Constant Mean Curvature Surfaces with Boundary in Hyperbolic Space

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Abstract. We study constant mean curvature compact surfaces immersed in hyperbolic space with non-empty boundary ($= H$-surfaces). We prove that the only H-surfaces with boundary circular and $0 \leqslant |H| \leqslant 1$, are the umbilical examples. When the surface is embedded, conditions to be umbilical are given. Finally, we characterize umbilical surfaces bounded by a circle among all H-discs with small area.

1. Introduction and Statements of Results

In this paper we ask for *compact* smooth surfaces *immersed* in hyperbolic space $\mathbb{H}^3$ with non-empty boundary and constant mean curvature H. We shall refer to them as *H-surfaces*. For example, if the boundary is a circle, the situation is very different than the Euclidean case: there is a family of compact umbilical surfaces with constant mean curvature H, where H starts from zero, grows until the maximum of the curvature of the circle (larger than one) and decreases towards one. It is natural to ask whether the umbilical surfaces are the only possible examples with boundary a circle. In this sense, the next two conjectures remain unsolved:

Conjecture 1. An embedded compact constant mean curvature surface in $\mathbb{H}^3$ with boundary a circle must be an umbilical surface.

Conjecture 2. An immersed constant mean curvature disc in $\mathbb{H}^3$ with boundary a circle must be an umbilical surface.

Recently, partial answers have been obtained in [5] and [17]. With respect to the Conjecture 2, it is proved in [2] that the umbilical surfaces are the only constant mean curvature *stable* discs immersed in $\mathbb{H}^3$ bounded by a circle: stable means that the second variation of the area is positive semidefinite for all compactly supported volume preserving variations. In the other hand, results on existence of H-graphs on domains of horospheres or geodesic planes have been done in [5], [16] and [18].

In this section, we fix some notations and we recall some properties about hyperbolic space. Let us consider the upper halfspace model of the hyperbolic three-space

$$\mathbb{H}^3 =: \mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3; z > 0\}$$

equipped with the metric

$$\langle , \rangle = ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

The hyperbolic space $\mathbb{H}^3$ has a natural compactification $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$, where $\partial_\infty \mathbb{H}^3$ can be identified with asymptotic classes of geodesics rays in $\mathbb{H}^3$. In the halfspace model of $\mathbb{H}^3, \partial_\infty \mathbb{H}^3 = \{z = 0\} \cup \{\infty\}$ is the one-point compactification of the xy-plane $\{z = 0\}$. In the following, any metric concept is refereed to the hyperbolic geometry: circle, distance, length, area,... whereas any Euclidean reference will be explicit.

We shall describe the complete umbilical surfaces of hyperbolic space. In particular, these surfaces have constant mean curvature since the second fundamental form is a constant multiple of its metric.

1. *Totally geodesic surfaces or geodesic planes.* In our model of hyperbolic space, they identify as vertical Euclidean planes and Euclidean hemispheres that orthogonally intersect the xy-plane. The mean curvature is zero. After an isometry of $\mathbb{H}^3$, any geodesic plane can be considered as

$$P(a) = \{(x, y, z) \in \mathbb{R}^3_+; x^2 + y^2 + z^2 = a^2\},$$

where $a > 0$.

2. *Equidistant spheres.* These are surfaces that equidist from a totally geodesic surface. In the upper halfspace model, they are tilted Euclidean planes transverse to the xy-plane and Euclidean spherical caps included in $\mathbb{R}^3_+$ that are not hemispheres and whose boundary is an Euclidean circle in the xy-plane. The mean curvature H satisfies $0 < |H| < 1$. So, if the Euclidean centre lies at $\mathbb{R}^3_+$, the mean curvature is positive when the orientation points inside. If the Euclidean centre lies in $\{z < 0\}$, then H is positive with respect to the orientation pointing outside.

3. *Horospheres.* Intrinsically, a horosphere is a geodesic sphere whose centre lies at infinity. In our model, these are horizontal planes $L(a) = \{z = a\}$ for each $a > 0$ and Euclidean spheres of $\mathbb{R}^3_+$ tangent to the xy-plane. The mean curvature H satisfies $|H| = 1$. If the horosphere writes as $L(a)$, H is positive for the orientation that points upward. In second case, H is positive if the orientation points inward.

4. *Hyperbolic spheres.* A hyperbolic sphere is the set of points that equidist from a fix point called centre. In our model of $\mathbb{H}^3$, they are Euclidean spheres which lie completely included in $\mathbb{R}^3_+$. The mean curvature satisfies $|H| > 1$ and H is positive when the chosen orientation points inward. If $\rho > 0$ denotes the hyperbolic radius of the sphere, the mean curvature is given by $H = \coth \rho$.

Among the isometries of $\mathbb{H}^3$, we emphasize two kinds associated with each point $p_0 \in \partial_\infty \mathbb{H}^3$. The first ones are the *hyperbolic translations* along the geodesic γ that orthogonally meets $\partial_\infty \mathbb{H}^3$ at the point p_0. From the Euclidean viewpoint, a hyperbolic translation is an Euclidean homothety centred at p_0. Every geodesic γ

uniquely determines a one-parameter group of hyperbolic translations where the Killing field extends the velocity vector field of γ. So, if we consider the geodesic $\gamma(t) = (0, 0, t)$, the associated hyperbolic translations are given by

$$h_t : (x, y, x) \mapsto t(x, y, z) \quad t > 0. \tag{1}$$

The other type of isometries are the *hyperbolic reflections*. Consider P a geodesic plane represented by an orthogonal hemisphere of $\mathbb{R}^3_+$ to the xy-plane and centred at p_0. Hyperbolic reflections with respect to P are Euclidean inversions centred at p_0 that fix the plane P. When P is a vertical plane, the corresponding hyperbolic reflections are Euclidean reflections with respect to P.

