Geometric PDEs in the presence of isolated singularities

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Abstract. This is a short course on the behavior of solutions to some geometric elliptic PDEs of Monge-Ampère type in two variables, in the presence of non-removable isolated singularities. We will describe local classification theorems around such an isolated singularity, as well as global classification theorems for the case of finitely many isolated singularities.

1. Introduction

Given a smooth solution $u = u(x, y)$ of an elliptic partial differential equation (PDE) on a punctured disc $D^* = \{(x, y) \in \mathbb{R}^2 : 0 < (x-a)^2 + (y-b)^2 < r^2\}$, a classical problem is to describe the behavior of u at the puncture $q = (a, b)$. The usual question in this setting is whether the singularity is removable, i.e. whether the solution $u \in C^2(D^*)$ actually extends smoothly to D. For some classical geometric equations like the *minimal graph equation*

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0,$$

any isolated singularity is automatically removable (see [**Ber**]). This results holds in a much wider context, and it has been generalized by many authors. For instance, very recently by Leandro and Rosenberg [**LeRo**] for the prescribed mean curvature equation associated to a Killing submersion. For many other quasilinear elliptic equations, isolated singularities are removable as long as u lies in the Sobolev space $H^1(D^*) \equiv W^{1,2}(D^*)$, see for instance [**GiTr**].

Contrastingly, some elliptic PDEs admit solutions $u(x, y)$ with non-removable isolated singularities, such that both the solution and its first derivatives are bounded around the singularity but u does not extend C^2 across it.

For example, let us consider the general elliptic Monge-Ampère equation in dimension two:

$$u_{xx}u_{yy} - u_{xy}^2 = F(x, y, u, u_x, u_y) \geq c > 0, \tag{1.1}$$

where F is a smooth function. Observe that if $u(x, y)$ is asolution to this equation, then its graph $z = u(x, y)$ is a locally convex surface in $\mathbb{R}^3$. In these conditions, any solution to (1.1) on a punctured disk extends continuously to the puncture, but

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this extension is not necessarily C^1-smooth at this point: it could have a *conical singularity* at the puncture (see Figure 1).

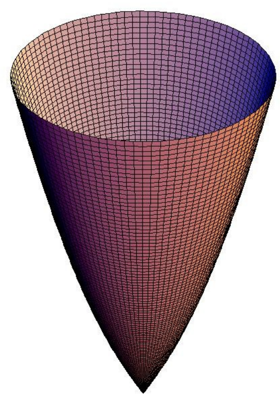

FIGURE 1. A solution to the elliptic Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 1$ with a conical singularity.

Two of the most classical and widely studied elliptic Monge-Ampère equations are the *Hessian one equation*

$$u_{xx}u_{yy} - u_{xy}^2 = 1 \tag{1.2}$$

and the *constant curvature equation*

$$u_{xx}u_{yy} - u_{xy}^2 = K(1 + u_x^2 + u_y^2)^2, \qquad K > 0, \tag{1.3}$$

where K is a positive constant. The first one is the simplest elliptic Monge-Ampère equation that one can consider, and its solutions are related to minimal surfaces in $\mathbb{R}^3$, flat surfaces in hyperbolic space $\mathbb{H}^3$, improper affine sphere in the affine 3-space, special Lagrangian surfaces in $\mathbb{C}^2$ or area preserving diffeomorphisms of the plane. The second equation describes graphs of constant positive curvature $K > 0$ in $\mathbb{R}^3$, and is also linked to harmonic maps into $\mathbb{S}^2$ and constant mean curvature surfaces in $\mathbb{R}^3$. Moreover, both equations admit solutions with non-removable isolated singularities of *conical type*, as explained above.

Our objective here is to study how to construct and classify solutions to the equations (1.2), (1.3) in the presence of conical isolated singularities. Let us make some brief remarks about this.

(1) For elliptic Monge-Ampère equations, there is a well developed theory of generalized solutions in the viscosity sense (see [**CaCa**] for a general introduction to viscosity solutions of fully non-linear elliptic PDEs). These viscosity solutions need not be C^2 smooth. However, this *viscosity* property fails to hold at a non-removable isolated singularity of an elliptic Monge-Ampère equation. Thus, an alternative procedure for studying this very natural situation is in order.

(2) The two Monge-Ampère equations that we will consider have an important feature: they have associated some geometrically defined holomorphic function with respect to a certain conformal structure associated to the equation. Hence, we shall use for our purposes complex analysis in a substantial way.

(3) Although the general Monge-Ampère equation (1.1) does not have an associated holomorphic data, it turns out that many of the arguments that we expose here still hold in great generality. Indeed, it is possible to associate to any solution of (1.1) a natural conformal structure so that, in terms of it, the coordinates of the graph to the solution satisfy a quasilinear elliptic system with good analytic properties. However, we shall restrict here to the most simple cases of equations (1.2) and (1.3). The general case is an ongoing project of the authors with their Ph.D. student Asun Jiménez [**GJM**].

We have organized the contents as follows.

In Section 2 we deal with the basic concepts regarding the Hessian one equation (1.2). In particular, we will explain how any solution to (1.2) has an underlying conformal structure, and two associated holomorphic functions with respect to this conformal structure. Also, we will describe how any solution to (1.2) can be recovered from these conformal data, and how this information gives a simple proof to the Jörgens theorem [**Jor1**]: *Any C^2 solution to* (1.2) *globally defined in $\mathbb{R}^2$ is a quadratic polynomial.*

In Section 3 we give a local classification of the isolated singularities of the Hessian one equation (1.2). First, we prove that an isolated singularity of (1.2) is removable if and only if its underlying conformal structure is that of a punctured disk (i.e. *not* that of an annulus). Then we prove that, if the singularity is non-removable, the gradient of the solution has a well defined limit at the singularity, which is a regular real-analytic strictly convex Jordan curve. And conversely, we show that any such curve arises associated to some isolated singularity of a solution to (1.2) in the above way. This local classification result was obtained by Aledo, Chaves and Gálvez [**ACG**].

In Section 4 we classify all the solutions to the Hessian one equation (1.2) that are globally defined on the plane, and have a finite number of singularities. In other words, we classify the C^2 solutions to (1.2) that are globally defined on a finitely punctured plane $\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$. In 1955 K. Jörgens [**Jor2**] solved the above problem for the case of *one* singularity, proving that the only such global solutions are equiaffine transformations of the radially symmetric solution

$$u(x,y) = \frac{1}{2}\left(r\sqrt{1+r^2} + \sinh^{-1}(r)\right), \qquad r := \sqrt{x^2+y^2}.$$

Here, an equiaffine transformation is an affine transformation of $\mathbb{R}^3$ of a certain type; these transformations preserve solutions to (1.2). For the case of $n > 1$ singularities, we will explain the following result by Gálvez, Martínez and Mira [**GMM**], which gives a full solution to the problem: *Any solution to* (1.2) *in a finitely punctured plane is uniquely determined (up to equiaffine transformations) by its underlying conformal structure. Conversely, any circular domain in $\mathbb{C}$ is the conformal structure of some global solution to* (1.2) *in a finitely punctured plane.* In particular, the moduli space of global solutions to (1.2) with $n > 1$ singularities is, modulo equiaffine transformations, a $(3n-4)$-dimensional family.

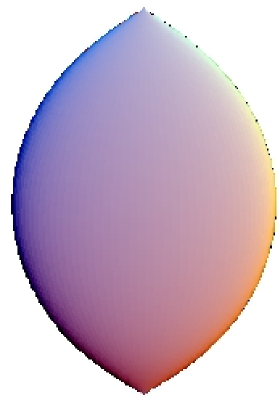

FIGURE 2. A rotational peaked sphere ($K = 1$) in $\mathbb{R}^3$

In Section 5 we turn our attention to the study of the constant curvature equation (1.3). More generally, we will deal there with some basic facts about surfaces of constant positive curvature $K > 0$ in $\mathbb{R}^3$. We will explain how these surfaces also have an associated conformal structure, and a holomorphic quadratic differential. Also, we will show that the Gauss map is a harmonic map into $\mathbb{S}^2$ for this conformal structure, and we will provide some representation formulas.

In Section 6 we analyze in detail the behavior of a solution to the constant curvature equation (1.3) at an isolated singularity. First, we prove that such a singularity is removable if and only if it has the extrinsic conformal structure of a punctured disk, if and only if the mean curvature of the graph is bounded around the singularity. Then, we give a complete classification of the non-removable isolated singularities of (1.3), which tells the following: *the limit unit normal of a solution to* (1.3) *at a non-removable isolated singularity is a real-analytic, regular, strictly convex Jordan curve in* $\mathbb{S}^2_+$*; and conversely, any such curve arises as the limit unit normal of exactly one isolated singularity to* (1.3).

