorem 2.1. Reflection across hyperplanes of this kind are Euclidean reflections. Then the centers of spheres can be parametrized by $(c(t), 0, \ldots, t)$, $t \in [t_1, t_2]$. To show that M is a hypersurface of revolution, it suffices to prove that c is a constant function.

By contradiction, assume that $c' \neq 0$ on $[t_1, t_2]$. Since the spheres of the foliation are Euclidean spheres, we take the same function f as in (2). Let us fix t with the variable λ as (3). Identity (14) gives

$$nrH(-1+\lambda^2)^{3/2} = g_0 + g_1\lambda + g_2\lambda^2,$$

where the coefficients g_i are functions on t. Squaring this identity, the leader coefficients give $H = 0$. Therefore, if $H \neq 0$, we obtain a contradiction unless $c' = 0$. In this case, M is a hypersurface of revolution and we have proved 1.

Now let us study the maximal case $H = 0$. Computing the right-hand side of (14), we have

$$\left(n - 1 + r'^2 - c'^2 + rr'' - \frac{rr'c''}{c'}\right) + \left(\frac{rc''}{c'} - 2r'\right)\lambda + (2-n)\lambda^2 = 0. \tag{15}$$

The 2-degree term in (15) gives $g_2 = n - 2 = 0$. Therefore, if $n \geqslant 3$, c' must vanish and M is a hypersurface of revolution again. This proves 2(a). Let us consider the case $n = 2$. The λ^1 and λ^0-terms in (15) give

$$g_1 = \frac{rc''}{c'} - 2r' = 0,$$

$$g_0 = 1 + r'^2 - c'^2 + rr'' - \frac{rr'c''}{c'} = 0.$$

We simplify the above equations by

$$rc'' - 2r'c' = 0, \tag{16}$$

$$1 - r'^2 + rr'' - c'^2 = 0. \tag{17}$$

A first integral of (16) is given by

$$c' = ar^2 \tag{18}$$

for a positive constant $a \in \mathbb{R}$. Substituting in (17), it follows that

$$1 - r'^2 + rr'' - a^2 r^4 = 0. \tag{19}$$

Let us integrate equation (19) by a similar approach to that given in [12, p. 87]. Consider $x = r^2$ and $y = (r^2)'$ as the new dependent and independent variables. A straightforward computation gives

$$(r^2)' = 2r\sqrt{a^2 r^4 + 2br^2 + 1},$$

for a constant $b \in \mathbb{R}$. Thus

$$\frac{dt}{dr} = \frac{1}{\sqrt{a^2 r^4 + 2br^2 + 1}}.$$

and (18) becomes

$$c(u) = a \int^u \frac{u^2}{\sqrt{a^2 u^4 + 2bu^2 + 1}}\, du. \tag{20}$$

In this way, the parametrization obtained is:

$$x(u,\theta) = a \int^u \frac{u^2}{\sqrt{a^2 u^4 + 2bu^2 + 1}}\, du + u\cos\theta,$$
$$y(u,\theta) = u\sin\theta,$$
$$z(u,\theta) = \int^u \frac{du}{\sqrt{a^2 u^4 + 2bu^2 + 1}}.$$

The integrals that appear in this parametrization come determined in terms of elliptic integrals.

We end Theorem 4.1 by presenting two examples.

Example 1. Firstly, we give the degenerate case by setting $a = 0$. The surface obtained is the *Lorentzian catenoid* and its parametrization is

$$x(u,\theta) = u\cos\theta,$$
$$y(u,\theta) = u\sin\theta,$$
$$z(u,\theta) = \frac{1}{\sqrt{2b}} \operatorname{arcsinh}(\sqrt{2b}u).$$

This surface is rotational and it is generated by the rotation of the curve $(2b^{-1/2}\sinh(\sqrt{2b}u), 0, u)$ with respect to the x_3-axis (see Figure 1). The Lorentzian catenoid is the only maximal spacelike surface of revolution in $\mathbf{L}^3$ with respect to a timelike rotation axis ([7]).

