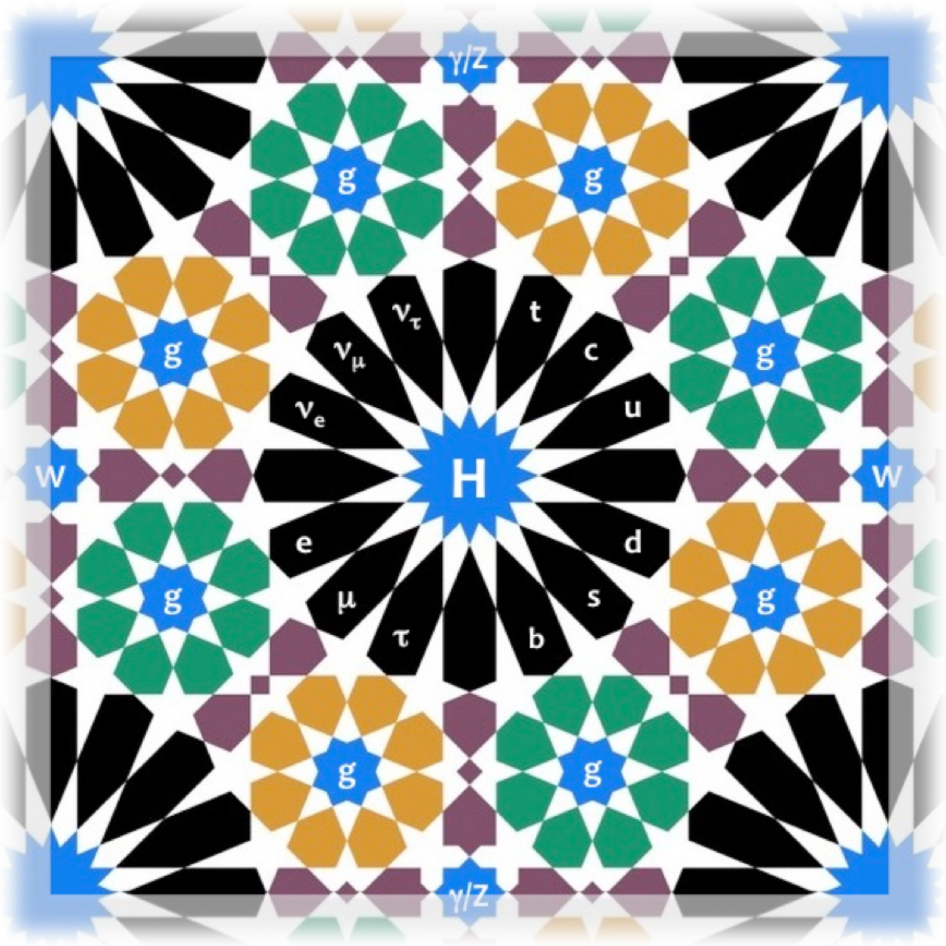
The Standard Model (part I)
Particles, quantum fields, symmetries. Electroweak interactions and their phenomenology



José Ignacio Illana

Departamento de Física Teórica y del Cosmos

FTAE High Energy Theory

UNIVERSIDAD DE GRANADA

[https://www.ugr.es/~jillana/SM1.php]

Outline

(#) *Exercises*

1. Particles, fields and symmetries
   - ▷ Basics: Poincaré symmetry
   - ▷ Particle physics with quantum fields
   - ▷ Global and gauge symmetries
     - Internal symmetries and the gauge principle
     - Quantization of gauge theories
     - Spontaneous Symmetry Breaking
2. The Standard Model
   - ▷ Electroweak interactions: one generation of quarks *or* leptons [strong int. in part II]
   - ▷ Electroweak SSB: Higgs sector, gauge boson and fermion masses
   - ▷ Additional generations: fermion mixings (quarks *vs* leptons)
3. Electroweak phenomenology
   - ▷ Input parameters, experiments, observables, precise predictions, global fits
   - ▷ Neutrinos [quark flavour physics in part II]
4. Tools

Bibliography

[1] **M. Maggiore**,
A Modern Introduction to Quantum Field Theory,
Oxford University Press, 2005.

[2] **M. D. Schwartz**,
Quantum Field Theory and the Standard Model,
Cambridge University Press, 2014.

[3] **M. E. Peskin and D. V. Schroeder**,
An Introduction to Quantum Field Theory,
Addison-Wesley, 1995.

[4] **A. Lahiri, P. B. Pal**,
A first book of Quantum Field Theory,
Narosa Publishing House, 2nd edition, 2005.

[5] **S. Pokorski**,
Gauge Field Theories,
Cambridge University Press, 2nd edition, 2000.

[6] **Particle Data Group**,
Review of Particle Properties,
[pdg.lbl.gov]

[0] **J. I. Illana**,
Teoría Cuántica de Campos,
[cuantización canónica]
[cuantización funcional]
Quantum Field Theory and the structure of the Standard Model,
[lecture notes]

1. Particles, fields and symmetries

Why Quantum Field Theory to describe Particle Physics?