At this moment, let us make precise the concept of H-surface with boundary. Let $\phi : \Sigma \to \mathbb{H}^3$ be an isometric immersion of a smooth compact surface Σ with boundary $\partial\Sigma \neq \emptyset$. Let Γ be a one-dimensional submanifold of $\mathbb{H}^3$. We say that Γ is the *boundary* of ϕ if ϕ maps diffeomorphically $\partial\Sigma$ onto Γ. We say that Σ is an H-*surface* if ϕ has constant mean curvature H, where $H \in \mathbb{R}$; if Γ is the boundary of ϕ, we will say that Σ is a H-surface with boundary Γ. We shall identify Σ with its image by ϕ and $\partial\Sigma$ with the curve Γ.

It may be worthwhile to say a few words about the proofs. The methods used in proving our results are essentially the *tangency principle* and a *flux formula*. Both techniques appear in the study of constant mean curvature surfaces. The tangency principle is based on the classical maximum principle for elliptic equations. Let Σ_1, Σ_2 be two surfaces in $\mathbb{H}^3$ and $p \in \Sigma_1 \cap \Sigma_2$ a common tangency point; i.e. $T_p\Sigma_1 = T_p\Sigma_2$ (if $p \in \partial\Sigma_1 \cap \partial\Sigma_2$, we assume also $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$). Let $U \subset T_p\Sigma_1 \cap T_p\Sigma_2$ be a neighbourhood of the origin and $f_1, f_2 : U \to \mathbb{R}$ smooth functions whose graphs are neighbourhoods of p on Σ_1 and Σ_2 respectively. We say that Σ_1 *is above* Σ_2 on U if $f_2 \leqslant f_1$ on U. Locally, an H-surface in $\mathbb{H}^3$ satisfies a second order quasilinear elliptic partial differential equation for which one can apply the Hopf maximum principle ([9]). We can state the following version of the maximum principle for H-surfaces (for instance, see [7] for details):

Proposition 1.1 (**Tangency Principle**) *Let Σ_1 and Σ_2 be oriented surfaces in $\mathbb{H}^3$ of mean curvature $H_1 \leqslant H_2$. If Σ_1 and Σ_2 have a point p of common tangency, either in the interior or in the (analytic) boundary, and Σ_1 lies above Σ_2 near p, then Σ_1 must coincide with Σ_2 in a neighbourhood of p*

We need also a certain "flux formula" which appears in the theory of constant mean curvature surfaces in $\mathbb{R}^3$ ([12],[13]). The analogous formula holds in the hyperbolic setting and it can be stated as follows (for embedded H-surfaces, see [11] and [17]):

Lemma 1.2 (**Flux Formula**) *Let Σ be a H-surface immersed in $\mathbb{H}^3$ with boundary Γ. If Y is a Killing vector field in $\mathbb{H}^3$, then*

$$\int_\Gamma \langle N \wedge \alpha', Y \rangle \, ds = -H \int_\Gamma \langle \alpha \wedge \alpha', Y \rangle \, ds, \tag{2}$$

where $\alpha = \alpha(s)$ is a parameterization of Γ and N is the corresponding Gauss map of Σ given by H.

If Γ lies in some geodesic plane P, and Σ is embedded, equation (2) can be written as

$$\int_{\Gamma} \langle \nu, Y \rangle \, ds = 2H \int_{\Omega} \langle \eta_{\Omega}, Y \rangle \, d\Omega, \tag{3}$$

where ν is the inner conormal along Γ, Ω is the bounded domain by Γ on P and η_{Ω} is the unit normal vector to Ω according the orientation of the cycle $\Sigma \cap \Omega$ that agrees with N on Σ. As application of the formula (3), we consider the case that the boundary is circular. Let Γ be a circle of radius ρ. After an isometry of $\mathbb{H}^3$, the centre of Γ lies in the z-axis and $\Gamma \subset P(a)$ for some $a > 0$. If we choose the Killing vector field defined by $Y(p) = p$, we have by (3)

$$2|H|\pi \sinh^2 \rho \leqslant 2\pi \sinh \rho \cosh \rho \Rightarrow |H| \leqslant \coth \rho. \tag{4}$$

The inequality (4) gets a necessary condition on the possible values of the mean curvature H for H-surfaces bounded by a circle of radius $\rho > 0$. The analogous result in the Euclidean ambient is given by HEINZ [8] and it states that $|H| \leqslant 1/r$ for any H-surface in $\mathbb{R}^3$ with boundary a circle of radius r.

Remark: A consequence of the flux formula is that given a number $H \in \mathbb{R}$ and a Jordan curve Γ, for any H-surface bounded by Γ, the number $|\int_{\Gamma} \langle \nu, Y \rangle|$ does not depend on Σ but only on the boundary Γ and the value of the mean curvature H. This number is called the flux of the surface in the direction determined by Y.

Now we summarize the most significant results obtained in this paper. In Section 2, we give a condition on the position of an embedded surface bounded by a circle to be umbilical (Corollary 2.3):

Let $\Sigma \subset \mathbb{H}^3$ be an embedded H-surface bounded by a circle. If Σ does not intersect the exterior of the domain determined by the circle, either in the corresponding boundary geodesic plane or in the boundary horosphere, then Σ is umbilical.

Following ideas from [15], in Section 3 we establish a result about uniqueness of H-surfaces included in solid hyperbolic cylinders and with boundary in a geodesic plane. As a consequence, it is proved one of the main results of this paper (Corollary 3.3) and that gives a partial answer to the conjectures established in the Introduction:

Any H-surface immersed in $\mathbb{H}^3$ bounded by a circle is umbilical provided that $0 \leqslant |H| \leqslant 1$.

This result has been shown independently by BARBOSA and EARP in a different approach (see [5]). For $|H| > 1$, we obtain umbilicity if Σ is included in some hyperbolic cylinder with radius $\operatorname{arc tanh}\left(\frac{1}{|H|}\right)$ (Theorem 3.5).

