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CONSTANT MEAN CURVATURE SURFACES BOUNDED BY A CIRCLE

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Introduction. The structure of the space of compact constant mean curvature surfaces with prescribed boundary is not known, even in the simplest case: when the boundary is a round circle with, for instance, unit radius. Heinz [**4**] found that a necessary condition for existence in this situation is that $|H| \leq 1$. The only known examples are the umbilical ones: the flat disc if $H=0$ and the two spherical caps with radius $1/|H|$ if $H \neq 0$; and some non-embedded surfaces of genus bigger than two whose existence was proved by Kapouleas in [**7**].

We shall consider a connected compact surface Σ and $\phi:\Sigma\to\mathbf{R}^3$ an immersion of constant mean curvature H such that $\phi:\partial\Sigma\to\phi(\partial\Sigma)$ is a diffeomorphism. We will say in this situation that Σ is an *H-surface with boundary* Γ, where $\Gamma=\phi(\partial\Sigma)$.

When the boundary is a circle of radius one, we shall suppose that it is in the z-plane, and we shall denote by S^1 the circle $\{(x,y,0)\in\mathbf{R}^3; x^2+y^2=1\}$. Given $0<|H|\leq 1$, the two spherical caps bounding S^1 are stable, but it is not known if they are the only ones bounded by S^1. There is no even answer to this question for immersed discs. In [**2**] it appears the question to find sufficient conditions of stability for a domain. Following ideas of Ruchert [**9**], Barbosa and do Carmo prove that if Σ is a simply-connected surface immersed in $\mathbf{R}^3$ with constant mean curvature and $\int_\Sigma |\sigma|^2\,d\Sigma<8\pi$, then Σ is stable, where σ denotes the second fundamental form. Also repeated with the problem of stability, Koiso [**8**] has proved that the spherical caps are the only surfaces with minimum area in the family of surfaces with constant volume and boundary S^1. In this paper we prove the following result.

Let $\phi:\Sigma\to\mathbf{R}^3$ be an immersion from a compact disc in $\mathbf{R}^3$ with constant mean curvature and such that $\phi(\partial\Sigma)$ is a circle of radius one. If σ is the second fundamental form and $\int_\Sigma |\sigma|^2\,d\Sigma\leq 8\pi$, then ϕ is umbilical.

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1. Preliminaries. Consider an immersed oriented surface $\phi : \Sigma \to \mathbf{R}^3$, and let N be a unit normal vector field. Given a normal variational vector field fN, where f is a smooth function on Σ with compact support, we define the area and volume functional:

$$A(t) = \int_\Sigma d\Sigma_t, \qquad V(t) = \int_{\Sigma\times[0,t]} \Phi^*(d, A_0).$$

Here $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbf{R}^3$ is a smooth variation of $\Sigma = \Phi(\Sigma \times 0)$ by immersed surfaces $\Sigma_t = \Phi(\Sigma \times t)$, coming from the variational vector field fN. Also $d\Sigma_t$ and dA_0 denote the area form of Σ induced by Φ_t and the volume form of $\mathbf{R}^3$. Thus $V(t)$ measures the algebraic volume between Φ_0 and Φ_t. It is well known that ϕ has constant mean curvature if and only if $A'(0) = 0$ for volume preserving normal variations.

We define stability of a constant mean curvature surface Σ to mean all compact proper domains $\Omega \subset \Sigma$ are relative minima of the area functional for volume preserving variations of Σ supported in Ω. In fact this is equivalent to $A''(0) \geq 0$ for all volume preserving variations of Σ. Barbosa and do Carmo [**2**] have shown that the study of stable surfaces begins with the quadratic form associated to the second variation

$$A''(0) =: Q(f) = \int_\Sigma (|\nabla f|^2 - |\sigma|^2 f^2)\, d\Sigma,$$

and their statement that a constant mean curvature compact surface is stable means

$$Q(f) = \int_\Sigma (|\nabla f|^2 - |\sigma|^2 f^2)\, d\Sigma \geq 0$$
$$\forall\, f \in C^\infty(\Sigma),$$
$$f = 0 \quad \text{on} \quad \partial\Sigma, \int_\Sigma f\, d\Sigma = 0.$$

According to this definition, the spherical caps are stable.