- QFT is the (only) way to reconcile Quantum Mechanics and Special Relativity

[−] Wave equations (relativistic or not) cannot account for changing # of particles. And the relativistic versions suffer pathologies:

* negative probability densities
* negative-energy solutions
* violation of causality

[+] Quantum fields:

* provide a natural framework (*Fock space* of multiparticle states)
* make sense of negative-energy solutions (*antiparticles*)
* solve causality problem (*Feynman propagator*)
* explains spin-statistics connection (*theorem*)
* arguably, solve the wave-particle duality puzzle (no particles, *only fields*)

Basics: Poincaré symmetry

Guided by symmetry

- **Relativistic fields** are *irreps* of Poincaré group (rotations, boosts, translations)

  scalar $\phi(x)$, vector $V_\mu(x)$, tensor $h_{\mu\nu}(x)$, ...
  Weyl $\psi_L(x)$, $\psi_R(x)$; Dirac $\psi(x)$, ...

- **Lagrangian** densities: **local** $\mathcal{L}(x) = \mathcal{L}(\phi, \partial_\mu \phi)$ (maybe several "ϕ_i", ψ, V_μ, ...)

  invariant under Poincaré transformations

– e.g. for a free Dirac field $\psi(x)$:

$$\boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu\ , \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

⋆ Field dynamics

⋆ Noether's theorem: (continuous) symmetry implies conservation laws

(*energy*, *momentum*, *angular momentum*)

Lagrangians | Dynamics (classical)

- Principle of least action: $\delta S = 0$ where $S = \int \mathrm{d}^4x\, \mathcal{L}(x)$

  $\Rightarrow$ Field EoM (E-L equations)

$$\delta S = \int \mathrm{d}^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right)$$

$$= \int \mathrm{d}^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i = 0 \quad , \quad \forall \phi_i$$

(integrating by parts and assuming fields vanish at boundary)

– e.g. EoM of a free Dirac field is the Dirac equation

$$\boxed{(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0}$$

$$\rightsquigarrow \quad \psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1,2} \left(a_{\boldsymbol{p},s} u^{(s)}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px} + b^*_{\boldsymbol{p},s} v^{(s)}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} \right)$$

with $p^2 = E_{\boldsymbol{p}}^2 - \boldsymbol{p}^2 = m^2$, $\quad (p\!\!\!/ - m)u(\boldsymbol{p}) = 0, \quad (p\!\!\!/ + m)v(\boldsymbol{p}) = 0.$

Lagrangians **Quantization** («particle» concept emerges)

- Impose canonical quantization rules:

  commutation/anticommutation of fields with conjugate momenta $\Pi_i(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)}$

  $$[\phi(t,\boldsymbol{x}), \Pi_\phi(t,\boldsymbol{y})] = \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{y}), \quad \{\psi(t,\boldsymbol{x}), \Pi_\psi(t,\boldsymbol{y})\} = \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

  so that the Hamiltonian is bounded from below.

  – e.g for a free fermion field, *anticommutation* is enforced! implying

  $$\{a_{\boldsymbol{p},r}, a^\dagger_{\boldsymbol{k},s}\} = \{b_{\boldsymbol{p},r}, b^\dagger_{\boldsymbol{k},s}\} = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})\delta_{rs}\,, \quad \{a_{\boldsymbol{p},r}, a_{\boldsymbol{k},s}\} = \cdots = 0$$

  – After *normal ordering* :: (all creation to left of annihilation opts) to subtract zero-point energy,

  $$H = \int \mathrm{d}^3x : \mathcal{H}(x) : = \int \frac{\mathrm{d}p^3}{(2\pi)^3} E_{\boldsymbol{p}} \sum_{s=1,2} (a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} + b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s})$$

  ⇒ Fields become operators that annihilate/create particles/antiparticles

  $$|0\rangle \;\text{(vacuum)}\,, \quad a^\dagger_{\boldsymbol{p},s}|0\rangle \;\text{(1 particle)}\,, \quad b^\dagger_{\boldsymbol{p},s}|0\rangle \;\text{(1 antiparticle)}\,, \quad \ldots$$

  ⇒ Multiparticle states symmetric/antisymmetric under exchange (spin-statistics!)

One-particle representations

- One-particle states are unitary irreps of the Poincaré group, so that

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\,\mathcal{P}^\dagger\mathcal{P}\,|\psi_2\rangle \quad \text{(invariant matrix elements)}$$

$\mathcal{P}$ are represented by unitary operators in this space, and the generators $\underbrace{J^i \text{ (rotations)}, K^i \text{ (boosts)}}_{J_{\mu\nu}=-J_{\nu\mu}}$, P^μ (translations) by Hermitian operators.

$$(J^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i})$$

- Rotations form a compact subgroup (its finite dimensional irreps are unitary). But Lorentz group and Poincaré group are non-compact. Therefore:

  The *unitary* representations of the Poincaré group are *infinite-dimensional*.