In Section 4 we show a uniqueness result so that an embedded H-surfaces with $|H| \leqslant 1$ and with boundary a convex curve is included in a horosphere. Finally, Section 5 is concerned with the study of constant mean curvature discs bounded by a circle Γ. For each $H > 1$, the two hyperbolic spherical caps (small and big one) determined by Γ in a hyperbolic sphere of radius $\rho = \operatorname{arc tanh}\left(\frac{1}{|H|}\right)$ are H-surfaces

bounded by Γ. We give the next characterization of hyperbolic spherical caps (Theorem 5.1):

Spherical caps are the only H-discs in $\mathbb{H}^3$, $|H| > 1$, bounded by a circle and with area less than the area of the big spherical cap with the same boundary and mean curvature H.

This result implies that the Conjecture 2 is true provided that the area of the *H*-disc is small in some sense. Finally, many of our results remain true in arbitrary dimension but, by simplicity in the exposure, we only have considered the hyperbolic three-dimensional space.

2. Embedded *H*-Surfaces with Planar Boundary

Let us consider an embedded *H*-surface in $\mathbb{H}^3$ with boundary. It is natural to know whether or not the surface inherits the symmetries of its boundary. In this sense, the most important tool is the ALEXANDROV reflection method [1]. Alexandrov used the tangency principle to compare a constant mean curvature closed surface of $\mathbb{H}^3$ with itself by a reflection process with respect to a one-parameter family of geodesic planes. As conclusion, the surface has symmetries with respect to each direction of $\mathbb{H}^3$ and hence, it must be round. The same technique can carry to the non-empty boundary case. So, when the surface Σ is bounded by a circle contained in a geodesic plane we have:

Proposition 2.1 *Let $\Sigma \subset \mathbb{H}^3$ be an embedded H-surface bounded by a circle Γ and let P be the geodesic plane containing Γ. If Σ lies in one of the two halfspaces determined by the boundary geodesic plane P, then Σ is an umbilical surface.*

It can see a detailed proof in [17]. The same statement of Proposition 2.1 holds if the circle is included in some horosphere *P* and the surface lies in one of the two halfspaces determined by *P*. In this case, after an isometry of $\mathbb{H}^3$, we can consider the horosphere *P* as $L(a)$, for some $a > 0$ and that $\Sigma \subset \{z \geqslant a\}$. We apply the Alexandrov method by vertical geodesic planes. Then it is not difficult to conclude that Σ is a surface of revolution and, since Σ is compact, that Σ is umbilical.

Therefore, if the surface is embedded it is important to assure that it lies in one of the two sides determined by the boundary geodesic plane or the boundary horosphere. As the Euclidean case (see [6]), if the boundary is convex and the surface is transverse to the boundary geodesic plane along the boundary of the surface, then it is included in a halfspace [17]. Other different situation is the following (see [10] for the Euclidean case):

Theorem 2.2 *Let Γ be a Jordan curve included in Π, where Π is a geodesic plane, an equidistant sphere or a horosphere of $\mathbb{H}^3$. Let Ω be the bounded domain by Γ in Π. If Σ is an embedded H-surface with boundary Γ and $\Sigma \cap \text{ext}(\Omega) = \emptyset$, then $\Sigma = \Omega \cup \Gamma$ or $\Sigma \cap \text{int}(\Omega) = \emptyset$ (ext and int mean the exterior and interior of Ω as subsets of Π). As conclusion, Σ is umbilical or it is included in one of the two halfspaces of $\mathbb{H}^3$ determined by Π.*

Proof: After a hyperbolic motion of $\mathbb{H}^3$, we can assume that

1. If Π is a geodesic plane, then $\Pi = P(a)$ for some $a > 0$.

2. If Π is an equidistant sphere, then Π is a spherical cap in $\mathbb{R}^3_+$ centred at $\{z > 0\}$ and whose boundary lies in the xy-plane.

3. If Π is a horosphere, $\Pi = L(a)$ for some $a > 0$.

Let us consider the foliation of $\mathbb{H}^3$ given by hyperbolic translations of Π, i.e., homotheties from the origin of $\mathbb{R}^3$ (see (1)):

$$\{\Pi(t) = h_t(\Pi); t > 0\}.$$

We call I the halfspace determined by Π defined by

$$I = \bigcup_{t > 1} \Pi(t).$$

Let us orient Π by a unit normal field so that Π has non negative mean curvature $h \geqslant 0$ (in the case that Π is a geodesic plane, we choose the orientation that points towards I). As we notice in Section 1, the mean curvature h satisfies $0 \leqslant h \leqslant 1$. Let us choose in each $\Pi(t)$ the orientation determined by the non-negative mean curvature.

To prove Theorem 2.2, let us assume that $\Sigma \cap \mathrm{int}\,(\Omega) \neq \varnothing$. Define $f : \Sigma \to \mathbb{R}$ by $f(p) = t$ where $p \in \Pi(t)$. Since Σ is compact, there exist $p_0, p_1 \in \Sigma$ such that f attains its maximum and minimum respectively. Let $t_0 = f(p_0)$ and $t_1 = f(p_1)$. Then $t_1 \leqslant 1 \leqslant t_0$. If $t_0 = t_1$, we have $\Sigma = \Omega \cup \Gamma$ and this proves the theorem. In another case, we will obtain a contradiction. Without loss of generality, we can assume that $t_0 > 1$ and $p_0, p_1 \notin \partial\Sigma$. Since Σ is compact, let S be an Euclidean sphere in $\mathbb{R}^3_+$ of sufficient big radius such that $\Sigma \subset B$, where B is the Euclidean ball bounded by S. Let $S^+ = \bigcup_{t \geqslant 1}(S \cap \Pi(t))$. Then

$$T = S^+ \cup K \cup \Sigma$$

is a closed embedded surface (not smooth on $\Gamma \cup \partial S$), where $K \subset \Pi$ is the domain bounded by the circle $S \cap \Pi$ and the curve Γ. Therefore T defines an interior domain W. Let us orient Σ by the unit normal field N that points towards I.