Let Σ be an H-surface with boundary S^1 and N its Gauss map. We represent by A the Weingarten endomorphism field corresponding to N. The Euclidean inner product $\langle , \rangle$ of $\mathbf{R}^3$ induces, by means of ϕ, a Riemannian metric ds^2 on Σ. Let ∇^2 and Δ be respectively the Hessian and Laplacian operators associated to that metric. We consider now the function $h : \Sigma \to \mathbf{R}^3$ given by

$$h = \langle \phi \wedge N, a \rangle, \quad a \in \mathbf{R}^3, |a| = 1,$$

that is, the normal component of the Killing field on $\mathbf{R}^3$ corresponding to the rotations around the vector a. For each $p \in \Sigma$ and $v \in T_p\Sigma$, we have

$$(dh)_p(v) = \langle (d\phi)_p(v) \wedge N(p), a \rangle - \langle \phi(p) \wedge A_p v, a \rangle.$$

From here, one can compute the Hessian $\nabla^2 h$ of the function h. In fact, if $p \in \Sigma$ and $u, v \in T_p\Sigma$,

$$\begin{aligned}(\nabla^2 h)_p(u,v) = &-\langle (d\phi)_p(v) \wedge A_p u, a \rangle - \langle (d\phi)_p(u) \wedge A_p v, a \rangle \\ &- \langle \phi(p) \wedge (\nabla A)_p(u,v), a \rangle - \langle A_p u, A_p v \rangle h(p),\end{aligned}$$

where ∇A is the covariant derivative of the endomorphism field A. Taking trace in this equality, we conclude that

$$\Delta h = -2\langle \phi \wedge \nabla H, a \rangle - |\sigma|^2 h.$$

Hence, in the constant mean curvature case, we have that the function h satisfies an elliptic equation which is nothing but the Jacobi equation corresponding to the second variation operator of the area functional. Moreover, the variation in Σ with variational vector field hN is the normal projection of the variational vector field in $\mathbf{R}^3$ associated with the 1-parameter family of rotations around to the straight line given by the vector a. Thus $\int_\Sigma h d\Sigma = 0$. In other words,

Lemma 1.1. *Let $\phi : \Sigma \to \mathbf{R}^3$ be an immersion of constant mean curvature and $N : \Sigma \to \mathbf{R}^3$ the Gauss map for ϕ. Then the function $h = \langle \phi \wedge N, a \rangle$ for any $a \in \mathbf{R}^3$ satisfies*

$$\Delta h + |\sigma|^2 h = 0,$$

σ being the second fundamental form of ϕ. Moreover, if $a = (0,0,1)$ and $\phi(\partial\Sigma)$ is the circle S^1, the function h vanishes along $\partial\Sigma$. If the function $h = 0$ on Σ, the surface is rotationally symmetric with respect to the axis a, i.e., it is umbilical.

The function h of Lemma 2.1 is an eigenfunction of the elliptic operator $L = \Delta + |\sigma|^2$ and then its nodal lines are piecewise smooth curves. Also, as $h = 0$ along $\partial\Sigma$, the boundary is a nodal line. If h has one nodal domain, this domain will be all the surface and then h will have sign. But $\int_\Sigma h\, d\Sigma = 0$ and then h is constantly zero in the whole surface, and, therefore, it is umbilical.

2. The main theorem. We are going to prove the main result of this paper.

Theorem 2.1. *Let Σ be an H-disc with boundary S^1, and let σ be the second fundamental form. If*

$$\int_\Sigma |\sigma|^2\, d\Sigma \leq 8\pi,$$

then the immersion is umbilical.

As we stated in the introduction, the hypotheses would imply that Σ is stable. In this theorem we assure that the surface is umbilical, i.e., a flat disc or a spherical cap.

