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Existence of constant mean curvature graphs in hyperbolic space

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Abstract. We give an existence result for constant mean curvature graphs in hyperbolic space $\mathbb{H}^{n+1}$. Let Ω be a compact domain of a horosphere in $\mathbb{H}^{n+1}$ whose boundary $\partial\Omega$ is mean convex, that is, its mean curvature $H_{\partial\Omega}$ (as a submanifold of the horosphere) is positive with respect to the inner orientation. If H is a number such that $-H_{\partial\Omega} < H < 1$, then there exists a graph over Ω with constant mean curvature H and boundary $\partial\Omega$. Umbilical examples, when $\partial\Omega$ is a sphere, show that our hypothesis on H is the best possible.

1 Introduction

We will start by considering the upper halfspace model for the hyperbolic space $\mathbb{H}^{n+1}$. That is

$$\mathbb{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\}$$

endowed with the metric

$$\langle , \rangle = \frac{dx_1^2 + \ldots + dx_{n+1}^2}{x_{n+1}^2}.$$

In this setting, any horosphere of $\mathbb{H}^{n+1}$, after a suitable isommetry, can be mapped on

$$L(c) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{H}^{n+1}; x_{n+1} = c\}$$

for some $c > 0$. Let $\Omega \subset L(c)$ be a compact domain of such a horosphere and u a smooth function defined on Ω. Then we will mean by the graph of the function u the hypersurface

$$\Sigma = \{(x, u(x)); x \in \Omega\}$$

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of $\mathbb{H}^{n+1}$. This graph Σ will have constant mean curvature H and boundary $\partial\Omega$ if and only if

$$(1) \qquad Q(u) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \frac{n}{u}\left(H - \frac{1}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{on } \Omega$$

$$(2) \qquad u = c \qquad \text{on } \partial\Omega$$

where ∇ and div denote the Euclidean gradient and divergence operators and $|.|$ is the Euclidean norm in $\mathbb{R}^n$. This assertion can be deduced from the following more general fact. If Σ is any (not necessarily graph) orientable hypersurface immersed into $\mathbb{H}^{n+1}$ and N is a unit normal field on Σ with respect to the hyperbolic metric, then $N'(p) = N(p)/p_{n+1}$, where $p \in \Sigma$ and $p_{n+1} = x_{n+1}(p)$ is an Euclidean unit normal field for Σ. Hence, the respective principal curvatures $k_i(p)$ and $k_i'(p)$ are related as follows

$$k_i(p) = p_{n+1}k_i'(p) + N_{n+1}'(p) \qquad 1 \leq i \leq n,$$

where $N_{n+1}' = x_{n+1}(N')$. So, the corresponding mean curvature functions H and H' satisfy

$$(3) \qquad H(p) = p_{n+1}H'(p) + N_{n+1}'(p).$$

Now, when Σ is a graph of a function u it is well known that the Euclidean mean curvature function H' is given by

$$nH' = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \qquad \text{on } \Omega,$$

with respect to the orientation $N' = (-\nabla u, 1)(1+|\nabla u|^2)^{\frac{-1}{2}}$ pointing to the positive side of the x_{n+1}-axis. Using (3) in this last equation we get (1) in the case where H is constant. Notice that, in (1), H is the mean curvature corresponding to the unit normal field pointing to the positive side of x_{n+1}. From now on, we will choose that orientation for all the graphs whose boundary is in a horosphere. With this choice, any domain of the horosphere $L(c)$ will have constant mean curvature 1.

The main purpose of this paper is to solve the Dirichlet problem (1)–(2) under some convexity conditions on Ω. Equation (1) is of quasilinear elliptic type and so the standard theory for this kind of equations can be applied [6, Part II].

Recently, existence and regularity theorems for constant mean curvature hypersurfaces in hyperbolic space have been obtained. When the asymptotic boundary is prescribed, existence of complete minimal hypersurfaces was studied in Anderson's papers [2, 3] and more later by Lin [11] and Hardt and Lin [7]. When the mean curvature is a non-zero constant, recent advances can be found in [1], [12] and [14]. The case of graphs, that we are interested in, was dealt with in [4] when the boundary is included in a hyperbolic hyperplane, and in [12] and [13] when it lies in a horosphere. In fact, if Ω is a compact domain in a horosphere and $\partial\Omega$ is mean convex, Nelli and Spruck constructed in [12] a graph with constant mean curvature H and boundary $\partial\Omega$ for each $0 < H < 1$. The estimates that they found there allowed them to construct graphs with constant mean curvature and prescribed boundary at the infinity. Their method of proof consists in blowing down from a minimal graph with boundary $\partial\Omega$ in a horosphere (whose existence is assured by the standard elliptic theory [6], as Lin pointed out in [11]), until arriving at the very domain Ω whose mean curvature is 1.