- Poincaré group has two Casimir operators (commute with all generators)

$$\boxed{m^2 = P_\mu P^\mu, \quad W_\mu W^\mu} \qquad W_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma \quad \text{(Pauli-Lubanski vector)}$$

  whose eigenvalues label the irreps. Lorentz invariant (choose convenient frame).

One-particle representations | Wigner classification

- Two cases, characterized by **mass m and spin j**

  – $m \neq 0$: choose $P^\mu = (m, 0, 0, 0) \Rightarrow W_\mu W^\mu = -m^2 j(j+1)$
  $\Rightarrow$ **massive particles of spin j** have $2j+1$ dof ($j_3 = -j, -j+1, \ldots, j$)
  because $SU(2)$ is the *little group* (transformations leaving P^μ invariant)

  – $m = 0$: choose $P^\mu = (\omega, 0, 0, \omega) \Rightarrow W_\mu W^\mu = -\omega^2[(J^1 + K^2)^2 + (J^2 - K^1)^2]$
  $\Rightarrow$ **massless particles of spin j** have 2 dof (helicity $h = \pm j$)
  because now $SO(2)$ is the *little group* (rotations in plane $\perp$ to P^μ)

- Note: To construct a **unitary field theory with** V_μ (contains both spin 0 and 1) one has to choose carefully the Lagrangian so that the *physical theory* **never excites**:

  – the spin-0 component (if massive)

  – neither the longitudinal spin-1 component (if massless) $\Leftrightarrow$ gauge invariance

Particle physics

Particle physics | S-matrix elements

- Observables (cross sections, decays widths) expressed in terms of S-matrix elements ($m \to n$ processes)

$$_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n | \boldsymbol{k}_1 \boldsymbol{k}_2 \cdots \boldsymbol{k}_m \rangle_{\text{in}}$$

(scalar fields/particles to simplify)

- Only free fields are related to particles/antiparticles ($a^\dagger_{\boldsymbol{p}}$, $b^\dagger_{\boldsymbol{p}}$).

We expect

$$\phi(x) \xrightarrow[t\to-\infty]{} Z_\phi^{1/2}\phi_{\text{in}}(x) \ , \quad \phi(x) \xrightarrow[t\to+\infty]{} Z_\phi^{1/2}\phi_{\text{out}}(x) \ ,$$

$\phi(x)$: interacting fields

$\phi_{\text{in}}(x)$, $\phi_{\text{out}}(x)$: free fields (before, after interaction)

Z_ϕ: *wave function* renormalization

- LSZ reduction formula relates S-matrix elements with the (Fourier transform of) vacuum expectation values of *time-ordered* field products (correlators):

$$\left(\prod_{i=1}^{m}\frac{\mathrm{i}\sqrt{Z_\phi}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{\mathrm{i}\sqrt{Z_\phi}}{p_j^2-m^2}\right)\,{}_{\text{out}}\langle \boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_n|\boldsymbol{k}_1\boldsymbol{k}_2\cdots\boldsymbol{k}_m\rangle_{\text{in}}$$

$$=\int\left(\prod_{i=1}^{m}\mathrm{d}^4x_i\ \mathrm{e}^{-\mathrm{i}k_ix_i}\right)\int\left(\prod_{j=1}^{n}\mathrm{d}^4y_j\ \mathrm{e}^{+\mathrm{i}p_jy_j}\right)\langle 0|\,T\{\underbrace{\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)}_{\text{interacting fields}}\}\,|0\rangle$$

The correlator = the Green's function of $m+n$ points $G(\boldsymbol{p}_1\cdots\boldsymbol{p}_n;\boldsymbol{k}_1\cdots\boldsymbol{k}_m)$

▷ Physical particles (asymptotic states) are on-shell ($p^2-m^2=0$).

For on-shell incoming and outgoing particles, the rhs of LSZ formula (correlator) will have **poles** that cancel those in the prefactor of the lhs, yielding a regular S-matrix element [*residues* of the correlator].

- The correlators can be expressed in terms of free fields "ϕ_0" :

$$\langle 0|\, T\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\langle 0|\, T\left\{\phi_0(x_1)\cdots\phi_0(x_n)\exp\left[\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\rm int}[\phi_0(x)]\right]\right\}|0\rangle}{\langle 0|\, T\left\{\exp\left[\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\rm int}[\phi_0(x)]\right]\right\}|0\rangle}$$

- In perturbation theory one expands the exponential and computes every correlator using *Wick's theorem* (all possible "contractions")

$$\textit{contraction} \equiv \overbrace{\phi(x)\phi}(y) = D_F(x-y) = \langle 0|\, T\{\phi(x)\phi(y)\}\,|0\rangle = \textit{Feynman propagator}$$

- Feynman diagrams/rules provide a systematic procedure to organize/compute the perturbative series in terms of *propagators* (and vertices)

