On curvature estimates for constant mean curvature surfaces

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Abstract. In this paper we review four curvature estimates for constant mean curvature surfaces in $\mathbb{R}^3$. They are the Schoen Curvature Estimate [**39**], the Choi-Schoen Curvature Estimate [**5**], the One-sided Curvature Estimate by Colding and Minicozzi [**14**] and the Curvature Estimate for CMC Disks by Meeks and Tinaglia [**33**].

Introduction

In the study of the geometry of surfaces in $\mathbb{R}^3$, estimates for the norm of the second fundamental form, $|A|$, are particularly remarkable. In fact, when $|A|$ is bounded the surface cannot bend too sharply and such estimates provide a very satisfying description of its local geometry. If, as usual, we let K_Σ and H denote respectively the Gaussian curvature and the mean curvature of Σ, then the Gauss equation gives that $|A|^2 - H^2 = -2K_\Sigma$. Thus when H is constant, in fact when H^2 is bounded, bounding the norm of the second fundamental form is equivalent to bounding the Gaussian curvature. Thus, we refer to such estimates as curvature estimates.

There are many classical and recent important results in the literature where curvature estimates for CMC surfaces are obtained assuming certain geometric conditions, see for instance [**2, 4, 5, 12, 14, 21, 33, 34, 38, 39, 40, 41, 44, 47**] et al.. In this paper we review four of them. They are the Schoen Curvature Estimate [**39**], the Choi-Schoen Curvature Estimate [**5**], the One-sided Curvature Estimate by Colding and Minicozzi [**14**] and the Curvature Estimate for CMC Disks by Meeks and Tinaglia [**33**].

Throughout this paper, $\mathcal{B}_r(x)$ and $B_r(x)$ will denote, respectively, the geodesic disk and the Euclidean ball of radius r centered at x.

1. Bounding $|A|$

Let Σ be a surface immersed in $\mathbb{R}^3$, not necessarily minimally immersed. Let $x \in \Sigma$ and assume that $\partial\mathcal{B}_R(x) \subset \Sigma\backslash\partial\Sigma$. When $\sup_{\mathcal{B}_R(x)} |A_\Sigma| = 0$, then $\mathcal{B}_R(x)$ is a flat disk of radius R.

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What happens when $R\sup_{\mathcal{B}_R(x)}|A_\Sigma| \leq \delta$?

Let N denote the unit normal to Σ and for any point $y \in \mathcal{B}_R(x)$, let γ_y denote the shortest geodesic connecting x and y. Then we have the following estimate,

$$\mathrm{dist}_{\mathbb{S}^2}(N(x),N(y)) \leq \int_{\gamma_y}|\nabla N| \leq \sup_{\mathcal{B}_R(x)}|\nabla N|\,\mathrm{dist}_\Sigma(x,y) \leq R\sup_{\mathcal{B}_R(x)}|A_\Sigma| \leq \delta$$

If $\delta < \frac{\pi}{2}$, then for any $y \in \mathcal{B}_R(x)$, $\mathrm{dist}_{\mathbb{S}^2}(N(x),N(y)) < \frac{\pi}{2}$ and $\mathcal{B}_R(x)$ is locally graphical over its tangent plane at x. Furthermore, if $\mathcal{B}_R(x)$ is graphical over $T_x\Sigma$, then

$$1+|\nabla u|^2 = \frac{1}{\langle N(x),N(y)\rangle^2} = \frac{1}{\cos^2\mathrm{dist}_{\mathbb{S}^2}(N(x),N(y))} \text{ and}$$

$$|\operatorname{Hess} u| \leq (1+|\nabla u|^2)|A_\Sigma| \leq (1+|\nabla u|^2)\frac{\delta}{R}.$$

Thus, $|\nabla u|^2 = \frac{1}{\cos^2\mathrm{dist}_{\mathbb{S}^2}(N(x),N(y))} - 1$ and a computation shows that if $\delta < \frac{\pi}{4}$ then

$$|\nabla u|^2 \leq 4\delta^2 \text{ and } |\operatorname{Hess} u| \leq (1+|\nabla u|^2)|A_\Sigma| \leq 5\frac{\delta}{R}.$$

In other words, given a point x in a surface Σ, a neighbourhood of x is graphical over the tangent plane of Σ at x. However, the size of such neighbourhood depends on x and, in general, it could be very small. The previous discussion shows that when the norm of the second fundamental form is bounded then the size of such neighbourhood is uniformly bounded from below independently of the point.

We conclude this brief section by proving a standard geometric fact about surfaces with bounded norm of the second fundamental form that will be needed later.

LEMMA 1.1. *Let Σ be a surface in $\mathbb{R}^3$, $p,q \in \Sigma$ and let $\gamma: [0,\lambda] \to \Sigma$ be a geodesic, parametrized by arclength, such that $\gamma(0) = p$ and $\gamma(\lambda) = q$. If for some $\alpha \geq 0$,*

$$\sup_{t\in[0,\lambda]}|A_\Sigma(\gamma(t))| \leq \frac{\alpha}{\lambda}$$

then $|q-p| \geq \lambda(1-\alpha)$.

PROOF. Let $k(t)$ denote the curvature in $\mathbb{R}^3$ of γ at $\gamma(t)$. Then, since γ is a geodesic, for any $t \in [0,\lambda]$

$$|k(t)| \leq |A(\gamma(t))| \leq \frac{\alpha}{\lambda}.$$

Since

$$\left|\frac{d}{dt}\langle\gamma'(t),\gamma'(0)\rangle\right| \leq |k(t)|$$

we have for all $t_0 \in [0,\lambda]$

$$\langle\gamma'(t_0),\gamma'(0)\rangle - 1 = \int_0^{t_0}\frac{d}{dt}\langle\gamma'(t),\gamma'(0)\rangle \implies$$

$$\langle\gamma'(t_0),\gamma'(0)\rangle \geq 1 - \int_0^{t_0}\left|\frac{d}{dt}\langle\gamma'(t),\gamma'(0)\rangle\right| \geq 1 - \int_0^{t_0}|k(\gamma(t))| \geq 1-\alpha\frac{t_0}{\lambda} \geq (1-\alpha).$$

Also

$$\langle\gamma(\lambda),\gamma'(0)\rangle - \langle\gamma(0),\gamma'(0)\rangle = \int_0^\lambda \frac{d}{dt}\langle\gamma(t),\gamma'(0)\rangle \implies$$
$$\langle q-p,\gamma'(0)\rangle = \langle\gamma(\lambda)-\gamma(0),\gamma'(0)\rangle \geq \lambda(1-\alpha).$$

This implies that $|q-p| \geq \lambda(1-\alpha)$. □

2. Schoen Curvature Estimate

In this section we review the Schoen Curvature Estimate.

THEOREM 2.1 (Schoen). *There exists a constant C such that the following holds. Let $\Sigma \subset \mathbb{R}^3$ be an orientable stable minimal surface. Take $x \in \Sigma$ and assume that $\mathcal{B}_R(x) \subset \Sigma$, $\partial\mathcal{B}_R(x) \subset \Sigma\backslash\partial\Sigma$. Then,*

$$\sup_{\mathcal{B}_{\frac{R}{2}}(x)} |A_\Sigma|^2 \leq CR^{-2}.$$

This estimate says that if a minimal surface is stable, then sufficiently away from the boundary the surface looks "nice" on a fixed scale; here the word "nice" refers to the discussion in Section 1.

In [**37**], Ros generalizes the Schoen Curvature Estimate by removing orientable from the hypotheses. In [**2**] Bérard and Hauswirth extend Schoen's work to stable, constant mean curvature surfaces with trivial normal bundle in space forms. In [**47**] Zhang generalizes the Schoen Curvature Estimate to stable, constant mean curvature surfaces (with trivial normal bundle) in a general 3-manifold. His estimate depends on the mean curvature, an upper bound on the sectional curvature, and on the covariant derivative of the curvature tensor of the ambient manifold (see also [**8**]). In [**38**] Rosenberg, Souam and Toubiana further generalize the Schoen Curvature Estimate to stable, constant mean curvature surfaces in a 3-manifolds assuming only a bound on the sectional curvature of the ambient manifold.