The surfaces $\Pi(t_0)$ and Σ are tangent at p_0 and the respective orientations agree at p_0. Therefore we can compare both surfaces at this point to conclude that $H \leqslant h$. If we have $H = h$, the tangency principle yields that Σ is included in Π. This is a contradiction because $p_0 \notin \Omega$. As conclusion, $H < h$. Now we compare Σ with $\Pi(t_1)$ at the point p_1. Since $N(p_1)$ points towards W, $N(p_1)$ agrees with the orientation on $\Pi(t_1)$ at p_1. As above, the tangency principle gives $H > h$, getting a contradiction again.   □

Theorem 2.2 has an immediate consequence when the boundary is circular. Given a circle in $\mathbb{H}^3$ there exists a geodesic plane and a horosphere that contains this circle. Proposition 2.1 and the subsequent comment given there allows us to prove that the Conjecture 1 posed in Section 1 is true in the following situation:

Corollary 2.3. *Let $\Sigma \subset \mathbb{H}^3$ be an embedded H-surface bounded by a circle. If Σ does not intersect the exterior of the domain determined by the circle, either in the*

corresponding boundary geodesic plane or in the boundary horosphere, then Σ is umbilical.

3. Uniqueness in Hyperbolic Cylinders

Let γ be a geodesic in $\mathbb{H}^3$. We define the *right hyperbolic cylinder* of radius $\rho > 0$ determined by γ as the set of points that lie at distance ρ with respect to γ. In general, if P is a geodesic plane, $\gamma \subset P$ and γ is a geodesic orthogonal to P, we call *hyperbolic cylinder* determined by Γ with respect to γ the set $C(\Gamma, \gamma)$ of hyperbolic translations of Γ along γ. For instance, let $P(a)$ be a geodesic plane, $\Gamma \subset P(a)$ and γ the geodesic $\gamma(t) = (0, 0, t)$, $t > 0$. The hyperbolic cylinder determined by Γ is given by

$$C(\Gamma, \gamma) = \bigcup_{t>0} h_t(\Gamma).$$

In particular, if Γ is a circle of radius $\rho > 0$ and γ is the geodesic such that Γ is invariant by any rotation that fix pointwise γ, $C(\Gamma, \gamma)$ is the right hyperbolic cylinder of radius ρ. If Ω is the bounded domain by a Jordan curve $\Gamma \subset P$, we call the *solid hyperbolic cylinder* determined by Γ with respect to γ the set $C(\Omega, \gamma)$ of hyperbolic translations of Ω along γ.

We need to give the definition of *Killing graph*. Let Ω be a domain of a geodesic plane P and γ a geodesic orthogonal to γ. A *Killing graph* on Ω associates each point $p \in \Omega$ with a point on the orbit through p of the hyperbolic translations along γ. In the upper halfspace model of $\mathbb{H}^3$, if γ is the z-axis and $P = P(a)$ for some $a > 0$, a Killing graph on Ω is a Euclidean radial graph of Ω from the origin of $\mathbb{R}^3$. If the boundary of the Killing graph lies in the geodesic plane P, the tangency principle (or Theorem 2.2) implies that the graph is included in one of the two halfspaces determined by P in $\mathbb{H}^3$. An easy application of the Hopf maximum principle assures that given $H \in \mathbb{R}$, there exists a unique H-Killing graph on Ω with boundary $\partial\Omega$.

With the purpose to simplify notations and proofs, we will assume in this section that, after an isometry of $\mathbb{H}^3$, the geodesic plane P is $P(1)$ and that the geodesic γ is the z-axis of $\mathbb{R}^3_+$. The uniqueness of H-Killing graphs with the same boundary can generalize when one of both surfaces is an H-Killing graph and the other one is a H-surface included in the solid hyperbolic cylinder determined by the boundary.

Theorem 3.1 *(**Uniqueness on Cylinders**) Let γ be a geodesic orthogonal to a geodesic plane P. Let $\Gamma \subset P$ be a Jordan curve and let $\Omega \subset P$ denote the bounded domain determined by Γ. Let G be a H-Killing graph on Ω with boundary Γ. If Σ is a H-surface in $\mathbb{H}^3$ with boundary Γ and included in $C(\Omega, \gamma)$, then $\Sigma = G$ or $\Sigma = G^*$, where G^* is the hyperbolic reflection of G with respect to P.*

Proof: We can assume that the geodesic plane is $P(1)$. Then the cylinder $C(\Omega, \gamma)$ is given by

$$C(\Omega, \gamma) = \{p \in \mathbb{R}^3_+; p/|p| \in \Omega\},$$

where $|p|$ denotes the Euclidean norm of the point p. Assume that G is over P and let us orient G to have $H \geqslant 0$. This orientation points down. By hyperbolic

translations h_t with respect to γ, we move upward G ($t\to\infty$), and we consider the family of H-Killing graphs on P defined by $\{G_t = h_t(G); t\geqslant 1\}$. Then

$$\partial G_t = h_t(\Gamma) \subset C(\Gamma, \gamma).$$

Since the surface Σ is compact, there exists $t_0 > 1$ such that $G_{t_0} \cap \Sigma = \emptyset$. Letting $t \to 1$, G_{t_0} touches Σ for the first time at some height t_1, $1 \leqslant t_1 < t_0$. If $t_1 > 1$, there is a tangent point between G_{t_1} and Σ. With our choice of orientation on G, the tangency principle assures that $G_{t_1} \subset \Sigma$ or $\Sigma \subset G_{t_1}$. But this is impossible because $\partial G_{t_1} \neq \Gamma$. Thus $t_1 = 1$. In this case, if there exists a tangent (interior or boundary) point, then tangency principle yields $G_1 = G = \Sigma$. In the other case, G is above Σ. With a similar reasoning and taking h_t for $0 < t < 1$ and small t, and subsequently increasing $t \to 1$, we have that $G^* = \Sigma$ (if there is a tangent point) or G^* is below Σ. As conclusion we have that $\Sigma = G$ or $\Sigma = G^*$ or, in another case, Σ is included in the 3-domain enclosed by $G \cup G^*$. In the last case and taking the Killing vector field $Y(p) = p$, we have the following strict inequality between the inner pointing conormal vector ν_Σ to $\nu_G \Sigma$ and G respectively along Γ:

$$|\langle \nu_\Sigma, Y \rangle| < \langle \nu_G, Y \rangle.$$

This is a contradiction with the flux formula (3) (see Remark in Section 1). Therefore the only possibilities are $\Sigma = G$ or $\Sigma = G^*$. $\qquad\qquad\square$