- **Note:** functional quantization (path integral) provides an *alternative* method

$$\langle 0|\, T\{\hat\phi(x_1)\cdots\hat\phi(x_n)\}\,|0\rangle = \frac{\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\ \mathrm{e}^{\mathrm{i}S}}{\int \mathcal{D}\phi\ \mathrm{e}^{\mathrm{i}S}}$$

perturbatively *or* not! (*lattice*)

- **Causality** requires $[\phi(x), \phi^\dagger(y)] = 0$ if $(x-y)^2 < 0$ (*spacelike* interval)

▷ Recall that a (free) field is a combination of positive and negative energy waves:

$$\phi(x) = \int \frac{\mathrm{d}p^3}{(2\pi)^3\sqrt{E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}px} \right)$$

▷ From the commutation relations of creation and annihilation operators:

$$[\phi(x), \phi^\dagger(y)] = \int \frac{\mathrm{d}p^3}{(2\pi)^3\sqrt{E_{\boldsymbol{p}}}} \int \frac{\mathrm{d}q^3}{(2\pi)^3\sqrt{E_{\boldsymbol{q}}}} \left(\mathrm{e}^{-\mathrm{i}(px-qy)} [a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] + \mathrm{e}^{\mathrm{i}(px-qy)} [b^\dagger_{\boldsymbol{p}}, b_{\boldsymbol{q}}] \right)$$

$$= \Delta(x-y) - \Delta(y-x)$$

where the first (second) contribution comes from particles (antiparticles) and

$$\Delta(x-y) = \int \frac{\mathrm{d}p^3}{(2\pi)^3 E_{\boldsymbol{p}}} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

▷ If $(x-y)^2 < 0$ choose frame where $x - y \equiv (0, \boldsymbol{r})$. Then

$$\Delta(x-y) = \Delta(y-x) \propto \frac{m}{r}\mathrm{e}^{-mr} \neq 0, \quad \text{for } mr \gg 1$$

Therefore:

$$\text{If only particles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) \neq 0 \text{ (!!)}$$

$$\text{If } \textit{both} \text{ particles and antiparticles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) - \Delta(y-x) = 0 \;(\checkmark)$$

- In fact the **Feynman propagator** contains *both* contributions:

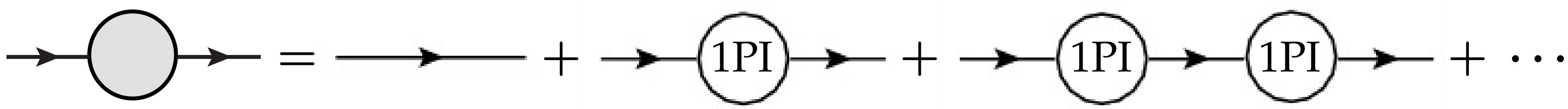
$$D_F(x-y) = \langle 0|\, T\{\phi(x)\phi^\dagger(y)\}\, |0\rangle = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)$$

  – Probability amplitude that particle created in y propagates to x, if $x^0 > y^0$

  – Probability amplitude that antiparticle created in x propagates to y, if $y^0 > x^0$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \quad \text{where} \quad \varepsilon \to 0^+ \text{(usually omitted)}$$

- Corrections to external legs (external propagators) can be resummed:

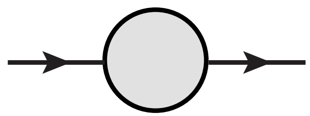


$$= \frac{\mathrm{i}}{p^2 - m_0^2} + \frac{\mathrm{i}}{p^2 - m_0^2}\left[-\mathrm{i}M^2(p^2)\right]\frac{\mathrm{i}}{p^2 - m_0^2} + \ldots$$

$$= \frac{\mathrm{i}}{p^2 - m_0^2 - M^2(p^2)} \qquad (m_0 = \text{mass in } \mathcal{L})$$

and Taylor expanding about $p^2 = m^2$ (*physical* mass):

$$p^2 - m_0^2 - M^2(p^2) = (p^2 - m^2)\left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)$$

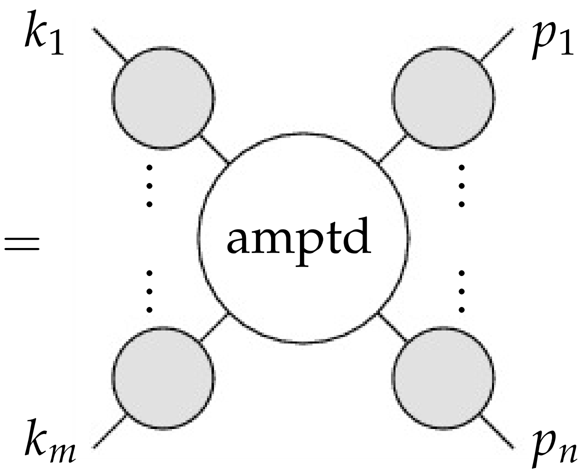
$\Rightarrow$  $= \dfrac{\mathrm{i}Z_\phi}{p^2 - m^2} +$ regular near $p^2 = m^2$

$$\text{with} \quad m^2 = m_0^2 + M^2(m^2)\,, \quad Z_\phi = \left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)^{-1}$$

