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Constant Mean Curvature Surfaces Foliated by Circles in Lorentz–Minkowski Space

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Abstract. We prove that a spacelike surface in $\mathbf{L}^3$ with nonzero constant mean curvature and foliated by pieces of circles in spacelike planes is a surface of revolution. When the planes containing the circles are timelike or null, examples of nonrotational constant mean curvature surfaces constructed by circles are presented. Finally, we prove that a nonzero constant mean curvature spacelike surface foliated by pieces of circles in parallel planes is a surface of revolution.

Mathematics Subject Classifications (1991): 53A10, 53C42.

Key words: surface of revolution, spacelike surface, mean curvature, foliation.

1. Introduction and Statements of Results

In this paper we investigate constant mean curvature spacelike surfaces in Lorentz–Minkowski three-dimensional space $\mathbf{L}^3$ that are foliated by pieces of circles. We say that a surface M is foliated by circles (resp. pieces of circles) if M is constructed by a smooth uniparametric family of circles (resp. pieces of circles). The planes containing the circles are called planes of the foliation. In Euclidean 3-space a classical result of Meusnier [12] states that the only minimal surface of revolution is the catenoid. In 1867, Riemann [16] found all complete minimal surfaces foliated by circles in parallel planes, which can be parametrized by one real parameter. Each one of these surfaces is a simply periodic embedded minimal surfaces defined by elliptic functions on a twice punctured rectangular torus. In the same time, Enneper [4, 5] proved that if a minimal surface is foliated by pieces of circles, then the foliation planes are actually parallel, and so the surface is one of Riemann type. A historical discussion about these investigations can be found in [13].

In Euclidean three-space, Nitsche proved that if a surface with nonzero constant mean curvature H is foliated by pieces of circles, then the planes containing the circles must be parallel [14]. Furthermore, he showed that the surface is rotational, that is, it is one of the surfaces discovered by Delaunay [3]. In arbitrary dimension, the study of minimal hypersurfaces foliated by spheres has been done in [7]. He

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proved that, as in the three-dimensional case, the hypersurface is rotational, that is, is the generalized catenoid.

We extend this kind of results to the Lorentzian setting. A surface in $\mathbf{L}^3$ is called *spacelike* if the induced metric is positive definite. From the physical viewpoint, constant mean curvature spacelike surfaces in Lorentzian spaces appear related with different problems in general relativity and have been the focus for a number of authors (a lengthy discussion can seen in [2, 11]). Maximal surfaces ($H = 0$) foliated by pieces of circles have been classified in [9]. In this case, it is proved that the foliation planes are parallel. Furthermore, besides rotational examples (that were studied in [8]), new examples of maximal surfaces are constructed. In this situation, there exists a family of singly periodic maximal surfaces foliated by pieces of circles in parallel planes that play the same role as Riemann surfaces in Euclidean space. A different approach to these periodic maximal surfaces can found in [10].

In this paper, we continue studying nonvanishing constant mean curvature spacelike surfaces in $\mathbf{L}^3$ foliated by pieces of circles. The paper is organized as follows. In Section 2, we give a quick review of some classical formulas of the local surface theory in $\mathbf{L}^3$ and a discussion about what it is the concept of circle that we will use. In Section 3, the following results are proved

THEOREM 1. *Let M be a spacelike surface in $\mathbf{L}^3$ with nonzero constant mean curvature foliated by pieces of circles. If the planes of the foliation are spacelike, then they are parallel planes.*

THEOREM 2. *Let M be a spacelike surface in $\mathbf{L}^3$ with nonzero constant mean curvature and foliated by pieces of circles in parallel planes. Then M is a surface of revolution.*

As a consequence of Theorems 1 and 2, we obtain

COROLLARY 1. *Let M be a spacelike surface in $\mathbf{L}^3$ with nonzero constant mean curvature and foliated by pieces of circles in spacelike planes. Then M is a surface of revolution.*

Section 3 is divided in three subsections according with the causal character of the foliation planes. Theorem 1 and Corollary 1 are the analogous ones with Euclidean case (cf. [14]). When the surface is foliated by pieces of circles in timelike or null planes, Theorem 1 is not true. In this paper, we exhibit examples of spacelike surfaces in $\mathbf{L}^3$ of nonzero constant mean curvature foliated by pieces of circles in nonparallel timelike or null planes. In particular, these examples are nonrotational. Theorem 2 is the extension to the Lorentz–Minkowski three-space of Lemma 2 in [14]. In this sense, the surfaces of revolution of constant mean curvature have been classified in [6]. In the maximal case, i.e. when the mean curvature vanishes on the surface M, it has been proved in [9] that if M is foliated by pieces of circles, the planes containing the circles must be parallel. In [10], Theorem 2 has been generalized to higher dimensions when the foliation is made by hyperspheres in spacelike hyperplanes.