The Schoen Curvature Estimate can be used to prove the following characterization of the plane that was proven independently by Do Carmo and Peng [**18**], Fisher-Colbrie and Schoen [**20**] and Pogorelov [**36**].

THEOREM 2.2. *The plane is the only complete and stable orientable minimal surface immersed in $\mathbb{R}^3$.*

PROOF. Let Σ be a complete and stable orientable minimal surface immersed in $\mathbb{R}^3$ and let $\widetilde{\Sigma}$ be its universal cover that is also stable. Letting R go to infinity in the Schoen Curvature Estimate we obtain that the norm of the second fundamental form of $\widetilde{\Sigma}$ is identically zero and thus Σ is a plane. □

Let us begin by recalling the definition of stability. Let Σ be an orientable minimal surface. Such surface Σ is said to be *stable* if it minimizes area up to second order, that is if for any smooth normal variation F_t with compact support

$$\frac{d^2}{dt^2}(\text{Area}\, F_t(\Sigma))\mid_{t=0} \geq 0.$$

This condition is equivalent to the following:

DEFINITION 2.3. An orientable minimal surface Σ is stable if for any $\phi \in C_0^\infty(\Sigma)$

$$\int_\Sigma |A_\Sigma|^2\phi^2 \leq \int_\Sigma |\nabla\phi|^2. \tag{2.1}$$

Equation (2.1) is called *stability inequality.*

Thus, assuming some integral information for $|A|$, the Schoen Curvature Estimate gives a point-wise estimate for $|A|$. One of the standard arguments to obtain point-wise estimates from integral information is via the following lemma.

LEMMA 2.4. *Let $u \in L^1(\Sigma)$ and suppose that* $\text{Area}\,(\Sigma)$ *is finite. Then,*

$$\sup_\Sigma |u| = \lim_{p\to\infty} \|u\|_{L^p(\Sigma)}.$$

Here $\sup_\Sigma |u|$ *denotes the essential supremum.*

PROOF. Clearly, $\|u\|_{L^p(\Sigma)} \leq \text{Area}\,(\Sigma)^{\frac{1}{p}} \sup_\Sigma |u|$, and thus

$$\limsup_{p\to\infty} \|u\|_{L^p(\Sigma)} \leq \sup_\Sigma |u|.$$

On the other hand if $\lambda < \sup_\Sigma |u|$, then $\text{Area}(\Sigma_\lambda := \{p \in \Sigma \mid |u(p)| \geq \lambda\}) > 0$ and $\|u\|_{L^p(\Sigma)} \geq \lambda(\text{Area}\,\Sigma_\lambda)^{\frac{1}{p}}$. Thus $\liminf_{p\to\infty} \|u\|_{L^p(\Sigma)} \geq \lambda$. □

In order to prove the Schoen Curvature Estimate we also need to recall some well-known properties of $|A|$: for a minimal hyper-surface in $\mathbb{R}^n$, $|A|$ satisfies the following partial differential inequality that is known as Simons' inequality [**43**],

$$\Delta|A|^2 \geq -2|A|^4 + 2\left(1 + \frac{2}{n+1}\right)|\nabla|A||^2. \tag{2.2}$$

Before proving the Schoen Curvature Estimate, it is important to note that in [**39**] Schoen proves a lower bound for the conformal factor of Σ, that is considerably more than a curvature estimate, and then a curvature estimate. The proof of the Schoen Curvature Estimate presented here uses a more direct argument; this argument, in fact an even more general version of it, can be found in [**41**]. The strategy is to estimate higher L^p norms of the norm of the second fundamental form using a standard Moser iteration technique and then apply Lemma 2.4.

In order to apply Lemma 2.4 and in other steps of the proof, we will also need the following area bound for stable geodesic disks.

LEMMA 2.5. *Let Σ be a stable minimal surface in $\mathbb{R}^3$. If $\mathcal{B}_s(x) \subset \Sigma$, $\partial\mathcal{B}_s(x) \subset \Sigma\backslash\partial\Sigma$, then*

$$\text{Area}\,\mathcal{B}_s(x) \leq \frac{4}{3}\pi s^2.$$

PROOF. If $\mathcal{B}_r(x)$ is not simply-connected for a certain $r \leq s$ then let $\widetilde{\Sigma}$ be the universal cover of Σ and let $\widetilde{x} \in \widetilde{\Sigma}$ be a lift of x. Then, $\text{Area}\,\mathcal{B}_s(x) \leq \text{Area}\,\mathcal{B}_s(\widetilde{x})$ and since the Gaussian curvature of a minimal surface immersed in $\mathbb{R}^3$ is non-positive, the exponential map is a diffeomorphism and $\mathcal{B}_r(\widetilde{x})$ is simply-connected for any $r \leq s$. Therefore, it suffices to prove the lemma when $\mathcal{B}_r(x)$ is simply-connected for any $r \leq s$.

Let $L(t)$ denote the length of $\partial\mathcal{B}_t(x)$. Then, the first variation of arc length gives

$$\frac{d}{dt}L(t) = \int_{\partial\mathcal{B}_t(0)} k_g$$

where k_g is the geodesic curvature. Thus, using Gauss-Bonnet theorem

$$\frac{dL}{dt}(t) = \int_{\partial\mathcal{B}_t(0)} k_g = 2\pi - \int_{\mathcal{B}_t(x)} K_\Sigma. \tag{2.3}$$

Integrating (2.3) twice, we get

$$\operatorname{Area}\mathcal{B}_s(x) - \pi s^2 = -\int_0^s\int_0^t\int_{\mathcal{B}_\rho(x)} K_\Sigma.$$

Furthermore,

$$\int_0^s\int_0^t\int_{\mathcal{B}_\rho(x)} K_\Sigma = \frac{1}{2}\int_0^s \frac{d^2}{dt^2}(s-t)^2\left(\int_0^t\int_{\mathcal{B}_\rho(x)} K_\Sigma\right) =$$
$$-\frac{1}{2}\int_0^s \frac{d}{dt}(s-t)^2\left(\int_{\mathcal{B}_t(x)} K_\Sigma\right) = \int_0^s\int_{\partial\mathcal{B}_t(x)} K_\Sigma\frac{(s-t)^2}{2}$$

and using the coarea formula,

$$\int_0^s\int_{\partial\mathcal{B}_t(x)} K_\Sigma\frac{(s-t)^2}{2} = \int_{\mathcal{B}_s(x)} K_\Sigma\frac{(s-t)^2}{2}.$$

Recalling that for a minimal surface $-K_\Sigma = \frac{|A|^2}{2}$, so far we have obtained that

$$\operatorname{Area}\mathcal{B}_s(x) - \pi s^2 = \int_{\mathcal{B}_s(x)} |A|^2\frac{(s-r)^2}{4}. \tag{2.4}$$

We are now going to use the stability inequality, that is equation (2.1), to bound the right hand side of equation (2.4). Letting $\phi = \frac{(s-r)}{2}$ in the stability inequality, we obtain

$$\int_{\mathcal{B}_s(x)} |A|^2\frac{(s-r)^2}{4} \leq \int_{\mathcal{B}_s(x)} \frac{1}{4} = \frac{\operatorname{Area}\mathcal{B}_s(x)}{4}.$$

This together with equation (2.4) finishes the proof of the lemma. $\square$

Finally, we recall the following theorem of Schoen-Simon-Yau of which we omit the proof [**41**].