Theorem 3.1 allows us to characterize umbilical surfaces in the family of H-surfaces bounded by a circle and such that $|H| \leqslant 1$. Firstly, we need the next lemma, which it is a direct consequence of the tangency principle when we compare a H-surface with horospheres of $L(a)$-type:

Lemma 3.2. *Let Γ be a Jordan curve and Σ a H-surface in $\mathbb{H}^3$ with boundary Γ such that $0 \leqslant |H| \leqslant 1$. Consider the upper halfspace model of $\mathbb{H}^3$. Then*

$$\min\{z(p); p \in \Sigma\} = \min\{z(p); p \in \Gamma\},$$

where $z(p)$ denotes the third coordinate in $\mathbb{R}^3_+$ of the point p.

Corollary 3.3. *Let $\Gamma \subset \mathbb{H}^3$ be a circle. Then any H-surface immersed in $\mathbb{H}^3$ with boundary Γ is umbilical provided that $0 \leqslant |H| \leqslant 1$.*

Proof: After an isometry, we assume that $\Gamma \subset P(1) \cap L(a)$ for some $a > 0$, i.e., the centre of Γ lies in the z-axis. As consequence of Lemma 3.2, Σ lies over $L(a)$.

Let us first consider the case $H = 0$. Because Σ is compact, there is a large number $t_0 > 1$ such that Σ is below $P(t_0)$. Now we translate $P(t_0)$ by hyperbolic translations along γ until to intersect Σ. If there is a (interior or boundary) tangent point, the tangency principle implies that Σ is included in some umbilical surface and, by analyticity, Σ is umbilical. Otherwise, we can move $P(t_0)$ until the boundary of Σ and then, Σ lies in the bounded domain by $P(1)$ and $L(a)$. In particular, Σ is contained in the solid hyperbolic cylinder with respect to γ determined by Γ. Let $G \subset P(1)$ be the domain bounded by $C(\Gamma, \gamma)$. Now Theorem 3.1 asserts that $\Sigma = G$ or $\Sigma = G^*$, and thus, Σ is umbilical.

Now we consider $0 < |H| \leqslant 1$. Up orientation, we assume that $H > 0$. Let $S \subset \mathbb{H}^3$ be the hyperbolic sphere with centre in the z-axis and with big radius so that the

bounded domain B_S defined by S verifies $\Sigma \subset B_S$. We move S by Euclidean translations in the positive direction of the z-axis until to touch Σ. Since the mean curvature of S is greater than 1, S touches Σ only at boundary points. We call E the (translated) sphere of S that intersects the first time with Σ. Then Σ lies inside B_E and since Γ is a circle, $E \cap \Sigma = \Gamma$. Furthermore, the curve Γ is included in the Euclidean lower hemisphere of E.

We consider the hyperbolic cylinder $C(\Gamma, \gamma)$. We have two possibilities about the intersection $C(\Gamma, \gamma) \cap E$. This intersection is formed by two different circles, one of them is Γ or, the intersection agrees with the circle Γ. In the first case, let Γ_1 be the other component of $C(\Gamma, \gamma) \cap E$. Notice that Γ and Γ_1 lie in Euclidean planes parallel to the xy-plane. Without loss of generality, assume that Γ_1 is above Γ with respect to the positive direction of the z-axis. Let us remark that $C(\Gamma, \gamma) = C(\Gamma_1, \gamma)$. Take the hyperbolic cylinder C with respect to γ and tangent to E. Consider Γ_2 is the intersection circle between E and the cylinder C. Then $C = C(\Gamma_2, \gamma)$ and

$$\Sigma \subset B_E \subset C(\Omega, \gamma),$$

where Ω is the bounded domain by Γ_2 in the geodesic plane that contains Γ_2. If $\Gamma_2 \subset L(a_2)$, let us consider the Killing graph on P defined by

$$G = E \cap \bigcup_{t \geqslant a_2} L(t).$$

The mean curvature of G is greater than 1 with the inward orientation and its boundary is the circle Γ_2.

Let us move G by hyperbolic translations h_t and letting $t \to 0$ until to reach Γ_1 at the first time $t = t_0$. Now, if $p \in h_{t_0}(G) \cap \Sigma$, then p is the image by h_{t_0} from some point $q \in \Gamma_1$. Exactly $h_{t_0}(\Gamma_1) = \Gamma$. Thus Σ lies included in the solid hyperbolic cylinder with respect to γ determined by Γ. On the other hand, if $\Gamma_1 \subset L(a_1)$ for some $a_1 > 0$, the set $J = E \cap \bigcup_{t \geqslant a_1} L(t)$ is a Killing graph on P. Moreover, $h_{t_0}(J)$ is a Killing graph with boundary Γ and included in $C(\Gamma_1, \gamma)$. By using Theorem 3.1, $h_{t_0}(J)$ agrees with Σ. In particular, Σ is umbilical since J is certainly umbilical.

In the case that $C(\Gamma, \gamma) \cap E = \Gamma$, $C(\Gamma, \gamma)$ is tangent to E along Γ. Moreover, and with the same notation, the cylinder $C(\Omega, \gamma)$ is the solid hyperbolic cylinder defined by Γ, $G \subset C(\Omega, \gamma)$ is a Killing graph on P and $\partial G = \Gamma$. Again, we conclude the proof using Theorem 3.1.   □

A similar reasoning can carry with a H-surface that is included inside of some ball of radius $\operatorname{arc tanh}\left(\frac{1}{|H|}\right)$ and such that the mean curvature satisfies $|H| > 1$.

