

# The role of minimal surfaces in the study of the Allen-Cahn equation

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ABSTRACT. In these lectures, we review some recent results on the existence of solutions of the Allen-Cahn equation  $\varepsilon^2 \Delta_g u + u - u^3 = 0$  in a Riemannian manifold  $(M, g)$ . In the case where the ambient manifold  $M$  is compact, we also provide a proof of the existence of such solutions whose zero set is close to a given minimal hypersurface.

Given a  $(n + 1)$ -dimensional Riemannian manifold  $(M, g)$ , we are interested in solutions of the semilinear elliptic equation

$$(0.1) \quad \varepsilon^2 \Delta_g u + (1 - u^2) u = 0,$$

which is known as the *Allen-Cahn equation* following [2]. Here  $\varepsilon > 0$  is a parameter. The ambient manifold  $M$  which might be compact or non compact but, for the sake of simplicity, we will assume that  $M$  does not have a boundary. This problem has its origin in the *gradient theory of phase transitions* [2] where one is interested in critical points of the energy

$$E_\varepsilon(u) := \frac{\varepsilon}{2} \int_M |\nabla^g u|_g^2 \, \text{dvol}_g + \frac{1}{4\varepsilon} \int_M (1 - u^2)^2 \, \text{dvol}_g,$$

and one easily sees that (0.1) is the Euler-Lagrange equation of  $E_\varepsilon$ . We will see in these lectures that the space of solutions of (0.1) is surprisingly rich and also that it has an interesting structure. As we will see, minimal hypersurfaces play a key role in the understanding of the set of solutions of (0.1). We will review the different technics leading to existence of solutions and we will also provide a simplified proof of an existence result which was first published in [44] and which is at the origin of most of the constructions in the field.

## 1. The role of minimal hypersurfaces

The relation between *sets of minimal perimeter* and critical points of  $E_\varepsilon$  was first established by Modica in [41]. Let us briefly recall the main results in this

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direction : If  $u_\varepsilon$  is a family of *local minimizers* of  $E_\varepsilon$  whose energy is controlled, namely for which

$$(1.1) \quad \sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon) < +\infty,$$

then, up to a subsequence,  $u_\varepsilon$  converges as  $\varepsilon$  tends to 0, in  $L^1$  to  $\mathbf{1}_\Gamma - \mathbf{1}_{\Gamma^c}$ , where  $\partial\Gamma$  has minimal perimeter. Here  $\mathbf{1}_\Gamma$  (resp.  $\mathbf{1}_{\Gamma^c}$ ) is the characteristic function of the set  $\Gamma$  (resp.  $\Gamma^c = M - \Gamma$ ). Moreover,

$$E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}} \mathcal{H}^n(\partial\Gamma).$$

For *critical points* of  $E_\varepsilon$  which satisfy (1.1), a related assertion is proven in [30]. In this case, the convergence of the interface holds with certain integer multiplicity to take into account the possibility of multiple transition layers converging to the set of minimal perimeter (or to the same minimal hypersurface).

These results provide a link between solutions of (0.1) and the theory of minimal hypersurfaces which has been exploited to construct nontrivial solutions of (0.1).. For example, solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [44] (see also [33] for the Euclidean case). Let us describe these results more carefully. We assume that  $(M, g)$  is a compact Riemannian  $(n + 1)$ -dimensional manifold without boundary (in the case where  $M$  has a boundary one requires that the solution  $u$  of (0.1) has 0 Neumann boundary condition) and that  $\Gamma \subset M$  is an oriented minimal hypersurface which *separates*  $M$  into two different connected components. Namely,  $\Gamma$  is the zero set of a smooth function  $f_\Gamma$  which is defined on  $M$  and for which 0 is a regular value. Then  $M \setminus \Gamma = M_+ \cup M_-$  where

$$M_+ := f_\Gamma^{-1}((0, \infty)) \quad \text{and} \quad M_- := f_\Gamma^{-1}((-\infty, 0)).$$

We also assume that,  $N$ , the unit normal vector field of  $\Gamma$  which is compatible with the given orientation, points towards  $M_+$  while  $-N$  points toward  $M_-$  ( $\Gamma$  might have many connected components).

We recall the definition of the Jacobi operator about  $\Gamma$

$$(1.2) \quad J_\Gamma := \Delta_{\hat{g}} + |A_\Gamma|^2 + \text{Ric}_g(N, N),$$

where  $\Delta_{\hat{g}}$  is the Laplace-Beltrami operator on  $\Gamma$  for  $\hat{g}$  the induced metric on  $\Gamma$ ,  $\text{Ric}_g$  denotes the Ricci tensor on  $(M, g)$  and  $|A_\Gamma|^2$  denotes the square of the norm of the shape operator defined by

$$\hat{g}(A_\Gamma X, Y) = g(\nabla_X^g Y, N),$$

for all  $X, Y \in T\Gamma$ . We will say that  $\Gamma$  is a *nondegenerate* minimal hypersurface if  $J_\Gamma$  has trivial kernel.

Given these definitions, we have the :

**THEOREM 1.1.** [44] *Assume that  $(M, g)$  is an  $(n + 1)$ -dimensional compact Riemannian manifold without boundary and  $\Gamma \subset M$  is a nondegenerate oriented minimal hypersurface such that  $M \setminus \Gamma = M_+ \cup M_-$  and  $N$  points toward  $M_+$  while  $-N$  points towards  $M_-$ . Then, there exists  $\varepsilon_0 > 0$  and for all  $\varepsilon \in (0, \varepsilon_0)$  there exists  $u_\varepsilon$ , critical point of  $E_\varepsilon$ , such that  $u_\varepsilon$  converges uniformly to 1 on compact subsets of  $M^+$  (resp. to  $-1$  on compact subsets of  $M^-$ ) and*

$$E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}} \mathcal{H}^n(\Gamma),$$

as  $\varepsilon$  tends to 0.

The proof of this result relies on an *infinite dimensional Liapunov-Schmidt reduction argument* which, in this context was first developed in [44]. As already mentioned, in the second part of these lectures, we will provide a complete, simple proof of this result. This proof is much simpler than the original proof in [44] and makes use of many ideas which have been developed by the different authors working in the field.

As far as multiple transition layers are concerned, given a minimal hypersurface  $\Gamma$  (subject to some additional property on the sign of the potential of the Jacobi operator about  $\Gamma$ , which holds on manifolds with positive Ricci curvature) and given an integer  $k \geq 1$ , solutions of (0.1) with multiple transitions layers near  $\Gamma$  were built in [18], in such a way that

$$E_\varepsilon(u_\varepsilon) \longrightarrow \frac{k}{\sqrt{2}} \mathcal{H}^n(\Gamma).$$

More precisely, we have the following difficult result :

**THEOREM 1.2.** [18] *Assume that  $(M, g)$  is a  $(n+1)$ -dimensional compact Riemannian manifold and  $\Gamma \subset M$  is a nondegenerate, oriented, connected minimal hypersurface such that  $M \setminus \Gamma = M_+ \cup M_-$  and  $N$  points toward  $M_+$  and  $-N$  points toward  $M_-$ . Further assume that*

$$|A_\Gamma|_g^2 + \text{Ric}_g(N, N) > 0,$$

*along  $\Gamma$ . Then, there exists a sequence  $(\varepsilon_j)_j$  of positive numbers tending to 0 and there exists  $u_j$ , critical point of  $E_{\varepsilon_j}$ , such that  $u_j$  converges uniformly to 1 on compact subsets of  $M^+$  (resp. to  $(-1)^k$  on compact subsets of  $M^-$ ) and*

$$E_{\varepsilon_j}(u_j) \longrightarrow \frac{k}{\sqrt{2}} \mathcal{H}^n(\Gamma),$$

*as  $j$  tends to  $\infty$ . In particular,  $u_j$  has  $k$  transition layers close to  $\Gamma$ .*

Observe that Theorem 1.1 provides solutions for *any*  $\varepsilon$  small while the later only produces solutions for *some sequence* of  $\varepsilon$  tending to 0. This new restriction is due to some subtle *resonance phenomena* which arises in the case of multiple interfaces.

## 2. Entire solutions to the Allen-Cahn equation in Euclidean space

In this section, we focus on the case where the ambient manifold is the Euclidean space. Hence, we are interested in *entire* solutions of

$$(2.1) \quad \Delta u + u - u^3 = 0,$$

in  $\mathbb{R}^{n+1}$ . Namely, solutions which are defined on all  $\mathbb{R}^{n+1}$ . We review some recent results and explain the different existence proofs which are available in this case.

**2.1. The de Giorgi's conjecture.** In dimension 1, solutions of (2.1) which have finite energy are given by translations of the function  $u_1$  which is the unique solution of the problem

$$(2.2) \quad u_1'' + u_1 - u_1^3 = 0, \quad \text{with} \quad u_1(\pm\infty) = \pm 1 \quad \text{and} \quad u_1(0) = 0.$$

In fact, the function  $u_1$  is explicitly given by

$$u_1(x) = \tanh\left(\frac{x}{\sqrt{2}}\right).$$

Observe that, for all  $\mathbf{a} \in \mathbb{R}^{n+1}$  with  $|\mathbf{a}| = 1$  and for all  $b \in \mathbb{R}$ , the function

$$(2.3) \quad u(\mathbf{x}) = u_1(\mathbf{a} \cdot \mathbf{x} + b),$$

solves (2.1).

A celebrated conjecture due to de Giorgi asserts that, in dimension  $n + 1 \leq 8$ , these solutions are the only ones which are bounded, non constant and monotone in one direction. In other words, if  $u$  is a (smooth) bounded solution of (2.1) and if  $\partial_{x_{n+1}} u > 0$  then  $u^{-1}(\lambda)$  is either a hyperplane or the empty set.

In dimensions 2 and 3, de Giorgi's conjecture has been proven in [25], [4] and (under some extra assumption) in the remaining dimensions in [48] (see also [21], [23]). When  $n + 1 = 2$ , the monotonicity assumption can even be replaced by a weaker stability assumption [28]. Finally, counterexamples in dimension  $n + 1 \geq 9$  have recently been built in [17], using the existence of non trivial minimal graphs in higher dimensions. We will return to this later on.

Let us briefly explain the proof of the *de Giorgi conjecture* in the two dimensional case, as given by A. Farina in [22].

**THEOREM 2.1.** [25] *The de Giorgi's conjecture is true in dimension 2.*

**PROOF.** [22] Assume that  $u$  satisfies (2.1). Then

$$(2.4) \quad \Delta \nabla u + (1 - 3u^2) \nabla u = 0.$$

It is convenient to identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and write

$$\nabla u = \rho e^{i\theta},$$

where  $\rho$  and  $\theta$  are real valued functions. Observe that we implicitly use the fact that  $\partial_{x_2} u > 0$  and hence we can choose the function  $\theta$  to take values in  $(0, \pi)$ . Elliptic estimates imply that  $\nabla u$  is bounded and hence so is  $\rho$ .