In Section 7 we study the case of *immersed conical singularities* of K-surfaces in $\mathbb{R}^3$. More specifically, we extend the above results on isolated singularities of solutions to the constant curvature equation (1.3) to the case where the surface is not a graph anymore. In doing so, we give a classification result similar to the one above, but with the difference that this time the limit unit normal is a real-analytic, immersed closed curve in $\mathbb{S}^2$ that is locally convex at regular points, but that may have a certain type of singular points of cuspidal type.

In Section 8 we deal with the global problem of classifying *peaked spheres* in $\mathbb{R}^3$. By definition, a peaked sphere is a closed convex K-surface that is everywhere regular except at a finite number of points. A regular peaked sphere is a round sphere, there are no peaked spheres with exactly one singularity, and a peaked sphere with exactly two singularities is a rotational sphere as in Figure 1. Using some results by Troyanov, Luo-Tian, Alexandrov and Pogorelov, we explain that the space of peaked spheres with $n > 2$ singularities is a $(3n - 6)$-dimensional family. Then, by using the previous local analysis at an isolated singularity of

(1.3), we provide some applications of this result to free boundary value problems for harmonic maps into $\mathbb{S}^2$, and for constant mean curvature surfaces in $\mathbb{R}^3$.

Finally, in Section 9, we will explain some related results of other theories, and we will propose several open problems.

2. The Hessian one equation: preliminaries

Let $u(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a solution to the Monge-Ampère equation (1.1) on a planar domain $D \subset \mathbb{R}^2$. Without loss of generality we shall assume that $u_{xx} > 0$. Then, it is easy to check that

$$d\sigma^2 = u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2$$

is a Riemannian metric on D. Moreover, if we consider the graph of $u(x,y)$,

$$\mathcal{S}_u = \{(x,y,u(x,y)\} \subset \mathbb{R}^3,$$

it turns out that $\mathcal{S}_u$ is a convex regular surface in $\mathbb{R}^3$, and its second fundamental form is given precisely by $d\sigma^2$.

Motivated by this, we give the following definition.

Definition 2.1. The Riemann surface structure induced on the surface by $d\sigma^2$ will be called the **underlying conformal structure** of the solution u to (1.1).

The above Riemann surface structure is important for the study of solutions to the Hessian one equation (1.2). Indeed, there are some natural complex functions associated to any solution to (1.2) that are holomorphic with respect to the underlying conformal structure.

Specifically, let $u(x,y)$ be a solution to (1.2) over a planar domain $D \subset \mathbb{R}^2$, and consider the function $G(x,y) : D \subset \mathbb{R}^2 \to \mathbb{C}$ given by

$$G(x,y) = s(x,y) + it(x,y) := x + u_x(x,y) + i(y + u_y(x,y)). \tag{2.1}$$

A computation using (1.2) shows that the Jacobian of the mapping $(x,y) \mapsto (s,t)$ is ≥ 2, and also that the metric $d\sigma^2$ associated to $u(x,y)$ is given by

$$d\sigma^2 = \frac{1}{2 + u_{xx} + u_{yy}}(ds^2 + dt^2).$$

Therefore, $G = s + it$ is a conformal parameter on D for the underlying conformal structure associated to u. In particular, G is holomorphic for this Riemann surface structure.

Moreover, it can be proved that the function $F : D \subset \mathbb{R}^2 \to \mathbb{C}$ given by

$$F(x,y) = x - u_x(x,y) + i(-y + u_y(x,y)) \tag{2.2}$$

is also holomorphic for the underlying conformal structure, since it satisfies the Cauchy-Riemann equations with respect to the conformal parameters (s,t) defined previously. Also, observe that

$$2(x+iy) = G + \overline{F}. \tag{2.3}$$

In particular, x and y are harmonic functions, and the Jacobian of $2(x+iy)$ with respect to a conformal parameter z is $|dG|^2 - |dF|^2 > 0$. This also tells that

$$4(dx^2 + dy^2) = |d(G + \overline{F})|^2 \leq (|dF| + |dG|)^2 \leq 4|dG|^2,$$

that is,

$$dx^2 + dy^2 \leq |dG|^2. \tag{2.4}$$

These two independent holomorphic functions F, G in (2.1), (2.2) can be used to provide a conformal representation formula for solutions to (1.2). The existence of this representation formula was used classically by Blaschke and Jörgens [**Jor1, Jor2**]. The general formulation that we write here is a restatement of a result by Ferrer, Martínez and Milán [**FMM**]. It is a consequence of the above formulas together with some standard integrability arguments.

THEOREM 2.2. *Let Σ denote a Riemann surface. Let $F, G : \Sigma \to \mathbb{C}$ be two holomorphic functions on Σ satisfying $|dF/dG| < 1$ and $dG \neq 0$.*

Then the map $\psi : \Sigma \to \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ given by

$$\psi = \left(G + \overline{F}, \frac{1}{4} \left\{ |G|^2 - |F|^2 + 2\mathrm{Re}(GF) \right\} - \mathrm{Re} \int F dG \right) \tag{2.5}$$

is the graph of a solution to the Hessian one equation (1.2) *as long as:*

(1) *$G + \overline{F}$ is one-to-one, and*
(2) *the integral $\int F\, dG$ has no real periods on Σ.*

Conversely, every graph in $\mathbb{R}^3$ of a solution to (1.2) *over a planar domain $D \subset \mathbb{R}^2$ is recovered in this way in terms of its underlying conformal structure, from the pair of holomorphic functions F, G given by* (2.1) *and* (2.2).

The above holomorphic functions F, G in (2.1), (2.2) were also used by K. Jörgens to classify all the entire solutions (i.e. C^2 solutions globally defined on $\mathbb{R}^2$) of the Monge-Ampère equation (1.2).

THEOREM 2.3 ([**Jor1**]). *Any entire solution to the Hessian one equation* (1.2) *is a quadratic polynomial.*

PROOF. Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 entire solution to (1.2). Then it has the conformal type of the disk $\mathbb{D}$ or the complex plane $\mathbb{C}$ for its underlying conformal structure Σ. However, (2.4) shows that $|dG|^2$ is a complete conformal flat metric on Σ, and hence Σ is conformally the complex plane $\mathbb{C}$. But now, as $|dF/dG| < 1$, we see by Liouville's theorem that the function dF/dG is constant.

As $dG \neq 0$, we can reparametrize locally the conformal domain around any point, so that $G(\zeta) = \zeta$, and hence $F(\zeta) = a\zeta + b$, where ζ is the new conformal parameter. A computation from (2.5) shows that the surface is the graph of a quadratic polynomial in our local domain. By analyticity, the same holds globally. □

It must be emphasized that any solution to the Hessian one equation (1.2) has the property that its graph $(x, y, u(x, y))$ is an improper affine sphere in the 3-dimensional affine space, with constant affine normal $(0, 0, 1)$. Conversely, any improper affine sphere with affine normal $(0, 0, 1)$ is locally the graph of a solution to (1.2). The above representation formula has been used to describe the global behaviour of improper affine spheres, in particular in the presence of certain types of *admissible singularities*. See [**ACG, FMM, Mar**] for more details about improper affine spheres and its study in terms of holomorphic functions.

3. Isolated singularities of $u_{xx}u_{yy} - u_{xy}^2 = 1$: local study

The first step in order to understand the behavior of a solution to (1.2) around a non-removable isolated singularity is to determine its underlying conformal structure. This is given by the following result:

LEMMA 3.1. *Let $u(x,y) : D \setminus \{q\} \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth solution to* (1.2). *Then u extends smoothly to q if and only if its underlying conformal structure is that of a punctured disk.*

PROOF. It is clear that if u extends smoothly, it is conformally a punctured disk. Conversely, assume that the underlying conformal structure of u is that of a punctured disk. Then the function $x + iu_y : \mathbb{D}^* \to \mathbb{C}$ is holomorphic for this structure, and its real part extends continuously to the origin in $\mathbb{D}^*$. Thus, $x + iu_y$ can be analytically extended to the whole $\mathbb{D}$. The same argument holds for $u_x + iy$. Then, F, G given by (2.1) and (2.2) extend holomorphically to the origin, and by (2.5) the singularity is removable. □

We deal next with the problem of classifying locally the solutions to (1.2) around a non-removable isolated singularity. For that, let us take into account the following facts:

(1) By Lemma 3.1, the underlying conformal structure of such a singularity is that of an annulus.
(2) We can assume without loss of generality that the solution $u(x,y)$ to (1.2) has the singularity at the origin, with $u(0,0) = 0$, simply by applying a translation to its graph $z = u(x,y)$ (Euclidean translations preserve solutions to (1.2)).
(3) Two solutions to (1.2) will be considered to be *equivalent* if they agree on an open set around the origin singularity. If this is the case, as by (2.5) the solutions to (1.2) are real analytic, then the two solutions agree everywhere on their common domain.
(4) A holomorphic change of coordinates in the underlying conformal domain of a solution does not change the solution itself, that is, the graph of the solution given by (2.5) remains invariant.
(5) By convexity, the graph of the solution to (1.2) has the following property: its gradient (u_x, u_y) extends to the isolated singularity as a closed convex curve in $\mathbb{R}^2$. In other words, there exists a limit tangent cone at the singularity, which is generated by a closed convex curve.