Finally, with the help of the Mathematica software, we present different drawings of constant mean curvature surfaces foliated by circles in nonparallel (timelike or null) planes. Moreover, in each one of the three cases, we show drawings of rotational constant mean curvature spacelike surfaces.

2. Preliminaries

Let $\mathbf{L}^3$ be the three-dimensional Lorentz–Minkowski space, that is, the space $\mathbf{R}^3$ endowed with the metric

$$\langle , \rangle = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2,$$

where (x_1, x_2, x_3) denote the coordinates in $\mathbf{R}^3$. A vector $v \neq 0$ is called spacelike (resp. timelike or null) if $\langle v, v\rangle > 0$ (resp. $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$). A plane in $\mathbf{L}^3$ is called spacelike (resp. timelike or null) if its Euclidean unit normals are timelike (resp. spacelike or null). Let M be a smooth surface immersed in $\mathbf{L}^3$. We say that M is spacelike if the induced metric on M is a Riemannian metric in each tangent plane. This is equivalent to each tangent plane is spacelike or that, locally, the unit normal vector is timelike in each point of M.

Let M be a connected immersed spacelike surface in $\mathbf{L}^3$. Since $\mathbf{L}^3$ has defined a globally unit timelike vector field, M is oriented. Take the unit normal vector field of M, i.e. a vector field ν along M which satisfies $\langle \nu, \nu\rangle = -1$. The mean curvature function of M is defined by

$$H = -\tfrac{1}{2}\mathrm{trace}\, \mathrm{d}\nu.$$

Let $\mathbf{X} = \mathbf{X}(u, v)$ be a local parametrization of M. We consider the first fundamental form

$$\mathrm{I} = \langle \mathrm{d}\mathbf{X},\ \mathrm{d}\mathbf{X}\rangle = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$$

and the second fundamental form

$$\mathrm{II} = -\langle \mathrm{d}\nu,\ \mathrm{d}\mathbf{X}\rangle = e\,\mathrm{d}u^2 + 2f\,\mathrm{d}u\,\mathrm{d}v + g\,\mathrm{d}v^2.$$

From the classical local theory (see, for example, [15]), the mean curvature is given by

$$H = \tfrac{1}{2}\mathrm{trace}_{\mathrm{I}}\,\mathrm{II} = \frac{eG - 2fF + gE}{2W^2}, \tag{1}$$

where

$$W = \sqrt{EG - F^2} > 0.$$

By using the Minkowski vector product $\wedge$, the formula (1) for the mean curvature H changes as follows

$$2HW^3 = E[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - 2F[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}] + G[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}] \overset{\text{def}}{=} P, \quad (2)$$

where [, ,] denotes the determinant of $\mathbf{L}^3$

$$[v_1, v_2, w] = \langle v_1 \wedge v_2, w\rangle \quad \forall w \in \mathbf{L}^3.$$

In this paper we will study spacelike surfaces of constant mean curvature, that is, H is a constant function on M. These surfaces are critical points of the area with respect to volume preserving spacelike variations. The surfaces with $H = 0$ on M are called also as maximal surfaces because they are maximal for the area integral [1].

To end this section, we deal with the concept of circle in $\mathbf{L}^3$. Our own definition of circle comes motivated from the Euclidean setting.

DEFINITION 1. A circle in $\mathbf{L}^3$ is the orbit of a point p out of a straight line l under the action of the group of rotations in $\mathbf{L}^3$ that leave l pointwise fixed.

Thus, a circle is included in an orthogonal plane to l. Depending on the causal character of l, we describe the circles of $\mathbf{L}^3$

(1) l is a timelike line. The group of the Lorentz motions $\{R_\theta; \theta \in \mathbf{R}\}$ that fix l pointwise is given by the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to an orthonormal basis (e_1, e_2, e_3), where e_3 spans the line l. The circles of $\mathbf{L}^3$ corresponding with this case are written as

$$\alpha(s) = c + r(\cos s\ e_1 + \sin s\ e_2), \quad (3)$$

with $r \neq 0$, $c \in l$.