Now, with these notations, (2.4) can be written as

$$\Delta \rho - |\nabla \theta|^2 \rho + (1 - 3\rho^2) \rho + i(\rho \Delta \theta + 2 \nabla \rho \cdot \nabla \theta) = 0.$$

In particular, the imaginary part of the left-hand side of the equation is identically equal to 0 and hence

$$\operatorname{div}(\rho^2 \nabla \theta) = 0.$$

As already mentioned,  $\theta \in (0, \pi)$  and the next Lemma (Liouville type result) implies that  $\theta$  is in fact a constant function. Therefore, the unit normal vector to the level lines of  $u$  is constant.  $\square$

In order for the proof to be complete, it remains to prove the following :

**LEMMA 2.1.** *Assume that  $\rho$  is a positive smooth, bounded function. Further assume that  $\theta$  is a bounded solution of*

$$(2.5) \quad \operatorname{div}(\rho^2 \nabla \theta) = 0,$$

*in  $\mathbb{R}^2$ . Then,  $\theta$  is the constant function.*

PROOF. Let  $\chi$  be a cutoff function which is identically equal to 1 in the unit ball and identically equal to 0 outside the ball of radius 2. For all  $R > 0$  we define

$$\chi_R = \chi(\cdot/R).$$

Observe that  $|\nabla\chi_R| \leq C R^{-1}$  for some constant  $C > 0$  independent of  $R > 0$ . We multiply (2.5) by  $\chi_R^2 \theta$  and integrate the result over  $\mathbb{R}^2$  to find

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \rho^2 \chi_R^2 dx = -2 \int_{\mathbb{R}^2} \theta \rho^2 \chi_R \nabla\theta \nabla\chi_R dx.$$

Now, the integral on the right-hand side is taken over the set of  $x \in \mathbb{R}^2$  such that  $|x| \in [R, 2R]$ . Using Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla\theta|^2 \rho^2 \chi_R^2 dx &\leq 2 \left( \int_{|x| \in [R, 2R]} \rho^2 \theta^2 |\nabla\chi_R|^2 dx \right)^{1/2} \\ &\quad \times \left( \int_{|x| \in [R, 2R]} |\nabla\theta|^2 \rho^2 \chi_R^2 dx \right)^{1/2}. \end{aligned}$$

Using the bound  $|\nabla\chi_R| \leq C R^{-1}$ , we conclude that

$$(2.6) \quad \int_{\mathbb{R}^2} |\nabla\theta|^2 \rho^2 \chi_R^2 dx \leq C \left( \int_{|x| \in [R, 2R]} |\nabla\theta|^2 \rho^2 \chi_R^2 dx \right)^{1/2}.$$

This in particular implies that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \rho^2 \chi_R^2 dx \leq C^2,$$

and hence

$$\lim_{R \rightarrow \infty} \int_{x \in [R, 2R]} |\nabla\theta|^2 \rho^2 \chi_R^2 dx = 0.$$

Inserting this information back in (2.6), we conclude that

$$\int_{\mathbb{R}^2} |\nabla\theta|^2 \rho^2 \chi_R^2 dx = 0,$$

which implies that  $\theta$  is the constant function.  $\square$

**2.2. Solutions obtained using the variational structure of the problem.** In this section we explain how the variational structure of the problem can be exploited to derive the existence of nontrivial solutions of (2.1) whose zero set is prescribed.

As already mentioned, the functions  $u(\mathbf{x}) = u_1(\mathbf{a} \cdot \mathbf{x} + b)$  are obvious solutions of (2.1). In dimension 2, nontrivial examples (whose nodal set is the union of two perpendicular lines) were built in [11] using the following strategy :

**THEOREM 2.2.** [11] *In dimension 2, there exists a solution of (2.1) whose zero set is the union of two perpendicular lines.*

PROOF. For all  $R > 0$  we define

$$\Omega_R := \{(x, y) \in \mathbb{R}^2 : x > |y| \text{ and } x^2 + y^2 < R^2\}.$$

Then one considers the energy

$$E_R(u) := \frac{1}{2} \int_{\Omega_R} |\nabla u|^2 dx + \frac{1}{4} \int_{\Omega_R} (1 - u^2)^2 dx.$$

Standard arguments of the calculus of variations implies that, for all  $R > 0$ , there exists a minimiser  $u_R \in H_0^1(\Omega_R)$  which can be assumed to be positive and bounded by 1. This minimiser is a smooth solution of (2.1) in  $\Omega_R$  which has 0 boundary data and is bounded. It is easy to cook up a *test function* to show that

$$(2.7) \quad E_R(u_R) \leq C R.$$

Indeed, just build a function which interpolate smoothly from 0 to 1 in a layer of size 1 around the boundary of  $\Omega_R$  and which is identically equal to 1 elsewhere. Obviously the energy of this function is then controlled by the length of  $\partial\Omega_R$  and hence is less than a constant times  $R$ . Since the energy of the trivial solution (identically equal to 0) is

$$E_R(0) = \frac{1}{4} \int_{\Omega_R} dx \geq \bar{C} R^2,$$

we conclude that  $u_R$  is certainly not identically equal to 0 for  $R$  large enough since otherwise we would have the inequalities

$$\bar{C} R^2 \leq E_R(0) \leq E_R(u_R) \leq C R,$$

which does not hold for  $R$  large enough.

Elliptic estimates together with Ascoli-Arzelà's theorem allow one to prove that, as  $R$  tends to  $\infty$ , this sequence of minimisers  $u_R$  converges (up to a subsequence and uniformly on compacts) to a solution of (2.1) which is defined in the quadrant  $\{(x, y) \in \mathbb{R}^2 : x > |y|\}$  and which vanishes on the boundary of this set. A solution  $u_2$  defined in the entire space is then obtained using odd reflections through the lines  $x = \pm y$ . The function  $u_2$  is a solution of (2.1), whose 0-level set is the union of the two axis.

Observe that we need to rule out the fact that  $u_2$  is the trivial solution (identically equal to 0). The proof is again by contradiction. If  $u_2 \equiv 0$ , then for  $R$  large enough, the solution  $u_R$  would be less than  $1/2$  on  $\Omega_{\bar{R}}$  (here  $\bar{R}$  is fixed large enough, independently of  $R$ ). However, arguing as above it would be possible to modify the definition of  $u_R$  on  $\Omega_{\bar{R}}$  while reducing its energy (this would contradict the fact that  $u_R$  is a minimiser of the energy).  $\square$

This construction can easily be generalized to obtain solutions with *dihedral symmetry* by considering, for  $k \geq 3$ , the corresponding solution within the angular sector

$$\left\{ (r \cos \theta, r \sin \theta) : r > 0, |\theta| < \frac{\pi}{2k} \right\},$$

and extending it by  $2k - 1$  consecutive odd reflections to yield an entire solution  $u_k$  (we refer to [27] for the details). The zero level set of  $u_k$  is constituted by  $2k$  infinite half lines with dihedral symmetry.

Following the same strategy, X. Cabre and J. Terra [5] have obtained a higher dimensional version of this construction and they are able to prove the :

**THEOREM 2.3.** [5] *There exist solutions of (2.1) which are defined in  $\mathbb{R}^{2m}$  and whose zero set is the minimal cone*

$$C_{m,m} := \{(x, y) \in \mathbb{R}^{2m} : |x| = |y|\}.$$

Again using similar arguments, it is proven in [20] that there exists solutions of (2.1) in  $\mathbb{R}^3$  whose zero set is a given helicoid. In dimension 3, given  $\lambda > 0$ , we define the helicoid  $H_\lambda$  to be the minimal surface parameterized by

$$\mathbb{R} \times \mathbb{R} \ni (t, \theta) \longmapsto \left( t \cos \theta, t \sin \theta, \frac{\lambda}{\pi} \theta \right) \in \mathbb{R}^3,$$

and, identifying  $\mathbb{R}^3$  with  $\mathbb{C} \times \mathbb{R}$ , we define the screw motion of parameter  $\lambda$  by

$$\sigma_\lambda^\alpha(z, t) = \left( e^{i\alpha} z, t + \frac{\lambda}{\pi} \alpha \right),$$

for all  $\alpha \in \mathbb{R}$ . Using the variational argument as above, solutions of the Allen-Cahn equation which are defined in  $\mathbb{R}^3$  and whose zero set is equal to  $H_\lambda$  are constructed in [20]. More precisely :

**THEOREM 2.4.** [20] *Assume that  $\lambda > \pi$ , then there exists an entire solution to the Allen-Cahn equation (2.1) which is bounded and whose zero set is equal to  $H_\lambda$ . This solution is invariant under screw motion of parameter  $\lambda$ , namely  $u \circ \sigma_\lambda^\alpha = u$ , for all  $\alpha \in \mathbb{R}$ .*

The constrain on  $\lambda$  is sharp since we also have the :

**THEOREM 2.5.** [20] *Assume that  $\lambda \leq \pi$ , then there are no nontrivial bounded entire solution of the Allen-Cahn equation (2.1) whose zero set is equal to  $H_\lambda$ .*

Observe that, in this last result, we do not assume that the solution is invariant under screw motion.

This result generalizes in higher dimensions to provide solutions of (2.1) which are defined in  $\mathbb{R}^{2m+3}$ , whose zero set is the  $(2m+2)$ -dimensional helicoid, [7], which is embedded in  $\mathbb{R}^{2m+3}$  as the

$$\begin{aligned} \mathbb{R} \times \mathbb{R} \times (S^m \times S^m) &\longrightarrow \mathbb{R}^{2m+3} \\ (t, \theta, (z_1, z_2)) &\longmapsto (t(\cos \theta z_1 - \sin \theta z_2), t(\sin \theta z_2 + \cos \theta z_1), \theta). \end{aligned}$$

**2.3. Entire solutions of the Allen-Cahn equation which are associated to embedded minimal surfaces or hypersurfaces.** Recently, there has been important existence results for entire solutions of (2.1) which are associated to complete noncompact embedded minimal surfaces in Euclidean space. All these solutions are counterparts, in the noncompact setting, of the solutions obtained in [44] in the compact setting. They all rely on the knowledge of complete noncompact minimal hypersurfaces and use the *infinite dimensional Liapunov-Schmidt reduction argument* introduced in [44]. Let us mention two important results along these lines.

There is a rich family of minimal surfaces in  $\mathbb{R}^3$  which are complete, embedded and have finite total curvature. Among these surfaces there is the catenoid, Costa's surface [8] and its higher genus analogues, and all  $k$ -ended embedded minimal surfaces studied by J. Perez and A. Ros [46].