Corollary 3.4. *Let Γ be a circle and Σ an H-surface in $\mathbb{H}^3$ with boundary Γ such that $|H| > 1$. If Σ is included in some hyperbolic ball determined by a H-hyperbolic sphere S, then Σ is umbilical.*

Proof: By an isometry, we can assume $\Gamma \subset P(1)$ and is centred on the z-axis. Firstly, let us move S by any horizontal translation and in the upward direction by hyperbolic translations with respect to γ. If there is a tangent point, this implies that Σ is umbilical. On the contrary case, one concludes that Σ is included in the ball defined by a H-sphere E, $E \cap \Sigma = \Gamma$ and such that Γ lies in the Euclidean lower

hemisphere of E. Hence, the subsequent arguments are similar as in Corollary 3.3 concluding that the surface is umbilical. □

Now let us consider the right hyperbolic cylinder of radius $\rho > 0$ with respect to the geodesic $\gamma(t) = (0, 0, t)$, $t > 0$. This cylinder is given by

$$C(\rho) = \{(x, y, z) \in \mathbb{R}_+^3; x^2 + y^2 = (\sinh^2 \rho)z^2\}$$

and the solid hyperbolic cylinder that determines is defined by

$$C(\rho)^* = \{(x, y, z) \in \mathbb{R}_+^3; x^2 + y^2 \leqslant (\sinh^2 \rho)z^2\}.$$

The cylinder $C(\rho)$ has mean curvature

$$H(\rho) = \frac{1}{2}(\tanh \rho + \coth \rho) > 1.$$

Let us observe that the map $\rho \mapsto H(\rho)$ is decreasing with respect to ρ. We extend Corollary 3.4 assuming that Σ is included in a solid hyperbolic cylinder.

Theorem 3.5. *Let Γ be a circle of radius ρ. Let Σ be a H-surface in $\mathbb{H}^3$ with boundary Γ and $|H| \geqslant 1$. If Σ lies included in a solid cylinder of radius* arc $\tan h\left(\frac{1}{|H|}\right)$, *then the surface is umbilical.*

Proof: Let $\rho = \operatorname{arc tanh}\left(\frac{1}{|H|}\right)$. After a motion of $\mathbb{H}^3$, we can assume that the solid cylinder writes as $C(\rho)^*$ described above. Let us denote by $S(t, s)$ the Euclidean sphere of radius s centred at the point $(0, 0, t)$, $t > s$, such that $S(t, s)$ is tangent to $C(\rho)$ and it is included in $C(\rho)^*$. We denote this sphere by $S(t)$. The mean curvature of $S(t)$ is t/s with the normal vector field pointing inward. Furthermore

$$\frac{t}{s} = \coth \rho = |H|,$$

i.e. the sphere $S(t)$ is a H-surface.

For each t, let $\Gamma(t) = S(t) \cap C(\rho)$. Define $\alpha(t) > 0$ the number that defines the horosphere $L(\alpha(t))$ such that $\Gamma(t) \subset L(\alpha(t))$. Set the following notation:

$$S^+(t) = S(t) \cap \bigcup_{m \geqslant \alpha(t)} L(m).$$

$$S^-(t) = S(t) \cap \bigcup_{m \leqslant \alpha(t)} L(m).$$

Since Σ is compact, for large t, $S^+(t) \cap \Sigma = \emptyset$. Let us take hyperbolic translations of $S^+(t)$ with respect to γ until to intersect Σ. If there is a tangent point, the tangency principle yields that Σ agrees with $S^+(t)$ in an open set. Thus Σ is umbilical. In the same way, for small t, $S^-(t) \cap \Sigma = \emptyset$ and move up $S^-(t)$ by translations until to touch Σ the first time. If this occurs at some tangent point, we have that Σ is umbilical again.

On the contrary, there exist two positive numbers t_1, t_2, $t_1 < t_2$, such that:

1. $S^-(t_1)$ intersects Σ at some point of Γ and Σ lies included in the Euclidean convex set of $C(\rho)^*$ determined by $S^-(t_1)$. Moreover, $S^-(t_1)$ is included in the Euclidean lower hemisphere of $S(t_1)$.

2. $S^+(t_2)$ intersects Σ at some point of Γ and Σ lies in the Euclidean convex set of $C(\rho)^*$ defined by $S^+(t_2)$. The surface $S^+(t_2)$ contains the Euclidean upper hemisphere of $S(t_2)$.

Following [3], we prove the following

Claim: $S^{-1}(t_1)$ and $S^+(t_2)$ do intersect (this is equivalent to $t_1 \geqslant t_2$).

Proof of the claim: Let P denote the geodesic plane that contains Γ.
 • If P is parallel to the z-axis, then P intersects $S^+(t_2)$ and $S^-(t_1)$ in semicircles of radius r, with $r \geqslant \rho$. If $t_1 < t_2$, this is impossible.
 • If P intersects $S^+(t_2)$ in a full circle, then it is evident that $S^-(t_1)$ intersects also $S^+(t_2)$.
 • We study the last case: P is not parallel to the z-axis and it does not intersect $S^+(t_2)$ neither $S^-(t_1)$ in full circles.

The plane P intersects $S^+(t_2)$ and $S^-(t_1)$ in arcs of circles S_2 and S_1 respectively of radius greater than ρ. If $t_1 < t_2$, we can find a horosphere $L(a)$ splitting $S^+(t_2)$ from $S^-(t_1)$ such that the centre O_2 of S_2 lies above $L(a)$ and the centre O_1 of S_1 lies below.

Let $p_1 = \Gamma \cap S^-(t_1)$ and $p_2 = \Gamma \cap S^+(t_2)$. As the radius of S_1 and S_2 are greater than ρ, the Euclidean halfline starting from p_1 and passing by the centre O of Γ contains to O_1 after O. In the same way, the line joining p_2 with O intersects O_2 after to across O. Thus O_1 lies above O_2, which is false and this case is impossible.