(2) l is a spacelike line. In this case, the group of rotations determined by l with respect to an orthonormal basis (e_1, e_2, e_3) of $\mathbf{L}^3$, where l is generated by e_1, is

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}.$$

The circles obtained are given by

$$\alpha(s) = c + r(\sinh s\ e_2 + \cosh s\ e_3), \quad (4)$$

where c and r are as above and

$$\langle e_2, e_2\rangle = 1 \quad \langle e_3, e_3\rangle = -1.$$

(3) l is a null line. Consider that l is spanned by the vector e_2+e_3, where (e_1, e_2, e_3) is an orthonormal basis of $\mathbf{L}^3$ and e_3 is a timelike vector. The group of rotations associated to l is

$$R_\theta = \begin{pmatrix} 1 & \theta & -\theta \\ -\theta & 1-\frac{1}{2}\theta^2 & \frac{1}{2}\theta^2 \\ -\theta & -\frac{1}{2}\theta^2 & 1+\frac{1}{2}\theta^2 \end{pmatrix}$$

and the circles are described as

$$\alpha(s) = c + se_1 + \tfrac{1}{2}rs^2(e_2+e_3), \tag{5}$$

with $r \neq 0$.

According to our definition, it is not hard to express each one of the above cases from Euclidean viewpoint. After an isometry of the ambient, we can assume that

(1) l is the x_3-axis. The circles obtained in (3) are actually Euclidean circles in parallel planes to the x_1x_2-plane.
(2) l is the x_1-axis. The circles (4) are Euclidean hyperbolas in parallel planes to the x_2x_3-plane.
(3) l is the line $\{x_2 = x_3, x_1 = 0\}$. In this case, the circles (5) are Euclidean parabolas in parallel planes to $x_2 - x_3 = 0$.

Finally, we have the following

DEFINITION 2. A surface M in $\mathbf{L}^3$ is a surface of revolution (or rotational surface) if there exists a straight line l such that M is invariant by the rotations that leave l pointwise fixed.

Therefore, a surface of revolution is formed by a uniparametric family of circles of $\mathbf{L}^3$ in parallel planes. Moreover, these planes are orthogonal to the axis of revolution l. Notice also that when the planes are null, they contain the axis l.

3. Proofs of Results

Without loss of generality and after a homothety of $\mathbf{L}^3$, we assume that $H = 1/2$ on M. Since M is foliated by circles, it is generated by a one-parameter family of pieces of circles. Denote by u the parameter of the foliation. Since Theorems 1 and 2 are local, we divide theirs proofs in three parts, depending on the causal character of the foliation planes. If these planes are spacelike, we will prove both theorems. In each one of the two remaining cases, we show examples of surfaces foliated by circles in nonparallel planes. Finally, and each one of three cases, we shall prove

Theorem 2. For this, and after a Lorentz motion, we can assume that the planes of the foliation are parallel to the x_1x_2-plane, x_2x_3-plane or $x_2 - x_3 = 0$-plane according if they are spacelike, timelike or null respectively. Then we shall prove that M is invariant by one of the rotation groups described in Section 2.

3.1. THE PLANES OF THE FOLIATION ARE SPACELIKE

Consider $\Gamma = \Gamma(u)$ an orthogonal smooth curve to each u-plane of the foliation and denote by u an arc-length parameter. Firstly, we will prove that the foliation is given by parallel planes. For this, it suffices to prove that Γ is a straight line. The proof is by contradiction. Let $\mathbf{t}$ be the unit tangent to Γ. Then $\mathbf{t}$ is a timelike vector and Γ is a timelike curve. Consider the Frenet frame for Γ: $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Here $\mathbf{n}$ and $\mathbf{b}$ are spacelike unit vectors. Each circle of the foliation writes as (3). Up to a change of coordinates in (3) and a change of the basis (e_1, e_2) by $(\mathbf{n}, \mathbf{b})$, a (local) parametrization of M is defined by

$$\mathbf{X}(u, v) = \mathbf{c} + r(\cos v\ \mathbf{n} + r \sin v\ \mathbf{b}),$$

where $r > 0$ and $\mathbf{c}$ are differentiable functions of u. Put

$$\mathbf{c}' = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}, \tag{6}$$

with α, β, γ smooth functions on u. Set the Frenet equations of Γ, that is, the expressions of the derivatives of $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ in terms of this basis

$$\begin{aligned}\mathbf{t}' &= \kappa\mathbf{n}\\ \mathbf{n}' &= \kappa\mathbf{t} + \sigma\mathbf{b}\\ \mathbf{b}' &= -\sigma\mathbf{n},\end{aligned}$$

where κ and σ are smooth on u. Notice that $\kappa \neq 0$ in some interval because Γ is not a straight line.

Taking account of (6), a computation of W^2 and P yields that both numbers are trigonometric polynomial on $\cos nv, \sin nv$. So there exist complex smooth functions on u, A_n and B_n, such that

$$W^2 = \sum_{n=-2}^{2} A_n(\cos nv\ + i \sin nv),$$

$$P = \sum_{n=-3}^{3} B_n(\cos nv\ + i \sin nv),$$

where $i = \sqrt{-1}$, $A_{-n} = \overline{A}_n$, $B_{-n} = \overline{B}_n$ ($\overline{A}_n$ and $\overline{B}_n$ denote the conjugate of A_n and B_n) and A_0, B_0 are real numbers.