The main result in [19] asserts that there exists solutions of (2.1) whose nodal set is close to a dilated version of any nondegenerate complete, noncompact minimal surface with finite total curvature. In other words, if one considers the equation with scaling

$$(2.8) \quad \varepsilon^2 \Delta u + u - u^3 = 0,$$

then the following result holds :

**THEOREM 2.6.** [19] *Given  $\Gamma$ , a nondegenerate complete, embedded minimal surface with finite total curvature, there exists  $\varepsilon_0 > 0$  and for all  $\varepsilon \in (0, \varepsilon_0)$  there exists  $u_\varepsilon$  solution of (2.8), such that  $u_\varepsilon^{-1}(0)$  converges uniformly on compacts to  $\Gamma$ .*

In this result, nondegeneracy refers to the fact that all bounded Jacobi fields of  $\Gamma$  come from the action of rigid motions on  $\Gamma$ .

Also, thanks to the result of Bombieri-de Giorgi-Giusti [3], it is known that there exists minimal graphs which are not hyperplanes in dimension  $n + 1 \geq 9$ . Following similar ideas, it is proved in [17] that one can construct entire solutions of (2.1) whose level sets are not hyperplanes, provided the dimension of the ambient space is  $n + 1 \geq 9$ .

**THEOREM 2.7.** [17] *In dimension 9, there exist solutions of the Allen-Cahn equation which are monotone in one direction and whose level sets are not hyperplanes.*

**2.4. Entire solutions of (2.1) which are associated to the Toda system.** We assume in this section that the dimension is equal to 2. There is yet another interesting construction of entire solutions of (2.1) which can be done in dimension 2. Surprisingly, this construction relates solutions of (2.1) whose zero set is the union of finitely many curves which are close to parallel lines with solutions of a *Toda system*.

**DEFINITION 2.1.** *We say that  $u$ , solution of (2.1), has  $2k$ -ends if, away from a compact set, its nodal set is given by  $2k$  connected curves which are asymptotic to  $2k$  oriented, disjoint, half lines  $\mathbf{a}_j \cdot \mathbf{x} + b_j = 0$ ,  $j = 1, \dots, 2k$  (for some choice of  $\mathbf{a}_j \in \mathbb{R}^2$ ,  $|\mathbf{a}_j| = 1$  and  $b_j \in \mathbb{R}$ ) and if, along these curves, the solution is asymptotic to either  $u_1(\mathbf{a}_j \cdot \mathbf{x} + b_j)$  or  $-u_1(\mathbf{a}_j \cdot \mathbf{x} + b_j)$ .*

Given any  $k \geq 1$ , one can prove the existence of a wealth of  $2k$ -ended solutions of (2.1). Moreover, one can show that these solutions belong to a smooth  $2k$ -parameter family of  $2k$ -ended solutions of (2.1). To state this result in precise way, we assume that we are given a solution  $q_1, \dots, q_k$  of the *Toda system*

$$(2.9) \quad c_0 q_j'' = e^{\sqrt{2}(q_{j-1} - q_j)} - e^{\sqrt{2}(q_j - q_{j+1})},$$

for  $j = 1, \dots, k$ , where  $c_0 = \frac{\sqrt{2}}{24}$  and we agree that

$$q_0 \equiv -\infty \quad \text{and} \quad q_{k+1} \equiv +\infty.$$

The *Toda system* (2.9) is a classical example of integrable system which has been extensively studied. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential. We refer to [32] and [42] for the complete description of the theory. Of importance is the fact that solutions of (2.9) can be described (almost explicitly) in terms of  $2k$  parameters. Moreover, if  $\mathbf{q}$  is a solution of (2.9), then the long term behavior (i.e. long term scattering) of the  $q_j$  at  $\pm\infty$  is well understood and it is known that, for all  $j = 1, \dots, k$ , there exist  $a_j^+, b_j^+ \in \mathbb{R}$  and  $a_j^-, b_j^- \in \mathbb{R}$ , all depending on  $q_1, \dots, q_k$ , such that

$$(2.10) \quad q_j(t) = a_j^\pm |t| + b_j^\pm + \mathcal{O}_{C^\infty(\mathbb{R})}(e^{-\tau_0 |t|}),$$

as  $t$  tends to  $\pm\infty$ , for some  $\tau_0 > 0$ . Moreover,  $a_{j+1}^\pm > a_j^\pm$  for all  $j = 1, \dots, k - 1$ .



Given  $\varepsilon > 0$ , we define the vector valued function  $q_{1,\varepsilon}, \dots, q_{k,\varepsilon}$ , whose components are given by

$$(2.11) \quad q_{j,\varepsilon}(x) := q_j(\varepsilon x) - \sqrt{2} \left( j - \frac{k+1}{2} \right) \log \varepsilon.$$

It is easy to check that the  $q_{j,\varepsilon}$  are again solutions of (2.9).

Observe that, according to the description of the asymptotics of the functions  $q_j$ , the graphs of the functions  $q_{j,\varepsilon}$  are asymptotic to oriented half lines at infinity. In addition, for  $\varepsilon > 0$  small enough, these graphs are disjoint and in fact their mutual distance is given by  $-\sqrt{2} \log \varepsilon + \mathcal{O}(1)$  as  $\varepsilon$  tends to 0.

It will be convenient to agree that  $\chi^+$  (resp.  $\chi^-$ ) is a smooth cutoff function defined on  $\mathbb{R}$  which is identically equal to 1 for  $x > 1$  (resp. for  $x < -1$ ) and identically equal to 0 for  $x < -1$  (resp. for  $x > 1$ ) and additionally  $\chi^- + \chi^+ \equiv 1$ . With these cutoff functions at hand, we define the 4 dimensional space

$$(2.12) \quad D := \text{Span} \{ x \mapsto \chi^\pm(x), x \mapsto x \chi^\pm(x) \},$$

and, for all  $\mu \in (0, 1)$  and all  $\tau \in \mathbb{R}$ , we define the space  $\mathcal{C}_\tau^{2,\mu}(\mathbb{R})$  of  $\mathcal{C}^{2,\mu}$  functions  $h$  which satisfy

$$\|h\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R})} := \|(\cosh x)^\tau h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty.$$

Keeping in mind the above notations, we have the :

**THEOREM 2.8.** [13] *For all  $\varepsilon > 0$  sufficiently small, there exists an entire solution  $u_\varepsilon$  of the Allen-Cahn equation (2.1) whose nodal set is the union of  $k$  disjoint curves  $\Gamma_{1,\varepsilon}, \dots, \Gamma_{k,\varepsilon}$  which are the graphs of the functions*

$$x \mapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon x),$$

where the functions  $h_{j,\varepsilon} \in \mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D$  satisfy

$$\|h_{j,\varepsilon}\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D} \leq C \varepsilon^\alpha.$$

for some constants  $C, \alpha, \tau, \mu > 0$  independent of  $\varepsilon > 0$ .

In other words, given a solution of the Toda system, we can find a one parameter family of  $2k$ -ended solutions of (2.1) which depend on a small parameter  $\varepsilon > 0$ . As  $\varepsilon$  tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions  $q_{j,\varepsilon}$ .

Going through the proof, one can be more precise about the description of the solution  $u_\varepsilon$ . If  $\Gamma \subset \mathbb{R}^2$  is a curve in  $\mathbb{R}^2$  which is the graph over the  $x$ -axis of some function, we denote by  $\text{dist}(\cdot, \Gamma)$  the signed distance to  $\Gamma$  which is positive in the upper half of  $\mathbb{R}^2 \setminus \Gamma$  and is negative in the lower half of  $\mathbb{R}^2 \setminus \Gamma$ . Then, we have the :

**PROPOSITION 2.1.** [13] *The solution of (2.1) provided by Theorem 2.8 satisfies*

$$\|e^{\varepsilon \hat{\alpha} |\mathbf{x}|} (u_\varepsilon - u_\varepsilon^*)\|_{L^\infty(\mathbb{R}^2)} \leq C \varepsilon^{\bar{\alpha}},$$

for some constants  $C, \bar{\alpha}, \hat{\alpha} > 0$  independent of  $\varepsilon$ , where

$$(2.13) \quad u_\varepsilon^* := \sum_{j=1}^k (-1)^{j+1} H(\text{dist}(\cdot, \Gamma_{j,\varepsilon})) - \frac{1}{2}((-1)^k + 1).$$

It is interesting to observe that, when  $k \geq 3$ , there are solutions of (2.9) whose graphs have no symmetry and our result yields the existence of entire solutions of (2.1) without any symmetry provided the number of ends is larger than or equal to 6.

**2.5. Stable solutions of the Allen-Cahn equation.** Recently, building on the recently developed tools and progress on the understanding of solutions of (2.1), there has been some interest in the understanding of *stable solutions* of (2.1). We start with the :

DEFINITION 2.2. *We will say that  $u$ , solution of (0.1), is stable if*

$$(2.14) \quad \int_{\mathbb{R}^{n+1}} (|\nabla\psi|^2 - \psi^2 + 3u^2\psi^2) dx \geq 0,$$

for any smooth function  $\psi$  with compact support.

In dimensions 2 and 3, the stability property is also a key ingredient in the proof of the de Giorgi's conjecture which is given in [4], [25] and, as observed by Dancer [10], the stability assumption is indeed a sufficient condition to classify solutions of (2.1) in dimensions 2 and 3 and prove that solutions satisfying (2.14) are given by (2.3) in these dimensions.

The monotonicity assumption in the de Giorgi conjecture implies the stability of the solution. Indeed, if  $u$  is a solution of (2.1) such that  $\partial_{x_{n+1}}u > 0$ , then  $u$  is stable in the above sense and the linearized operator

$$L := -(\Delta + 1 - 3u^2),$$

satisfies maximum principle. More generally, observe that if  $\phi > 0$  is a solution of

$$L\phi = 0,$$

one can multiply this equation by  $\phi^{-1}\psi^2$  and integrate the result by part to get

$$\int_{\mathbb{R}^{n+1}} (|\nabla\psi|^2 - \psi^2 + 3u^2\psi^2) dx = \int_{\mathbb{R}^{n+1}} |\nabla\psi - \phi^{-1}\psi\nabla\phi|^2 dx \geq 0,$$

and hence  $u$  is stable in the sense of Definition 2.2. In the case where  $u$  is monotone in the  $x_{n+1}$  direction, this argument can be applied with  $\phi = \partial_{x_{n+1}}u$  to prove that monotone solutions of (2.1) are stable. Standard arguments imply that  $L$  also satisfy the maximum principle.