Particle physics | Perturbation theory | LSZ (rewritten)

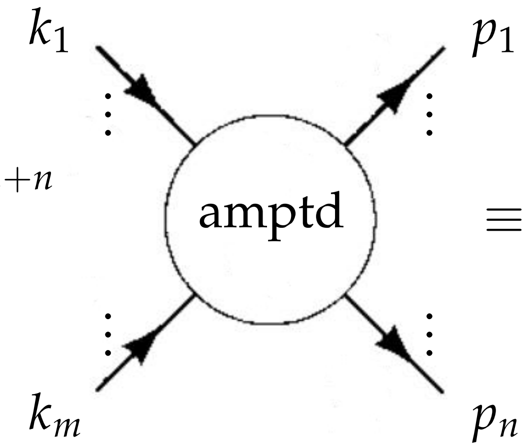
▷ Then we may factor out external legs from *amputated* diagrams:

$$G(\boldsymbol{p}_1 \cdots \boldsymbol{p}_n; \boldsymbol{k}_1 \cdots \boldsymbol{k}_m) =$$



and express the LSZ formula in a simpler form:

$${}_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n | \boldsymbol{k}_1 \boldsymbol{k}_2 \cdots \boldsymbol{k}_m \rangle_{\text{in}} = \left(\sqrt{Z}\right)^{m+n}$$



$$\equiv (2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_j k_j \right) \mathrm{i}\mathcal{M}$$

- Feynman rules require integration over loop momenta resulting *sometimes* in divergent expressions.

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\mathcal{M}^{(1)}}_{\text{divergent?}} + \dots$$

(the loop expansion is also an expansion in powers of $\hbar$: *quantum* corrections)

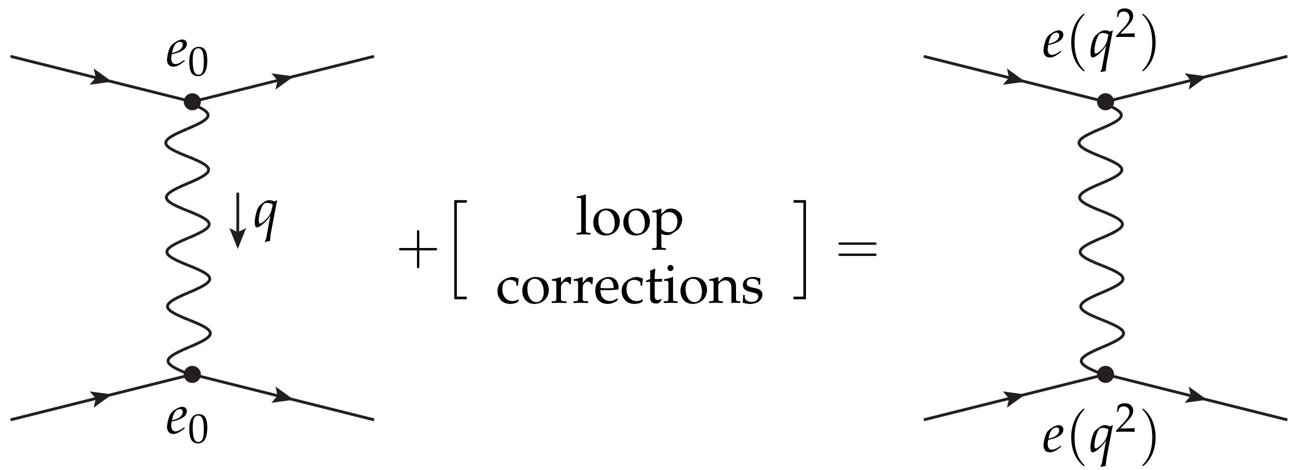
- **Regularization** and **renormalization** needed to make sense of these divergences.

▷ One assumes that fields and parameters in the Lagrangian (*bare*) must be redefined order by order in terms of new ones (*renormalized*) so that physical predictions are finite

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\widehat{\mathcal{M}}^{(1)}}_{\text{finite}} + \dots$$

▷ As a consequence, renormalized *coupling constants run* (depend on a scale)

e.g.



$e_0 =$ *bare* coupling $\qquad e(q^2) =$ *renormalized* coupling

$q^2 =$ *renormalization scale* (at which e is “measured”)

Note: e is not an observable

Global and gauge symmetries

Internal symmetries | free Lagrangian

- In addition to **spacetime** (Poincaré) symmetries, the free Lagrangian

$$\text{(Dirac)} \qquad \boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \,, \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

$\Rightarrow$ Invariant under **internal** global U(1) phase transformations:

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta}\psi(x) \,, \quad Q,\ \theta \text{ (constants)} \in \mathbb{R}$$

$\Rightarrow$ By Noether's theorem, divergentless current:

$$\mathcal{J}^\mu = Q\,\overline{\psi}\gamma^\mu\psi \,, \quad \partial_\mu \mathcal{J}^\mu = 0$$

and a conserved «charge»

$$\mathcal{Q} = \int \mathrm{d}^3x\, \mathcal{J}^0, \quad \partial_t \mathcal{Q} = 0$$