Let $\mathcal{A}$ denote the space of solutions to (1.2) having the origin as a non-removable singularity, modulo equivalence as explained above. The following result fully describes the class $\mathcal{A}$ in terms of their limit tangent cones at the singularity. This is a result by Aledo, Chaves and Gálvez [**ACG**]. Similar ideas were also used in a previous paper by the authors [**GaMi**], where a local classification of non-removable singularities was studied and solved for isolated singularities of flat surfaces in the hyperbolic 3-space $\mathbb{H}^3$.

THEOREM 3.2 ([**ACG**]). *There exists a bijective explicit correspondence between the class $\mathcal{A}$ and the class of regular, real analytic, strictly convex Jordan curves in $\mathbb{R}^2$.*

PROOF. Let $u(x,y)$ be a solution to (1.2) having a non-removable singularity at the origin, with $u(0,0) = 0$. Up to conformal equivalence, we may assume by Lemma 3.1 that its underlying conformal structure is that of a quotient strip $\Omega/(2\pi\mathbb{Z})$, where $\Omega := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < r_0\}$ for some $r_0 > 0$. So, we may parametrize the graph of $u(x,y)$ in terms of the conformal coordinate $z = s + it \equiv$

(s,t), that is, we may consider the 2π-periodic map

$$(x, y, u(x, y))(s, t) : \Omega \subset \mathbb{C} \equiv \mathbb{R}^2 \to \mathbb{C},$$

and assume that the real axis corresponds to the singularity, i.e. this map extends continuously to $\mathbb{R}$ with $x(s,0) = y(s,0) = 0$ for all $s \in \mathbb{R}$.

At this point, we may extend the harmonic functions $x(s,t)$ and $y(s,t)$ to $\mathbb{R} \times [-r_0, r_0] \equiv \{z \in \mathbb{C} : -r_0 < \mathrm{Im} z < r_0\}$, by $x(s,-t) = -x(s,t)$ and $y(s,-t) = -y(s,t)$. By the mean value theorem for harmonic functions, these extended functions are real analytic and harmonic on this larger domain. Now, since $x + iu_y$ and $u_x + iy$ are holomorphic, we conclude that u_x, u_y also extend analytically to $\mathbb{R} \times [-r_0, r_0]$, and are 2π-periodic.

This allows us to consider the real analytic curve

$$V(s) = (u_x, u_y)(s, 0) : \mathbb{R}/2\pi\mathbb{Z} \equiv \mathbb{S}^1 \to \mathbb{R}^2,$$

that is, the limit gradient of the solution to (1.2) at the singularity. Let us remark that the curve $V(s)$ determines completely the solution $u(x,y)$. Indeed, from $x(s,0) = 0 = y(s,0)$ we see that $V(s)$ gives the value of the holomorphic functions F, G in (2.1), (2.2) along the real axis (thus, globally by holomorphic continuation), and that the solution $u(x,y)$ is recovered in terms of F, G by (2.5).

We wish to prove now that the real analytic closed curve $V : \mathbb{S}^1 \to \mathbb{R}^2$ is regular and strictly convex.

For that, we define the harmonic function

$$\omega := \log \left| \frac{dF}{dG} \right|^2.$$

A computation shows that, for points on the real axis,

$$\omega_z(s, 0) = -4i||V'(s)||^2 \kappa^V(s),$$

where κ^V denotes the curvature of the planar curve $V(s)$. If $V'(s_0) \neq 0$ and $\kappa^V(s_0) \neq 0$ for some $s_0 \in \mathbb{R}$, then the harmonic function ω has more than one nodal curve passing through s_0 (the real axis being one of them). This would imply that there are points in Ω with $|dF| = |dG|$, which is impossible as we already explained. Thus, the curve is strictly locally convex at regular points.

Finally, we need to check that the curve $V(s)$ is everywhere regular. First, observe that by convexity, the curve V bounds a convex set of the plane, and it is non-constant. Thus, there is some direction $w_0 = (a,b) \in \mathbb{S}^1 \subset \mathbb{R}^2$ for which the supporting lines to this convex set in that direction intersect the curve $V(s)$ at two regular points, say $V(s_1), V(s_2)$, where $s_1, s_2 \in [0, 2\pi)$. Also, from the Cauchy-Riemann equations applied to $x + iu_y$ and $u_x + iy$, we see that

$$(ax_t + by_t)(s, 0) = (-a(u_y)_s + b(u_x)_s)(s, 0).$$

Hence, $ax_t + by_t$ vanishes at s_1 and s_2. But as $V'(s) = ((u_x)_s, (u_y)_s)$, we see that $ax_t + by_t$ must also vanish at any singular point of the curve $V(s)$.

Therefore, if $V(s)$ had some singular point, the 2π-periodic harmonic function $ax(s,t) + by(s,t)$, which has the real axis as a nodal curve, would be crossed by at least *three* other nodal curves, at points $s_1, s_2, s_3 \in [0, 2\pi)$. But as the set $ax + by = 0$ in the (x,y)-plane consists of two straight lines intersecting at the origin, the existence of a third nodal curve would contradict the property that the map $(x(s,t), y(s,t))$ gives a diffeomorphism from $[0, 2\pi) \times (0, r_0]$ to a punctured neighborhood of the origin in the (x, y)-plane. Hence, $V(s)$ is everywhere regular, as we wished to show.

The converse part, that is, that every regular strictly convex Jordan curve V in $\mathbb{R}^2$ can be realized as the limit unit normal of some non-removable isolated singularity to (1.2), is obtained by reversing all the above arguments. This is quite easy, so we will omit the details. □

4. Isolated singularities of $u_{xx}u_{yy} - u_{xy}^2 = 1$: global study

The trivial solutions to (1.2) are the quadratic polynomials with quadratic part $ax^2 + 2bxy + cy^2$ satisfying $ac - b^2 = 4$. By Jörgens' theorem, they are the unique C^2 solutions to (1.2) globally defined on $\mathbb{R}^2$. The simplest solution other than these polynomials is the rotational example

$$u(x,y) = \frac{1}{2}\left(r\sqrt{1+r^2} + \sinh^{-1}(r)\right), \qquad r := \sqrt{x^2+y^2}, \tag{4.1}$$

which is globally defined and C^2 on $\mathbb{R}^2 \setminus \{(0,0)\}$, and has a non-removable singularity at the origin. Motivated by Jörgens' theorem, it is natural to ask if (4.1) is the unique solution to (1.2) on the once punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$. This problem was also solved by Jörgens in the 1950s. The answer is essentially affirmative, but we need to take into account that the equation (1.2) is invariant under equiaffine transformations of the space:

Lemma 4.1. *Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be an equiaffine transformation of the form*

$$\Phi\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad a_{11}a_{22} - a_{21}a_{12} = 1. \tag{4.2}$$

Then, if $u(x,y)$ is a solution to (1.2), the function $u^*(x^*, y^*)$ given by

$$\Phi(x, y, u(x,y)) = (x^*, y^*, u^*(x^*, y^*))$$

is a new solution to (1.2) in terms of the variables x^*, y^*.

Therefore, we shall be interested in obtaining the classification result for solutions to (1.2) *modulo equiaffine transformations*. In other words, two solutions to (1.2) differing only by an equiaffine transformation will be considered to be *equivalent*.

With this, we have the following uniqueness result for the solution (4.1).

Theorem 4.2 ([**Jor2**]). *Any C^2 solution to (1.2) globally defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ is an equiaffine transformation of the rotational example (4.1) (or a quadratic polynomial).*

We will give a proof to this theorem later on, as a consequence of a more general result. Specifically, the previous results motivate the following problem.

> *What are the solutions to the Hessian one equation (1.2) that are globally defined and C^2 over a finitely punctured plane?*

The solutions to this problem will have the strongest possible regularity among global solutions to (1.2), other than the quadratic polynomials. Indeed, they will be real analytic everywhere except for a finite number of points, at which the solution will extend continuously but will not be C^1 (recall: if it extended C^1, the singularity would be removable). They also constitute the C^2 solutions to (1.2) that are defined on the largest possible sets of $\mathbb{R}^2$, that is, $\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$.

So, we will assume now that $u(x, y)$ is a C^2 solution to (1.2) on a finitely punctured plane $\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$, and so that u does not extend smoothly across any of the *singularities* $p_1, \dots, p_n$. We shall call such a function a *global solution of* (1.2) *with* n *singularities* .

The first step for analyzing the existence and behavior of such a solution is to control its underlying conformal structure. First, we know by Lemma 3.1 that this structure is that of an annulus around any singularity. Moreover, from (2.4) and a classical argument by Osserman, it is easy to prove (see [**FMM**]) that the conformal structure at infinity of the solution is that of a punctured disk.