As a consequence, in dimension 9, the monotone solutions constructed by del Pino, Kowalczyk and Wei [17] provide some non trivial stable solutions of the Allen-Cahn equation. It is shown in [45] that :

THEOREM 2.9. [45] *Assume that  $n + 1 = 2m \geq 8$ . Then, there exist bounded, stable solutions of (2.1) whose level sets are not hyperplanes.*

In fact, we can be more precise and we prove that the zero set of our solutions are asymptotic to

$$C_{m,m} := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |x| = |y|\},$$

which is a minimal cone in  $\mathbb{R}^{2m}$  usually referred to as Simons' cone. The proof of Theorem 2.9 strongly uses the fact that Simons' cone is a minimizing minimal hypersurface and hence is stable but also uses the fact that this cone is *strictly area minimizing* (we refer to [29] or [45] for a definition).

As already mentioned, in dimension  $n + 1 = 2m$ , with  $m \geq 1$ , Cabré and Terra [5] have found solutions of the Allen-Cahn equation whose zero set is exactly given by the cone  $C_{m,m}$ . When  $m \geq 1$ , these solutions generalize the so called *saddle* solutions which have been found by Dang, Fife and Peletier [11] in dimension 2. The proof of this result makes use of a variational argument in the spirit of [11].

Moreover, the same authors have proven that, when  $m = 2$  or  $3$ , the solutions they find is unstable [6].

This result is in fact a corollary of a more general result :

**THEOREM 2.10.** [45] *Assume that  $n+1 \geq 8$  and that  $C$  is a minimizing cone in  $\mathbb{R}^{n+1}$ . Then, there exist bounded solutions of (2.1) whose zero sets are asymptotic to  $C$  at infinity.*

We refer to [29] or [45] for a definition of *minimizing cone*. The solutions of (2.1) constructed in these theorems are not unique and in fact they arise in families whose dimension can be computed.

In view of the de Giorgi's conjecture, the results of Dancer and the above result, the following statement seems natural : *Assume that  $u$  is a bounded, stable solution of (2.1) and that  $n+1 \leq 7$ , then the level sets of  $u$  should be hyperplanes.*

This question parallels the corresponding well known conjecture concerning the classification of stable, embedded minimal hypersurface in Euclidean space : *Stable, embedded minimal hypersurfaces in Euclidean space  $\mathbb{R}^{n+1}$  are hyperplanes as long as  $n+1 \leq 7$ .* This latter problem is still open except when the ambient dimension is equal to 3 where the result of Fischer-Colbrie and Schoen [24] guaranties that affine planes are the only stable embedded minimal surface in  $\mathbb{R}^3$ .

### 3. Proof of Theorem 1.1

As already mentioned, the proof of Theorem 1.1 and many constructions in the field make use of an *infinite dimensional Lyapunov-Schmit reduction argument* which originally appeared in [44]. We present here a rather detailed proof which is much simpler than the original proof since it uses many ideas which have been developed by all the different authors working on the subject or on closely related problems : del Pino, Kowalczyk, Wei, Brendle, Malchiodi, Montenegro, ..... We believe that the technical tools have now evolved sufficiently so that they can be presented in a rather simple and synthetic way.

**3.1. Local coordinates near a hypersurface and expression of the Laplacian.** The first important tool is the use of *Fermi coordinates* to parameterize a neighborhood of a given hypersurface which is embedded in  $M$ .

In this section, we assume that  $n \geq 1$  and that  $\Gamma$  is an oriented smooth hypersurface embedded in a compact  $(n+1)$  dimensional Riemannian manifold  $(M, g)$ . We assume that  $\Gamma$  separates  $M$  into two different connected components in the sense that  $\Gamma$  is the zero set of a smooth function  $f_\Gamma$  for which 0 is a regular value. We define the *Fermi coordinates* about  $\Gamma$  and then, we provide some asymptotic expansion of the Laplace-Beltrami operator in Fermi coordinates about  $\Gamma$ .

We denote by  $N$  the unit normal vector field on  $\Gamma$  which defines the orientation of  $\Gamma$ . We make use of the exponential map to define

$$(3.1) \quad Z(y, z) := \text{Exp}_y(z N(y)),$$

where  $y \in \Gamma$  and  $z \in \mathbb{R}$ . The implicit function theorem implies that  $Z$  is a local diffeomorphism from a neighborhood of a point  $(y, 0) \in \Gamma \times \mathbb{R}$  onto a neighborhood of  $y \in M^{n+1}$ .

**REMARK 3.1.** *In the case where the ambient manifold is the Euclidean space, we simply have*

$$Z(y, z) = y + z N(y),$$

for  $y \in \Gamma$  and  $z \in \mathbb{R}$ .

Given  $z \in \mathbb{R}$ , we define  $\Gamma_z$  by

$$\Gamma_z := \{Z(y, z) \in M : y \in \Gamma\}.$$

Observe that for  $z$  small enough (depending on the point  $y \in \Gamma$  where one is working),  $\Gamma_z$  restricted to a neighborhood of  $y$  is a smooth hypersurface which will be referred to as the *hypersurface parallel to  $\Gamma$  at height  $z$* . The induced metric on  $\Gamma_z$  will be denoted by  $g_z$ .

The following result is a consequence of Gauss' Lemma. It gives the expression of the metric  $g$  on the domain of  $M$  which is parameterized by  $Z$ .

LEMMA 3.1. *We have*

$$Z^* \mathring{g} = g_z + dz^2,$$

where  $g_z$  is considered as a family of metrics on  $T\Gamma$ , smoothly depending on  $z$ , which belongs to a neighborhood of  $0 \in \mathbb{R}$ .

PROOF. It is easier to work in local coordinates. Given  $y \in \Gamma$ , we fix local coordinates  $x := (x_1, \dots, x_n)$  in a neighborhood of  $0 \in \mathbb{R}^n$  to parameterize a neighborhood of  $y$  in  $\Gamma$  by  $\Phi$ , with  $\Phi(0) = y$ . We consider the mapping

$$\tilde{F}(x, z) = \text{Exp}_{\Phi(x)}(z N(\Phi(x))),$$

which is a local diffeomorphism from a neighborhood of  $0 \in \mathbb{R}^{n+1}$  into a neighborhood of  $y$  in  $M$ . The corresponding coordinate vector fields are denoted by

$$X_0 := \tilde{F}_*(\partial_z) \quad \text{and} \quad X_j := \tilde{F}_*(\partial_{x_j}),$$

for  $j = 1, \dots, n$ . The curve  $x_0 \mapsto \tilde{F}(x_0, x)$  being a geodesic we have  $g(X_0, X_0) \equiv 1$ . This also implies that  $\nabla_{X_0}^g X_0 \equiv 0$  and hence we get

$$\partial_z g(X_0, X_j) = g(\nabla_{X_0}^g X_0, X_j) + g(\nabla_{X_0}^g X_j, X_0) = g(\nabla_{X_0}^g X_j, X_0).$$

The vector fields  $X_0$  and  $X_j$  being coordinate vector fields we have  $\nabla_{X_0}^g X_j = \nabla_{X_j}^g X_0$  and we conclude that

$$2 \partial_z g(X_0, X_j) = 2 g(\nabla_{X_j}^g X_0, X_0) = \partial_{x_j} g(X_0, X_0) = 0.$$

Therefore,  $g(X_0, X_j)$  does not depend on  $z$  and since on  $\Sigma$  this quantity is 0 for  $j = 1, \dots, n$ , we conclude that the metric  $g$  can be written as

$$g = g_z + dz^2,$$

where  $g_z$  is a family of metrics on  $\Gamma$  smoothly depending on  $z$  (this is nothing but Gauss' Lemma).  $\square$

The next result expresses, for  $z$  small, the expansion of  $g_z$  in terms of geometric objects defined on  $\Gamma$ . In particular, in terms of  $\mathring{g}$  the induced metric on  $\Gamma$ ,  $\mathring{h}$  the second fundamental form on  $\Gamma$ , which is defined by

$$\mathring{h}(t_1, t_2) := -\mathring{g}(\nabla_{t_1}^g N, t_2)$$

and in terms of the square of the second fundamental form which is the tensor defined by

$$\mathring{h} \otimes \mathring{h}(t_1, t_2) := \mathring{g}(\nabla_{t_1}^g N, \nabla_{t_2}^g N),$$

for all  $t_1, t_2 \in T\Gamma$ . Observe that, in local coordinates, we have

$$(\mathring{h} \otimes \mathring{h})_{ij} = \sum_{a,b} \mathring{h}_{ia} \mathring{g}^{ab} \mathring{h}_{bj}.$$

With these notations at hand, we have the :

LEMMA 3.2. *The induced metric  $g_z$  on  $\Gamma_z$  can be expanded in powers of  $z$  as*

$$g_z = \mathring{g} - 2z \mathring{h} + z^2 \left( \mathring{h} \otimes \mathring{h} + g(R_g(\cdot, N), \cdot, N) \right) + \mathcal{O}(z^3),$$

where  $R_g$  denotes the Riemannian tensor on  $(M, g)$ .

PROOF. We keep the notations introduced in the previous proof. By definition of  $\mathring{g}$ , we have

$$g_z = \mathring{g} + \mathcal{O}(z).$$

We now derive the next term the expansion of  $g_z$  in powers of  $z$ . To this aim, we compute

$$\partial_z g(X_i, X_j) = g(\nabla_{X_i}^g X_0, X_j) + g(\nabla_{X_j}^g X_0, X_i),$$

for all  $i, j = 1, \dots, n$ . Since  $X_0 = N$  on  $\Gamma$ , we get

$$\partial_z \bar{g}_z |_{z=0} = -2\mathring{h},$$

by definition of the second fundamental form. This already implies that

$$g_z = \mathring{g} - 2\mathring{h}z + \mathcal{O}(z^2).$$

Using the fact that the  $X_0$  and  $X_j$  are coordinate vector fields, we can compute

$$\partial_z^2 g(X_i, X_j) = g(\nabla_{X_0}^g \nabla_{X_i}^g X_0, X_j) + g(\nabla_{X_0}^g \nabla_{X_j}^g X_0, X_i) + 2g(\nabla_{X_i}^g X_0, \nabla_{X_j}^g X_0).$$

By definition of the curvature tensor, we can write

$$\nabla_{X_0}^g \nabla_{X_j}^g X_0 = R_g(X_0, X_j) + \nabla_{X_j}^g \nabla_{X_0}^g X_0 + \nabla_{[X_0, X_j]}^g X_0,$$

which, using the fact that  $X_0$  and  $X_j$  are coordinate vector fields, simplifies into

$$\nabla_{X_0}^g \nabla_{X_j}^g X_0 = R_g(X_0, X_j) + \nabla_{X_j}^g \nabla_{X_0}^g X_0.$$

Since  $\nabla_{X_0}^g X_0 \equiv 0$ , we get

$$\nabla_{X_0}^g \nabla_{X_j}^g X_0 = R_g(X_0, X_j) X_0.$$

Inserting this into (3.2) yields

$$\partial_z^2 g(X_i, X_j) = 2g(R_g(X_0, X_i) X_0, X_j) + 2g(\nabla_{X_i}^g X_0, \nabla_{X_j}^g X_0).$$

Evaluation at  $x_0 = 0$  gives

$$\partial_z^2 g_z |_{z=0} = 2g(R(N, \cdot) N, \cdot) + 2g(\nabla^g N, \nabla^g N).$$

The formula then follows at once from Taylor's expansion.  $\square$

Similarly, the mean curvature  $H_z$  of  $\Gamma_z$  can be expressed in term of  $z$ ,  $\mathring{g}$  and  $\mathring{h}$ . We have the :

LEMMA 3.3. *The following expansion holds*

$$H_z = \text{Tr}_{\mathring{g}} \mathring{h} + \left( \text{Tr}_{\mathring{g}} \mathring{h} \otimes \mathring{h} + \text{Ric}_g(N, N) \right) z + \mathcal{O}(z^2),$$

for  $z$  close to 0.