Internal symmetries | free Lagrangian

- For a free fermion quantum field:

  ⇒ The Noether charge is an operator:*

$$\mathcal{Q} = Q \int \mathrm{d}^3x \; :\overline{\psi}\gamma^0\psi: \; = Q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} - b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s} \right)$$

$$\mathcal{Q}\; a^\dagger_{\boldsymbol{k},s} |0\rangle = +Q\; a^\dagger_{\boldsymbol{k},s} |0\rangle \;\; (\text{particle}) \,, \quad \mathcal{Q}\; b^\dagger_{\boldsymbol{k},s} |0\rangle = -Q\; b^\dagger_{\boldsymbol{k},s} |0\rangle \;\; (\text{antiparticle})$$

* normal ordering prescription for fermionic operators

$$: a_{\boldsymbol{p},r} a^\dagger_{\boldsymbol{q},s} : \, \equiv -a^\dagger_{\boldsymbol{q},s} a_{\boldsymbol{p},r} \,, \quad : b_{\boldsymbol{p},r} b^\dagger_{\boldsymbol{q},s} : \, \equiv -b^\dagger_{\boldsymbol{q},s} b_{\boldsymbol{p},r}$$

The gauge principle | gauge symmetry dictates interactions

- To make $\mathcal{L}_0$ invariant under local ≡ gauge transformations of U(1):

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\psi(x)\,, \quad \theta = \theta(x) \in \mathbb{R}$$

perform the minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eQA_\mu \quad \text{(covariant derivative)}$$

where a gauge field $A_\mu(x)$ is introduced transforming as:

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \quad \Leftarrow \quad \boxed{D_\mu\psi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta(x)}D_\mu\psi} \quad \overline{\psi}\not{D}\psi \text{ inv.}$$

(1)

⇒ The new Lagrangian contains **interactions** between ψ and A_μ:

$$\boxed{\mathcal{L}_{\text{int}} = -e\,Q\,\overline{\psi}\gamma^\mu\psi A_\mu} \quad \propto \begin{cases} \text{coupling} & e \\ \text{charge} & Q \end{cases}$$

$$(= -e\,\mathcal{J}^\mu A_\mu)$$

The gauge principle | gauge invariance dictates interactions

- Dynamics for the gauge field ⇒ add gauge invariant kinetic term:

$$\text{(Maxwell)} \qquad \boxed{\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad \Leftarrow \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$

– The full U(1) gauge invariant Lagrangian for a fermion field $\psi(x)$ reads:

$$\mathcal{L}_{\text{sym}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_1) \quad \text{(QED)}$$

– The same applies to a complex scalar field $\phi(x)$:

$$\mathcal{L}_{\text{sym}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{(sQED)}$$

- A general gauge symmetry group G is an *compact* N-dimensional Lie group

$$\mathrm{g} \in G\,, \quad \mathrm{g}(\boldsymbol{\theta}) = \mathrm{e}^{-\mathrm{i}T_a\theta^a}\,, \quad a = 1,\ldots,N$$

$$\theta^a = \theta^a(x) \in \mathbb{R}\,, \quad T_a = \text{Hermitian generators}\,, \quad [T_a, T_b] = \mathrm{i}f_{abc}T_c \quad \text{(Lie algebra)}$$

$$\text{structure constants:} \quad f_{abc} = 0 \quad \text{Abelian}$$

$$f_{abc} \neq 0 \quad \text{non-Abelian}$$

⇒ *Unitary* finite-dimensional irreducible representations:

$g(\boldsymbol{\theta})$ represented by $U(\boldsymbol{\theta})$

$$d \times d \text{ matrices}: \quad U(\boldsymbol{\theta}) \text{ [given by } \{T_a\} \text{ algebra representation]}$$

$$d\text{-multiplet}: \quad \Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x)\,, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

- Examples:

G	N	Abelian	
U(1)	1	Yes	
SU(n)	n^2-1	No	($n \times n$ unitary matrices with det $= 1$)

  - U(1): 1 generator (q), one-dimensional irreps only
  - SU(2): 3 generators

    $f_{abc} = \epsilon_{abc}$ (Levi-Civita symbol)

    * Fundamental irrep ($d = 2$): $T_a = \frac{1}{2}\sigma_a$ (3 Pauli matrices)
    * Adjoint irrep ($d = N = 3$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$
  - SU(3): 8 generators

    $f^{123} = 1,\ f^{458} = f^{678} = \frac{\sqrt{3}}{2},\ f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367} = \frac{1}{2}$

    * Fundamental irrep ($d = 3$): $T_a = \frac{1}{2}\lambda_a$ (8 Gell-Mann matrices)
    * Adjoint irrep ($d = N = 8$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$

(for SU(n): f_{abc} totally antisymmetric)