THEOREM 2.6. *Let $\Sigma^{n-1}\subset\mathbb{R}^n$ be an orientable stable minimal hyper-surface. There exist constants $C(n,p)$ such that for all $p\in\left[2, 2+\sqrt{\frac{2}{n-1}}\right)$ and any non-negative Lipschitz function Φ with compact support*

$$\int_\Sigma |A_\Sigma|^{2p}\Phi^{2p} \leq C(n,p)\int_\Sigma |\nabla\Phi|^{2p}.$$

We are now ready to prove the Schoen Curvature Estimate.

Proof of Theorem 2. It suffices to show that $|A|(x)\leq CR^{-2}$ and, by rescaling, it suffices to consider the case $R=1$. Let Σ be a compact minimal surface and Φ be a smooth function on Σ. The Sobolev embedding theorem [**23**] together with Holder inequality says that that for $p\in[1,2)$

$$\left(\int_\Sigma |\Phi|^{\frac{2p}{2-p}}\right)^{\frac{p-2}{2p}} \leq C_p \left(\int_\Sigma |\nabla\Phi|^p\right)^{\frac{1}{p}} \leq C_p \operatorname{Area}(\Sigma)^{\frac{2-p}{2p}} \left(\int_\Sigma |\nabla\Phi|^2\right)^{\frac{1}{2}}. \tag{2.5}$$

Using Holder inequality, raising to the power 2 and letting $r = \frac{p}{2-p}$ we obtain that

$$\left(\int_\Sigma |\Phi|^{2r}\right)^{\frac{1}{r}} \leq C_r \operatorname{Area}(\Sigma)^{\frac{1}{r}} \int_\Sigma |\nabla\Phi|^2 \tag{2.6}$$

for $r \in [1, \infty)$.

Let g be a nonnegative, smooth function on Σ such that

$$\Delta g \geq -2|A|^2 g. \tag{2.7}$$

Multiplying each side of equation (2.7) by $\xi^2 g^{2q-1}$ where ξ is a cut-off function gives

$$\xi^2 g^{2q-1}\Delta g \geq -2|A|^2\xi^2 g^{2q}.$$

Integrating by parts over Σ we get

$$-\int_\Sigma \xi^2 g^{2q-1}\Delta g = \int_\Sigma \nabla(\xi^2 g^{2q-1})\nabla g = \int_\Sigma (\nabla\xi^2)\frac{\nabla g^{2q}}{2q} + (2q-1)\int_\Sigma \xi^2 g^{2q-2}|\nabla g|^2.$$

Thus

$$\int_\Sigma (\nabla\xi^2)\frac{\nabla g^{2q}}{2q} + (2q-1)\int_\Sigma \xi^2 g^{2q-2}|\nabla g|^2 \leq 2\int_\Sigma |A|^2\xi^2 g^{2q}.$$

Computing $|\nabla(\xi g^q)|^2$ we obtain

$$|\nabla(\xi g^q)|^2 = |\nabla\xi|^2 g^{2q} + q^2\xi^2 g^{2q-2}|\nabla g|^2 + \frac{1}{2}\nabla\xi^2\nabla g^{2q}.$$

Thus

$$|\nabla(\xi g^q)|^2 \leq |\nabla\xi|^2 g^{2q} + q\left((2q-1)\xi^2 g^{2q-2}|\nabla g|^2 + \frac{\nabla\xi^2\nabla g^{2q}}{2q}\right).$$

Finally

$$\int_\Sigma |\nabla(\xi g^q)|^2 \leq \int_\Sigma |\nabla\xi|^2 g^{2q} + 2q\int_\Sigma |A|^2\xi^2 g^{2q}.$$

Using equation (2.6) with $\Phi = \xi g^q$ and $r = 3$ we get

$$\begin{aligned}\left(\int_\Sigma \xi^6 g^{6q}\right)^{\frac{1}{3}} \leq & C(\operatorname{Area}\Sigma)^{\frac{1}{3}}\int_\Sigma |\nabla(\xi g^q)|^2 \\ \leq & C(\operatorname{Area}\Sigma)^{\frac{1}{3}}\left(\int_\Sigma |\nabla\xi|^2 g^{2q} + q\int_\Sigma |A|^2\xi^2 g^{2q}\right)\end{aligned}$$

where C is a universal constant. Using Holder inequality with $p = \frac{3}{2}$ and $q = 3$

$$\left(\int_\Sigma |\nabla\xi|^2 g^{2q} + q\int_\Sigma |A|^2\xi^2 g^{2q}\right) \leq (\text{Area}\,\Sigma)^{\frac{1}{3}}\left(\int_\Sigma |\nabla\xi|^3 g^{3q}\right)^{\frac{2}{3}} + q\left(\int_\Sigma \xi^3|A|^6\right)^{\frac{1}{3}}\left(\int_\Sigma \xi^{\frac{3}{2}} g^{3q}\right)^{\frac{2}{3}}.$$

Let ξ be a function which is one in $\mathcal{B}_a(x)$, zero outside $\mathcal{B}_{a+s}(x)$ and $|\nabla\xi| \leq \frac{1}{s}$ then,

$$\left(\int_{\mathcal{B}_a(x)} g^{6q}\right)^{\frac{1}{6q}} \leq \left(C(\text{Area}\,\mathcal{B}_{a+s}(x))^{\frac{1}{3}}\right)^{\frac{1}{2q}} \cdot \left((\text{Area}\,\mathcal{B}_{a+s}(x))^3 s^{-2} + q\left(\int_{\mathcal{B}_{a+s}(x)} |A|^6\right)^{\frac{1}{3}}\right)^{\frac{1}{2q}}\left(\int_{\mathcal{B}_{a+s}(x)} g^{3q}\right)^{\frac{1}{3q}}.$$

Let $q = s^{-2} = 2^i$ and denote by $a_0 = \sum_{i=1}^{\infty} 2^{-\frac{i}{2}}$, $a_1 = \sum_{i=1}^{\infty}\frac{i}{2^i}$. Suppose that $\text{Area}\,\mathcal{B}_{a+a_0}(x)$ and $\int_{\mathcal{B}_{a+a_0}(x)}|A|^6$ are bounded then,

$$\begin{aligned}\left(\int_{\mathcal{B}_a} g^{3(2^{i+1})}\right)^{\frac{1}{3(2^{i+1})}} &\leq C^{\frac{1}{3(2^i)}} 2^{\frac{i+1}{2^{i+1}}}\left(\int_{\mathcal{B}_{a+2^{-\frac{i}{2}}}} g^{3(2^i)}\right)^{\frac{1}{3(2^i)}} \\ &\leq C^{\frac{1}{3(2^i)}+\frac{1}{3(2^{i-1})}} 2^{\frac{i+1}{2^{i+1}}+\frac{i}{2^i}}\left(\int_{\mathcal{B}_{a+2^{-\frac{i}{2}}+2^{-\frac{i-1}{2}}}} g^{3(2^{i-1})}\right)^{\frac{1}{3(2^{i-1})}} \\ &\leq (C)^{a_1}\left(\int_{\mathcal{B}_{a+a_0}} g^3\right)^{\frac{1}{3}}.\end{aligned}$$

In the previous sequence of inequalities, C might not be the same constant but it essentially depends only on $\text{Area}\,\mathcal{B}_{a+a_0}(x)$ and $\int_{\mathcal{B}_{a+a_0}(x)}|A|^6$. Applying Lemma 2.4 then gives that there exists a constant C depending on $\text{Area}\,\mathcal{B}_{a+a_0}(x)$ and $\int_{\mathcal{B}_{a+a_0}(x)}|A|^6$ such that

$$g(x) \leq \sup_{\mathcal{B}_a(x)} g \leq C\left(\int_{\mathcal{B}_{a+a_0}} g^3\right)^{\frac{1}{3}}.$$