A straightforward computation of W^2 and P leads to the values of A_n, B_n:

$$A_0 = r^2(-\alpha^2 + r'^2 + \tfrac{1}{2}(\beta^2 + \gamma^2 - \kappa^2 r^2)) \tag{7}$$

$$A_1 = r^2(-\alpha\kappa r + \beta r') + i(\gamma r^2 r') \tag{8}$$

$$A_2 = \tfrac{1}{4}r^2(\beta^2 - \gamma^2 - \kappa^2 r^2) + i(\tfrac{1}{2}\beta\gamma r^2) \tag{9}$$

$$\begin{aligned} B_0 \ =\ & r^2(-\alpha^3 + \alpha\beta^2 + \alpha\gamma^2 + \alpha' r r' + 2\beta\kappa r r' + \alpha r'^2 - \alpha r r'' + \\ & +\gamma\kappa r^2\sigma - \tfrac{1}{2}\beta'\kappa r^2 - \tfrac{1}{2}5\alpha\kappa^2 r^2 + \tfrac{1}{2}\beta\kappa' r^2) \end{aligned} \tag{10}$$

$$\begin{aligned} B_1 \ =\ & \tfrac{1}{4}r^2(2\alpha'\beta r - 2\alpha\beta' r - 8\alpha^2\kappa r + 3\beta^2\kappa r + 3\gamma^2\kappa r - 3\kappa^3 r^3 + \\ & +4\alpha\beta r' + 2\kappa' r^2 r' + 6\kappa r r'^2 - 2\kappa r^2 r'' + 2\alpha\gamma r\sigma) + \\ & +\tfrac{1}{2} i r^2(\alpha'\gamma r - \alpha\gamma' r + 2\alpha\gamma r' - \alpha\beta r\sigma + \kappa r^2 r'\sigma) \end{aligned} \tag{11}$$

$$\begin{aligned} B_2 \ =\ & \tfrac{1}{4}r^3(-\beta'\kappa r - 5\alpha\kappa^2 r + \beta\kappa' r + 6\beta\kappa r') + \\ & +\tfrac{1}{2}r^3 i(-\gamma'\kappa r + \gamma\kappa' r + 6\kappa\gamma r') \end{aligned} \tag{12}$$

$$B_3 = \tfrac{1}{4}r^3\kappa(\beta^2 - \gamma^2 - \kappa^2 r^2) + i(\tfrac{1}{2}\beta\gamma\kappa r^3). \tag{13}$$

On the other hand, squaring (2) and since $H = 1/2$, we have the identity

$$(W^2)^3 = P^2. \tag{14}$$

This equation imposes a series of relations between A_n and B_n and that can be easily checked directly

$$A_2^3 = B_3^2, \tag{15}$$

$$3A_1 A_2^2 = 2B_2 B_3, \tag{16}$$

$$3A_0 A_2^2 + 3A_1^2 A_2 = B_2^2 + 2B_1 B_3, \tag{17}$$

$$A_0^2 A_2 + 3A_0 A_1^2 + 6|A_1|^2 A_2 + 3A_2|A_2|^2 = B_1^2 + 2B_0 B_2 + 2B_3\overline{B}_1. \tag{18}$$

It follows from (13) that $B_3 = r\kappa A_2$. Using (15), we have that $A_2^2(A_2 - r^2\kappa^2) = 0$. There are two possibilities

- *First case.* $A_2 = 0$. By (9), we obtain $\gamma = 0$ and $\beta = \pm r\kappa$. Now (17) and (12) lead $B_2 = 0$ and $\alpha = \pm r'$. Substituting this into (7) and (8), we get $A_0 = A_1 = 0$. Thus $W = 0$, which is a contradiction.

- *Second case.* $A_2 = \kappa^2 r^2$. Then $B_3 = \kappa^3 r^3$. Now (9) gives $\gamma = 0$ and $\beta^2 = 4\kappa^2 + \kappa^2 r^2$. Equation (16) shows that $B_2 = 3r\kappa A_1/2$. Identities (8), (12) give

$$\alpha\kappa^2 + \beta\kappa' - \beta'\kappa = 0 \tag{19}$$

and $A_1 = r^2(-\alpha\kappa r + \beta r')$. Define a function y by setting

$$y = \frac{\beta}{\kappa}.$$

Therefore, $y^2 = 4 + r^2$. By (11) and the equation (18) we have

$$4\kappa^2 + \kappa^2 r^2 + 2r'^2 = 0.$$

In particular, this implies that $\kappa = 0$, getting a contradiction.