By uniformization, the underlying conformal structure of the solution $u(x, y)$ is that of the complex plane $\mathbb{C}$ with a finite set of disjoint disks removed. Up to conformal equivalence, we can view this region as a *once punctured bounded circular domain*, that is, a bounded domain $\Omega \setminus \{z_0\}$ where $\partial\Omega$ is a finite collection of disjoint circles (one of them enclosing the rest), and $z_0 \in \Omega$.

We can now state the classification of all such solutions to (1.2).

Theorem 4.3 ([**GMM**]). *Any once punctured bounded circular domain with* n *boundary circles can be realized as the conformal structure of a unique (up to equaffine transformations) global solution to* (1.2) *with* n *singularities.*

In particular, for $n = 1$ *we recover Theorem 4.2, and for* $n > 1$ *we obtain that the space of global solutions to* (1.2) *with* n *singularities can be identified, modulo equaffine transformations, with an open set of* $\mathbb{R}^{3n-4}$.

Proof. We use the representation formula in Theorem 2.2. Let $\Omega \setminus \{z_0\} \subset \mathbb{C}$ denote a once punctured bounded circular domain. It is then classically known (see Ahlfors' book [**Ahl**]) that there exist two unique holomorphic functions $p, q : \Omega \setminus \{z_0\} \subset \mathbb{C} \to \mathbb{C}$ such that:

(1) p (resp. q) is a biholomorphism from $\Omega \setminus \{z_0\}$ onto a vertical (resp. horizontal) slit domain in $\mathbb{C}$ with n slits, i.e. $\mathbb{C}$ minus n disjoint vertical (resp. horizontal) segments.
(2) p (resp. q) has at z_0 a simple pole of residue 1.

If we define now

$$G(z) = p(z) + q(z), \qquad F(z) = p(z) - q(z), \tag{4.3}$$

we see that $G+\overline{F}$ is constant on each circle $C_1, \dots, C_n$ in $\partial\Omega$, i.e. $(G+\overline{F})|_{C_k} = p_k \in \mathbb{R}^2 \equiv \mathbb{C}$. By the maximum principle and the condition at z_0 we get $|dF| < |dG|$. So, a simple topological argument shows that $G + \overline{F}$ gives a diffeomorphism from $\Omega \setminus \{z_0\}$ onto $\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$.

Also, using that $\text{Re}p(z)$ and $\text{Im}q(z)$ are both constant along each boundary circle C_k, we get

$$\text{Re} \int_{C_k} F dG = \text{Re} \int_{C_k} (p\, dq - q\, dp) = -\text{Re} \int_{C_k} d(p\bar{q}) = 0.$$

Applying now Theorem 2.2 we see that F, G define a solution $u(x, y)$ to (1.2). This solution is globally defined on the finitely punctured plane $\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$, as explained above.

So, we have constructed for every punctured bounded circular domain in $\mathbb{C}$ a global solution with n singularities to (1.2), which we call a *canonical solution.* We need to prove now that any other global solution with n singularities to (1.2) only differs by an equiaffine transformation from one of those *canonical solutions*.

So, let $(x, y, u(x, y))$ be a global solution of (1.2) with n singularities, and let $\Omega \setminus \{z_0\}$ be its associated once punctured bounded circular domain. Hence, $x + iy : \Omega \setminus \{z_0\} \to \mathbb{C}$ is constant along each boundary circle of $\partial\Omega$. If we apply to it the equiaffine transformation Φ given by (4.2), its associated pair of holomorphic functions F, G turn into the pair F^*, G^* given by the following relation:

$$\begin{pmatrix} G^* + F^* \\ G^* - F^* \end{pmatrix} = \begin{pmatrix} a_{11} & ia_{12} \\ -ia_{21} & a_{22} \end{pmatrix} \begin{pmatrix} G + F \\ G - F \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where $c_1, c_2 \in \mathbb{C}$. Also, we know that G has a simple pole at the puncture z_0, and that by the regularity condition $|dF| < |dG|$, F has at most a simple pole at z_0. So, G^* and F^* also have at most a simple pole at z_0. Besides, $\text{Res}(G, z_0) \neq \pm\text{Res}(F, z_0)$ since that equality would imply by (2.3) that x or y is bounded, which is not true.

Taking the above facts into account, it is easy to choose the coefficients a_{ij} so that

$$\text{Res}(G^* + F^*, z_0) = \text{Res}(G^* - F^*, z_0) =: \alpha \neq 0.$$

At last, by making and adequate dilation and rotation of the bounded circular domain Ω, we may assume that $\alpha = 2$.

As $x^* + iy^*$ is constant on each boundary component of $\partial\Omega$, by (2.3) we see that $G^* + F^*$ (resp. $G^* - F^*$) has constant imaginary (resp. real) part along each boundary component of $\partial\Omega$. But recalling now equation (4.3) together with the definition and uniqueness of the holomorphic functions p, q, we conclude that $(x^*, y^*, u^*(x^*, y^*))$ is a canonical solution. This concludes the proof.

The final two assertions of the theorem are now immediate:

(1) If $n = 1$, the domain Ω is just a disk, and the functions p, q are given by

$$p(z) = \frac{1}{z} - z, \qquad q(z) = \frac{1}{z} + z.$$

It follows then immediately from (2.5) that any global solution with exactly one singularity is the rotational example (4.1), or an equiaffine transformation of it.

(2) If $n > 1$, the space of global solutions with n singularities, modulo equiaffine transformations, is in bijective correspondence with the conformal equivalence classes of once punctured bounded circular domains with n boundary components. As this second class is known to be a $(3n - 6)$-dimensional family, the result follows.

$\square$

In the case $n = 2$, the circular domain Ω is an annulus and it is also possible to describe explicitly all the global solutions to (1.2) with exactly two singularities. The description is given in terms of Jacobi theta functions, see [**GMM**] for the details.

5. Surfaces of constant curvature in $\mathbb{R}^3$

We now turn our attention to the Monge-Ampère equation (1.3), which describes constant curvature graphs in $\mathbb{R}^3$. To start, we will consider the more general case of immersed surfaces.

Let $\psi : M^2 \to \mathbb{R}^3$ denote an immersed surface with constant curvature $K > 0$ in $\mathbb{R}^3$. Up to a dilation, we shall assume that $K = 1$. Such surfaces will be called from now on *K-surfaces* in $\mathbb{R}^3$.

By changing orientation if necessary, the second fundamental form II of the immersion ψ is positive definite, and thus it induces a conformal structure on M^2. This structure will be called the *extrinsic conformal structure* of the surface ψ. When the surface is a graph, it satisfies (1.3) for $K = 1$ and this extrinsic conformal structure agrees with the underlying conformal structure of the solution, as introduced in Definition 2.1.

In this way, we can regard the surface as an immersion $X : \Sigma \to \mathbb{R}^3$ from a Riemann surface Σ such that, if $z = u + iv$ is a complex coordinate on Σ and $N : \Sigma \to \mathbb{S}^2$ denotes the unit normal of the surface, then $\langle X_u, N_u\rangle = \langle X_v, N_v\rangle < 0$ and $\langle X_u, N_v\rangle = 0$. The condition $K = 1$ implies

$$X_u = N \times N_v, \qquad X_v = -N \times N_u, \tag{5.1}$$

which in particular yields that $N : \Sigma \to \mathbb{S}^2$ is a harmonic map into $\mathbb{S}^2$, and the immersion X satisfies the equation

$$X_{uu} + X_{vv} = 2X_u \times X_v. \tag{5.2}$$

By convexity, N is a local diffeomorphism, and X is uniquely determined by N up to translations.

In terms of a conformal parameter $z = u+iv$ for the second fundamental form, the fundamental forms of a K-surface X are given by

$$\left\{ \begin{array}{rcl} \langle dX, dX\rangle & = & Q\, dz^2 + 2\mu|dz|^2 + \bar{Q}d\bar{z}^2, \\ -\langle dX, dN\rangle & = & 2\rho|dz|^2, \\ \langle dN, dN\rangle & = & -Q\, dz^2 + 2\mu|dz|^2 - \bar{Q}d\bar{z}^2, \end{array} \right. \tag{5.3}$$

where by definition

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

and

(1) $Qdz^2 := \langle X_z, X_z\rangle dz^2 = -\langle N_z, N_z\rangle dz^2$ is a holomorphic quadratic differential on Σ, which we call the *extrinsic Hopf differential* of the surface.
(2) $\mu, \rho : \Sigma \to (0, \infty)$ are smooth positive real functions, which by the condition $K = 1$ satisfy the relation

$$\rho^2 = \mu^2 - |Q|^2. \tag{5.4}$$

From Q and ρ we can define another relevent geometric function, namely, the real analytic function ω given by

$$\rho = |Q| \sinh \omega. \tag{5.5}$$

This function is well defined, and positive when $Q \neq 0$, and takes the value $\omega = +\infty$ at the isolated zeros of Q. Moreover, it satisfies the equation

$$\omega_{z\bar{z}} + |Q| \sinh \omega = 0 \qquad \text{(or, equivalently, } \omega_{z\bar{z}} + \rho = 0). \tag{5.6}$$

Finally, the mean curvature of X is given by any of these formulas:

$$H = \frac{\rho\,\mu}{\mu^2 - |Q|^2} = \frac{\mu}{\rho} = \coth(\omega). \tag{5.7}$$

For later use, let us also state here the following boundary regularity result by Jacobowsky.