Proof. We have three possibilities:

a) The function h vanishes in whole Σ and, in this case, Σ is a flat disc or a spherical cap.

b) $h \not\equiv 0$ and the function h has exactly one nodal domain. Then h has sign, but this is impossible because $\int_\Sigma h d\Sigma = 0$.

c) $h \not\equiv 0$ on Σ and it has more than one nodal domain. In this case we will get a contradiction. The hypotheses tell us that one nodal domain, named, for example, Ω, verifies $\int_\Omega |\sigma|^2\, d\Sigma \leq 4\pi$.

We consider the metric $d\bar{s}^2 = |\sigma|^2/2ds^2$. The surface $(\Sigma, d\bar{s}^2)$ has Gauss curvature $\overline{K} \leq 1$ and the equality holds if and only if $\phi : (\Sigma, ds^2) \to \mathbf{R}^3$ is an umbilical immersion. Let $v = h_{|\Omega}$ be the restriction on Ω of h. From Lemma 2.1, we have

$$\begin{aligned} \overline{\Delta} v + 2v = 0 \quad &\text{on} \quad \Omega, \\ v = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}$$

where $\overline{\Delta}$ is the Laplacian operator with respect to the $d\bar{s}^2$ metric. Because v has sign in Ω and it is an eigenfunction of the Laplacian $\overline{\Delta}$, the function v is an eigenfunction for the first eigenvalue with Dirichlet boundary condition. If we denote by $\bar{\lambda}_1(\Omega)$ the first eigenvalue of the Laplacian with Dirichlet condition in $(\Omega, d\bar{s}^2)$, we have

$$\bar{\lambda}_1(\Omega) = 2.$$

On the other hand,

$$\int_\Omega d\overline{\Sigma} = \int_\Omega \frac{|\sigma|^2}{2}\, d\Sigma \leq 2\pi.$$

Let Ω^* be a geodesic disc in the unit sphere $S^2(1)$ with

$$\operatorname{area}(\Omega^*) = \operatorname{area}(\Omega, d\bar{s}^2).$$

We compare the first eigenvalue of Ω^* with the first eigenvalue of the hemisphere: since $\operatorname{area}(\Omega^*) \leq 2\pi$, we obtain $\lambda_1(\Omega^*) \geq 2$. Because Ω^* is simply-connected and Ω is a subset of a simply-connected surface, the result on comparison of the first eigenvalue in [**1**] leads to

$$\lambda_1(\Omega^*) \leq \bar{\lambda}_1(\Omega)$$

and the equality is true if $(\Omega, d\bar{s}^2)$ has constant Gauss curvature equal to 1. Then $\lambda_1(\Omega^*) = 2 = \bar{\lambda}_1(\Omega)$ and $\overline{K} \equiv 1$. But it means that $\phi(\Sigma)$ is an umbilical surface and then $h \equiv 0$ on Σ, which gives a contradiction. □

It is clear that the function h gives information about an H-surface immersed with the boundary a circle. If h changes sign and if the surface is stable, from the Courant theorem [**3**], h has exactly two nodal domains. The following proposition studies the topology of the nodal domains of the function h.

Proposition 2.2. *Let Σ be a stable H-disc in $\mathbf{R}^3$ with the boundary a circle. Then the surface is a flat disc, a spherical cap or the two nodal domains of h are simply-connected.*

Proof. Consider that the surface is not umbilical. Let Ω_1, Ω_2 be the nodal domains. To prove that they are simply-connected, it is sufficient to prove that there is a nodal line intersecting $\partial\Sigma$.