In this paper, we will obtain C^0 and C^1-estimates for constant mean curvature graphs in hyperbolic space whose boundary is $\partial\Omega$ in a horosphere, which allow us to proof that one can blow up from the domain Ω to get graphs with fixed boundary $\partial\Omega$ and constant mean curvature, including the minimal one and even with negative constant mean curvature. Concretely, we will prove

Theorem 1.1 *Let Ω be a compact domain included in a horosphere of $\mathbb{H}^{n+1}$ such that $\partial\Omega$ is mean convex. Let H be a real number such that $-H_{\partial\Omega} < H \leq 1$. Then there exists a graph over Ω with boundary $\partial\Omega$ and constant mean curvature H.*

Notice that if we think in $\mathbb{H}^{n+1}$ as the upper halfspace model an equivalent formulation of this result is as follows

Theorem 1.1 *Let Ω be a compact domain included in the horosphere $L(c) = \{x \in \mathbb{H}^{n+1}; x_{n+1} = c\}$, $c > 0$ such that $\partial\Omega$ is (Euclidean) mean convex. If H is a real number such that $-cH'_{\partial\Omega} < H \leq 1$, where $H'_{\partial\Omega}$ is the Euclidean mean curvature of $\partial\Omega$ in $L(c)$, then there exists a graph with constant mean curvature H and boundary $\partial\Omega$.*

Our hypothesis on H cannot be improved. In fact, one may consider umbilical examples to see that, if $\partial\Omega$ is a round $(n-1)$-sphere with Euclidean mean curvature $k > 0$ lying in $L(c)$, there are no umbilical graphs on Ω spanning $\partial\Omega$ with constant mean curvature $H \leq -kc$.

The plan of the paper is as follows. In Sect. 2 we analyse graphs on horospheres in the Minkowski model for hyperbolic space and we apply the maximum principle for constant mean curvature hypersurfaces to discuss some configurations of embedded hypersurfaces with boundary in a horosphere. In sections 3 and 4 we derive height and gradient estimates for graphs of constant mean curvature, or H-graphs, on horospheres. Finally in Sect. 5, we state the existence theorem and a kind of uniqueness result.

2 Preliminaries

A fundamental tool to study equation (1) is the Hopf maximum principle ([6, Theorem 9.2] and [8]). The following geometric consequence of that maximum principle has been already used (see [5] for details and definitions):

Proposition 2.1 (Tangency principle) *Let Σ_1 and Σ_2 be oriented hypersurfaces in $\mathbb{H}^{n+1}$ with constant mean curvature $H_1 \leq H_2$. If Σ_1 and Σ_2 have a point p,* *either in the interior or in the (analytic) boundary, where they are tangent, and Σ_1 lies above Σ_2 near p, then Σ_1 and Σ_2 must coincide in a neighbourhood of p.*

As a corollary of this, one has uniqueness for the Dirichlet problem (1)–(2) corresponding to constant mean curvature graphs with boundary in a horosphere. In fact, we can generalize this uniqueness to more general situations. So it will be possible to obtain conditions assuming that a compact hypersurface whose boundary is in a horosphere is included in some of the two domains determined in $\mathbb{H}^{n+1}$ by the horosphere. Following ideas from [9] we have the following result which will be useful in the sequel.

Proposition 2.2 *Let Σ be an oriented compact hypersurface immersed in $\mathbb{H}^{n+1}$ with constant mean curvature H and whose boundary Γ is a closed embedded submanifold of a horosphere $L(c)$. Let Ω be the bounded domain determined by Γ in $L(c)$. We have*

1. *If $|H| \leq 1$, then Σ is included in $L(c)^+ = \{x \in \mathbb{H}^{n+1}; x_{n+1} \geq c\}$.*
2. *If $|H| \geq 1$, Σ is embedded and $\Sigma \cap (L(c) - \Omega) = \emptyset$, then Σ is included in $L(c)^+$ or $L(c)^- = \{x \in \mathbb{H}^{n+1}; x_{n+1} \leq c\}$.*

Proof. 1. Consider the case $|H| \leq 1$. Take a horosphere $L(t)$, $t < c$, with t small enough that $L(t) \cap \Sigma = \emptyset$. Now, we increase t to touch Σ the first time. Since each horosphere has constant mean curvature 1 with respect to the unit normal field pointing upwards and $|H| \leq 1$, the tangency principle 2.1 implies that this tangent contact is obtained for $t = c$ and so $\Sigma \subset L(c)^+$.