PROOF. The mean curvature appears in the first variation of the volume form of parallel hypersurfaces, namely

$$H_z = -\frac{1}{\sqrt{\det g_z}} \frac{d}{dz} \sqrt{\det g_z}$$

The result then follows at once from the expansion of the metric  $g_z$  in powers of  $z$  together with the well known formula

$$\det(I + A) = 1 + \operatorname{Tr} A + \frac{1}{2} ((\operatorname{Tr} A)^2 - \operatorname{Tr} A^2) + \mathcal{O}(\|A\|^3),$$

where  $A \in M_n(\mathbb{R})$ . □

REMARK 3.2. *In the case where the ambient manifold is the Euclidean space, we simply have*

$$g_z = \mathring{g} - 2z \mathring{h} + z^2 \mathring{h} \otimes \mathring{h},$$

and this implies that we also have an explicit formula for the expression of the mean curvature of  $\Gamma_z$  in terms of  $z$ . Indeed, we have, for  $z$  close enough to 0,

$$H_z = \sum_{j=0}^{\infty} \operatorname{Tr}_{\mathring{g}} \underbrace{(\mathring{h} \otimes \dots \otimes \mathring{h})}_{j \text{ times}} z^j.$$

This last expression follows from the fact, in the proof of Lemma 3.3, we can use the formula

$$\frac{d}{dz} \det(I - zA) = \operatorname{Tr} \left( \sum_{j=0}^{\infty} z^j A^{j+1} \right)$$

where  $A \in M_n(\mathbb{R})$  and  $z \in \mathbb{R}$  is small. Now, we can always assume that, at the point where the computation is performed,  $g_{ij} = \delta_{ij}$ . In this case we can identify  $\mathring{h}$  with a symmetric matrix  $A$  and  $\mathring{k}$  with  $A^2$ . In particular, we can write

$$H_z = -\frac{1}{\det(I - zA)} \frac{d}{dz} \det(I - zA) = \operatorname{Tr} \left( \sum_{j=0}^{\infty} z^j A^{j+1} \right),$$

for all  $z$  small enough.

Recall that the Laplace Beltrami operator is given by

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_{x_i} \left( g^{ij} \sqrt{|g|} \partial_{x_j} \right).$$

in local coordinates. Therefore, in a tubular neighborhood of  $\Gamma$ , the Euclidean Laplacian in  $\mathbb{R}^{n+1}$  can be expressed in Fermi coordinates by the (well-known) formula

$$(3.3) \quad \Delta_{g_e} = \partial_z^2 - H_z \partial_z + \Delta_{g_z}.$$

**3.2. Construction of an approximate solution.** In this section, we use the Fermi coordinates which have been introduced in the previous section and rephrase the equation we would like to solve in a tubular neighborhood of  $\Gamma$ . We also build an approximate solution to (0.1) whose nodal set is equal to  $\Gamma$ .

We define

$$u_\varepsilon := u_1(\cdot/\varepsilon)$$

and

$$\dot{u}_\varepsilon(z) := u_1'(z/\varepsilon) \quad \ddot{u}_\varepsilon(z) := u_1''(z/\varepsilon),$$

where  $u_1$  is the solution of (2.2)

We agree that  $\Gamma$  is a smooth, compact, minimal hypersurface which is embedded in  $M$  and we use the notations introduced in the previous section for the Fermi coordinates about  $\Gamma$ .

Given any (sufficiently small) smooth function  $\zeta$  defined on  $\Gamma$ , we define  $\Gamma_\zeta$  to be the normal graph over  $\Gamma$  for the function  $\zeta$ . Namely

$$\Gamma_\zeta := \{\text{Exp}_y(\zeta(y)N(y)) \in M : y \in \Gamma\}.$$

We keep the notations of the previous section and, in a tubular neighborhood of  $\Gamma$  we write

$$Z^*u(y, z) = \bar{u}(y, z - \zeta(y)),$$

where  $\zeta$  is a (sufficiently small) smooth function defined on  $\Gamma$ . It will be convenient to denote by  $t$  the variable

$$t := z - \zeta(y).$$

Using the expression of the Laplacian in Fermi coordinates which has been derived in (3.3), we find with little work that the equation we would like to solve can be rewritten as

$$(3.4) \quad \varepsilon^2 \left[ (1 + \|d\zeta\|_{g_z}^2) \partial_t^2 \bar{u} + \Delta_{g_z} \bar{u} - (H_\zeta + \Delta_{g_z} \zeta) \partial_t \bar{u} - 2(d\zeta, d\partial_t v)_{g_z} \right]_{|z=t+\zeta} + \bar{u} - \bar{u}^3 = 0,$$

for  $t > 0$  close to 0 and  $y \in \Gamma$ . Some comments are due about the notations. In this equation and below all computations of the quantities between the square brackets [ ] are performed using the metric  $g_z$  defined in Lemma 3.2 and considering that  $z$  is a parameter, and once this is done, we set  $z = t + \zeta(y)$ .

We set

$$\bar{u}(y, t) := u_\varepsilon(t) + v(y, t),$$

in which case, the equation (3.4) becomes

$$\mathfrak{N}(v, \zeta) = 0,$$

where we have defined

$$(3.5) \quad \mathfrak{N}(v, \zeta) := \left[ (\varepsilon^2(\partial_t^2 + \Delta_{g_z}) + 1 - 3u_\varepsilon^2) v - \varepsilon(\Delta_{g_z} \zeta + H_z)(\dot{u}_\varepsilon + \varepsilon \partial_t v) + \|d\zeta\|_{g_z}^2 (\dot{u}_\varepsilon + \varepsilon^2 \partial_t^2 v) - 2\varepsilon^2 (d\zeta, d\partial_t v)_{g_z} \right]_{|z=t+\zeta} + v^3 + 3u_\varepsilon v^2.$$

Observe that, when  $v \equiv 0$  and  $\zeta \equiv 0$ , we simply have

$$(3.6) \quad \mathfrak{N}(0, 0) = -\varepsilon H_t \dot{u}_\varepsilon.$$

Also recall that

$$(3.7) \quad H_t = \left( \text{Tr}_g \dot{h} \otimes \dot{h} + \text{Ric}_g(N, N) \right) t + \mathcal{O}(t^2).$$

Observe that we have implicitly used the fact that  $\Gamma$  is a minimal hypersurface and hence  $H_0 = \text{Tr}_{\dot{g}} \dot{h} = 0$ .

In particular, this implies that, there exists a constant  $C > 0$  such that

$$(3.8) \quad |\mathfrak{N}(0, 0)| \leq C \varepsilon^2.$$

Similar estimates can be derived for the partial derivatives of  $\mathfrak{N}(0, 0)$ .

We have the :

LEMMA 3.4. *For all  $k, k' \geq 0$ , there exists a constant  $C_{k, k'} > 0$  such that*

$$(3.9) \quad |\nabla^{k'} \partial_t^k \mathfrak{N}(0, 0)|_{\dot{g}} \leq C_{k, k'} \varepsilon^{2-k},$$

*in the neighborhood of  $\Gamma$  which is parameterized by  $Z$ .*

Given a function  $f$  which is defined in  $\Gamma \times \mathbb{R}$ , we define  $\Pi$  to be the  $L^2$ -orthogonal projection on  $\dot{u}_\varepsilon$ , namely

$$\Pi(f) := \frac{1}{\varepsilon c} \int_{\mathbb{R}} f(y, t) \dot{u}_\varepsilon(t) dt,$$

where the normalization constant

$$c := \frac{1}{\varepsilon} \int_{\mathbb{R}} \dot{u}_\varepsilon^2(t) dt = \int_{\mathbb{R}} (u_1')^2(t) dt.$$

Of importance for us, will be the  $L^2$ -projection of  $\mathfrak{N}(0, 0)$  over  $\dot{u}_\varepsilon$ . The crucial observation is that

$$(3.10) \quad \int_{\mathbb{R}} H_t \dot{u}_\varepsilon^2 dt = \mathcal{O}(\varepsilon^3),$$

because of parity. Using this property, we conclude that :

LEMMA 3.5. *For all  $k \geq 0$ , there exists a constant  $C_k > 0$  such that*

$$(3.11) \quad |\nabla^k \Pi(\mathfrak{N}(0, 0))|_{\dot{g}} \leq C_k \varepsilon^3,$$

*in  $\Gamma$ .*

The function  $\bar{u}_\varepsilon$ , which is defined by

$$(3.12) \quad Z^* \bar{u}_\varepsilon(y, t) := u_\varepsilon(t),$$

in a neighborhood of  $\Gamma$ , will be used to define an approximate solution to our problem.

**3.3. Analysis of the model linear operator.** In this section, we analyze the operator

$$(3.13) \quad L_\varepsilon := \varepsilon^2 (\partial_t^2 + \Delta_{\dot{g}}) + 1 - 3u_\varepsilon^2,$$

which is acting on functions defined on the product space  $\Gamma \times \mathbb{R}$ , which is naturally endowed with the product metric

$$\dot{g} + dt^2.$$

First, we will recall some standard injectivity result which is the key result. Then, we will use this results to obtain an *a priori* estimate for solutions of  $L_\varepsilon w = f$ , when the functions  $w$  and  $f$  are defined in appropriate weighted spaces. The proof of the *a priori* estimate is by contradiction. Finally, application of standard results in functional analysis will provide the existence of a right inverse for the operator  $L_\varepsilon$  acting on some special infinite codimension function space.