Lemma 5.1 ([**Jac**]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial\Omega$ is C^∞. Let $X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ be a solution to the Dirichlet problem*

$$X_{uu} + X_{vv} = 2\,X_u \times X_v \qquad \text{in } \Omega, \qquad\qquad X = \varphi \qquad \text{on } \partial\Omega,$$

where $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$. Assume that $X_k \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and also that each X_k lies in the Sobolev space $H^1(\Omega) \equiv W^{1,2}(\Omega)$, for $k = 1, 2, 3$. Then $X_k \in C^\infty(\overline{\Omega})$.

6. Isolated singularities of the constant curvature equation

We focus next on the local classification of isolated singularities to (1.3). So, to start, we will only consider the case of constant curvature graphs. The more general case of immersed isolated singularities of K-surfaces in $\mathbb{R}^3$ will be explained in the next section.

6.1. Characterization of removable singularities. The first step is to understand in terms of the underlying conformal structure and the geometric data on the graph when an isolated singularity is removable. The next result was basically proved by Heinz and Beyerstedt [**HeBe**], but we sketch here a different proof by the authors and L. Hauswirth [**GHM**] which is also suitable for the general case of immersed isolated singularities (see Theorem 7.2). Indeed, we will only be using here that the surface is embedded and has finite area around the singularity, and not that it is a graph. A similar result holds even without the embeddedness and finite area conditions, see the next section.

Theorem 6.1. *Let $u(x,y) : D^* \to \mathbb{R}$ denote a C^2 solution of the constant curvature equation (1.3) for $K = 1$, with an isolated singularity at the puncture $q = (a,b) \in D$. The following conditions are equivalent.*

(i) *The isolated singularity q is removable.*
(ii) *The mean curvature of the graph of u is bounded around the singularity.*
(iii) *The graph of u has around the singularity the underlying conformal structure of a punctured disk.*

Proof. It is obvious that (i) implies (iii). Let us prove first that (iii) implies (i).

If the graph has the underlying conformal structure of the punctured disk $\mathbb{D}^*$, then as it has finite area, it holds

$$\int_{\mathbb{D}^*} (\langle N_u, N_u\rangle + \langle N_v, N_v\rangle)\, dudv < \infty,$$

i.e. $N \in \mathbb{H}^1(\mathbb{D}^*, \mathbb{S}^2) \equiv W^{1,2}(\mathbb{D}^*, \mathbb{S}^2)$. So, by Helein's regularity theorem [**Hel**] for harmonic maps into $\mathbb{S}^2$, N can be harmonically extended to $\mathbb{D}$. The surface X is then extended accordingly by means of (5.1). If dN is non-singular at 0, then X is immersed at 0, and so the singularity is removable. In contrast, if dN is singular at 0, a result by Wood [**Woo**] gives that, as dN needs to be non-singular in $\mathbb{D}^*$, N has a branch point at z_0. That is, there are local coordinates (x,y) and

(u,v) on Σ and $\mathbb{S}^2$ centered at z_0 and $N(z_0)$, respectively, such that N has the form $u+iv=(x+iy)^k$, for some integer $k>1$. Thus N is non-injective on a neighborhood of the singularity, which contradicts convexity.

The proof that (iii) implies (ii) is an application of the maximum principle for the function $\omega:\mathbb{D}^*\to(0,\infty]$ given by (5.5).

Denote $\omega_0=\min\{\omega(\zeta):|\zeta|=1\}$. Let $\{r_n\}$ be a strictly decreasing sequence of real numbers $r_n\in(0,1)$, tending to 0. Let h_n denote the unique harmonic function on the annulus

$$\mathbb{A}_n=\{z\in\mathbb{C}:r_n\leq|z|\leq 1\},$$

with the Dirichlet conditions $h_n=\omega_0$ on $\mathbb{S}^1$ and $h_n=0$ on $\{\zeta:|\zeta|=r_n\}$. Since by (5.6), we have $\omega_{z\bar{z}}\leq 0$, then by the maximum principle we get

$$0\leq h_n(z)\leq h_{n+1}(z)\leq\omega(z),\qquad\forall\ n\in\mathbb{N}\text{ and }z\in\mathbb{C}\text{ with }r_n\leq|z|\leq 1.$$

Thereby, we see that $\{h_n\}$ converge to some harmonic function h on $\mathbb{D}^*\cup\mathbb{S}^1$ which is constantly equal to ω_0 on $\mathbb{S}^1$. But as h is bounded, we deduce that $h(z)\equiv\omega_0$ on $\overline{\mathbb{D}}$. So, we get $\omega(z)\geq\omega_0$ for every $z\in\overline{\mathbb{D}}$. This implies from (5.7) that H is bounded on $\mathbb{D}^*$, as wished.

Finally, to prove that (ii) implies (iii) we argue by contradiction. Assume that the underlying conformal structure around the singularity is that of an annulus. We denote this conformal parametrization of the graph by X, and the conformal coordinate of the annulus $\mathbb{A}$ by $z=u+iv$. Since H is bounded and the graph has finite area around the singularity (by convexity), we get by (5.3)

$$\int_{D^*}H\,dA=\int_{\mathbb{A}}\frac{\mu}{\rho}\,\rho\,dudv=\int_{\mathbb{A}}(\langle X_u,X_u\rangle+\langle X_v,X_v\rangle)dudv<\infty.$$

Now, since X satisfies (5.2), Lemma 5.1 shows that $X(u,v)$ can be extended smoothly to the boundary of $\mathbb{A}$, so that it is constant there. But by (5.3) we get then that $\mu^2-|Q|^2=0$ on this boundary. And as

$$\frac{|Q|^2}{\mu^2-|Q|^2}=H^2-1<\infty$$

on $\mathbb{A}$ since H is bounded by hypothesis, we deduce that Q vanishes on this boundary curve. Thus $Q=0$ everywhere, i.e. the graph is a piece of a round sphere with the underlying conformal structure of an annulus and that is constant on a boundary curve of the annulus. This is impossible, and finishes the proof. □

6.2. The classification theorem. In order to classify the isolated singularities of the constant curvature equation (1.3), we need to explain a couple of details first:

1. (1) We will consider that two solutions to (1.3) are *equivalent* if, possibly after a translation in the space, their graphs agree on an open set. This is a very natural equivalence, since the solutions to (1.3) are real analytic when parametrized in terms of their underlying conformal structure.
2. (2) By convexity, if a solution to (1.3) has an isolated singularity, then its graph has a *limit unit normal* at the singularity, which consists of a convex (but maybe not strictly convex, in principle) Jordan curve in $\mathbb{S}^2$.
3. (3) Any convex Jordan curve on $\mathbb{S}^2$ lies on a hemisphere, that we can assume without loss of generality to be the upper hemisphere $\mathbb{S}^2_+$.

The next theorem gives the local classification of the isolated singularities of (1.3) up to equivalence in accordance with the previous comment. It basically tells that the limit unit normal at such an isolated singularity is a regular real analytic strictly convex Jordan curve, and that any such curve can be realized as the limit unit normal of a unique isolated singularity. Again, the construction also works for isolated singularities of immersed K-surfaces, not necessarily graphs, see Section 7.

THEOREM 6.2 ([**GHM**]). *Let α be a regular, real analytic strictly convex Jordan curve in $\mathbb{S}^2_+$. Then, there is exactly one solution to* (1.3) *having a non-removable singularity at the origin, and whose limit unit normal at the singularity is α.*

Conversely, the limit unit normal of any non-removable isolated singularity to (1.3) *is a regular, real analytic, strictly convex Jordan curve in $\mathbb{S}^2_+$. Thus, any non-removable isolated singularity of* (1.3) *is equivalent to one of the singularities constructed above.*

PROOF. Let $\mathcal{S}$ be the graph of a non-removable isolated singularity of a solution to (1.3). By Theorem 6.1, its underlying conformal structure is that of an annulus. Hence, we may parametrize $\mathcal{S}$ by a map $X : \mathcal{U}^+ \subset \mathbb{C} \to \mathbb{R}^3$, so that $\mathcal{S} = X(\mathcal{U}^+)$, where $\mathcal{U}^+ := \{z \in \mathbb{C} : 0 < \mathrm{Im} z < \delta\}$ for some $\delta > 0$, and X is 2π-periodic. Moreover, X is real analytic and extends continuously to the boundary $\mathbb{R} \subset \partial\mathcal{U}^+$ with $X(u, 0) = 0$ for all $u \in \mathbb{R}$.