Let ν be the interior conormal along S^1. We orient $\partial\Sigma$ and Σ such that $\{\phi_*(t), \nu, N\}$ and $\{\phi_*(t), \phi, a\}$ are positive oriented frames of $\mathbf{R}^3$, where t is a unit field tangent to $\partial\Sigma$. Since $h_{|\partial\Sigma} = 0$, we have $(dh)_p(t_p) = 0$ for each $p \in \partial\Sigma$. On the other hand,

$$(dh)_p\nu_p = \langle \nu_p \wedge N(p), a\rangle + \langle \phi(p) \wedge dN_p\nu_p, a\rangle = \sigma(t,\nu)(p).$$

In the following step, we prove the map from $\partial\Sigma$ to S^1 given by $p \mapsto \sigma(t,\nu)(p)$ has at least one zero. For that, we assume that $\sigma(t,\nu)$ don't vanish on $\partial\Sigma$. Then neither point of the boundary is umbilical nor are $\{t,\nu\}$ principal directions along $\partial\Sigma$. We represent by k_1 and k_2 the two principal curvature functions on Σ corresponding to the choice of unit normal field N. Remember that they are continuous functions such that $k_1 \leq H \leq k_2$ and the equalities occur only at the umbilical points. Corresponding to these principal curvatures we have two fields of line elements (unidimensional distributions) on Σ, namely, $\mathcal{D}_i = \ker(A - k_iI)$, $i = 1, 2$, which have a finite number of singularities since the surface is not umbilical [**6**]. Then $\mathcal{D}_i$ are transverse to the boundary. By using the Poincare-Hopf theorem, the index of $\mathcal{D}_i$ in each umbilical point is not positive [**5**, pp. 137–139]. Then the sum of index is not positive, in contradictions with that Σ has Euler characteristic 1.

Therefore $\sigma(t,\nu)$ vanishes at least once. If $\sigma(t,\nu)$ is not constant zero, it changes sign. In this case, and because $\partial\Sigma$ is a closed curve, $\sigma(t,\nu)$ has at least two zeros.

If $p \in \partial\Sigma$ is a point where $\sigma(t,\nu)$ vanishes, then the point p is a critical point of the eigenfunction h. Now we notice that either another nodal line different from $\partial\Sigma$ starts at the point p or h is identically zero on Σ. In fact, if h does not change its sign in each neighborhood of the point p, there would exist a neighborhood U of p in Σ where h would be, for instance, nonnegative. So

$$\Delta h_{|U} = -|\sigma|^2 h_{|U} \leq 0,$$

and, using the maximum principle, h cannot attain its minimum 0 either at an interior point or at p because $(dh)_p = 0$ unless that h

were identically zero on U and hence on Σ, in contradiction with the hypotheses. $\square$

To finish, we remark that there is another notion of stability, which is not equivalent with the above one. More precisely, we define *strong stability* with the same definition as the one given in the preliminaries, but omitting the condition "volume preserving." Then the quadratic form associated for the second variation agrees, but the functions f do not verify necessary the condition $\int_\Sigma f\, d\Sigma = 0$. With this definition, a domain on the sphere which contains a hemisphere is not strong stable and a domain included in a hemisphere is: it is an easy consequence of the Rayleigh's characterization of the first eigenvalue of the Laplacian with Dirichlet condition and the property of decreasing of the first eigenvalue with respect to the inclusion of sets. We ask for strongly stable H-surfaces with boundary a unit circle, and we shall characterize the small spherical cap of radius $1/|H|$ as the only strongly stable H-surface with boundary a unit circle. This result is a consequence of the properties of the function $h = \langle \phi \wedge N, a \rangle$.

Theorem 2.3. *The only strongly stable H-surface with boundary S^1 is the flat disc and the small spherical cap of radius $1/|H|$.*

Proof. We consider the function h. If h is a constant zero in the surface, the surface is umbilical. In other cases, we will get a contradiction.

If h is not a constant zero, h changes sign because $\int_\Sigma h\, d\Sigma = 0$. Therefore, the function h is an eigenfunction of the operator $L = \Delta + |\sigma|^2$ for the Dirichlet boundary condition and with eigenvalue $\lambda = 0$. As $Q(f) \geq 0$ for any smooth function f vanishing in the boundary, then $\lambda = \lambda_1(\Sigma) = 0$ and h is the first eigenfunction. Then h does not change sign and we have a contradiction. $\square$

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