In conclusion, we achieve that the orthogonal curve Γ to the foliation planes is a straight line. It follows that the planes of the foliation are parallel. This concludes Theorem 1.

Now, we shall prove that M is surface of revolution. After a motion of $\mathbf{L}^3$, we can assume that the foliation planes are parallel to the x_1x_2-plane. Then M can be parametrized locally as

$$\mathbf{X}(u, v) = (a + r\cos v, b + r\sin v, u), \tag{20}$$

where $a = a(u), b = b(u)$ are differentiable functions. In order to show that M is rotational, we prove that a and b are constant functions. Again, let us compute W^2 and P:

$$\begin{aligned} W^2 &= \frac{r^2}{2}(a'^2 - b'^2)\cos 2v + (r^2 a'b')\sin 2v + \\ &\quad + (2r^2 r'a')\cos v + (2r^2 r'b')\sin v + r^2\left(-1 + r'^2 + \frac{a'^2 + b'^2}{2}\right), \\ P &= r^2(-ra'' + 2a'r')\cos v + r^2(2r'b' - rb'')\sin v + \\ &\quad + r^2(-1 + a'^2 + b'^2 + r'^2 - rr''). \end{aligned}$$

In this situation, we have

$$A_2 = \tfrac{1}{4}r^2(a'^2 - b'^2 + 2ia'b'), \qquad B_3 = 0.$$

From (15), we obtain $A_2 = 0$. Therefore $a' = b' = 0$ and the curve formed by the centers of circles of the foliation is an orthogonal straight line to the x_1x_2-plane. In conclusion, M is surface of revolution and this proves Theorem 2 for this case.

By (14), the function r in (20) that determines M is ruled by the differential equation

$$-1 + r'^2 - rr'' = r(-1 + r'^2)^{3/2}, \tag{21}$$

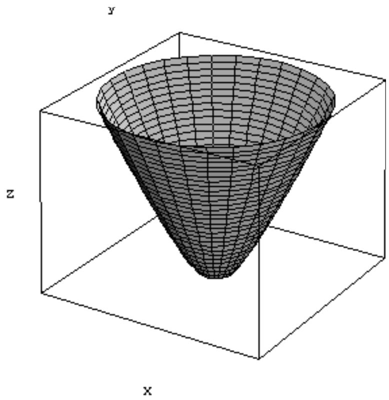

Figure 1. A surface of revolution with $H = \frac{1}{2}$ and the x_3 as axis of revolution. The foliation is given by Euclidean circles in parallel planes to the x_1x_2-plane.

where $r'^2 > 1$ (see also [6]). In Figure 1 we draw a rotational spacelike surface for $H = \frac{1}{2}$ with initial conditions $r(0) = 1$, $r'(0) = 2$ in Equation (21).

3.2. THE PLANES OF THE FOLIATION ARE TIMELIKE

Related with Theorem 1, we exhibit an example of a spacelike surface M of constant mean curvature $H = 1/2$ foliated by circles in nonparallel timelike planes. In our example, the planes of the foliation will be orthogonal, in the Lorentz sense, to the vector $\mathbf{t} = (1, u, u)$. Define $\mathbf{n}$ by setting $\mathbf{n} = \mathbf{t}'$. Then $\mathbf{n}$ is a null vector. Let $\mathbf{b}$ be the only null vector such that

$$\langle \mathbf{b}, \mathbf{n} \rangle = 1 \quad \langle \mathbf{b}, \mathbf{t} \rangle = 0 \quad [\mathbf{t}, \mathbf{n}, \mathbf{b}] = 1.$$

In this case, $\mathbf{n}$ and $\mathbf{b}$ belong to each plane of the foliation. With the notation of (4), there exists a differentiable function $\lambda = \lambda(u) \neq 0$ such that

$$e_2 = \lambda \mathbf{n} + \frac{1}{2\lambda}\mathbf{b}, \qquad e_3 = \lambda \mathbf{n} - \frac{1}{2\lambda}\mathbf{b}.$$