If Σ is a minimal surface, then $\Delta|A|^2 \geq -2|A|^4$. Therefore setting g in the previous discussion equal to $|A|^2$ and $\int_{\mathcal{B}_{a+a_0}} g^3 = \int_{\mathcal{B}_{a+a_0}}|A|^6$, we obtain that there exists a constant C depending on $\text{Area}\,\mathcal{B}_{a+a_0}(x)$ and $\int_{\mathcal{B}_{a+a_0}(x)}|A|^6$ such that

$$|A|^2(x) \leq \sup_{\mathcal{B}_a(x)}|A|^2 \leq C\left(\int_{\mathcal{B}_{a+a_0}}|A|^6\right)^{\frac{1}{3}}. \tag{2.8}$$

Thus, after choosing a such that $a + a_0 \leq 1$, the curvature estimate follows from equation (2.8), if we can show that Area $\mathcal{B}_1$ and $\int_{\mathcal{B}_1} |A|^6$ are bounded independently of Σ. In Lemma 2.5 we have provided a bound for Area $\mathcal{B}_1$. Taking $p = 3$ and a standard cut-off function, Theorem 2.6 shows that $\int_{\mathcal{B}_1} |A|^6$ is bounded. This finishes the proof of Theorem 2. □

3. Choi-Schoen Curvature Estimate

The Choi-Schoen Curvature Estimate says that if the total curvature of a geodesic minimal disk is sufficiently small, then the curvature of the disk is bounded in the interior and it decays like the inverse square of the distance of the point to the boundary. Once again, in a sense that has been described in Section 1, it says that if the total curvature is small then there is a fixed scale where the surface looks "nice" or, in other words, wherever the surface does not look "nice" there must be a gain in total curvature. Note however that the helicoid has the simplest topology, simply-connected, but it has infinite total curvature.

THEOREM 3.1 (Choi-Schoen Curvature Estimate). *There exists $\varepsilon_1 > 0$ such that the following holds. Let Σ be a surface immersed in $\mathbb{R}^3$, $x \in \Sigma$ and $\mathcal{B}_{r_0}(x) \subset \Sigma$, $\partial\mathcal{B}_{r_0}(x) \subset \Sigma \backslash \partial\Sigma$. If there exists $\delta \in [0, 1]$ such that*

$$\int_{\mathcal{B}_{r_0}(x)} |A_\Sigma|^2 < \delta\varepsilon_1$$

then for all $0 < \sigma \leq r_0$ and $y \in \mathcal{B}_{r_0-\sigma}(x)$

$$\sigma^2 |A_\Sigma|^2(y) \leq \delta.$$

In [**4**] Bourni and Tinaglia generalize the Choi-Schoen Curvature Estimate to surfaces with sufficiently small $W^{1,p}$ norm of the mean curvature and show that such condition is optimal. Among other things, they use a generalized version of Simons' inequality due to Ecker and Huisken [**19**].

We begin by proving certain general results about minimal submanifolds. The first result is a Mean Value Property (see for instance Proposition 1.16 in [**7**]). Let $\mathcal{B}_r$ and B_r denote, respectively, $\mathcal{B}_r(0)$ and $B_r(0)$.

LEMMA 3.2 (Mean Value Property). *Let Σ be a k-dimensional minimal submanifold immersed in $\mathbb{R}^n$ and containing the origin and let f be a non-negative C^1 function on Σ then*

$$\frac{d}{dr}\left(r^{-k}\int_{B_r\cap\Sigma} f\right) = \frac{d}{dr}\int_{B_r\cap\Sigma} f\frac{|x^N|^2}{|x|^{k+2}} + r^{-k-1}\int_{B_r\cap\Sigma} x\cdot\nabla f, \tag{3.1}$$

where x^N denotes the normal component of x, and for $0 < s < t$

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f = \int_{(B_t\backslash B_s)\cap\Sigma} f\frac{|x^N|^2}{|x|^{k+2}} + \int_s^t r^{-k-1}\int_{B_r\cap\Sigma} x\cdot\nabla f. \tag{3.2}$$

Proof. Using the formula

$$\int \operatorname{div}_\Sigma X = -\int X \cdot H = 0$$

with the vector field $X(x) = \gamma(|x|)f(x)$, where $\gamma \in C^1(\mathbb{R})$ is such that, for some $r > 0$, $\gamma(t) = 1$ for $t \leq r/2$, $\gamma(t) = 0$ for $t \geq r$ and $\gamma'(t) \leq 0$, we get

$$\begin{aligned}\frac{d}{dr}\left(r^{-k}\int_{B_r\cap\Sigma}\phi\left(\frac{|x|}{r}\right)f\right) &= r^{-k}\frac{d}{dr}\int_{B_r\cap\Sigma} f\frac{|x^N|^2}{|x|^2}\phi\left(\frac{|x|}{r}\right)\\ &\quad + r^{-k-1}\int_{B_r\cap\Sigma} x\cdot\nabla f\phi\left(\frac{|x|}{r}\right)\end{aligned}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(|x|/r) = \gamma(|x|)$ (cf. equation 18.1 in [**42**]). Then (3.1) follows after letting ϕ in the above formula increase to the characteristic function of $(-\infty, 1)$ and (3.2) follows by integrating (3.1) from s to t. □

Remark 3.3. The first term on the RHS of (3.1) and (3.2) in *Lemma 3.1* are positive. For the second term on the RHS of (3.2) we note that for any C^1 function h on Σ, integration by parts yields

$$\int_s^t r^{-k-1}\int_{B_r\cap\Sigma} h = \frac{1}{k}\int_{B_t\cap\Sigma} h\left(\frac{1}{r_s^k}-\frac{1}{t^k}\right)$$

where $r_s = \max\{|x|, s\}$, and furthermore

$$x\cdot\nabla f = \operatorname{proj}_{T_xM} x\cdot\nabla f = \frac{1}{2}\nabla|x|^2\cdot\nabla f = -\frac{1}{2}\nabla(r^2-|x|^2)\cdot\nabla f.$$

Thus, integrating by parts, we get the following two estimates, as a corollary of *Lemma 3.1*, which we will need later:

$$\frac{d}{dr}\left(r^{-k}\int_{B_r\cap\Sigma} f\right) \geq \frac{1}{2}r^{-k-1}\int_{B_r\cap\Sigma}(r^2-|x|^2)\Delta_\Sigma f \tag{3.3}$$

and

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f \geq \frac{1}{k}\int_{B_t\cap\Sigma} x\cdot\nabla f\left(\frac{1}{r_s^k}-\frac{1}{t^k}\right) \tag{3.4}$$

where $r_s = r_s(x) = \max\{|x|, s\}$.

Using Lemma 3.2 we obtain the following Mean Value Inequality (see for instance Corollary 1.17 in [**7**]).

Lemma 3.4 (Mean Value Inequality). *Let Σ be a minimal hyper-surface immersed in $\mathbb{R}^n$ containing the origin and such that $B_1(0)\cap\partial\Sigma = \emptyset$. Let also f be a non-negative function on Σ such that*

$$\Delta_\Sigma f \geq -\lambda_1 f \tag{3.5}$$

for some $\lambda_1 \geq 0$. Then

$$f(0) \leq \omega_{n-1}^{-1} e^{\frac{\lambda_1}{2}}\int_{B_1\cap\Sigma} f$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Proof. Define

$$g(t) = t^{-(n-1)} \int_{B_t \cap \Sigma} f$$

then using (3.3) and (3.5)

$$g'(t) \geq -\frac{\lambda_1}{2} t g(t) \geq -\frac{\lambda_1}{2} g(t).$$

Hence,

$$g'(t) + \frac{\lambda_1}{2} t g(t) \geq 0 \implies (g(t) e^{\frac{\lambda_1}{2} t})' \geq 0.$$

After integrating from 0 to 1:

$$\omega_{n-1} f(0) = \lim_{t \to 0^+} g(t) \leq e^{\frac{\lambda_1}{2}} g(1) = e^{\frac{\lambda_1}{2}} \int_{B_1 \cap \Sigma} f.$$

□

Using Simons' inequality and the Mean Value Inequality we now prove Theorem 3.1.