2. Suppose now that $|H| \geq 1$. Take an Euclidean hemisphere S upon the horosphere $L(c)$ whose boundary disc D is contained in $L(c)$ and is large enough that $\Omega \subset \text{int}(D)$ and that the domain B determined by $S \cup D$ verifies $\Sigma \cap L(c)^+ \subset B$. Thus $\Sigma \cup S \cup (D - \Omega)$ is a closed hypersurface embedded in $\mathbb{H}^{n+1}$ and, so, determines an interior domain, say W. Choose a unit normal field B for Σ in such a way that N points into W at each point. It turns out that, if there are points of the hypersurface Σ in both $L(c)^+$ and $L(c)^-$, then N takes the same value at the points where the x_{n+1} coordinate attains its maximum and minimum respectively. Reversing N if necessary, we can concluded that the unit normal field (for which $H \geq 1$) takes the same value at the highest and at the lowest point of the hypersurface. Lowering a horosphere to the highest point or pushing it up to the lowest one we obtain a contradiction using the tangency principle. Thus the hypersurface lies in one of $L(c)^+$ or $L(c)^-$. □

Using the same tangency principle, when the considered hypersurface Σ is a graph and taking the settled (upwards) orientation, we may sharpen the result above.

Proposition 2.3 *Let Σ be a graph over a domain Ω in a horosphere $L(c)$ of $\mathbb{H}^{n+1}$. Suppose that Σ has constant mean curvature H (with respect to the upwards orientation). Then*

1. *$H \leq 1$ is equivalent to $\Sigma \subset L(c)^+$ (the "interior" domain).*
2. *$H \geq 1$ is equivalent to $\Sigma \subset L(c)^-$ (the "exterior" domain).*

We can obtain, also as another consequence from the tangency principle the next Corollart about monotonicity with respect to H of graphs with constant mean curvature H in $\mathbb{H}^{n+1}$ whose boundary lies in a horosphere.

Corollary 2.4 (Monotonicity) *Let Ω_1, Ω_2 be two compact domains in the horosphere $L(c)$ of $\mathbb{H}^{n+1}$ such that $\Omega_1 \subset int(\Omega_2)$. Consider two graphs Σ_1 and Σ_2 over Ω_1 and Ω_2 with constant mean curvatures H_1 and H_2 (with respect to the upwards orientation) and boundaries $\partial\Omega_1$ and $\partial\Omega_2$. If $H_1 \geq H_2$, then we have that $u_1 \leq u_2$ on Ω_1. In particular, if $\Omega_1 = \Omega_2$, then $u_1 \leq u_2$.*

Proof. Suppose, without less of generality, that the point $(0, c) \in \Omega_1$, where $0 \in \mathbb{R}^n$ and consider the group of hyperbolic translations (Euclidean homotheties) given by

$$T_s : (x_1 \ldots, x_{n+1}) \longmapsto \mathrm{e}^s(x_1, \ldots, x_{n+1})$$

where $s \in \mathbb{R}$. Take s big enough that $T_s(\Sigma_2)$ does not intersect Σ_1. Now we bring back $T_s(\Sigma_2)$ to its original position by decreasing s until the first so with $T_{s_0}(\Sigma_2) \cap \Sigma_1 \neq \emptyset$. Since Σ_1 is a graph on $\Omega_1 \subset\subset \Omega_2$, $T_s(\partial\Omega_1)$ does not touch Σ_1 for any $s \neq 0$. On the other hand, the tangency principle 2.1 forbids an interior contact point in $\Sigma_1 \cap T_{s_0}(\Sigma_2)$ because $H_1 \geq H_2$. Then $s_0 \leq 0$ and Σ_2 is over Σ_1 on Ω_1. □

Remark 1 An alternative proof of this Corollary 2.4 could be done from the theory of quasilinar elliptic equations. In fact, if u and v are solutions of (1), one can apply the maximum principle to the difference function $u - v$ which satisfies a linear elliptic equation. This is achieved in [10, (5.10)] when u and v determine constant mean curvature graphs over domains in hyperbolic hyperplanes. In the case of graphs over horospheres, the same argument works if $H \leq 1$. But this last inequality is a consequence, in our case, from Proposition 2.3.

Once we have established these preliminaries, we may come back to consider the Dirichlet problem (1)–(2) that we wanted to solve. In order to find these H-graphs we will apply a version of continuity method. From the standard theory for quasilinear elliptic equations [6], one concludes existence of H-graphs over a given domain Ω in the horosphere $L(c)$ of $\mathbb{H}^{n+1}$, for H lying in some interval I by proving that

$$J = \{H \in I; \text{there exists an } H\text{-graph on } \Omega \text{ whose boundary is } \partial\Omega\}$$

is a nonempty, open and closed set. We will achieve it as follows:

1. If $1 \in I$, then $J \neq \emptyset$ because the very domain Ω is a graph with constant mean curvature 1.
2. The implicit function theorem for elliptic partial differential equations assures that, if we prove that the operator Q in (1) is invertible, then we can solve the Dirichlet problem (1)–(2) for H in some interval around of any H_0 for

which we have a solution. In this way, the openess of J is reduced to check the invertibility of Q.