Then a parametrization of M is given by (see (4))

$$\mathbf{X}(u, v) = \mathbf{c} + r\left(\lambda e^v \mathbf{n} - \frac{1}{2\lambda} e^{-v} \mathbf{b}\right),$$

where $r > 0$ and $\mathbf{c}$ are smooth functions. We consider the following change of coordinates:

$$v \to \lambda(u) e^v, \qquad u \to u.$$

Then the parametrization $\mathbf{X}$ writes as

$$\mathbf{X}(u,v)=\mathbf{c}+rv\mathbf{n}-\frac{r}{2v}\mathbf{b}.$$

In our case, let

$$\mathbf{n}=(0,1,1),\qquad \mathbf{b}=\left(-u,\frac{1-u^2}{2},-\frac{1+u^2}{2}\right),$$

$$\mathbf{c}=(1,u,u),\qquad r=\sqrt{5}.$$

Then the surface is parametrized by

$$\mathbf{X}(u,v)=\left(1+\frac{\sqrt{5}u}{2v},u-\frac{\sqrt{5}(1-u^2)}{4v}+\sqrt{5}v,u+\frac{\sqrt{5}(1+u^2)}{4v}+\sqrt{5}v\right)$$

has $1/2$ as mean curvature and it is not difficult to notice that the foliation planes are not parallel. This surface is drawn in Figure 2.

To end this subsection, we prove Theorem 2 for the timelike case. So, let M be a constant mean curvature spacelike surface foliated by pieces of circles in parallel timelike planes. After a motion in $\mathbf{L}^3$, we can suppose that these planes are parallel to the x_2x_3-plane. In this case, we can parametrize the surface by

$$\mathbf{X}(u,v)=(u,a+r\sinh v,b+r\cosh v),$$

where a and b are differentiable functions on u. To conclude that M is rotational it suffices to prove that a and b are constant. A computation of W^2 and P shows that both numbers can be expressed as trigonometric polynomial on $\cosh nv$ and $\sinh nv$. Exactly, we have

$$\begin{aligned}W^2 \;=\; & -\tfrac{1}{2}r^2(a'^2+b'^2)\cosh 2v+(r^2a'b')\sinh 2v-\\ & -2r^2r'b'\cosh v+2r^2r'a'\sinh v+r^2\left(1-r'^2+\frac{a'^2-b'^2}{2}\right),\\ P \;=\; & r^2(-2r'b'+rb'')\cosh v+r^2(2r'a'-ra'')\sinh v+\\ & +r^2(1-r'^2+rr''+a'^2-b'^2).\end{aligned}$$

Putting

$$w=-iv,$$

the same relations (15)–(18) hold again. In our case, $A_2=-r^2(a'^2+b'^2)/4$ and $B_3=0$. By (15), we obtain $a'=b'=0$ and this means that M is rotational.

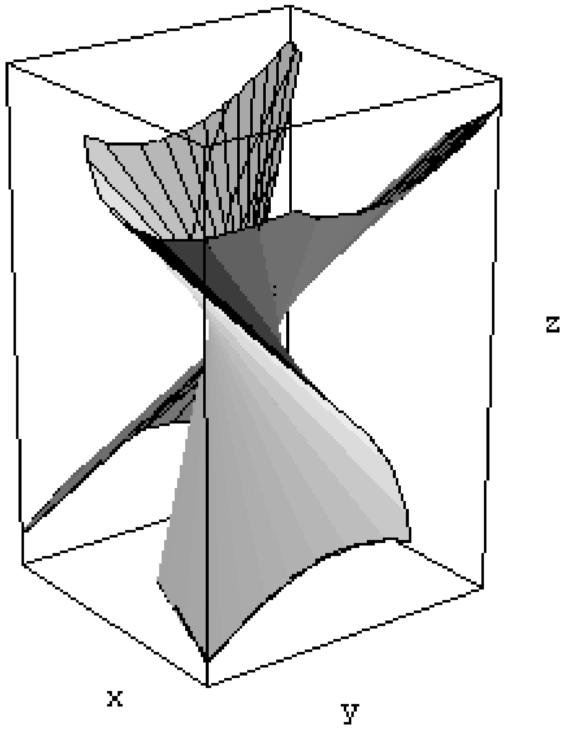

Figure 2. A spacelike surface foliated by pieces of circles in nonparallel timelike planes and with constant mean curvature $H = \frac{1}{2}$.

Since $a' = b' = 0$, the expressions of W^2 and P and the identity (14) gives us the next differential equation for r

$$1 - r'^2 + rr'' = r(1 - r'^2)^{3/2}, \tag{22}$$

where $r'^2 < 1$. These surfaces are studied in [6]. In Figure 3 we present a drawing of one of them for the initial conditions $r(0) = 1$, $r'(0) = 0.2$ in (22).