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of G:

$$\mathcal{L}_0 = \overline{\Psi}(\mathrm{i}\not\partial - m)\Psi \,, \qquad \Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x) \,, \quad \boldsymbol{\theta} = \boldsymbol{\theta}(x) \in \mathbb{R}$$

substitute the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu \,, \quad \widetilde{W}_\mu \equiv T_a W_\mu^a$$

where a gauge field $W_\mu^a(x)$ per generator is introduced, transforming as:

$$\widetilde{W}_\mu(x) \mapsto \widetilde{W}'_\mu(x) = \underbrace{U\widetilde{W}_\mu(x)U^\dagger}_{\text{adjoint irrep}} - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \quad \Leftarrow \quad \boxed{D_\mu\Psi \mapsto UD_\mu\Psi} \quad \overline{\Psi}\not{D}\Psi \text{ inv.}$$ (1)

$\Rightarrow$ The new Lagrangian contains interactions between Ψ and W_μ^a:

$$\boxed{\mathcal{L}_{\text{int}} = g\, \overline{\Psi}\gamma^\mu T_a \Psi W_\mu^a} \quad \propto \begin{cases} \text{coupling} & g \\ \text{charge} & T_a \end{cases}$$

$$(= g\, \mathcal{J}_a^\mu W_\mu^a)$$

- Dynamics for the gauge fields ⇒ add gauge invariant kinetic terms:

(Yang-Mills) $$\mathcal{L}_{\rm YM} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\} = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}$$

$$\widetilde{W}_{\mu\nu} \equiv T_a W^a_{\mu\nu} \equiv D_\mu \widetilde{W}_\nu - D_\nu \widetilde{W}_\mu = \partial_\mu \widetilde{W}_\nu - \partial_\nu \widetilde{W}_\mu - ig[\widetilde{W}_\mu, \widetilde{W}_\nu] \;\Leftrightarrow\; \widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

$$\Rightarrow \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$$ (2)

⇒ $\mathcal{L}_{\rm YM}$ contains cubic and quartic self-interactions of the gauge fields W^a_μ:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$$

$$\mathcal{L}_{\rm cubic} = -\frac{1}{2} g f_{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) W^{b,\mu} W^{c,\nu}$$

$$\mathcal{L}_{\rm quartic} = -\frac{1}{4} g^2 f_{abe} f_{cde}\, W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu}$$

Quantization **propagators** (particle interpretation of field correlators)

- The (Feynman) propagator of a scalar field:

$$D_F(x-y) = \langle 0|\, T\{\phi(x)\phi^\dagger(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

(Feynman *prescription* $\varepsilon \to 0^+$)

is a Green's function of the Klein-Gordon operator:

$$(\Box_x + m^2)D_F(x-y) = -\mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{D}_F(p) = \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}$$

- The propagator of a fermion field:

$$S_F(x-y) = \langle 0|\, T\{\psi(x)\overline{\psi}(y)\}\,|0\rangle = (\mathrm{i}\partial\!\!\!/_x + m)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

is a Green's function of the Dirac operator:

$$(\mathrm{i}\partial\!\!\!/_x - m)S_F(x-y) = \mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{S}_F(p) = \frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\varepsilon}$$

Quantization of gauge theories | propagators

- HOWEVER a gauge field propagator cannot be defined unless $\mathcal{L}$ is modified:

(e.g. modified Maxwell) $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

Euler-Lagrange: $$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu = 0$$

– In momentum space the propagator is the inverse of:

$$-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \quad \Rightarrow \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\Rightarrow$ Note that $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular!

$\Rightarrow$ One may argue that $\mathcal{L}$ above will not lead to Maxwell equations …

unless we fix a (Lorenz) gauge where: (remove redundancy)

$$\partial^\mu A_\mu = 0 \quad \Leftarrow \quad A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu\Lambda \ \text{ with } \ \partial^\mu\partial_\mu\Lambda \equiv -\partial^\mu A_\mu$$

Quantization of gauge theories | gauge fixing (Abelian case)

- The extra term is called Gauge Fixing:

$$\boxed{\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$\Rightarrow$ modified $\mathcal{L}$ equivalent to Maxwell Lagrangian just in the gauge $\partial^\mu A_\mu = 0$

$\Rightarrow$ the ξ-dependence always cancels out in physical amplitudes

- Several choices for the gauge fixing term (simplify calculations): R_ξ gauges

$$(\text{'t Hooft-Feynman gauge}) \quad \xi = 1 : \quad \widetilde{D}_{\mu\nu}(k) = -\frac{\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}$$

$$(\text{Landau gauge}) \quad \xi = 0 : \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right]$$

Quantization of gauge theories | gauge fixing (non-Abelian case)

- For a non-Abelian gauge theory, the gauge fixing terms:

$$\boxed{\mathcal{L}_{\rm GF} = -\sum_a \frac{1}{2\xi_a}(\partial^\mu W_\mu^a)^2}$$

allow to define the propagators:

$$\widetilde{D}^{ab}_{\mu\nu}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_a)\frac{k_\mu k_\nu}{k^2}\right]$$