As the singularity has finite area we have

$$\int_{\mathcal{U}^+} (\langle X_u, X_u\rangle + \langle X_v, X_v\rangle) du\, dv < \infty,$$

and so we see that $X \in H^1(\mathcal{U}^+, \mathbb{R}^3) \equiv W^{1,2}(\mathcal{U}^+, \mathbb{R}^3)$. In these conditions Lemma 5.1 ensures that X extends smoothly up to the boundary. It follows then that the extension of X to

$$\mathcal{U} := \{z \in \mathbb{C} : -\delta < \mathrm{Im}\, z < \delta\} \tag{6.1}$$

given by $X(u, -v) = -X(u, v)$ is a real analytic map with $X(u, 0) = 0$. Also, it can be proved that the unit normal $N(u, v)$ can be extended to $\mathcal{U}$ by $N(u, v) = N(u, -v)$. Thus, denoting $\alpha(u) := N(u, 0)$, we see that α is real analytic and 2π periodic.

Let us prove now that α is a regular strictly convex Jordan curve. First, observe that the zeros of Q are isolated on $\mathcal{U}$, and that on $\mathbb{R}$ we have $||\alpha'(u)||^2 = |Q(u, 0)|$. Also, observe that the function ρ in (5.3) vanishes on Ω exactly at the points in the real axis (since in the general case, it vanishes at the singular points of X). This tells that the function ω given by (5.5) also satisfies $\omega(u, 0) = 0$. Moreover, a computation using (5.5), (5.1) and the condition $X_u(u, 0) = 0$ provides the relation

$$\omega_v(u, 0) = ||\alpha'(u)||k_\alpha(u), \tag{6.2}$$

where here $k_\alpha(u)$ stands for the geodesic curvature in $\mathbb{S}^2$ of $\alpha(u)$. This equation implies that

$$\lim_{u\to u_0} ||\alpha'(u)||k_\alpha(u) = c_0 = c_0(u_0) \in \mathbb{R},$$

exists for every $u_0 \in \mathbb{R}$, and we want to ensure that $c_0 \neq 0$. For that, observe that if $c_0 = 0$ for some u_0, then $\nabla\omega(u_0) = 0$. Now, as ω satisfies the elliptic PDE (5.6), this implies that there are at least two nodal curves of ω passing through u_0 (one of which is the real line). But as the zeros of ω are singular points of the surface, this contradicts the fact that X is regular on $\mathcal{U}\setminus\mathbb{R}$, i.e. the fact that the singularity is isolated.

Finally, if $\alpha'(u_0)=0$ at some point, the condition $c_0\neq 0$ given above shows that α has a cusp type singularity at u_0. But as α is convex (and so embedded), this is not possible. Hence α must be a regular curve, as wished.

Conversely, let $\alpha:\mathbb{R}/(2\pi\mathbb{Z})\to\mathbb{S}^2$ be a real analytic, closed regular curve in $\mathbb{S}^2_+$. It follows then from the Cauchy-Kowalevsky theorem applied to the equation for harmonic maps into $\mathbb{S}^2$ that there exists a unique harmonic map $N(u,v)$ defined on an open set $\mathcal{U}\subset\mathbb{R}^2\equiv\mathbb{C}$ containing $\mathbb{R}$, such that

$$N(u,0)=\alpha(u),\qquad N_v(u,0)=0, \tag{6.3}$$

for all $u\in\mathbb{R}$. Moreover, as α is 2π-periodic, so is N (by uniqueness).

Let $X:\mathcal{U}\to\mathbb{R}^3$ be the map given by the representation formula (5.1) in terms of the harmonic map $N:\mathcal{U}/2\pi\mathbb{Z}\to\mathbb{S}^2$ with initial conditions (6.3), where $\mathcal{U}\subset\mathbb{C}$ is given by (6.1) for some $\delta>0$. It is then immediate that $X(u,0)=p$ for some $p\in\mathbb{R}^3$, so it follows from (5.1) and the periodicity of N that X is also 2π-periodic. Since $\alpha(u)$ is regular and convex, the map $\omega:\Omega\to\mathbb{R}$ given by (5.5) satisfies (6.2). Hence, $\omega_v(u,0)\neq 0$ for all $u\in\mathbb{R}$, and this implies that $X:\mathcal{U}^+/2\pi\mathbb{Z}\to\mathbb{R}^3$ is regular, by taking a smaller $\delta>0$ in the definition of $\mathcal{U}$, if necessary, and where $\mathcal{U}^+:=\mathcal{U}\cap\{\operatorname{Im} z>0\}$.

So, X is a $K=1$ surface in $\mathbb{R}^3$ that is regularly immersed around p, but that does not extend smoothly across the point (since it has the conformal type of an annulus). To finish the theorem we need to show that X is a graph.

For that, we use the *Legendre transform* (see [**LSZ**])

$$\mathcal{L}_X=\left(\frac{-N_1}{N_3},\frac{-N_2}{N_3},-X_1\frac{N_1}{N_3}-X_2\frac{N_2}{N_3}-X_3\right):\Omega^+\to\mathbb{R}^3, \tag{6.4}$$

where $X=(X_1,X_2,X_3)$ and $N=(N_1,N_2,N_3)$. It is classically known that $\mathcal{L}_X$ can be defined for convex multigraphs in the x_3-axis direction, so that $\mathcal{L}_X$ is also a convex multigraph in the x_3-axis direction. The interior unit normal of $\mathcal{L}_X$ is

$$\mathcal{N}_{\mathcal{L}}=\frac{1}{\sqrt{1+X_1^2+X_2^2}}\left(-X_1,-X_2,1\right). \tag{6.5}$$

In our case, as α lies on $\mathbb{S}^2_+$ it is clear that $X|_{\mathcal{U}^+}$ is a multigraph in the x_3-axis direction. So, if $\mathcal{L}_X:\mathcal{U}^+/2\pi\mathbb{Z}\to\mathbb{R}^3$ denote its Legendre transform, $\mathcal{L}_X(\mathbb{R}/2\pi\mathbb{Z})$ is a regular convex Jordan curve in the x_1,x_2-plane, and the unit normal of $\mathcal{L}_X$ along $\mathbb{R}$ is $(0,0,1)$, constant.

Therefore, $\mathcal{L}_X$ lies in the upper half-space $\mathbb{R}^3_+$, and there is some $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$ the intersection $\Upsilon_\varepsilon=\mathcal{L}_X(\mathcal{U}/2\pi\mathbb{Z})\cap\{x_3=\varepsilon\}$ is a regular convex Jordan curve. Consider now $S_{\varepsilon_1,\varepsilon_2}$ the portion of $\mathcal{L}_X$ that lies in the slab between the planes $\{x_3=\varepsilon_1\}$ and $\{x_3=\varepsilon_2\}$, where $0<\varepsilon_2<\varepsilon_1<\varepsilon_0$. Then, as $S_{\varepsilon_1,\varepsilon_2}$ is convex and the curves Υ_ε are convex Jordan curves, we get that the unit normal $\mathcal{N}_{\mathcal{L}}$ of $\mathcal{L}_X$ in this slab is a global diffeomorphism onto its image in $\mathbb{S}^2$. Letting $\varepsilon_1\to 0$ and choosing $\delta>0$ sufficiently small, we get that $\mathcal{N}_{\mathcal{L}}$ is a global diffeomorphism from $\mathcal{U}^+/2\pi\mathbb{Z}$ onto its spherical image in $\mathbb{S}^2$.

Consequently, by (6.5), $X(\mathcal{U}^+)$ is a graph over a region in the x_1,x_2-plane. This concludes the proof of the theorem. $\square$

The above theorem provides a correspondence between the space of real analytic convex Jordan curves on $\mathbb{S}^2$ and the space of non-removable isolated singularities of (1.3), up to equivalence.

It is remarkable that all these isolated singularities of graphs are *conical singularities*. Recall that, by definition, a conformal Riemannian metric $\lambda|dz|^2$ on $\mathbb{D}^*$ has a *conical singularity* of angle $2\pi\theta$ at 0 if

$$\lambda = |z|^{2\beta} f \, |dz|^2,$$

where f is a continuous positive function on $\mathbb{D}$ and $\beta = \theta - 1 > -1$.

In our case, it can be proved that *if $\mathcal{S}$ is the graph of a solution to* (1.3) *having an isolated singularity at the point p, then its induced metric has a conical singularity at p.* Moreover, the angle of this conical singularity is easy to compute: it is just $2\pi - \mathcal{A}(\alpha) \in (0, 2\pi)$, where $\mathcal{A}(\alpha)$ is the area of the convex region of $\mathbb{S}^2_+$ bounded by the convex Jordan curve α.