3.3. THE PLANES OF THE FOLIATION ARE NULL

We construct an example of a spacelike surface with constant mean curvature $H = \frac{1}{2}$ foliated by circles in nonparallel null planes. In this case, a parametrization of the surface is given by (see (5)):

$$\mathbf{X}(u, v) = \mathbf{c} + rv^2\mathbf{t} + v\mathbf{n},$$

where $r > 0$, $\mathbf{c}$ are differentiable functions on u and

$$\langle \mathbf{n}, \mathbf{n} \rangle = 1 \quad \langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{t}, \mathbf{n} \rangle = 0.$$

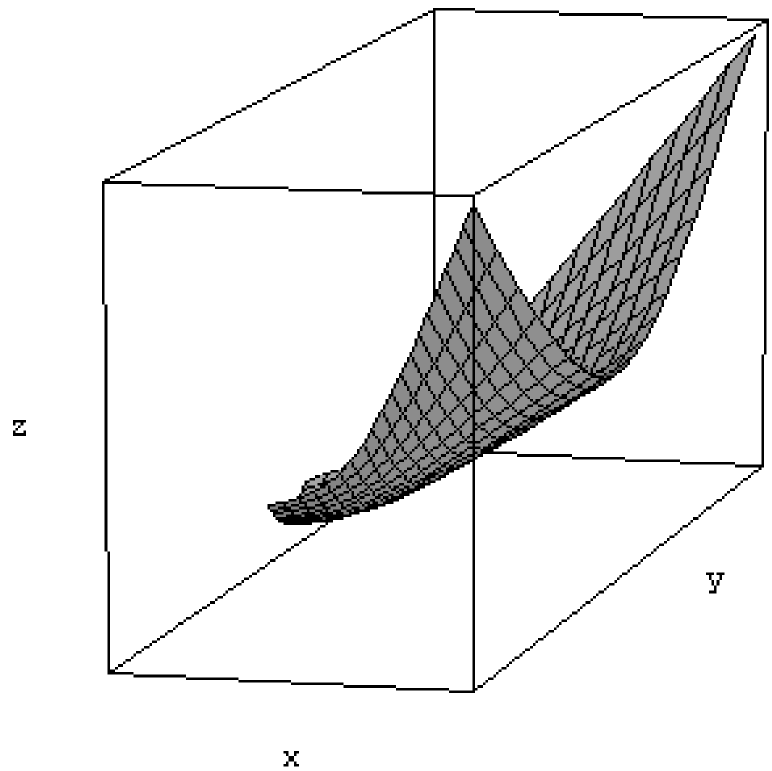

Figure 3. A surface of revolution with $H = \frac{1}{2}$ and the x_1 as axis of revolution. The foliation is given by hyperbolas in parallel planes to the x_2x_3-plane.

Consider $\mathbf{b}$ the only null vector such that $\langle \mathbf{n}, \mathbf{b} \rangle = 1$ and $[\mathbf{t}, \mathbf{n}, \mathbf{b}] = 1$. Let

$$\mathbf{t} = \left(u, \frac{1-u^2}{2}, -\frac{1+u^2}{2}\right), \qquad \mathbf{n} = (1, -u, -u), \qquad \mathbf{b} = (0, 1, 1).$$

$$\mathbf{c} = (u, -\tfrac{1}{2}u^2, -\tfrac{1}{2}u^2).$$

Again, a computation in (2) leads that the spacelike surface M parametrized by

$$\mathbf{X}(u, v) = \left(u + v + \frac{uv^2}{2}, -\frac{u^2}{2} - uv + \frac{v^2 - u^2v^2}{8}, -\frac{u^2}{2} - uv - \frac{v^2 + u^2v^2}{8}\right).$$

has constant mean curvature $H = \frac{1}{2}$ and the above reasoning shows that M is foliated by circles in null planes. An easy computation proves that the foliation is given by nonparallel planes. This surface is shown in Figure 4.

When the surface is foliated by pieces of circles in parallel planes, we prove that M is rotational and so, Theorem 2 for this case. As above, we assume that $H = \frac{1}{2}$.

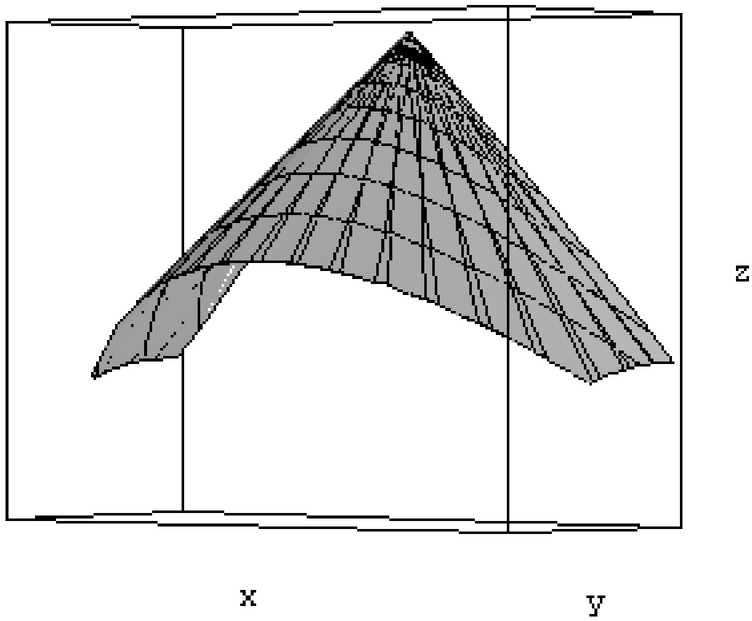

Figure 4. A spacelike surface foliated by pieces of circles in nonparallel null planes and with constant mean curvature $H = \frac{1}{2}$.