3. Closedness of I will be a consequence from obtaining a priori $C^{2,\alpha}$-estimates for any solution of (1)–(2). But the properties of divergence type quasilinear elliptic equations and Schauder theory guarantee that these $C^{2,\alpha}$-estimates come from uniform C^0 and C^1-estimates (cf. [6]).

3 Height estimates

In order to obtain the height and gradient estimates that we need to prove existence of constant mean curvature graphs, it will be convenient to leave the upper halfspace model for $\mathbb{H}^{n+1}$ that we have utilized before and consider the hyperbolic space as a hyperquadric in a Lorentz space. In this way, the induced metric on the graph takes a more manageable form.

So, we will represent by $\mathbb{L}^{n+2}$ the vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + \ldots + dx_{n+1}^2$$

and the hyperbolic space will be identified with

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = -1, p_0 > 0\}$$

equipped with the (Riemannian) induced metric from $\mathbb{L}^{n+2}$. In this setting horospheres, hyperplanes and spheres can be obtained intersecting $\mathbb{H}^{n+1}$ with affine hyperplanes of $\mathbb{L}^{n+2}$. For example, any horosphere is given by

$$L(\tau) = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = \tau\}$$

where $a \in \mathbb{L}^{n+2}$ is a nonzero lightlike vector, that is, $\langle a, a\rangle = 0$, and τ is a positive number. When one fixes that vector a and moves $\tau \in \mathbb{R}^+$ one obtains a foliation of $\mathbb{H}^{n+1}$ by means of horospheres having the same point at the infinity. It is easy to see that

$$\xi(p) = -p - \frac{1}{\tau}a \tag{4}$$

is a unit normal field on $L(\tau)$ with respect to which the horosphere has constant mean curvature 1. So, the ”interior” domain $L(\tau)^+$ determined by the horosphere is given by $\{p \in \mathbb{H}^{n+1}; \langle p, a\rangle \leq \tau\}$ and the ”exterior” one is $L(\tau)^- = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle \geq \tau\}$.

Now let Ω be a compact domain in the horosphere $L(\tau)$ and $f \in C^\infty(\Omega)$. Here the graph Σ of the function f is (recall the form of the geodesics of $\mathbb{H}^{n+1}$ in this Minkowski model)

$$\Sigma = \{\cosh f(q) q + \sinh f(q)\xi(q); q \in \Omega\}.$$

That is, taking into account (4),

$$\Sigma = \{\mathrm{e}^{-f(q)} q - \frac{1}{\tau} \sinh f(q) a; q \in \Omega\}. \tag{5}$$

Then, the points of Σ belong to $L(\tau)^+$ or $L(\tau)^-$ according to f is ≥ 0 or ≤ 0. Thus, if we choose the orientation of Σ as we pointed out in the section above (upwards orientation in the upper half-space model) we have that the corresponding position vector p and Gauss map N satisfy

$$\langle p, a \rangle = \tau e^{-f} \qquad \langle N, a \rangle = \frac{-\tau e^{-f}}{\sqrt{1 + e^{2f} |\nabla f|^2}} \tag{6}$$

From that, it is not difficult to see that the two functions u and f which determine a graph over a domain in a horosphere corresponding to the two settings (upper halfspace and Minkowski models) are related as follows

$$u = \frac{1}{\langle p, a \rangle} = \frac{1}{\tau} e^f .$$

In this way, to obtain C^0 and C^1 estimates for u is equivalent to do the same for f.

Suppose now that Σ is any hypersurface (not necessarily graph) immersed into $\mathbb{H}^{n+1}$ viewed in this Minkowski frame. If p stands for the position vector function on Σ in $\mathbb{L}^{n+2}$, we have the known equation (see [10] for instance)

$$\Delta p = np + nHN \tag{7}$$

where Δ is the Laplacian of the induced metric. Moreover, when the mean curvature function H of Σ is constant, we obtain (see [10] again)

$$\Delta N = -nHp - |\sigma|^2 N , \tag{8}$$

where σ is the second fundamental form of the immersion. From these equations, we will start to get height estimates for graphs in the hyperbolic space.