7. Immersed isolated singularities

From a geometrical point of view, it is very natural to deal with surfaces rather than graphs. So, in this section we will briefly explain the generalization of Theorems 6.1 and 6.2 to the case of isolated singularities of immersed surfaces of constant curvature $K > 0$ in $\mathbb{R}^3$.

We will state the results without proofs, in part because our main concern in these notes is the study of graphs, and in part because the key ideas in these proofs were already presented in Section 6.

The definition of an isolated singularity for an immersed surface in $\mathbb{R}^3$ is the following one.

Definition 7.1. Let $\psi : D^* \to \mathbb{R}^3$ denote an immersion of a punctured disc D^* into $\mathbb{R}^3$, and assume that ψ extends continuously to D. Then, the surface ψ is said to have an *isolated singularity* at $p = \psi(q) \in \mathbb{R}^3$.

If ψ is an embedding around q, p will be called an *embedded isolated singularity.* The singularity is called *extendable* if ψ and its unit normal N extend smoothly to D, and *removable* if it is extendable and $\psi : D \to \mathbb{R}^3$ is an immersion.

Let us point out that, for the case of strictly locally convex surfaces in $\mathbb{R}^3$, any surface having an embedded isolated singularity is actually a graph around the singularity (see [**GaMi**]). Thus, for K-surfaces in $\mathbb{R}^3$, this case is already covered by the results in the previous section.

The following removable singularity theorem for K-surfaces in $\mathbb{R}^3$ provides a generalization of Theorem 6.1.

Theorem 7.2 ([**GHM**]). *Let $\psi : D^* \to \mathbb{R}^3$ be an immersed K-surface with an isolated singularity at $p = \psi(q)$. The following conditions are equivalent.*

(i) *The isolated singularity p is extendable.*
(ii) *The mean curvature of ψ is bounded around the singularity.*
(iii) *ψ has around the singularity the extrinsic conformal structure of a punctured disk.*
(iv) *The singularity p is removable, or it is a branch point.*

The local classification of non-removable isolated singularities of solutions to (1.3) given in Theorem 6.2 can also be suitably extended to the case of immersed singularities of K-surfaces in $\mathbb{R}^3$. For that, we will need the following terminology.

Definition 7.3. A non-extendable isolated singularity of a K-surface in $\mathbb{R}^3$ is called an *immersed conical singularity* if the surface has finite area around the singularity.

Definition 7.4. A smooth map $\alpha : I \subset \mathbb{R} \to \mathbb{S}^2 \subset \mathbb{R}^3$ is called a *locally convex curve with admissible cusps* if, for every $s \in I$, the quantity $||\alpha'(s)||k_\alpha(s)$ is a non-zero real number. Here $k_\alpha(s)$ is the geodesic curvature of α in $\mathbb{S}^2$, i.e.

$$k_\alpha(s) = \frac{\langle \alpha''(s), J\alpha'(s)\rangle}{||\alpha'(s)||^3}$$

where J denotes the complex structure of $\mathbb{S}^2$.

Any regular locally strictly convex curve in $\mathbb{S}^2$ satisfies this property, since for regular points the condition of the definition is just that $k_\alpha \neq 0$.

Once here, Theorem 6.2 can be generalized to the case of immersed conical singularities as follows:

Theorem 7.5 ([**GHM**]). *Any immersed conical singularity of a K-surface in $\mathbb{R}^3$ has a well defined limit unit normal at the singularity, which is a real analytic closed locally convex curve with admissible cusps in $\mathbb{S}^2$.*

Conversely, given any such curve α, and an arbitrary point $p \in \mathbb{R}^3$, there is a unique K-surface in $\mathbb{R}^3$ with an immersed conical singularity at p and whose limit unit normal at the singularity is α.

Here, as usual, we are identifying two surfaces with the same isolated singularity if they overlap over a non-empty regular open set.

8. Peaked spheres in $\mathbb{R}^3$

Definition 8.1. A *peaked sphere* in $\mathbb{R}^3$ is a closed convex surface $S \subset \mathbb{R}^3$ (i.e. the boundary of a bounded convex set of $\mathbb{R}^3$) that is a regular surface everywhere except for a finite set of points $p_1, \dots, p_n \in S$, and such that $S \setminus \{p_1, \dots, p_n\}$ has constant curvature 1.

The points $p_1, \dots, p_n$ are called the *singularities* of the peaked sphere S.

Equivalently, a peaked sphere can also be defined as an embedding

$$\phi : \mathbb{S}^2 \setminus \{q_1, \dots, q_n\} \to \mathbb{R}^3$$

of constant curvature 1, such that ϕ extends continuously to $\mathbb{S}^2$. If ϕ does not C^1-extend across q_j, then $p_j := \phi(q_j) \in \mathbb{R}^3$ is a singularity of $S := \phi(\mathbb{S}^2) \subset \mathbb{R}^3$. These singularities are actually conical singularities, according to the definition in the previous section.

Peaked spheres are the most natural K-surfaces in $\mathbb{R}^3$ from several points of view. Indeed, as the only complete K-surfaces are round spheres, in general K-surfaces have singularities. So, the case that there is only a finite number of them is the most regular situation to be considered. Also, those isolated singularities can be thought of as *ends* of K-surfaces in $\mathbb{R}^3$, since this theory does not admit complete ends (for instance, the exterior Dirichlet problem for (1.3) does not have a solution).

A peaked sphere without singularities is a round sphere, and there are no peaked spheres with exactly one singularity. Also, any peaked sphere with two singularities is rotational, by a simple application of Alexandrov reflection principle. The case for more than two singularities is explained in the following theorem.

THEOREM 8.2. *Let Λ denote a conformal structure of $\mathbb{S}^2$ minus n points, $n > 2$, and let $\theta_1, \dots, \theta_n \in (0,1)$. Then, a necessary and sufficient condition for the existence of a peaked sphere $S \subset \mathbb{R}^3$ with n singularities $p_1, \dots, p_n$ of given conic angles $2\pi\theta_1, \dots, 2\pi\theta_n$, and such that Λ is the conformal structure of $S \setminus \{p_1, \dots, p_n\}$ for its intrinsic metric, is that*

$$n - 2 < \sum_{j=1}^{n} \theta_j < n - 2 + \min_j\{\theta_j\}. \tag{8.1}$$

Moreover, any peaked sphere in $\mathbb{R}^3$ is uniquely determined up to rigid motions by the conformal structure of $S \setminus \{p_1, \dots, p_n\}$ and by the cone angles $2\pi\theta_1, \dots, 2\pi\theta_n$.

In particular, the space of peaked spheres in $\mathbb{R}^3$ with $n > 2$ singularities is a $3n - 6$ parameter family, modulo rigid motions.

The theorem above is a consequence of three theorems. First, the intrinsic classification of cone metrics of constant positive curvature on $\mathbb{S}^2$ whose cone angles lie in $(0, 2\pi)$ by Troyanov [**Tro**] and Luo-Tian [**LuTi**]. Second, the general isometric embedding theorem of singular metrics of non-negative curvature in $\mathbb{R}^3$ by Alexandrov. And third, Pogorelov's local regularity result for these Alexandrov embeddings (see [**BuSh**] for the details).

Thus, peaked spheres are well understood from an intrinsic point of view. From an extrinsic point of view, the results of the previous sections show that their extrinsic conformal structure is that of a bounded circular domain, and that their unit normal has vanishing normal derivative along any of its boundary curves. Taking into account that this unit normal is a harmonic diffeomorphism, we can conclude some analytic consequences.

We will say that a harmonic map $g : \Omega \cup \partial\Omega \to \mathbb{S}^2$ from a bounded circular domain into $\mathbb{S}^2$ is a solution to the *Neumann problem for harmonic diffeomorphisms* if it is a diffeomorphism onto its image, and satisfies along each boundary curve that

$$\left.\frac{\partial g}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \qquad (\mathbf{n} \text{ is the exterior normal derivative along } \partial\Omega). \tag{8.2}$$

Observe that harmonicity is a conformal invariant, so the circular domain is giving a conformal equivalence class rather than a specific symmetric domain.

From the above classification of peaked spheres and the local results on isolated singularities, we obtain:

THEOREM 8.3 ([**GHM**]). *A harmonic map $g : \Omega \to \mathbb{S}^2$ is a solution to the Neumann problem for harmonic diffeomorphisms if and only if it is the Gauss map of a peaked sphere in $\mathbb{R}^3$, with respect to its extrinsic conformal structure.*

As a consequence, the spaces of harmonic maps into $\mathbb{S}^2$ that solve the above Neumann problem for some bounded circular domain with $n > 2$ boundary components is a $3n - 6$ dimensional family (here the circular domain Ω is not *fixed; only the number n is).*

As a consequence, as peaked spheres with two singularities are rotational, we obtain:

COROLLARY 8.4. Let $\mathbb{A}(r, R)$ be the annulus $\{z : r < |z| < R\}$. Then, the Neumann problem for harmonic diffeomorphisms $g : \mathbb{A}(r, R) \to \mathbb{S}^2$ has a solution if and only if $R/r > e^{\pi}$.