3.3.1. *The injectivity result.* We collect some basic information about the spectrum of the operator

$$(3.14) \quad L_0 := -(\partial_t^2 + 1 - 3u_1^2),$$

which arises as the linearized operator of (2.1) about  $u_1$  and which is acting on functions defined in  $\mathbb{R}$ . All the informations we need are included in the :

LEMMA 3.6. *The spectrum of the operator  $L_0$  is the union of the eigenvalue  $\mu_0 = 0$ , which is associated to the eigenfunction*

$$w_0(t) := \frac{1}{\cosh^2(\frac{t}{\sqrt{2}})},$$

*the eigenvalue  $\mu_1 = \frac{3}{2}$ , which is associated to the eigenfunction*

$$w_1(t) := \frac{\sinh(\frac{t}{\sqrt{2}})}{\cosh^2(\frac{t}{\sqrt{2}})},$$

*and the continuous spectrum which is given by  $[2, \infty)$ .*

PROOF. The fact that the continuous spectrum is equal to  $[2, \infty)$  is standard. The fact that the bottom eigenvalue is 0 follows directly from the fact that the equation for  $u_1$  is autonomous and hence the function  $u_1' = \partial_t u_1$  is in the  $L^2$ -kernel of  $L_0$ . Since this function is positive, it has to be the eigenfunction associated to the lowest eigenvalue of  $L_0$ . Direct computation shows that  $\mu_1$  is an eigenvalue of  $L_0$  and, finally, it is proven in [43] that  $\mu_0 = 0$  and  $\mu_1 = 3/2$  are the only eigenvalues of  $L_0$ .  $\square$

Observe that this result implies that the quadratic form associated to  $L_0$  is definite positive when acting on functions which are  $L^2$ -orthogonal to  $u_1'$ . More precisely, we have

$$(3.15) \quad \int_{\mathbb{R}} (|\partial_t w|^2 - w^2 + 3u_1^2 w^2) dt \geq \frac{3}{2} \int_{\mathbb{R}} w^2 dt,$$

for all function  $w \in H^1(\mathbb{R})$  satisfying the orthogonality condition

$$(3.16) \quad \int_{\mathbb{R}} w(t) u_1'(t) dt = 0.$$

As already mentioned, the discussion to follow is based on the understanding of the bounded kernel of the operator

$$(3.17) \quad L_* := \partial_t^2 + \Delta_{\mathbb{R}^n} + 1 - 2u_1^2,$$

which is acting on functions defined on the product space  $\mathbb{R} \times \mathbb{R}^n$ . This is the contain of the following :

LEMMA 3.7. *Assume that  $w \in L^\infty(\mathbb{R} \times \mathbb{R}^n)$  satisfies  $L_* w = 0$ . Then  $w$  only depends on  $t$  and is collinear to  $u_1'$ .*

PROOF. The original proof of this Lemma, which is based on Fourier transform in  $\mathbb{R}^n$ , can be found in [44]. We give here a much simpler proof which is borrowed from [17]. First, we observe that, by elliptic regularity theory, the function  $w$  is smooth and we decompose

$$w(t, y) = c(y) u_1'(t) + \bar{w}(t, y),$$

where  $\bar{w}(\cdot, y)$  satisfies (3.16) for all  $y \in \mathbb{R}^n$ . Inserting this decomposition into the equation satisfied by  $w$ , we find

$$u'_1 \Delta_{\mathbb{R}^n} c + (\partial_t^2 + 1 - 2u_1^2) \bar{w} + \Delta_{\mathbb{R}^n} \bar{w} = 0.$$

Multiplying this equation by  $u'_1$  and integrating the result over  $t \in \mathbb{R}$ , we conclude easily that

$$\Delta_{\mathbb{R}^n} c = 0$$

since  $L_0 u'_1 = 0$  and since  $\Delta_{\mathbb{R}^n} \bar{w}$  is  $L^2$ -orthogonal to the function  $u'_1$ . Since  $w$  is a bounded function, so is the function  $c$  and hence, we conclude that  $c$  is the constant function.

Next, we prove that  $\bar{w} \equiv 0$ . Since we have proven that  $c$  is the constant function, we can now write

$$(3.18) \quad (\partial_t^2 + 1 - 2u_1^2) \bar{w} + \Delta_{\mathbb{R}^n} \bar{w} = 0$$

We claim that, for any  $\sigma \in (0, \sqrt{2})$ , the function  $\bar{w}$  is bounded by a constant times  $(\cosh s)^{-\sigma}$ . Indeed, in the equation (3.18), the potential, which is given by  $1 - 3u_1^2$ , tends to  $-2$  as  $|t|$  tends to  $\infty$ . Using this property, we conclude that the function

$$W(y, t) := e^{-\sigma|t|} + \eta \cosh(\delta t) \sum_{i=1}^n \cosh(\delta y_i),$$

satisfies  $L_* W < 0$  in the region where  $|t| \geq t_*$ , provided  $t_* > 0$  is fixed large enough (depending on  $\sigma$ ). Since  $\bar{w}$  is bounded, we conclude that

$$|\bar{w}| \leq \|\bar{w}\|_{L^\infty} e^{\sigma t_*} \left( e^{-\sigma|t|} + \eta \cosh(\delta t) \sum_{i=1}^n \cosh(\delta y_i) \right),$$

when  $|t| \geq t_*$ . Letting  $\eta$  tend to 0, this implies that

$$|\bar{w}| \leq \|\bar{w}\|_{L^\infty} e^{-\sigma(|t| - t_*)},$$

for  $|t| \geq t_*$  and this completes the proof of the claim.

Multiplying the equation satisfied by  $\bar{w}$  by  $\bar{w}$  and integrating the result over  $\mathbb{R}$  (and ot over  $\mathbb{R}^n$ ), we find that

$$\int_{\mathbb{R}} (|\partial_t \bar{w}|^2 - \bar{w}^2 + 3u_1^2 \bar{w}^2) dt + \int_{\mathbb{R}} \bar{w} \Delta_{\mathbb{R}^n} \bar{w} dt = 0.$$

Using the identity

$$2\bar{w} \Delta_{\mathbb{R}^n} \bar{w} = \Delta_{\mathbb{R}^n} \bar{w}^2 - 2|\nabla \bar{w}|^2$$

together with Lemma 3.6, we conclude that the function

$$V(y) := \int_{\mathbb{R}} \bar{w}^2(t, y) dt,$$

satisfies

$$\Delta_{\mathbb{R}^n} V - \frac{3}{4} V = \int_{\mathbb{R}} |\nabla \bar{w}|^2 dt \geq 0.$$

We define  $E_1$  to be the first eigenvalue of  $-\Delta_{\mathbb{R}^n}$ , with 0 Dirichlet boundary condition, in the ball of radius 1. The associatd eigenvalue will be denoted by  $\lambda_1$ . In particular

$$(3.19) \quad \Delta_{\mathbb{R}^n} E_1 = -\lambda_1 E_1$$

Then  $E_R(x) := E_1(x/R)$  is the first eigenfunction of  $-\Delta_{\mathbb{R}^n}$ , with 0 Dirichlet boundary condition, in the ball of radius  $R$  and the associated eigenvalue is given by  $\lambda_1 R^{-2}$ .

We multiply (3.18) by  $E_R$  and integrate by parts the result over  $B_R$ , the ball of radius  $R$  in  $\mathbb{R}^n$ . We get

$$\left(\lambda_1 R^{-2} - \frac{3}{4}\right) \int_{B_R} V E_R dx + \int_{\partial B_R} \partial_r E_R V da \geq 0.$$

Choosing  $R$  large enough and using the fact that  $V \geq 0$ , we conclude that  $V \equiv 0$  in  $B_R$ . Therefore  $V \equiv 0$  on  $\mathbb{R}^n$ .  $\square$

**3.3.2. The a priori estimate.** We are now in a position to analyze the operator  $L_\varepsilon$  which has been defined in (3.13) and which is acting on Hölder weighted spaces which we now define. We define on  $\Gamma \times \mathbb{R}$ , the scaled metric

$$g_\varepsilon := \varepsilon^2 (\dot{g} + dt^2).$$

With these notations in mind, we can state the :

**DEFINITION 3.1.** *For all  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ , the space  $\mathcal{C}_\varepsilon^{k, \alpha}(\Gamma \times \mathbb{R})$  is the space of functions  $w \in \mathcal{C}_{\text{loc}}^{k, \alpha}(\Gamma \times \mathbb{R})$  where the Hölder norm is computed with respect to the scaled metric  $g_\varepsilon$ .*

In other words, if  $w \in \mathcal{C}_\varepsilon^{k, \alpha}(\Gamma \times \mathbb{R})$ , then

$$|\nabla^a \partial_t^b w(y, t)|_{\dot{g}} \leq C \|w\|_{\mathcal{C}_{\varepsilon, \nu}^{k, \alpha}(\Gamma \times \mathbb{R})} \varepsilon^{-a-b} d_\Gamma(y)^\nu,$$

provided  $a + b \geq k$ . Hence, taking partial derivatives, we loose powers of  $\varepsilon$ .

We shall work in the closed subspace of functions satisfying the orthogonality condition

$$(3.20) \quad \int_{\mathbb{R}} w(t, y) u_1'(t) dt = 0,$$

for all  $y \in \Gamma$ .

We have the following :

**PROPOSITION 3.1.** *There exist constants  $C > 0$  and  $\varepsilon_0 > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$  and for all  $w \in \mathcal{C}_\varepsilon^{2, \alpha}(\Gamma \times \mathbb{R})$  satisfying (3.20), we have*

$$(3.21) \quad \|w\|_{\mathcal{C}_\varepsilon^{2, \alpha}(\Gamma \times \mathbb{R})} \leq C \|L_\varepsilon w\|_{\mathcal{C}_\varepsilon^{0, \alpha}(\Gamma \times \mathbb{R})}.$$

**PROOF.** Observe that, by elliptic regularity theory, it is enough to prove that

$$\|w\|_{L^\infty(\Gamma \times \mathbb{R})} \leq C \|L_\varepsilon w\|_{L^\infty(\Gamma \times \mathbb{R})}.$$

The proof of this result is by contradiction. We assume that, for a sequence  $\varepsilon_i$  tending to 0 there exists a function  $w_i$  such that

$$\|w_i\|_{L^\infty(\Gamma \times \mathbb{R})} = 1,$$

and

$$\lim_{i \rightarrow \infty} \|L_{\varepsilon_i} w_i\|_{L^\infty(\Gamma \times \mathbb{R})} = 0$$

For each  $i \in \mathbb{N}$ , we choose a point  $x_i := (t_i, y_i) \in \mathbb{R} \times \Gamma$  where

$$|w_i(t_i, y_i)| \geq 1/2.$$

Arguing as in the proof of Lemma 3.7, one can prove that the sequence  $t_i$  tends to 0 and more precisely that  $|t_i| \leq C \varepsilon_i$ . Indeed, the function

$$(t, y) \mapsto 1,$$

can again be used as a super-solution for the problem and this shows that necessarily  $|t_i| \leq t_* \varepsilon_i$ .