Theorem 3.1 *Let Γ be a closed $(n-1)$-dimensional submanifold of $\mathbb{H}^{n+1}$ and $a \in \mathbb{L}^{n+2}$. Then there exists a constant $C_1 = C_1(\Gamma, a)$ depending only on Γ and a, such that if Σ is a compact hypersurface bounded by Γ and its mean curvature function satisfies $H^2 \leq 1$, we have*

$$\sup_{p \in \Sigma} |\langle p, a \rangle| \leq C_1 .$$

Proof. Let $\{e_0, e_1, \ldots, e_{n+1}\}$ be the canonical basis of $\mathbb{L}^{n+2}$. If N is the Gauss map of Σ, then for any $p \in \Sigma$

$$\langle HN(p) + p, HN(p) + p \rangle = H^2 - 1 \leq 0.$$

Then by continuity and since $\langle p, e_0 \rangle = -p_0 < 0$, we have

$$\langle HN(p) + p, e_0 \rangle \leq 0.$$

Thus from (7), $\Delta \langle p, e_0 \rangle \leq 0$ and the maximum principle implies that

$$0 > -p_0 = \langle p, e_0\rangle \geq \min_{p\in\Gamma}\langle p, e_0\rangle := B_1 \tag{9}$$

and B_1 depends only on Γ. Now, we take $1 \leq i \leq n+1$. Then

$$-\langle p, e_0\rangle^2 + \langle p, e_i\rangle^2 \leq \langle p, p\rangle = -1$$

and from (9)

$$|\langle p, e_i\rangle| \leq \sqrt{\langle p, e_0\rangle^2 - 1} \leq \sqrt{B_1^2 - 1} := B_2.$$

Finally, let $a = a_0 e_0 + \ldots + a_{n+1}e_{n+1}$. Then

$$|\langle p, a\rangle| \leq |a_0||B_1| + B_2(|a_1| + \ldots + |a_{n+1}|) := C_1.$$

□

Remark 2 Theorem 3.1 has a different geometric proof in the upper halfspace model. Since the hypersurface Σ is compact, we consider a sphere S in $\mathbb{H}^{n+1}$ that contains Σ in its inside. The mean curvature of S is greater than one with respect to the orientation pointing inside. So, we move S in a fix direction until S touches Σ. Since $H^2 \leq 1$, the tangency principle assures that this only occurs at boundary points. We get the a priori bounds of Σ in some coordinate moving S in that direction. Also, it is important to recall that the above reasoning says that any compact hypersurface in $\mathbb{H}^{n+1}$ with $H^2 \leq 1$ has non empty boundary (see [5]).

As consequence of Theorem 3.1, we obtain C^0-estimates for graphs on horospheres with constant mean curvature H, when H belongs to the interval $[0, 1]$.

Corollary 3.2 *Let Ω be a compact domain a horosphere of $\mathbb{H}^{n+1}$. Then there exists a constant $C_2 = C_2(\Omega)$ depending only on Ω, such that if Σ is a graph of a function f on Ω with constant mean curvature H, $H \in [0, 1]$, and with boundary $\partial\Omega$, we have*

$$0 \leq f \leq C_2.$$

Proof. Let $L(\tau)$ be the horosphere containing Ω. Recall that with the chosen orientation for graphs, we have that Σ is included in $L(\tau)^+$, and then, $f \geq 0$. By rotating coordinates and with the notation of Theorem 3.1, we assume that the lightlike vector that defines $L(\tau)$ is $a = -e_0 - e_1$. For any $p \in \Sigma$,

$$(p_1 - p_0)(p_1 + p_0) = -p_0^2 + p_1^2 \leq \langle p, p\rangle = -1. \tag{10}$$

Since $\langle p, a\rangle = p_0 - p_1 \geq 0$, we have $p_1 \leq p_0$. The inequality (10) and the fact that $B_1 \leq -p_0$ imply

$$\langle p, a\rangle = p_0 - p_1 \geq \frac{1}{p_1 + p_0} \geq \frac{1}{-2B_1} := B_3.$$

From (5)

$$f \leq \log\frac{\tau}{B_3} := C_2.$$

□

Finally we analyse the case $H \notin [0,1]$.