In that case, the solution is unique and radially symmetric.

We also point out that the parallel CMC surface to a peaked sphere provides a solution to a free boundary problem for CMC surfaces, in which the surface is asked to meet a configuration of $n \geq 2$ spheres of an adequate radius tangentially along each sphere.

9. Further results and open problems

We close these notes with some related results of different geometric theories, together with some open problems and some possible lines of further inquiry.

9.1. Constant mean curvature surfaces in Minkowski space. Let $\mathbb{L}^3$ denote the Minkowski 3-space, that is, $\mathbb{R}^3$ endowed with the Lorentzian metric given in canonical coordinates by

$$\langle , \rangle = dx^2 + dy^2 - dz^2.$$

A graph $u = u(x, y)$ in $\mathbb{L}^3$ has an induced Riemannian metric if and only if

$$u_x^2 + u_y^2 < 1 \tag{9.1}$$

at every point. In that case, the graph is said to be *spacelike*. Besides, a spacelike graph has constant mean curvature H if it satisfies the following quasilinear PDE, together with the spacelike condition (9.1), which is actually the ellipticity condition of the equation.

$$(1 - u_y^2)u_{xx} + 2u_x u_y u_{xy} + (1 - u_x^2)u_{yy} = 2H(1 - u_x^2 - u_y^2)^{3/2}. \tag{9.2}$$

The above equation admits isolated singularities, called *conelike singularities*. In these singularities there is not a well defined tangent plane, and the ellipticity condition (9.1) does not hold, i.e. $u_x^2 + u_y^2$ tends to 1 at those points.

For the case of *maximal graphs*, that is, the case $H = 0$, these singularities are well studied objects. A complete classification of entire maximal graphs with an arbitrary finite number of singularities was obtained by Fernández, López and Souam in [**FLS1**]. Their result generalizes previous theorems by Calabi [**Cal**] (*the only entire C^2 maximal graphs in $\mathbb{L}^3$ are planes*) and Kobayashi [**Kob**] (*the only entire maximal graphs in $\mathbb{L}^3$ with exactly one singularity are rotational examples*). There are many other interesting works on the classification of maximal graphs with conelike singularities, see [**FLS2, Fer, Kly, KlMi**].

In the case $H \neq 0$, not much is known as regards conelike singularities. One of the reasons is that there is a very large family of entire solution to (9.2), so the motivation is not as obvious as in the case $H = 0$, in which entire solutions are just linear functions. However, in the view of the previous results, the following two questions appear as natural and interesting:

(1) *Is it possible to give a local classification theorem for conelike singularities of solutions to* (9.2)*, in terms of their gradient curve at the singularity?*
(2) *Is it possible to classify the entire solutions to* (9.2) *with a finite number of conelike singularities?*

Let us also remark that CMC surfaces with singularities in Minkowski space $\mathbb{L}^3$ are closely related to *regular* CMC surfaces in Riemannian homogeneous 3-manifolds with a 4-dimensional isometry group. This follows from the works of Fernández and Mira [**FeMi1, FeMi2**] and Daniel [**Dan**], in which they study the existence and geometric applications of harmonic Gauss maps into the Poincaré disk for CMC surfaces of *critical mean curvature* in these homogeneous spaces. For the reader interested in this rapidly developing theory of CMC surfaces in homogeneous manifolds, we refer to [**FeMi3**] and the lecture notes [**DHM**].

9.2. Flat surfaces in hyperbolic 3-space. The theory of flat surfaces (i.e. surfaces of zero Gaussian curvature) in hyperbolic 3-space $\mathbb{H}^3$ has many analogies with that of K-surfaces in $\mathbb{R}^3$. For instance, flat surfaces in $\mathbb{H}^3$ are locally convex, and they admit a conformal representation in terms of two holomorphic functions with respect to their extrinsic conformal structure (see [**GMMi**]). Also, the complete flat surfaces in $\mathbb{H}^3$ are well known after the Volkov-Vladimirova-Sasaki theorem (see [**Sas, VoVl**]): the only ones are horospheres and hyperbolic cylinders.

It is then natural to study from a global perspective the geometry of flat surfaces in $\mathbb{H}^3$ in the presence of singularities. The study of flat surfaces with wave-front singularities in $\mathbb{H}^3$ was started by Kokubu, Umehara and Yamada [**KUY**], and continued afterwards in several other papers, among which we may quote [**KRSUY**]. Those singularities generically form curves on the surface, but again there exist isolated conical singularities for flat surfaces in $\mathbb{H}^3$.

A complete classification of the isolated singularities of flat surfaces in $\mathbb{H}^3$ was obtained by the authors in [**GaMi**], in terms of the behavior of the limit unit normal at the singularity. Thus, the local problem is fully solved, but there are some important global problems that remain open.

In [**CMM**], Corro, Martínez and Milán classified the complete embedded flat surfaces in $\mathbb{H}^3$ with exactly two singularities and one end (for the case of one singularity, it is easy to see that the only examples are rotational). For that, they used the relationship between flat surfaces in $\mathbb{H}^3$ and the solutions to the Hessian one equation (1.2), that comes from the following fact: if we take local coordinates (x, y) on a flat surface in $\mathbb{H}^3$ such that its first fundamental form is written as $I = dx^2 + dy^2$, then there exists a solution $u(x, y)$ to the Hessian one equation (1.2) such that the second fundamental form of the surface is given by

$$II = u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2.$$

Still, the following important problems remain unanswered.

(1) *Are there* compact *flat surfaces in* $\mathbb{H}^3$ *with only a finite number of singularities?*
(2) *Is is possible to classify the complete embedded flat surfaces in* $\mathbb{H}^3$ *with one end and a finite number* $n > 2$ *of singularities?*

9.3. Global problems for peaked spheres in $\mathbb{R}^3$. Recall that a peaked sphere in $\mathbb{R}^3$ is a closed convex K-surface that is everywhere regular except for a finite number of points, which are the singularities of the sphere. The extrinsic conformal structure of such a peaked sphere is that of a bounded circular domain in $\mathbb{C}$.

The following three problems are important open questions in what regards the geometry and classification of peaked spheres in $\mathbb{R}^3$.

(1) *Which bounded circular domains in* $\mathbb{C}$ *are realizable as the extrinsic conformal structure of a peaked sphere in* $\mathbb{R}^3$*? Is a peaked sphere uniquely determined by its extrinsic conformal structure?*

(2) *Find necessary and sufficient conditions for a set of points $p_1, \dots, p_n \in \mathbb{R}^3$ to be realized as the set of singularities of a peaked sphere in $\mathbb{R}^3$. Are two peaked spheres with the same singularities $p_j \in \mathbb{R}^3$ necessarily the same?*
(3) *Can one realize any conformal metric of constant curvature 1 on $\mathbb{S}^2$ with a finite number of conical singularities as the intrinsic metric of an immersed $K = 1$ surface in $\mathbb{R}^3$?*

The first problem has a strong connection with the Neumann problem for harmonic diffeomorphisms. Indeed, a classification of peaked spheres in terms of their extrinsic conformal structure would solve completely the Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$. Problem 2 is connected with the free boundary problem for CMC surfaces mentioned above, in the case that one wishes to prescribe the centers of the spheres (and not just the number of spheres and their common radius). In any case, it is clear that an arbitrary configuration of n points will not be in general the singular set of a peaked sphere in $\mathbb{R}^3$.

As regards the third problem, let us point out that the results by Alexandrov and Pogorelov show that these metrics are realized as the intrinsic metric of peaked spheres, *provided* all conical angles are in $(0, 2\pi)$, but there are many other abstract cone metrics. Such an isometric realization in $\mathbb{R}^3$ must necessarily be non-embedded. It must be emphasized that, by our local study, any conical singularity of arbitrary angle can be realized as an immersed $K = 1$ surface in $\mathbb{R}^3$ in many different ways.

Let us also remark that a complete classification for conformal metrics of positive constant curvature on $\mathbb{S}^2$ with n conical singularities remains open if $n > 3$ (see [**UmYa, Ere**] for the case of three conical singularities).

9.4. More general PDEs. A final goal of the approach considered here regarding the description of non-removable isolated singularities is to check its validity for the case of general elliptic PDEs in two variables, that are either elliptic equations of Monge-Ampère type, or degenerate elliptic quasilinear equations. It seems that several of the ideas coming from geometry or complex analysis that we have explained here for some specific geometric PDEs can be useful in a much wider context.

This is an ongoing project of the authors with their Ph.D. student Asun Jiménez [**GJM**].

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