Proof of Theorem 3.1. Note first that we can assume that $\delta > 0$, since else the theorem is trivially true. Set $F = (r_0 - r)^2 |A|^2$ on $\mathcal{B}_{r_0}$, where $r(x) = |x|$, and let δ_0 be the maximum value of F and x_0 the point where this maximum is attained. Assume, for a contradiction, that $\delta_0 > \delta$ and pick σ so that

$$\sigma^2 |A|^2(x_0) = \frac{\delta}{4}.$$

Then:

$$2\sigma \leq r_0 - r(x_0) \text{ and } \frac{1}{2} \leq \frac{r_0 - r}{r_0 - r(x_0)} \leq \frac{3}{2}\ , \ \forall x \in \mathcal{B}_\sigma(x_0)$$

and

$$(r_0 - r(x_0))^2 \sup_{\mathcal{B}_\sigma(x_0)} |A|^2 \leq 4F(x_0) \implies \sup_{\mathcal{B}_\sigma(x_0)} |A|^2 \leq 4|A|^2(x_0) = \sigma^{-2}\delta.$$

Let $\widetilde{\Sigma} = \eta_{x_0, \frac{\sigma}{4}}(\mathcal{B}_{r_0})$ (where $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$ that is a rescaling and a translation) and let $\widetilde{A}$ be the second fundamental form of $\widetilde{\Sigma}$. Then

$$\sup_{\mathcal{B}_4 \subset \widetilde{\Sigma}} |\widetilde{A}|^2 \leq \frac{\sigma^2}{16} \sup_{\mathcal{B}_\sigma(x_0) \subset \Sigma} |A|^2 \leq \frac{\delta}{16} < \frac{1}{16} \text{ and } |\widetilde{A}|^2(0) = \frac{\delta}{64}. \tag{3.6}$$

Note that by $\mathcal{B}_r$ we now denote the geodesic balls of radius r in $\widetilde{\Sigma}$ centered at the origin. Let $\widetilde{\Sigma}_0$ be the connected component of $B_1 \cap \mathcal{B}_2$ containing the origin. Then, $\widetilde{\Sigma}_0$ has its boundary contained in ∂B_1, see Lemma 1.1.

In what follows, we focus our analysis on $\widetilde{\Sigma}_0$. Abusing the notations, we omit the tildes and set $\Sigma = \widetilde{\Sigma}_0$. Simons' inequality with $|A| < 1$ implies:

$$\Delta |A|^2 \geq -2|A|^2.$$

Hence the inequality in the assumptions of Lemma 3.4 is satisfied with $\lambda_1 = 2$. Applying Lemma 3.4 we obtain

$$\pi |A(0)|^2 \leq e \left(\int_\Sigma |A|^2 \right) \leq e \delta \varepsilon_0. \tag{3.7}$$

Thus, picking ε_0 sufficiently small, so that $|A(0)|^2 < \frac{\delta}{64}$, contradicts (3.6) and proves the theorem.

Note that in equation (3.7) we have used the fact that the total curvature is rescale invariant. With an abuse of notation, let us reintroduce the tildes to denote the surfaces and quantities obtained after rescaling so that $\widetilde{\Sigma} = \eta_{x_0, \frac{\sigma}{4}}(\mathcal{B}_{r_0})$. Then,

$$\int_{\widetilde{\Sigma}_0} |\widetilde{A}|^2 \leq \int_{\widetilde{\Sigma}} |\widetilde{A}|^2 = \int_{\mathcal{B}_{r_0}} |A|^2 \leq \delta\varepsilon_0.$$

□

4. Colding-Minicozzi One-sided Curvature Estimate

In this section we discuss the Colding-Minicozzi One-sided Curvature Estimate [**14**]. This is one of the main results in Colding-Minicozzi Theory [**6, 11, 12, 13, 14**]. As this very short overview certainly does not do justice to their pioneering work, we refer the reader to the surveys [**10, 15, 16**] written by Colding and Minicozzi.

The One-sided Curvature Estimate says the following.

THEOREM 4.1. *There exists $\varepsilon > 0$ such that the following holds. Let Σ be a minimal disk embedded in $\mathbb{R}^3$ contained in $B_{2R}(0) \cap \{z > 0\}$ with $\partial\Sigma \subset \partial B_{2R}(0)$ and let Σ' be any component of $\Sigma \cap B_R(0)$.*

$$\text{If } \Sigma' \cap B_{\varepsilon R}(0) \neq \emptyset, \text{ then } \sup_{\Sigma'} |A_\Sigma| \leq \frac{1}{R}.$$

In other words, Theorem 4.1 says that if an embedded minimal disk lies on one side of a plane but sufficiently close to it, then the curvature is bounded at interior points.

The One-sided Curvature Estimate has been an extremely useful tool in tackling results that before had seemed unapproachable. For instance, in [**29**] Meeks and Rosenberg prove that

> The plane and the helicoid are the only simply-connected minimal surfaces properly embedded in $\mathbb{R}^3$.

See also [**3**]. In [**17**] Colding and Minicozzi tackle the Calabi-Yau Conjectures and prove that

> A complete minimal surface of finite topology embedded in $\mathbb{R}^3$ is properly embedded.

In [**33**] Meeks and Tinaglia prove a curvature estimate for disks embedded in $\mathbb{R}^3$ with nonzero constant mean curvature, see the next section. They then use this result to show that

> Round spheres are the only complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with nonzero constant mean curvature.

The aforementioned results are just a few instances where the One-sided Curvature Estimate, and more generally Colding-Minicozzi Theory, has been applied.

It is important to mention that Colding and Minicozzi also prove an intrinsic version of the One-sided Curvature Estimate, that is the embedded minimal disk can be replaced by an embedded minimal geodesic disk with no additional hypotheses on the geometry of its boundary [**17**].

THEOREM 4.2. *There exists $\varepsilon > 0$ such that the following holds. Let Σ be a minimal disk embedded in $\mathbb{R}^3$ contained in $\{z > 0\}$. If $\mathcal{B}_{2R}(x) \subset \Sigma \backslash \partial\Sigma$ and $|x| < \varepsilon R$, then*

$$\sup_{\mathcal{B}_R(x)} |A_\Sigma| \leq \frac{1}{R}.$$

Brief idea of the proof. Using the One-sided Curvature Estimate (extrinsic version), Colding and Minicozzi show that on a simply-connected embedded minimal geodesic disk there is a relation between intrinsic and extrinsic distances, that is a so-called chord-arc bound. In other words, while for any two points on a surface the intrinsic distance always bounds the extrinsic distance, in [**17**] they prove that on a simply-connected embedded minimal geodesic disk a somewhat reverse property holds. Thus, given a simply-connected minimal geodesic disk, while a priori there are no information on the boundary of suck disk, in fact its intersection with a sufficiently small extrinsic ball is compact. Finally, one can apply the One-sided Curvature Estimate to such compact intersection to obtain an intrinsic version. □

Moreover, the plane in the One-sided Curvature Estimate can be replaced by a minimal surface. Namely, the following result is also true.