Example 2. To give another example, we put $a = c = 1$. The parametrization of the corresponding surface M is

$$x(u,\theta) = u - \arctan u + u\cos\theta,$$
$$y(u,\theta) = u\sin\theta,$$
$$z(u,\theta) = \arctan u, \quad u \in (0,\infty),\ \theta \in \mathbb{R},$$

and its picture appears in Figure 2. This surface M is asymptotic to the plane $z = \pi/2$, that is, at this height, M has a flat end. Moreover the circles of the foliation converge to the straight-line $L_1 = \{x = -\pi/2, z = \pi/2\}$ as $u \to \infty$: for each point $(-\pi/2, y, \pi/2) \in L_1$, it suffices to take the sequence $\{u, \pi - y/u)\}$ to prove that

$$\lim_{u\to\infty} \left(x\left(u, \pi - \frac{y}{u}\right), y\left(u, \pi - \frac{y}{u}\right), z\left(u, \pi - \frac{y}{u}\right)\right) = \left(-\frac{\pi}{2}, y, \frac{\pi}{2}\right).$$

Thus, the reflection principle yields a new maximal surface by reflecting M across L_1.

If we consider the minus sign in (20), we obtain a surface M' that is congruent to M. More precisely, M' is the reflection of M across the origin. Denote $M^* = M \cup M'$ (see Figure 3). This surface lies in the slab $|z| < \pi/2$, with two flat ends at $\{z = \pm\pi/2\}$ and one singularity at the origin. In fact, the surface M is a fundamental domain of a simply periodic embedded maximal surface $\tilde{M}$ in $\mathbf{L}^3$ obtained by successive 180°-rotations across the straight-lines

$$L_n = \left\{x = \left(n - \tfrac{1}{2}\right)\pi, z = \left(n - \tfrac{1}{2}\right)\pi\right\}, \quad n \in \mathbb{Z}.$$

The properties of $\tilde{M}$ are summarized as follows:

- $\tilde{M}$ intersects horizontal planes in lines at integer heights:

$$\tilde{M} \cap \left\{z = \left(n - \tfrac{1}{2}\right)\pi\right\} = L_n, \quad n \in \mathbb{Z}.$$

- $\tilde{M}$ has flat ends at $z = (n - \frac{1}{2})\pi$, $n \in \mathbb{Z}$.
- $\tilde{M}$ is invariant under translations of the vector $(-\pi, 0, \pi)$ and under reflections across the lines L_n.
- $\tilde{M}$ presents singularities at the points $\{(-n\pi, 0, n\pi); n \in \mathbb{Z}\}$. □

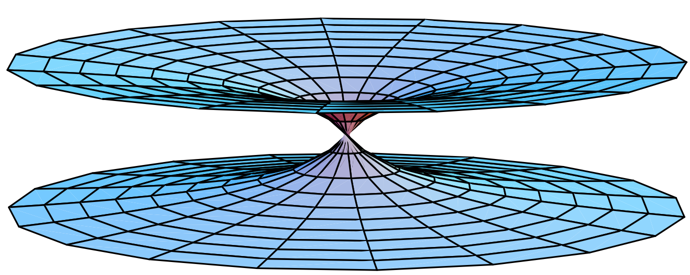

Figure 1. The Lorentzian catenoid

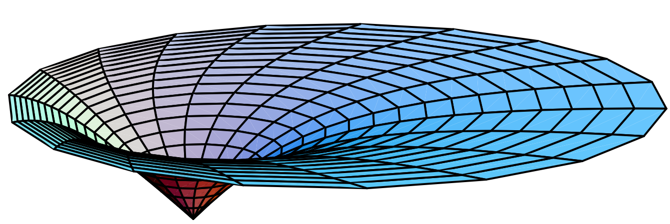

Figure 2. A piece M of a Riemann Lorentzian example

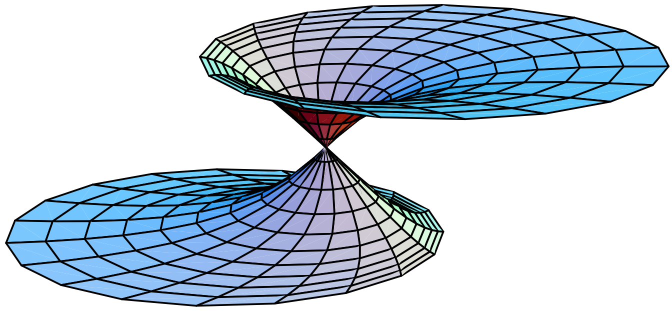

Figure 3. The Riemann Lorentzian example $M^* = M \cup M'$

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