Theorem 3.3 *Let be a number $H \notin [0,1]$. There exists a constant $C_3 = C_3(H)$ depending only on H such that if Σ is the graph of a function f on any compact domain Ω in a horosphere of $\mathbb{H}^{n+1}$ whose boundary is $\partial\Omega$, we have*

1. *if $H < 0$,*
$$0 \leq f \leq C_3.$$
2. *if $H > 1$,*
$$C_3 \leq f \leq 0.$$

Proof. Let $L(\tau) = \{p \in \mathbb{L}^{n+1}; \langle p, a\rangle = \tau\}$ be the horosphere where Ω is. Using formulas (7) and (8) we conclude

$$\Delta(H\langle p,a\rangle + \langle N,a\rangle) = -(|\sigma|^2 - nH^2)\langle N,a\rangle \geq 0. \tag{11}$$

Then the maximum principle gives

$$H\langle p,a\rangle + \langle N(p),a\rangle \leq \max_{\partial\Sigma}(H\tau + \langle N,a\rangle) \leq H\tau. \tag{12}$$

Now we have

$$\begin{aligned} 0 &= |a|^2 = |a^T|^2 + \langle N,a\rangle^2 - \langle p,a\rangle^2 \geq \langle N,a\rangle^2 - \langle p,a\rangle^2 \\ &= (\langle N,a\rangle + \langle p,a\rangle)(\langle N,a\rangle - \langle p,a\rangle), \end{aligned}$$

where a^T is the tangent part of a on Σ. Since $\langle N(p),a\rangle - \langle p,a\rangle \leq 0$, we have

$$\langle N(p),a\rangle + \langle p,a\rangle \geq 0. \tag{13}$$

From (12) and (13)

$$(H-1)\langle p,a\rangle \leq H\tau. \tag{14}$$

Since $\langle p,a\rangle = \tau e^{-f(p)}$, the inequality (14) yields the desired C^0-estimates of f if we set

$$C_3 := \log\frac{H-1}{H}.$$

□

4 Gradient estimates

In the analysis of (1)–(2) and in the search of a priori $C^{2,\alpha}$-estimates, it is necessary to establish a priori gradient estimates for solutions of the Dirichlet problem. We follow working in the Minkowski model and we will need some convexity assumptions on the domain Ω.

Theorem 4.1 *Let Ω be a compact domain of a horosphere such that $\partial\Omega$ is mean convex. Let Σ be a graph on Ω bounded by $\partial\Omega$ with constant mean curvature H. If*

$$-H_{\partial\Omega} < H \leq 1,$$

then there exists a constant $C_4 = C_4(\Omega, H)$ depending only on Ω and H such that

$$\sup_{\Omega} |\nabla f| \leq C_4.$$

Proof. From equation (11), there exists $q \in \partial\Sigma$ such that

$$H\langle p, a\rangle + \langle N, a\rangle \leq H\tau + \langle N(q), a\rangle. \tag{15}$$

Moreover at the point q,

$$H\langle \nu_q, a\rangle + \langle (dN)_q \nu_q, a\rangle \leq 0,$$

where ν is the interior conormal along $\partial\Sigma$. On the other hand, since $H \leq 1$ we know by Sect. 2 that $\langle p, a\rangle \leq \tau$ and then, in any point of $\Gamma := \partial\Omega$, $\langle \nu, a\rangle \leq 0$. So we have

$$(H - \sigma(\nu_q, \nu_q))\langle \nu_q, a\rangle \leq 0.$$

This inequality and since $\langle \nu_q, a\rangle \leq 0$ imply that

$$\sum_{i=1}^{n-1} \sigma(v_i, v_i) \geq (n-1)H, \tag{16}$$

where $\{v_1, \ldots, v_{n-1}\}$ an orthonormal frame in the tangent space $T_q\partial\Sigma$. At the boundary points we have

$$\sigma(v_i, v_i) = \sigma^{\Gamma}(v_i, v_i)\langle N(q), \eta(q)\rangle - \frac{\langle N, a\rangle}{\tau} \tag{17}$$

where σ^{Γ} denotes the second fundamental form of Γ as submanifold of $L(\tau)$ and η is the unit normal field to Γ in $L(\tau)$ that points inside. If $\xi = -p - \frac{1}{\tau}a$ is the unit normal field on $L(\tau)$,

$$1 = \langle \xi, \xi\rangle = \langle \nu, \xi\rangle^2 + \langle N, \xi\rangle^2 = \frac{\langle \nu, a\rangle^2}{\tau^2} + \langle N, \xi\rangle^2.$$

This formula joint with the equality

$$1 = \langle N, N\rangle = \langle N, \eta\rangle^2 + \langle N, \xi\rangle^2$$

assures that

$$\langle N, \eta\rangle = \frac{\langle \nu, a\rangle}{\tau}.$$

From (16), (17) and since H_{Γ} is positive, we have at the point q

$$\langle N(q), a\rangle \leq H_{\Gamma}(q)\langle \nu_q, a\rangle - \tau H. \tag{18}$$

First, we consider the case that $0 < H < 1$. Then inequalities (15) and (18) yield

$$H\langle p,a\rangle + \langle N(p),a\rangle \leq 0 \tag{19}$$

for any $p \in \Sigma$. As a consequence of Corollary 3.2 and (19)

$$\langle N(p),a\rangle \leq -H\langle p,a\rangle \leq -H\tau \mathrm{e}^{-C_2} := B_4 < 0$$

and B_4 depends only on Ω and H.