From the claim, we conclude that Σ lies included completely in a hyperbolic ball whose boundary is a H-hyperbolic sphere. Now we apply Corollary 3.4. $\square$

4. Surfaces with Boundary in a Horosphere

In this section we ask for embedded H-surfaces Σ bounded by a Jordan curve Γ included in a horosphere $L(a)$ and with constant mean curvature $H \in [0,1]$. By the tangency principle, we know (Lemma 3.2) that they lie over $L(a)$. Let $\Omega \subset L(a)$ be the bounded domain by Γ. Since Σ is embedded, $\Sigma \cup \Omega$ determines a bounded domain W in $\mathbb{H}^3$. We denote by N the Gauss map given by H. In [18] it has been established that if Ω is mean convex as submanifold of $L(a)$, then for each $H \in [0,1]$ there exists a H-graph on Ω. In this section, by *graph on a horosphere* we mean the following: if Ω is a domain of a horosphere, for each point $p \in \Omega$ we associate a point on the geodesic passing by p orthogonal to the horosphere. So, if we consider a horosphere of the type $L(a)$, a graph on a domain $\Omega \subset L(a)$ is an Euclidean graph on Ω.

The graph whose existence is assured in [18] has positive mean curvature when the orientation points outside W. Moreover, it is proved that this graph is the only embedded H-surface with boundary Γ, $H \in (0,1)$ and such that the orientation points outside W. In this section, we study the case that H is positive and the corresponding Gauss map N points towards W. If H is sufficiently small, we prove

that there is uniqueness in the family of embedded H-surfaces with boundary Γ and whose Gauss map points towards the domain W.

Theorem 4.1. *Let $\Gamma \subset L(a)$ be a Jordan strictly convex curve and let $\Omega \subset L(a)$ be the bounded domain by Γ. Then there is a number $H_0(\Gamma)$ depending only on Γ, $0 < H_0(\Gamma) < 1$, such that for each H with $0 \leqslant |H| < H_0(\Gamma)$, there exists an only embedded H-surface in $\mathbb{H}^3$ with boundary Γ and with the property that the Gauss map points towards the interior of the domain bounded by $\Sigma \cup \Omega$.*

Proof: Let $a = (0, 0, 1)$ and let $(,)$ be the Euclidean metric on $\mathbb{R}^3$. We consider

$$A = \{v \in \mathbb{R}^3; 0 < (v, a) < 1, |v| = 1\}$$

and Q_v any Euclidean plane orthogonal to v. For each $v \in A$, $P_v = Q_v \cap \mathbb{R}^3_+$ is an equidistant sphere if $(v, a) \neq 0$ and P_v is a geodesic plane if $(v, a) = 0$. Moreover, if $\alpha \in (0, \pi/2]$ is the angle between the vectors v and a, the mean curvature of P_v is $(v, a) = \cos(\alpha) \geqslant 0$, where the orientation chosen on P_v comes from the vector field $N(x, y, z) = zv$. For each $\alpha \in (0, \pi/2]$ we define the set

$$A_\alpha = \{v \in \mathbb{R}^3; (v, v) = 1, (v, a) = \cos(\alpha)\}.$$

At each point $p \in \Gamma$ and $v \in A_\alpha$, we put P_v touching Γ at p. Since Γ is convex in $L(a)$, P_v leaves Γ in one of the two halfplanes of $L(a)$ determined by $P_v \cap L(a)$. Let p_v be the intersection point between P_v, the xy-plane and the Euclidean plane spanning by the vectors a and v. Let us define $\Gamma_\alpha = \{p_v; v \in A_\alpha\}$. Let us observe that $\Gamma_{\pi/2} = \Gamma$. Because Γ is strictly convex and by continuity, there exists $\alpha_0 \in (0, \pi/2)$ such that Γ_α is an Euclidean convex curve on the xy-plane for all $\alpha \in [\alpha_0, \pi/2]$. Let

$$H_0(\Gamma) = \cos(\alpha_0).$$

After a horizontal translation, we consider the origin of $\mathbb{R}^3$ inside the domain determined by Γ_{α_0} in the xy-plane. Let Σ_1 and Σ_2 be two H-surfaces with boundary Γ, $|H| \leqslant H_0(\Gamma)$, and such that theirs Gauss maps N_i point towards the bounded domains W_i determined by $\Sigma_i \cup \Omega$, $i = 1, 2$. Let us take all planes P_v where $v \in A_{\alpha_0}$. These planes have mean curvature $H_0(\Gamma)$. Let us move each plane P_v from the infinity towards Σ_i by horizontal translations and following the direction determined by the Euclidean orthogonal projection of v on the xy-plane. The tangency principle gets that each surface Σ_i lies in the Euclidean convex domain $V \subset \mathbb{R}^3_+$ that lies above $L(a)$ and determined by all planes P_v and the horosphere $L(a)$.

Let us consider the set K of all lines joining the origin with each point of Γ and let $W \subset \{z \geqslant a\}$ be the Euclidean convex domain determined by K and $L(a)$. Then for each $i = 1, 2$

$$W_i \subset V \subset W.$$

Let us move Σ_1 by hyperbolic translations h_t with $t \geqslant 1$. For t sufficiently large, $h_t(\Sigma_1) \cap \Sigma_2 = \emptyset$. Notice that W_2 is included in the domain that lies over $L(a)$ and bounded by $h_t(\Sigma_1)$ and the piece of K between Γ and $h_t(\Gamma)$. Now we move the surface Σ_1 by translations h_t until to intersect Σ_2 in some time $t_0 \geqslant 1$. If there exists

a tangent point, then N_1 agrees with N_2 at this point because both orientations point towards V and thus $t_0 = 1$ and $\Sigma_1 = \Sigma_2$. This would conclude the proof.