After a motion in $\mathbf{L}^3$, we can assume that the planes of the foliation are parallel to the plane $x_2 - x_3 = 0$. Then a local parametrization is given by

$$\mathbf{X}(u, v) = \left(a + v, b + u + r\frac{v^2}{2}, b - u + r\frac{v^2}{2} \right),$$

where a, b and $r > 0$ are differentiable functions on u. If the function a is constant, then M is a surface of revolution. Calculating the expressions of W^2 and P we have

$$W^2 = 2(r' - 2r^2)v^2 - 4ra'v + 4b' := A_2v^2 + A_1v + A_0.$$

$$\begin{aligned} P &= (4rr' - r'')v^2 + (4r'a' + 2ra'')v - 2ra'^2 - 8rb' - \\ &\quad -2b'' := B_2v^2 + B_1v + B_0. \end{aligned}$$

The relation (14) implies

$$A_2^3 = 0, \tag{23}$$

$$3A_1^2A_2 + 3A_0A_2^2 = B_2^2, \tag{24}$$

$$A_1^3 + 6A_0A_1A_2 = 2B_1B_2, \tag{25}$$

$$A_0^3 = B_0^2. \tag{26}$$

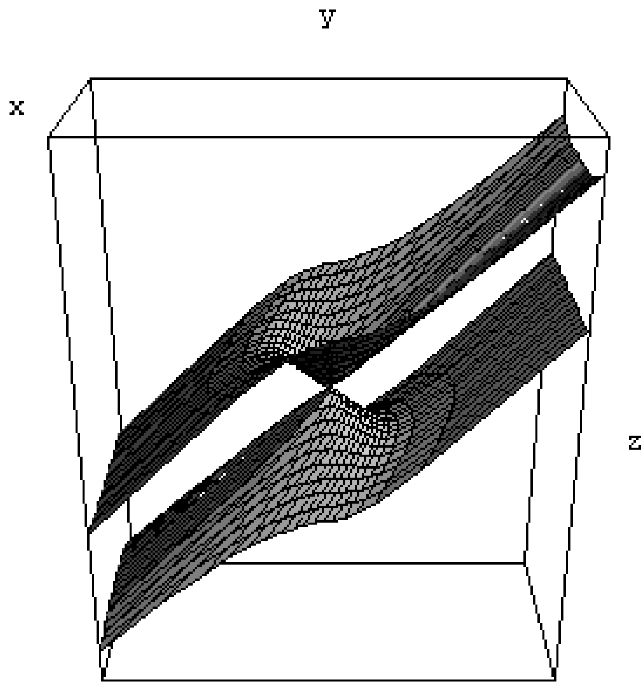

Figure 5. A surface of revolution with $H = \frac{1}{2}$ and the line $\{x_2 = x_3, x_1 = 0\}$ as axis of revolution. The foliation is given by parabolas in parallel planes to the $\{x_2 - x_3 = 0\}$-plane.

From (23) we obtain

$$r' = 2r^2. \tag{27}$$

Therefore, $r = 1/(-2u + \mu_1)$, for some constant μ_1. From (24), we have $B_2 = 0$ and equation (25) gets $A_1 = 0$. From the expression of W^2, this implies $a' = 0$. Therefore, M is rotational. Moreover, using (26), we get

$$4b'^{3/2} = 4rb' + b''. \tag{28}$$

Since $W^2 = A_0 = 4b' > 0$, we have that $b' > 0$. The differential equation (28), joint (27), describes all rotational constant mean curvature surfaces foliated by parabolas. For $\mu_1 = 0$, the solutions of (28) are

$$b(u) = \frac{u}{2(\mu_2 - u^2)} - \frac{\operatorname{arctahn}\frac{u}{\sqrt{\mu_2}}}{2\sqrt{\mu_2}} + \mu_3, \quad \mu_2, \mu_3 \in \mathbf{R}.$$

The surface obtained with $\mu_2 = 4$ and $\mu_3 = 0$ is shown in Figure 5.

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