Now let us study the case $-H_\Gamma < H \leq 0$. As $\langle a,a\rangle = 0$, we have

$$\langle \nu,a\rangle = -\sqrt{\tau^2 - \langle N,a\rangle^2}.$$

Then (18) gives

$$(1+H_\Gamma(q)^2)\langle N(q),a\rangle^2 + 2\tau H\langle N(q),a\rangle + \tau^2(H^2 - H_\Gamma(q)^2) \geq 0. \tag{20}$$

Some of the two roots of the left side is positive because $H \leq 0$. As the function $\langle N,a\rangle$ is negative because Σ is a graph, we have

$$\langle N(q),a\rangle \leq \frac{\tau}{1+H_\Gamma(q)^2}\left(H - H_\Gamma(q)\sqrt{1+H_\Gamma(q)^2 - H^2}\right). \tag{21}$$

The hypothesis on H assures that the right side in (21) is negative. Now (15), (21) and the fact that $-H_\Gamma < H \leq 0$ imply

$$\begin{aligned}\langle N,a\rangle &\leq H(\tau - \langle p,a\rangle) + \frac{\tau}{1+H_\Gamma(q)^2}\left(H - H_\Gamma\sqrt{1+H_\Gamma(q)^2 - H^2}\right)\\ &\leq \frac{\tau}{1+H_\Gamma(q)^2}\left(H - H_\Gamma(q)\sqrt{1+H_\Gamma(q)^2 - H^2}\right) := B_5 < 0,\end{aligned}$$

where B_5 depends only on Ω and H.

Therefore we are able to find negative constants B_4 and B_5 depending only on H and Ω such that

$$\begin{aligned}&\langle N,a\rangle \leq B_4 < 0 \quad \text{on } \Omega, \text{ if } 0 < H < 1.\\ &\langle N,a\rangle \leq B_5 < 0 \quad \text{on } \Omega, \text{ if } -H_\Gamma < H \leq 0.\end{aligned}$$

Now it is easy to conclude from (6), Corollary 3.2 and Theorem 3.3 that there exists a constant C_4, depending only on Ω and H such that $|\nabla f| \leq C_4$. □

5 The main result

In Sect. 4, Theorem 4.1 gives us a priori gradient estimates for a constant mean curvature graph defined in a mean convex domain of a horosphere when the mean curvature satisfies some assumptions with respect to the convexity of the boundary. This fact joint with the height estimates established in Sect. 3, allow us to use the standard theory of existence for the Dirichlet problem (1)–(2).

Theorem 5.1 *Let Ω be a compact domain of a horosphere such that $\partial\Omega$ is mean convex. Then if $H_{\partial\Omega}$ is the mean curvature function of $\partial\Omega$ with respect to the inner normal, and H is a number such that*

$$-H_{\partial\Omega} < H \leq 1,$$

there exists a graph on Ω of constant mean curvature H and boundary $\partial\Omega$.

Proof. Let us consider the set

$$J = \{H \in [-H_{\partial\Omega}, 1]; \text{the problem (1)–(2) can be solved for } H\}.$$

Since Ω is a domain of a horosphere, the number 1 belongs to J. Corollary 3.2 and Theorem 4.1 show that J is closed.

Finally we prove that J is an open set. For this, if $H \in J$, one would be able to solve the Dirichlet problem in some interval around H. Let Σ be a H-graph with boundary $\partial\Omega$ and $\Sigma = \text{graph}(u)$. Define

$$H : C^{2,\alpha}(\Sigma) \longrightarrow C^{\alpha}(\Sigma)$$

mapping each $v \in C^{2,\alpha}(\Sigma)$ in the mean curvature of the graph defined by v. The map H between both Banach spaces has as its differential the linearized operator of the mean curvature:

$$L = dH = \Delta + |\sigma|^2 - n.$$

We work in the Minkowski model. The kernel of this operator is trivial because

$$L\langle N, a\rangle = -n(H\langle p, a\rangle + \langle N, a\rangle) \geq 0, \quad \text{and} \quad \langle N, a\rangle < 0$$

The first inequality is a consequence from (19), in the case $0 < H < 1$, and is trivially true when $H \leq 0$ (recall that $\langle p, a\rangle \geq 0$ and $\langle N, a\rangle < 0$). Hence L is a Fredholm operator of index zero and L is a isomorphism. The implicit function theorem assures the Dirichlet problem can be solved around the value H. □

As we have pointed out in the Introduction, Theorem 5.1 was showed in [12] for $0 < H < 1$. However it is worthwhile to remark the geometric sense of our proof. In contrast to [12], we begin the continuity method with the given geometric solution of (1)–(2) for $H = 1$: the very domain $\Omega \subset L(c)$. After this and thanks to Corollary 3.2 and Theorem 4.1, we can ”blow up” from the 1-graph Ω until we reach the minimal solution $H = 0$ and, after this, until $H > -H_{\partial\Omega}$.