Now, we use

$$y \in T_{y_i} \Gamma \mapsto \text{Exp}_{y_i}^\Gamma(y) \in \Gamma$$

the exponential map on  $\Gamma$ , at the point  $y_i$ , and we define

$$\tilde{w}_i(y, t) := w_i(\varepsilon_i t, \text{Exp}_{y_i}^\Gamma(\varepsilon_i y))$$

which is defined on  $T_{y_i} \Gamma \times \mathbb{R}$ .

Using elliptic estimates together with Ascoli's Theorem, we can extract subsequences and pass to the limit in the equation satisfied by  $\tilde{w}_i$ . We find that  $\tilde{w}_i$  converges, uniformly on compacts to  $\tilde{w}$  which is a non trivial solution of

$$(\partial_t^2 + 1 - 3u_1^2 + \Delta_{\mathbb{R}^n}) \tilde{w} = 0$$

In addition  $\tilde{w}$  satisfies (3.16) and  $\tilde{w} \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ . This clearly contradicts the result of Lemma 3.7. The proof of the result is therefore complete.  $\square$

**3.3.3. The surjectivity result.** The final result of this section is the surjectivity of the operator  $L_\varepsilon$  acting on the space of functions satisfying (3.20).

**PROPOSITION 3.2.** *There exists  $\varepsilon_0 > 0$ , such that, for all  $\varepsilon \in (0, \varepsilon_0)$  and for all  $f \in \mathcal{C}_\varepsilon^{0,\alpha}(\Gamma \times \mathbb{R})$  satisfying (3.20), there exists a unique function  $w \in \mathcal{C}_\varepsilon^{2,\alpha}(\Gamma \times \mathbb{R})$  which also satisfies (3.20) and which is a solution of*

$$L_\varepsilon w = f,$$

in  $\Gamma \times \mathbb{R}$ .

**PROOF.** We use the variational structure of the problem and consider the functional

$$F(w) := \int_{\Gamma \times \mathbb{R}} (\varepsilon^2 (|\partial_t w|^2 + |\nabla w|_g^2) - w^2 + 3u_\varepsilon^2 w^2) \, d\text{vol}_g \, dt,$$

acting on the space of functions  $w \in H_0^1(\Gamma \times \mathbb{R})$  which satisfy (3.20) for a.e.  $y \in \Gamma$ .

Thanks to Lemma 3.6, we know that

$$F(w) \geq \frac{3}{2} \int_{\Gamma \times \mathbb{R}} w^2 \, dt \, d\text{vol}_g$$

Now, given  $f \in L^2(\Gamma \times \mathbb{R})$ , we can apply Lax-Milgram's Theorem to obtain a weak solution of  $L_\varepsilon w = f$  in  $H_0^1(\Gamma \times \mathbb{R})$ . It is then enough to apply elliptic regularity to conclude.  $\square$

**3.4. Study of a strongly coercive operator.** This short section is devoted to the mapping properties of the operator

$$\mathcal{L}_\varepsilon := \varepsilon^2 \Delta - 2$$

Certainly this operator satisfies the maximum principle and solvability of the equation  $\mathcal{L} w = f$  and obtention of the estimates boils down to the construction of appropriate super-solutions. In particular, we have

PROPOSITION 3.3. *There exists a constant  $C > 0$  such that*

$$(3.22) \quad \|w\|_{\mathcal{C}_\varepsilon^{2,\alpha}(M)} \leq C \|\mathcal{L}_\varepsilon w\|_{\mathcal{C}_\varepsilon^{0,\alpha}(M)},$$

where  $\mathcal{C}_\varepsilon^{k,\alpha}(M)$  is the Hölder space on  $M$  using the scaled metric  $\varepsilon^2 g$ .

PROOF. This is a consequence of standard elliptic estimates applied on geodesic balls of radius  $\varepsilon$ .  $\square$

**3.5. The nonlinear scheme.** We describe in this section the nonlinear scheme we are going to use to perturb the approximate solution into a genuine solution of (0.1).

3.5.1. *Some useful cutoff functions.* We will need various cutoff functions in our construction. Therefore, for  $j = 1, \dots, 6$ , we define the cut-off function  $\chi_j$  by

$$Z^* \chi_j(y, t) := \begin{cases} 1 & \text{when } |t| \leq c_* \left(1 - \frac{2j-1}{100}\right) \\ 0 & \text{when } |t| \geq c_* \left(1 - \frac{2j-2}{100}\right) \end{cases}$$

where  $c_* > 0$  is chosen small enough so that  $Z$  is a diffeomorphism from the set  $\{(y, t) \in \Gamma \times \mathbb{R} : |t| \leq 2c_*\}$  onto its image. We define  $\Omega_j$  to be the support of  $\chi_j$ .

3.5.2. *A one parameter family of approximate solutions.* Building on the analysis we have done in the previous sections, the approximate solution  $\tilde{u}_\varepsilon$  is defined by

$$\tilde{u}_\varepsilon := \begin{cases} \chi_1 \bar{u}_\varepsilon & \text{when } \Omega_1 \\ \pm(1 - \chi_1) & \text{when } \mathbb{R}_\pm^{n+1} \setminus \Omega_1 \end{cases}$$

where  $\pm$  corresponds to whether the point belongs to  $\mathbb{R}_\pm^{n+1}$ . Here the function  $\bar{u}_\varepsilon$  is the one defined in (3.12), namely

$$Z^* \bar{u}_\varepsilon(y, t) := u_\varepsilon(t).$$

Observe that  $\bar{u}_\varepsilon$  is exponentially close to  $\pm 1$  at infinity and hence, it is reasonable to graft it to the constant functions  $\pm 1$  away from some neighborhood of  $\Gamma$ .

3.5.3. *An infinite dimensional family of diffeomorphisms.* Given a function  $\zeta \in \mathcal{C}^{2,\alpha}(\Gamma)$ , we define a diffeomorphism  $D_\zeta$  of  $M$  as follows.

$$Z^* D_\zeta(y, t) = Z(y, t - \chi_2(y, t) \zeta(y))$$

in  $\Omega_2$  and

$$D_\zeta = Id$$

in  $M \setminus \Omega_0$ . It is easy to check that this is a diffeomorphism of  $M$  provided the norm of  $\zeta$  is small.

Also, observe that, the inverse of  $D_\zeta$  can be written as

$$Z^* D_\zeta^{-1}(y, t) = Z(y, t - \chi_2(y, t) \zeta(y) + \zeta(y, t, \zeta(y)) \zeta(y)^2)$$

in  $\Omega_2$ , where  $(y, t, z) \mapsto \zeta(y, t, z)$  is a smooth function defined for  $z$  small (this follows at once from the inverse function theorem applied to the function

$$t \mapsto t - \chi_2(y, t) z,$$

and details are left to the reader).

3.5.4. *Rewriting the equation.* First, given a function  $\zeta \in \mathcal{C}^{2,\alpha}(\Gamma)$ , small enough, we use the diffeomorphism  $D_\zeta$ , we write  $u = \bar{u} \circ D_\zeta$  so that the equation

$$\varepsilon^2 \Delta_g u + u - u^3 = 0,$$

can be rewritten as

$$\varepsilon^2 (\Delta_g \bar{u} \circ D_\zeta) \circ D_\zeta^{-1} + \bar{u} - \bar{u}^3 = 0$$

Observe that, in the coordinates  $(y, t)$ , namely when  $\chi_2 \equiv 0$  and hence  $D_\zeta$  is just given by  $Z^* D_\zeta(y, t) = Z(y, t - \zeta(y))$ , this equation is precisely  $\mathfrak{N}(v, \zeta) = 0$  where  $\mathfrak{N}$  is given by (3.5). Also observe that this equation is already nonlinear in  $\zeta$  and its partial derivatives. But,  $\zeta$  never appears in the composition of the function  $\bar{u}$ .

Now, we look for a solution of this equation as a perturbation of  $\tilde{u}_\varepsilon$ , and hence, we define

$$\bar{u} := \tilde{u}_\varepsilon + v,$$

so that the equation we need to solve can now be written as

$$(3.23) \quad \varepsilon^2 (\Delta_g v \circ D_\zeta) \circ D_\zeta^{-1} + v - 3\tilde{u}_\varepsilon^2 v = E_\varepsilon(\zeta) + Q_\varepsilon(v),$$

where

$$E_\varepsilon(\zeta) := \varepsilon^2 (\Delta_g \tilde{u}_\varepsilon \circ D_\zeta) \circ D_\zeta^{-1} + \tilde{u}_\varepsilon - \tilde{u}_\varepsilon^3,$$

is the error corresponding to the fact that  $\tilde{u}_\varepsilon$  is an approximate solution and

$$Q_\varepsilon(v) := \phi^3 + 3\tilde{u}_\varepsilon v^2,$$

collects the nonlinear terms in  $v$ .

Finally, in order to solve (3.23), we use a very nice trick which was already used in [17]. This trick amounts to decompose the function  $v$  into two equations and, instead of solving (3.23), solve a coupled system. At first glance this might look rather counterintuitive but, as we will see, this strategy allows one to use directly the results we have proven in the previous sections. Therefore, we set

$$v := \chi_4 v^\sharp + v^\flat,$$

where the function  $v^\flat$  solves

$$\begin{aligned} (\varepsilon^2 \Delta_g - 2) v^\flat &= (1 - \chi_4) \left[ \varepsilon^2 \left( \Delta_g (v^\flat \circ D_\zeta) \circ D_\zeta^{-1} - \Delta_g v^\flat \right) \right. \\ &\quad \left. + 3(\tilde{u}_\varepsilon^2 - 1) v^\flat - E_\varepsilon(\zeta) - Q_\varepsilon(\chi_4 v^\sharp + v^\flat) \right] \\ &\quad + \varepsilon^2 \left( (\Delta_g (\chi_4 v^\sharp) \circ D_\zeta) - \chi_4 \Delta_g (v^\sharp \circ D_\zeta) \right) \circ D_\zeta^{-1} \end{aligned}$$

For short, the right hand side will be denoted by  $N_\varepsilon(v^\flat, v^\sharp, \zeta)$  so that this equation reads

$$(3.24) \quad (\varepsilon^2 \Delta_g - 2) v^\flat = N_\varepsilon(v^\flat, v^\sharp, \zeta).$$