In another case, we shall obtain a contradiction. In this case, the first intersection point occurs when $t_0 = 1$ and Σ_1 lies strictly above Σ_2 (with respect to the positive z-axis). Doing the same reasoning with Σ_2, there would exist $t_1 > 1$ such that $h_{t_1}(\Sigma_2)$ touches Σ_1 at some interior tangent point. Again, tangency principle assures that $h_{t_1}(\Sigma_2) \subset \Sigma_1$ or $\Sigma_1 \subset h_{t_1}(\Sigma_2)$, which is impossible because $\partial h_{t_1}(\Sigma_2) = h_{t_1}(\Gamma) \neq \Gamma$. $\qquad\square$

5. Discs with Bounded Area

In the study of compact H-surface of $\mathbb{H}^3$ bounded by a circle, Corollary 3.3 characterizes, for $|H| \leqslant 1$, the umbilical surfaces as the only H-surfaces immersed in $\mathbb{H}^3$ bounded by a circle. We consider in this section the case $|H| > 1$, and we prove that the Conjecture 2 is true when the area of the surface is small in some sense. The idea of the proof comes from [14], where MONTIEL and the author established the corresponding Euclidean result.

Let Σ be a H-surface in $\mathbb{H}^3$ with boundary a circle Γ and $|H| > 1$. If the surface is a topological disc, an isoperimetric inequality proved by BARBOSA and DO CARMO in [4] gives the next relation between the length L of Γ and the area A of Σ:

$$L^2 \geqslant 4\pi A - (H^2 - 1)A^2.$$

Moreover, the equality holds if and only if Σ is an umbilical surface. If $\rho > 0$ is the radius of Γ, then $L = 2\pi \sinh \rho$. Thus, the area of Σ verifies

$$(H^2 - 1)A^2 - 4\pi A + 4\pi^2 \sinh^2 \rho \geqslant 0.$$

From this inequality, we have:

$$A \leqslant A_- =: \frac{2\pi}{H^2 - 1}\left(1 - \sqrt{\cosh^2 \rho - h^2 \sinh^2 \rho}\right)$$

or

$$A \geqslant A_+ =: \frac{2\pi}{H^2 - 1}\left(1 + \sqrt{\cosh^2 \rho - H^2 \sinh^2 \rho}\right).$$

Exactly, the numbers A_- and A_+ agree with the areas of the two H-spherical caps bounded by Γ. Moreover, the equalities hold if and only if the surface is umbilical. By (4)

$$\cosh^2 \rho - H^2 \sinh^2 \rho \geqslant 0.$$

Now we are in conditions to prove the following result:

Theorem 5.1. *Let Γ be a circle in $\mathbb{H}^3$ and $H \in \mathbb{R}$ such that $|H| > 1$. Then the umbilical surfaces are the only H-discs immersed in $\mathbb{H}^3$ bounded by a circle Γ and such the area A of the surface verifies $A \leqslant A_+$.*

Proof: After an isometry, we suppose that $\Gamma \subset L(1)$ is a circle of radius ρ. If $A = A_+$ the surface Σ is umbilical [4]. In another case, $A \leqslant A_-$. Since the surface is

a topological disc, the Gauss-Bonnet theorem yields

$$2\pi = \int_\Sigma K\,d\Sigma + \int_\Gamma k_g(s)\,ds,$$

where $k_g(s)$ is the geodesic curvature along Γ. Since $K \leqslant H^2 - 1$ on Σ, we obtain

$$2\pi \leqslant A(H^2 - 1) + \int_\Gamma k_g(s)\,ds \leqslant A_-(H^2 - 1) + \int_\Gamma k_g(s)\,ds,$$

or equivalently,

$$2\pi\sqrt{\cosh^2\rho - H^2\sinh^2\rho} \leqslant \int_\Gamma k_g(s)\,ds. \qquad (5)$$

Squaring (5), using Schwarz inequality and the fact $L = 2\pi\sinh\rho$, we get

$$4\pi^2\left(\cosh^2\rho - H^2\sinh^2\rho\right) \leqslant 2\pi\sinh\rho \int_\Gamma k_g^2(s)\,ds. \qquad (6)$$

Let α be an arc-length parameterization of Γ. As Γ is included in $L(1)$, the geodesic curvature of Γ is determined by

$$k_g = \frac{\langle \nu, a\rangle - \langle \nu, a\rangle}{\sinh^2\rho} + \langle \nu, a\rangle = \frac{-\langle \nu, \alpha\rangle + \cosh^2\rho\langle \nu, a\rangle}{\sinh^2\rho},$$

where ν denotes the inner conformal along Γ. Since the boundary is a circle of radius ρ, we obtain

$$\cosh^2\rho\langle \nu, a\rangle = \langle \nu, \alpha\rangle \pm \sqrt{-\sinh^2\rho\langle \nu, \alpha\rangle^2 + \sinh^2\rho\cosh^2\rho}.$$

Thus

$$k_g = \pm\frac{\sqrt{\cosh^2\rho - \langle \nu, \alpha\rangle^2}}{\sinh\rho}.$$

If we put this identity into (6), we have

$$4\pi^2\left(\cosh^2\rho - H^2\sinh\rho\right) \leqslant 2\pi\sinh\rho \int_\Gamma \frac{\cosh^2\rho - \langle \nu, \alpha\rangle^2}{\sinh^2\rho}\,ds$$

$$= 4\pi^2\cosh^2\rho - \frac{2\pi}{\sinh\rho}\int_\Gamma \langle \nu, \alpha\rangle^2\,ds \qquad (7)$$

For the Killing vector field $Y(p) = p$ and by the virtue of the flux formula (3), we have

$$(2\pi|H|\sinh^2\rho)^2 = \left(\int_\Gamma \langle \nu, \alpha\rangle\,ds\right)^2 \leqslant 2\pi\sinh\rho \int_\Gamma \langle \nu, \alpha\rangle^2\,ds,$$

i.e.

$$2\pi H^2\sinh^3\rho \leqslant \int_\Gamma \langle \nu, \alpha\rangle^2\,ds.$$

By using this inequality into (7), we obtain that the equality holds in (7). Therefore we have identities in (6) and (5). In particular, $A = A_-$. As conclusion, the surface is umbilical. $\qquad\square$

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