COROLLARY 4.3. There exist $c, \varepsilon > 0$ such that the following holds. Let Σ_1 and Σ_2 be disjoint minimal surfaces embedded in $\mathbb{R}^3$ contained in $B_{cR}(0)$ with $\partial\Sigma_i \subset \partial B_{cR}(0)$ for $i := 1, 2$. Assume also that Σ_1 is a disk, $\Sigma_2 \cap B_{\varepsilon R}(0) \neq \emptyset$, and let Σ' be any component of $\Sigma_1 \cap B_R(0)$.

$$\text{If } \Sigma' \cap B_{\varepsilon R}(0) \neq \emptyset, \text{ then } \sup_{\Sigma'} |A_\Sigma| \leq \frac{1}{R}.$$

PROOF. Under these hypotheses, using a result of Meeks and Yau [**35**] and a linking argument one can show that there exists a stable embedded minimal disk Σ_s such that $\Sigma_s \cap B_{\varepsilon R}(0) \neq \emptyset$ and $\partial\Sigma_s \cap \partial B_{cR}(0)$. Thus, if c is taken sufficiently large then, using the Schoen Curvature Estimate for stable surfaces, see Section 2, Σ_s is rather flat away from the boundary and can play the role of the plane $\{z = 0\}$ in the One-sided Curvature Estimate. This role will be clear once we give an idea of the proof of the curvature estimate. □

Before giving a rough idea of the proof of the One-sided Curvature Estimate, it is interesting to point out how Theorem 4.1 can be thought of an effective version of the Half-space Theorem by Hoffman and Meeks [**22**] for properly embedded minimal disks. The Half-space Theorem is a very beautiful result that certainly concerns a much more general class of complete surfaces but considering it in this less general case might give an intuition on why one expects the One-sided Curvature Estimate to be true.

THEOREM 4.4 (Half-space Theorem). *Let Σ be a minimal surface properly immersed in $\mathbb{R}^3$ and contained in a half-space. Then, Σ must be a flat plane.*

Proof when Σ is properly embedded. Without loss of generality, we can assume $\Sigma \subset \{z > 0\}$. Given a point $p \in \Sigma$ there certainly exists an R_p such that $p \in \Sigma \cap B_{\varepsilon R}(0)$ for any $R > R_p$; here ε is the small value given by the One-sided Curvature Estimate. Since the intersection $\Sigma \cap B_R(0)$ is compact, applying the One-sided Curvature Estimate and letting R go to infinity gives that $|A| \equiv 0$. $\square$

We now sketch some of the basic principles behind the proof of the One-sided Curvature Estimate. First of all, it can be shown that Theorem 4.1 is equivalent to the following theorem, see Figure 4.

THEOREM 4.5. *There exist $C, \lambda, \varepsilon > 0$ such that the following holds. Let Σ be a minimal disk embedded in $\mathbb{R}^3$ contained in $B_\lambda(0) \cap \{z > 0\}$ with $\partial\Sigma \subset \partial B_\lambda(0)$ then*

$$\sup_{\Sigma \cap B_\varepsilon(0)} |A_\Sigma| \leq C.$$

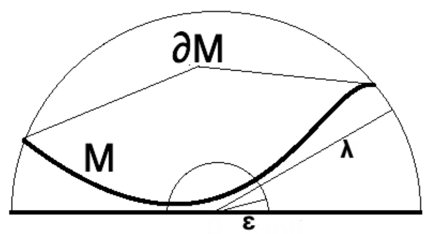

FIGURE 1. One-sided Curvature Estimate

In view of Theorem 4.5, arguing by contradiction, there exists a sequence of embedded minimal disks, Σ_n, contained in a half-space, with boundary going to infinity, $\partial\Sigma_n \subset \partial B_n(0)$, and norm of the second fundamental form arbitrary large at a point p_n converging to the origin, $\sup_{\Sigma_n \cap B_{\frac{1}{n}}(0)} |A| > n$. The strategy to prove the One-sided Curvature Estimate is to show that under such hypotheses there exists a sequence of points in Σ_n where the curvature is large and use that to prove that these points move downwards faster than they move sideways. That being the case, the surface must intersect the plane $\{z = 0\}$ giving a contradiction. For this, Colding and Minicozzi carefully study the structure of an embedded minimal disk at points where the curvature is large [**11, 12, 13, 14**]. In what follows we will give a very brief idea of how they manage to describe the location in space of these points but we will not address their existence. Let p_n^i denote the points of such sequence.

We first introduce the following definition.

DEFINITION 4.6. An N-valued graph over an annulus $D_{r_2} \backslash D_{r_1}$ is a graph over $(r_1, r_2) \times [-N\pi, N\pi] \subset \mathbb{C}^*$ where $\mathbb{C}^*$ is the universal cover of $\mathbb{C}$.

In [**11**] they prove the following.

THEOREM 4.7 (Basic Structure Theorem). *Given $N, \Omega > 0$ there exist C_1, C_2, C_3 such that the following holds. Let Σ be an embedded minimal disk contained in*

$B_{C_2R}(0)$ with $\partial\Sigma \subset \partial B_{C_2R}(0)$. If

$$\sup_{\mathcal{B}_s(0)} |A_\Sigma| \leq 2|A_\Sigma|(0) = 2\frac{C_1}{s}$$

then there exists an N-valued graph $\Sigma_g \subset \Sigma \cap \{z^2 \leq \Omega^2(x^2+y^2)\}$ over $D_R \backslash D_s$ with gradient $< \Omega$ and separation $\geq C_3 s$ over ∂D_s.

In other words, for each point where the curvature is large there exists a highly-sheeted multi-valued graph, Σ_g, i.e. there exists a flat annulus centered at that point over which the surface spirals, much like a helicoid. Applying this result to the sequence of points p_n^i we obtain a sequence of multi-valued graphs Σ_g^i. Moreover, since the boundaries of the disks Σ_n are going to infinity, the outer boundaries of the annuli over which the Σ_g^i's are defined are also going to infinity. In other words, each Σ_g^i is a large, highly-sheeted multi-valued graph. It is by using these multi-valued graphs that Colding and Minicozzi are able to locate the position of p_n^i in space.

Let us assume that Σ_g^{i+1} lies below Σ_g^i. While this is already a very delicate argument, it is still not sufficient to determine that the sequence of points must move downwards. A multi-valued graph behaves, in some sense, much like a helicoid, it is however NOT a helicoid. In other words, a priori the multi-valued graph does not have to extend horizontally parallel to the plane $\{z = 0\}$. Thus, even if p_{i+i} lies in the component below Σ_g^i, it could still be the case that p_{i+1} is well above p_i. The Basic Structure Theorem can be improved to show that if the number of sheets of a multi-valued graph is sufficiently large, i.e. N is sufficiently large, the growth of some sheets is sub-linear. Thus, if the number of p_n^i's grows linearly then, comparing the two growths, one obtains that these points must eventually move downwards and below the plane, giving a contradiction.

This ends our brief description of the proof of the One-sided Curvature Estimate. Notice that what they prove is precisely that under certain conditions, once the curvature of an embedded minimal disk is sufficiently large, then it cannot stop being large and the points where it is large are forced to move in one direction. Adding the hypothesis of being contained in a half-space to this description then gives the One-sided Curvature Estimate.

In [**34**] Meeks and Tinaglia generalize the One-sided Curvature Estimate to disks embedded in $\mathbb{R}^3$ with bounded constant mean curvature. The proof of their generalization applies in an essential manner results and techniques that can be found in the papers [**9, 14, 17**] by Colding and Minicozzi, [**27**] by Meeks, [**28**] by Meeks, Perez and Ros, [**29, 30**] by Meeks and Rosenberg and their previous papers [**31, 32, 33**]. Except for the results in [**28, 32**], all of the other results depend on the One-sided Curvature Estimates for minimal disks given by Colding and Minicozzi in [**14**]. Among other things, in the next section we give an idea on how to prove such generalization.

5. Curvature Estimate for CMC Disks

In this section we review a curvature estimate for simply-connected surfaces embedded in $\mathbb{R}^3$ with constant mean curvature by Meeks and Tinaglia [**33**]. To simplify the statements, we define an *H-disk* to be a simply-connected surface embedded in $\mathbb{R}^3$ with nonzero constant mean curvature equal to H.