REMARK 3.3. *Observe that the right hand side vanishes when  $\chi_4 = 1$  and hence it can be written as the product of a function with  $(1 - \chi_5)$ . In particular, we know from Proposition 3.3 that if*

$$(\varepsilon^2 \Delta - 2) w = f,$$

then

$$\|w\|_{\mathcal{C}_{\varepsilon, \nu}^{2,\alpha}(\mathbb{R}^{n+1})} \leq C \|f\|_{\mathcal{C}_{\varepsilon, \nu}^{0,\alpha}(\mathbb{R}^{n+1})}.$$

In the case where  $w$  is a solution of

$$(\varepsilon^2 \Delta - 2) w = (1 - \chi_5) f,$$

we see that  $(\varepsilon^2 \Delta - 2)w = 0$  wherever  $\chi_5 = 1$  and it is easy to check that we have the better estimate

$$\|\chi_6 w\|_{C_{\varepsilon, \nu}^{2, \alpha}(\mathbb{R}^{n+1})} \leq C \varepsilon \|f\|_{C_{\varepsilon, \nu}^{0, \alpha}(\mathbb{R}^{n+1})},$$

provided  $\varepsilon$  is small enough. This follows easily from the construction of suitable barrier functions for  $\varepsilon^2 \Delta - 2$ , in a neighborhood of any point  $x^0 \in \mathbb{R}^{n+1}$ , for example, one can use

$$x \mapsto \sum_{i=1}^{n+1} \cosh \left( \sqrt{2} \frac{(x_i - x_i^0)}{\varepsilon} \right),$$

as a barrier. We can summarize this discussion by saying that

$$(3.25) \quad \|w\|_{\tilde{C}_{\varepsilon, \nu}^{2, \alpha}(\mathbb{R}^{n+1})} \leq C \varepsilon \|f\|_{C_{\varepsilon, \nu}^{0, \alpha}(\mathbb{R}^{n+1})},$$

where, by definition

$$\|v\|_{\tilde{C}_{\varepsilon, \nu}^{2, \alpha}(\mathbb{R}^{n+1})} := \varepsilon^{-1} \|\chi_6 v\|_{C_{\varepsilon, \nu}^{2, \alpha}(\mathbb{R}^{n+1})} + \|v\|_{C_{\varepsilon, \nu}^{2, \alpha}(\mathbb{R}^{n+1})}.$$

Taking the difference between the equation satisfied by  $v$  and the equation satisfied by  $v^b$ , we find that it is enough that  $v^\sharp$  solves,

$$\begin{aligned} \varepsilon^2 \Delta_g(v^\sharp \circ D_\zeta) \circ D_\zeta^{-1} + v^\sharp - 3 \bar{u}_\varepsilon^2 v^\sharp &= E_\varepsilon(\zeta) + Q_\varepsilon(\chi_4 v^\sharp + v^b) - 3(\bar{u}_\varepsilon^2 - 1)v^b \\ &+ \varepsilon^2 \left( \Delta_g(v^b \circ D_\zeta) \circ D_\zeta^{-1} - \Delta_g v^b \right), \end{aligned}$$

in the support of  $\chi_4$ . Since we only need this equation to be satisfied on the support of  $\chi_4$ , we can as well solve the equation

$$(3.26) \quad \begin{aligned} L_\varepsilon v^\sharp - \varepsilon J_\Gamma \zeta \dot{u}_\varepsilon &= \chi_3 \left[ \varepsilon^2 \left( \Delta_g(v^\sharp \circ D_\zeta) \circ D_\zeta^{-1} + v^\sharp - 3 \bar{u}_\varepsilon^2 v^\sharp - L_\varepsilon v^\sharp \right) \right. \\ &+ \varepsilon^2 \left( \Delta_g(v^b \circ D_\zeta) \circ D_\zeta^{-1} - \Delta_g v^b \right) \\ &\left. + E_\varepsilon(\zeta) - \varepsilon J_\Gamma \zeta \dot{u}_\varepsilon + Q_\varepsilon(\chi_4 v^\sharp + v^b) - 3(\bar{u}_\varepsilon^2 - 1)v^b \right], \end{aligned}$$

where the operator  $L_\varepsilon$  is the one defined in (3.13). Here we have implicitly used the fact that

$$\chi_3 \tilde{u}_\varepsilon = \chi_3 \bar{u}_\varepsilon$$

For short, the right hand side will be denoted by  $M_\varepsilon(v^b, v^\sharp, \zeta)$  so that this equation reads

$$(3.27) \quad \varepsilon^2 L_\varepsilon v^\sharp - \varepsilon J_\Gamma \zeta \dot{u}'_\varepsilon = M_\varepsilon(v^b, v^\sharp, \zeta)$$

This equation can be projected over the space of functions satisfying (3.20) and the set of functions of the form  $u'_1$  times a function defined on  $\Gamma$ . Let us denote by  $\Pi$  the orthogonal projection on  $u'_1$ , namely

$$\Pi(f) := \frac{1}{\varepsilon c} \int_{\mathbb{R}} f(y, t) \dot{u}_\varepsilon(t) dt$$

where

$$c := \frac{1}{\varepsilon} \int_{\mathbb{R}} \dot{u}_\varepsilon^2(t) dt = \int_{\mathbb{R}} (u'_1)^2(t) dt,$$

and by  $\Pi^\perp$  the orthogonal projection on the orthogonal of  $\dot{u}_\varepsilon$ , namely

$$\Pi^\perp(f) := f - \Pi(f) \dot{u}_\varepsilon$$

We further assume that  $v^\sharp$  satisfies (3.20). Then (3.28) is equivalent to the system

$$(3.28) \quad L_\varepsilon v^\sharp = \Pi^\perp \left[ M_\varepsilon(v^b, v^\sharp, \zeta) \right],$$

and

$$(3.29) \quad -\varepsilon J_\Gamma \zeta = \Pi \left[ M_\varepsilon(v^b, v^\sharp, \zeta) \right].$$

**3.6. The proof of Theorem 1.1.** We summarize the above discussion by saying that we are looking for a solution of

$$\varepsilon^2 \Delta u + u - u^3 = 0,$$

of the form

$$u = \left( \tilde{u}_\varepsilon + \chi_4 v^\sharp + v^b \right) \circ D_\zeta,$$

where the function  $v^\sharp$  is defined on  $\Gamma \times \mathbb{R}$ , the function  $v^b$  is defined in  $M$  and the function  $\zeta$  defined on  $\Gamma$ , and satisfy

$$(3.30) \quad (\varepsilon^2 \Delta_g - 2) v^b = N_\varepsilon(v^b, v^\sharp, \zeta)$$

$$(3.31) \quad L_\varepsilon v^\sharp = \Pi^\perp \left[ M_\varepsilon(v^b, v^\sharp, \zeta) \right],$$

and

$$(3.32) \quad -\varepsilon J_\Gamma \zeta = \Pi \left[ M_\varepsilon(v^b, v^\sharp, \zeta) \right],$$

Closer inspection of the construction of the approximate solution shows that :

LEMMA 3.8. *The following estimates hold*

$$\|N_\varepsilon(0, 0, 0)\|_{C_\varepsilon^{0,\alpha}(M)} + \|\Pi^\perp(M_\varepsilon(0, 0, 0))\|_{C_\varepsilon^{0,\alpha}(\Gamma \times \mathbb{R})} \leq C \varepsilon^2.$$

Moreover

$$\|\Pi(M_\varepsilon(0, 0, 0))\|_{C^{0,\alpha}(\Gamma)} \leq C \varepsilon^3.$$

PROOF. Since  $v^\sharp = 0$ ,  $v^b = 0$  and  $\zeta = 0$ , the estimate follow from the understanding of

$$E_\varepsilon(0) = \varepsilon^2 \Delta_g \tilde{u}_\varepsilon + \tilde{u}_\varepsilon - \tilde{u}_\varepsilon^3,$$

But, we have already seen that

$$\varepsilon^2 \Delta_g \bar{u}_\varepsilon + \bar{u}_\varepsilon - \bar{u}_\varepsilon^3 = -\varepsilon H_t \dot{u}_\varepsilon.$$

The estimates then follow at once from (3.9).  $\square$

We also need the



LEMMA 3.9. *There exists  $\delta > 0$  such that the following estimates hold*

$$\begin{aligned} & \|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C_\varepsilon^{0,\alpha}(M)} \\ & \leq C \varepsilon^\delta \left( \|v_2^\flat - v_1^\flat\|_{C_\varepsilon^{2,\alpha}(M)} + \|v_2^\sharp - v_1^\sharp\|_{C_\varepsilon^{2,\alpha}(\Gamma \times \mathbb{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Gamma)} \right) \\ & \|\Pi^\perp(M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1))\|_{C_\varepsilon^{0,\alpha}(\Gamma \times \mathbb{R})} \\ & \leq C \varepsilon^\delta \left( \|v_2^\flat - v_1^\flat\|_{\bar{C}_\varepsilon^{2,\alpha}(M)} + \|v_2^\sharp - v_1^\sharp\|_{C_\varepsilon^{2,\alpha}(\Gamma \times \mathbb{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Gamma)} \right) \end{aligned}$$

and

$$\begin{aligned} & \|\Pi(M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1))\|_{C^{0,\alpha}(\Gamma)} \\ & \leq C \varepsilon^\delta \left( \|v_2^\flat - v_1^\flat\|_{\bar{C}_\varepsilon^{2,\alpha}(M)} + \|v_2^\sharp - v_1^\sharp\|_{C_\varepsilon^{2,\alpha}(\Gamma \times \mathbb{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Gamma)} \right). \end{aligned}$$

PROOF. The proof is rather technical but does not offer any real difficulty. Observe that, in the last two estimates, the use of the norm  $\|v^\flat\|_{\bar{C}_\varepsilon^{2,\alpha}(\mathbb{R}^{n+1})}$  instead of  $\|v^\flat\|_{C_{\varepsilon,\nu}^{2,\alpha}(\mathbb{R}^{n+1})}$  is crucial to estimate the term  $-3(\bar{u}_\varepsilon^2 - 1)v^\flat$  in the definition of  $M_\varepsilon(v^\flat, v^\sharp, \zeta)$ .  $\square$

We use the result of Proposition 3.2 and Proposition 3.3, to rephrase the solvability of (3.30)-(3.32) as a fixed point problem.

Theorem 2.10 is now a simple consequence of the application of a fixed point theorem for contraction mapping which leads to the existence of a unique solution

$$\mathbf{u}_\varepsilon = (\bar{u}_\varepsilon + \chi_4 v^\sharp + v^\flat) \circ D_\zeta$$

where

$$\|v^\flat\|_{\bar{C}_\varepsilon^{2,\alpha}(M)} + \|v^\sharp\|_{C_\varepsilon^{2,\alpha}(\Gamma \times \mathbb{R})} \leq C \varepsilon^3,$$

and

$$\|\zeta\|_{C^{2,\alpha}(\Gamma)} \leq C \varepsilon^2.$$

We leave the details for the reader.

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