Finally, as a corollary of Theorem 5.1, we will give a certain uniqueness result for embedded constant mean curvature hypersurfaces with boundary in a horosphere that generalizes, in some sense, another one that appears in [12]. Let Σ be an embedded compact hypersurface of constant mean curvature H and with boundary in a horosphere. Let us consider the upper halfspace model for $\mathbb{H}^{n+1}$, $L(c) = \{x_{n+1} = c\}$ the horosphere containing $\partial\Sigma$ and Ω the bounded domain by $\partial\Sigma$ in $L(c)$. Assume that Σ is included in the "interior" domain $L(c)^+$ (for instance, if $H^2 \leq 1$, this inclusion is assured by Proposition 2.2). Then $\Sigma \cup \Omega$ divides $L(c)^+$ in two domains, and one of them is bounded, say W. Since Σ is an oriented hypersurface, its unit normal field N points towards W or $\mathbb{H}^{n+1} - W$ in the whole of Σ. Spruck and Nelli [12] have proved that if $0 < H < 1$ and N points to $\mathbb{H}^{n+1} - W$, then Σ is the graph given by Theorem 5.1 . With these considerations and with stronger hypothesis on Σ, we extend this result for negative values of H. For this, we need the next definition. Let Ω be a compact domain in $L(c)$ and $(p, 0) \in \mathbb{R}^n \times \{0\}$. We call the *vertical hyperbolic cylinder* determined by Ω and $(p, 0)$ the set given by

$$\cup_{s>0} T_s(\Omega),$$

where T_s denotes the hyperbolic translation from the point $(p, 0)$, i.e., T_s is the Euclidean homothety of Ω from $(p, 0)$ with ratio e^s (see Corollary 2.4). With this definition, we prove the following uniqueness result:

Corollary 5.2 *Let $\Omega \subset L(c)$ be a mean convex compact domain star-shaped with respect to some interior point (p, c) of Ω and let H be a number such that*

$$-H_{\partial\Omega} < H \leq 0.$$

Let Σ be a compact embedded hypersurface with constant mean curvature H that lies in $L(c)^+$ and included in the vertical hyperbolic cylinder determined by Ω and the point $(p, 0)$. If its Gauss map N points towards the exterior of the bounded domain determined by $\Sigma \cup \Omega$, then Σ is the graph obtained in Theorem 5.1.

Proof. Let G be the H-graph bounded by $\partial\Omega$ given by Theorem 5.1. Notice that the orientation on G is that one pointing upwards. Also, denote by W the bounded domain in $\mathbb{H}^{n+1}$ determined by $\Sigma \cup \Omega$. With a similar reasoning as in Corollary 2.4, using hyperbolic translations and since the orientation on Σ points towards $\mathbb{H}^{n+1} - W$, Σ lies below G.

Now we consider the point $(p, c) \in \Omega$ with respect to which Ω is star-shaped. Let $\{h_t; t \geq 0\}$ be the horizontal homotheties from (p, c), where we suppose that h_0 is the identity and $h_t(\Omega) \subset h_s(\Omega)$ if $s < t$. Then we have a foliation of the domain Ω by $\{h_t(\partial\Omega)\}$. Moreover, each domain $\Omega_t := h_t(\Omega)$ is mean convex and with mean curvature $H_{\partial\Omega_t}$ satisfying

$$0 < H_{\partial\Omega} < H_{\partial\Omega_t} \qquad t > 0. \tag{22}$$

From (22), let G_t be the H-graph on Ω_t with boundary $\partial\Omega_t$ whose existence is assured by Theorem 5.1. From the monotonocity and since Ω_t converges to (p, c), if t is big enough, G_t is included in the domain W. Now we let $t \to 0$

until touching Σ a first time t_0. If $t_0 > 0$, this happens between G_{t_0} and Σ at some interior point q because

$$\partial G_{t_0} = \partial \Omega_{t_0} \subset \mathrm{int}(\Omega).$$

But the tangency principle gives a contradiction: if N_{t_0} denotes the unit normal field on G_{t_0}, then $N(q) = N_{t_0}(q)$ because they point towards $\mathbb{H}^{n+1} - W$. Therefore t_0 must be 0 and so, Σ lies above G. This fact and the first part of the proof imply that G and Σ agree. □

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