THEOREM 5.1. *Given $\delta > 0$ $H_0 > 0$, there exists a constant $K = K(\delta, H_0)$ such that for any H-disk M, $H \geq H_0$*

$$\sup_{\{p\in M \mid d_M(p,\partial M)\geq\delta\}} |A_M| \leq K.$$

Theorem 5.1 says that if a point on an H-disk, with H bounded from below, is sufficiently away from the boundary (intrinsically) then the curvature is bounded at that point. Note that the curvature estimate does not depend on an upper-bound for H. In order to prove Theorem 5.2 we first prove the curvature estimate below that depends on the exact value for H and then we argue to improve this result.

LEMMA 5.2. *Given $\delta > 0$, there exists a constant $K = K(\delta)$ such that for any H-disk M*

$$\sup_{\{p\in M \mid d_M(p,\partial M)\geq\frac{\delta}{H}\}} |A_M| \leq KH.$$

Assuming this weaker curvature estimate, the following Radius Estimate for H-disks follows, see Figure 5.

THEOREM 5.3 (Radius Estimate). *There exists a constant R such that any H-disk M has radius less than $\frac{R}{H}$. In other words,*

$$\sup_{p\in M} d_M(p,\partial M) \leq \frac{R}{H}.$$

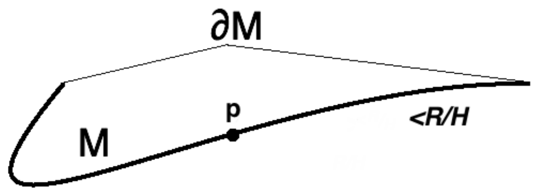

FIGURE 2. Radius Estimate

In 1951, Hopf [**24**] proved that a compact immersed sphere of constant mean curvature in $\mathbb{R}^3$ must be round. In 1956, Alexandrov [**1**] proved that the only closed (compact without boundary) surfaces of constant mean curvature embedded in $\mathbb{R}^3$ are round spheres. These two results about closed surfaces in $\mathbb{R}^3$ with constant mean curvature lead naturally to the following longstanding problem:

> **Are round spheres the only complete, simply-connected surfaces with nonzero constant mean curvature embedded in $\mathbb{R}^3$?**

In other words, what happens if the compactness hypothesis is replaced by completeness? Note that "simply-connected" and "embedded" are both necessary hypotheses in order to characterize the round sphere; cylinders are simple counter-examples. The answer to this question is yes and it is the crowning result of the systematic study of the geometry of H-disks done by Meeks and Tinaglia in [**31, 33, 34**]. This characterization of round spheres is an immediate consequence of the Radius Estimate. In other words

THEOREM 5.4. *Round spheres are the only complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with nonzero constant mean curvature.*

PROOF. Let M be such a surface then, because of the Radius Estimate, M is compact. That is, M is topologically a sphere embedded in $\mathbb{R}^3$ with constant mean curvature. Using either Hopf Theorem or Alexandrov Theorem gives that M is a round sphere. $\square$

Theorem 5.4 was proven by Meeks in [**26**] assuming that the surface is properly embedded in $\mathbb{R}^3$. Being proper implies that if the intrinsic distance between two points on the surface goes to infinity, the extrinsic distance goes to infinity as well.

This new characterization of the round sphere, together with previous results of Colding-Minicozzi [**17**] and Meeks-Rosenberg [**29**] for minimal surfaces, completes the classification of complete, embedded, simply-connected surfaces with constant mean curvature:

Planes, spheres and helicoids are the only complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with constant mean curvature.

We now prove the Radius Estimate assuming Lemma 5.2.

Proof of Theorem 5.3. Arguing by contradiction, there exists a sequence of 1-disks M_n with radii going to infinity. Theorem 5.2 implies that the sequence of 1-disks $\Sigma_n = \{p \in M_n \mid d_M(p, \partial M) > 1\}$ have uniformly bounded norm of the second fundamental form. A standard compactness argument shows that Σ_n converges C^2 to a complete surface Σ_∞ "almost embedded" in $\mathbb{R}^3$ with bounded norm of the second fundamental form, mean curvature equal to one and genus zero. Being the limit of a sequence of embedded surfaces, Σ_∞ cannot have a transverse self-intersection but it could still intersect itself tangentially. However, if $p \in \Sigma_\infty$ is such a point, then the unit normal vectors at p point in the opposite direction. Since this is the only way in which the limit surface might fail to be embedded, we say that such surface is "almost embedded." A simple example that clearly illustrates how this could happen is the following one: consider a sequence of pairs of spherical caps such that the distance between the pairs is going to zero. If the concavities of the spherical caps are facing each other then they cannot approach each other without intersecting. If the concavities face the same direction then, in the limit, there will be only a single embedded spherical cap. If the concavities are facing opposite directions and not facing each other then in the limit there will be two spherical caps tangent at one point.

If Σ_n converges to Σ_∞ with multiplicity greater than two, then it can be shown that the universal cover of Σ_∞ would be stable, in the sense of (2.1). Since there are no complete stable surfaces immersed in $\mathbb{R}^3$ with nonzero constant mean curvature, we have obtained a contradiction and thus we also know that Σ_n converges to Σ_∞

with multiplicity at most two, in particular finite. Similarly, we can use the non existence of complete stable surfaces immersed in $\mathbb{R}^3$ with nonzero constant mean curvature to show that Σ_∞ must be proper. If not, then a standard compactness argument shows that $\overline{\Sigma}_\infty - \Sigma_\infty$ is a complete surface almost embedded in $\mathbb{R}^3$ with nonzero constant mean curvature and its universal cover is stable. Such surface does not exist and this contradiction gives that Σ_∞ is proper.

In [**32**] Meeks and Tinaglia prove that a proper surface almost embedded in $\mathbb{R}^3$ with finite genus and bounded norm of the second fundamental form contains an embedded Delaunay surface D_∞ at infinity. While this was shown in [**25**] to be true for properly embedded surface of finite topology by Kusner, Korevaar and Solomon, here the full generality of the results in [**32**] is needed.

Claim 5.5. The sequence Σ_n cannot converge to D_∞ with finite multiplicity.

Proof of the claim. Arguing by contradiction, suppose that Σ_n converges to D_∞ with finite multiplicity. Consider a closed geodesic γ on D_∞ and let γ_n be a sequence of closed curves in Σ_n converging to it. Since γ is a closed geodesic in an embedded Delaunay surface of constant mean curvature one, γ is contained in a ball of radius one. Therefore γ_n, which is the boundary of a disk D_n in Σ_n, is also contained in a ball of radius one for n large. We claim that D_n is contained in a ball of radius three and thus cannot converge to D_∞. More generally, if γ is a closed curve contained in $B_r(0)$ that is the the boundary of an H-disk D_H, then D_H is contained in $B_{r+\frac{2}{H}}(0)$. Let $p \in D_H$ be the point that is furthest away from the boundary, then the normal vector at p, N_p, is pointing toward the origin. Let Π_t be the plane perpendicular to N_p at distance t from the origin. A standard argument using the Alexandrov reflection principle shows that the connected component of $D_H - [D_H \cap \Pi_{\frac{|p|+r}{2}}]$ containing p is graphical over $\Pi_{\frac{|p|+r}{2}}$. Standard height estimates for graphs with constant mean curvature give that the distance from p to $\Pi_{\frac{|p|+r}{2}}$, that is $\frac{|p|+r}{2}$, is at most $\frac{1}{H}$. Thus, $|p| + r \leq \frac{2}{H}$ which proves the claim. □

Therefore, thanks to Claim 5.5 we have obtained a contradiction and the Radius Estimate holds. □

We now use the Radius Estimate to improve the curvature estimate in Lemma 5.2, that is to prove Theorem 5.1.

Proof of Theorem 5.1. Arguing by contradiction, suppose that the theorem fails for some $\delta, H_0 > 0$. In this case there exists a sequence of H_n-disks, M_n with $H_n > H_0$ and points $p_n \in M_n$ satisfying:

$$\delta \leq d_{M_n}(p_n, \partial M_n), \tag{5.1}$$

$$n \leq |A_{M_n}|(p_n). \tag{5.2}$$

Rescale these disks by H_n to obtain the sequence of 1-disks $\widehat{M_n} = H_n M_n$, with the related sequence of points $\widehat{p}_n = H_n p_n$. By definition of these disks and points, by equations (5.1) and (5.2) and since $H_n \geq H_0$, we have

$$\delta H_0 \leq \delta H_n \leq d_{\widehat{M}_n}(\widehat{p}_n, \partial \widehat{M}_n), \tag{5.3}$$

$$\frac{n}{H_n} \leq |A_{\widehat{M}_n}|(\widehat{p}_n). \tag{5.4}$$

Note that equation (5.3) together with Lemma 5.2 with $H = 1$ gives that $|A_{\widehat{M}_n}|(\widehat{p}_n) \leq K(\delta H_0)$.

By Theorem 5.3, $d_{M_n}(\widehat{p}_n, \partial\widehat{M}_n) \leq R$, for the universal constant R, and so, by equation (5.3), $H_n \leq \frac{R}{\delta}$. Finally, by equation (5.4) we obtain

$$\frac{\delta}{R}n \leq \frac{n}{H_n} \leq |A_{\widehat{M}_n}|(\widehat{p}_n) \leq K(\delta H_0),$$

which is false for n sufficiently large. This contradiction completes the proof of the theorem. □

Thus, it remains to prove Lemma 5.2 and, by rescaling, it suffices to prove it for $H = 1$. The steps to prove Lemma 5.2 are essentially two. The first step is to prove a chord-arc bound for 1-disks. Just like in the proof of the Intrinsic 1-sided Curvature Estimate of Colding and Minicozzi, the up-shot of such chord-arc bound is that in order to prove the Intrinsic Curvature Estimate it is sufficient to prove an extrinsic version, the second step.

We first discuss the second step, that is the Extrinsic Curvature Estimate for 1-disks.

THEOREM 5.6. *Given $\varepsilon < 1$ there exist $C = C(\varepsilon) > 0$ such that the following holds. Let M be an H-disk containing the origin. If $\partial M \cap B_{\frac{\varepsilon}{2H}}(0) = \emptyset$, then*

$$|A_M|(0) \leq CH.$$

Idea of the proof of Theorem 5.6. It suffices to prove the result for $H = 1$. Arguing by contradiction, suppose that for some $\varepsilon < 1$ there exists a sequence of 1-disks, M_n, with $\partial M_n \cap B_{\frac{\varepsilon}{2}}(0) = \emptyset$ and norm of the second fundamental form arbitrary large at the origin, $|A|(0) > n$. Using the uniqueness of the helicoid, it can be shown that nearby the origin, on the scale of the norm of the second fundamental form, the surface can be approximated by a helicoid (see also [**45, 46**]). This is because using a rescaling argument one obtains a sequence of H_n-disks with boundary going to infinity, bounded norm of the second fundamental form which is one at the origin and H_n going to zero. Such sequence converges to a non-flat, simply-connected minimal surface M_∞ embedded in $\mathbb{R}^3$. The surface M_∞ must be properly embedded [**17**] and thus a helicoid [**29**].

One can use this local picture on the scale of the norm of the second fundamental form around the origin to extend this small "helicoid" with constant mean curvature one to a larger, meaning on a fixed scale, multi-valued graph. To do this, the first step is to find a closed curve γ satisfying certain hypotheses and bounding a disk on M_n. Then, using a result of Meeks and Yau [**35**] gives a stable minimal disk Σ_{min} with boundary γ and disjoint from M_n. Finally, thanks to the properties of its carefully chosen boundary and results in [**14**], one can show that Σ_{min} contains a large multi-valued graph. The curvature of M_n is bounded between the sheets of Σ_{min} and this gives that M_n itself contains a multi-valued graph on a fixed scale. However, as the norm of the second fundamental form goes to infinity, this description gives a sequence of graphs with constant mean curvature one which are contained in arbitrarily small slabs. This is not possible and proves the theorem. □

Note that arguing as in the proof of Theorem 5.3 one can prove the following Extrinsic Radius Estimate for 1-disks, Corollary 5.7, and then use such radius estimate to improve the curvature estimate, Corollary 5.8.

COROLLARY 5.7 (Extrinsic Radius Estimate). There exists a constant R_0 such that the following holds. Let M be an H-disk then

$$\sup_{p\in M} d_{\mathbb{R}^3}(p,\partial M)\leq \frac{R_0}{H}.$$

COROLLARY 5.8. Given $\delta>0$ $H_0>0$, there exists a constant $K=K(\delta,H_0)$ such that for any H-disk M, $H\geq H_0$

$$\sup_{\{p\in M \mid d_{\mathbb{R}^3}(p,\partial M)\geq\delta\}} |A_M|\leq K.$$

We now discuss the first step in the proof of the Intrinsic Curvature Estimate, that is the chord-arc bound. The steps of the proof of the chord-arc bound for 1-disks are essentially the same as the ones used by Colding and Minicozzi to prove the chord-arc bound in the minimal case [**17**]. However, a key result that is needed to carry-out such a straightforward generalization is a generalization of the Colding-Minicozzi One-sided Curvature Estimate discussed in the previous section. This latter generalization is nontrivial and applies in an essential manner the Colding-Minicozzi One-sided Curvature Estimate for minimal disks and results depending on it. See the last paragraph of the previous section. The One-sided Curvature Estimate for H-disks says the following.

THEOREM 5.9. *There exist $C,\varepsilon>0$ such that the following holds. Let M be an H-disk embedded in $\mathbb{R}^3$. If $M\cap B_{\frac{1}{2}}(0)\cap\{z=0\}=\emptyset$ and $\partial M\cap B_{\frac{1}{2}}(0)\cap\{z>0\}=\emptyset$, then*

$$\sup_{M\cap B_\varepsilon(0)\cap\{z>0\}} |A_M|\leq C.$$

Let us first compare Corollary 5.8 and Theorem 5.9. Corollary 5.8 gives a curvature estimate that does not require that the surface is contained in a half-space. However, the estimate depends on a lower bound for the value of the mean curvature and it becomes worse as such lower bound goes to zero. The estimate in Theorem 5.9 is independent of the value of the mean curvature. This independence is essential because in proving chord-arc bounds for H-disks we employ rescaling arguments where the mean curvature goes to zero.

Idea of the proof of Theorem 5.9. As for the proof of the Colding-Minicozzi One-sided Curvature Estimate, arguing by contradiction, there exists a sequence of embedded disks, M_n, contained in a half-space with bounded constant mean curvature, boundary going to infinity, $\partial M_n\in\partial B_n(0)$, and norm of the second fundamental form arbitrary large at a point p_n converging to the origin, $\sup_{M_n\cap B_{\frac{1}{n}}(0)}|A|>n$. Note that thanks to Corollary 5.8 the mean curvature H_n of M_n must be going to zero.

Arguing similarly to the proof of Theorem 5.6 but using also the fact that $M_n\cap\{z=0\}=\emptyset$, it can be shown that around the point p_n and on the scale of the norm of the second fundamental form, the surface can be approximated by a vertical helicoid. Let p_n' denote a point in M_n which is on the axis of such vertical helicoid.

Let $N \colon M_n \to \mathbb{S}^2$ denote the Gauss map and consider the connected component γ_n of the pre-image of the equator, $\mathbb{S}^2 \cap \{z = 0\}$, containing p'_n. In other words, for any point $p \in \gamma$, the tangent plane to M_n at p is vertical. In fact, it can be shown that the unit tangent vector to γ at each point $p \in \gamma$ is vertical and thus that the curve γ must intersect the $\{z = 0\}$ plane. This contradiction finishes the proof of the generalization of the One-sided Curvature Estimate for H-disks. $\square$

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