HOWEVER, unlike the Abelian case, this is not the end of the story …

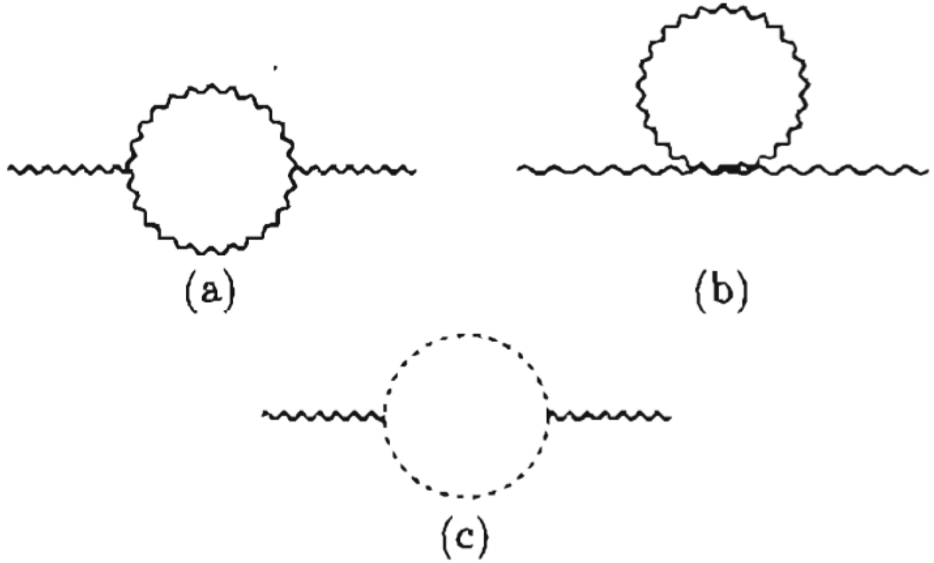
Quantization of gauge theories | Faddeev-Popov ghosts

- Add Faddeev-Popov *ghost* fields $c_a(x)$, $a = 1, \dots, N$: ('t Hooft-Feynman gauge)

$$\boxed{\mathcal{L}_{\rm FP} = (\partial^\mu \bar{c}_a)(D_\mu^{\rm adj})_{ab} c_b = (\partial^\mu \bar{c}_a)(\partial_\mu c_a - g f_{abc} c_b W_\mu^c)} \quad \Leftarrow \quad D_\mu^{\rm adj} = \partial_\mu - \mathrm{i} g T_c^{\rm adj} W_\mu^c$$

Computational trick: *anticommuting* scalar fields, just in loops as virtual particles

$\Rightarrow$ Faddeev-Popov ghosts needed to preserve gauge symmetry: ③



Self Energy

$$= \Pi_{\mu\nu} = \mathrm{i}(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$ ④

Ward identity: $k^\mu \Pi_{\mu\nu} = 0$

with

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\varepsilon}$$

[(−1) sign for closed loops! (like fermions)]

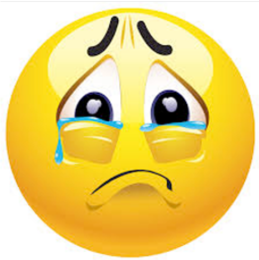
- Then the full quantum Lagrangian is

$$\boxed{\mathcal{L}_{\rm sym} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}}$$

⇒ Note that in the case of a massive vector field

$$\text{(Proca)} \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu$$

it is not gauge invariant!!!



What about the gauge principle???

– The propagator is:

5

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right)$$

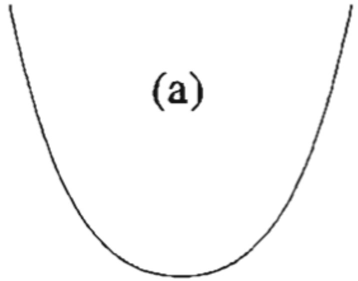
Spontaneous Symmetry Breaking — discrete symmetry

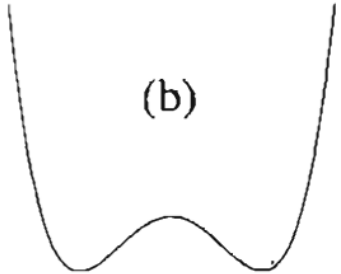
- Consider a real scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad \text{invariant under} \quad \phi \mapsto -\phi$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$





$\mu^2,\ \lambda \in \mathbb{R}$ (Real/Hermitian Hamiltonian) and $\lambda > 0$ (existence of a ground state)

(a) $\mu^2 > 0$: min of $V(\phi)$ at $\phi = 0$

(b) $\mu^2 < 0$: min of $V(\phi)$ at $\phi = v \equiv \pm\sqrt{\frac{-\mu^2}{\lambda}}$, in QFT $\langle 0|\,\phi\,|0\rangle = v \neq 0$ (VEV)

– A quantum field must have $v = 0$

$$a\,|0\rangle = 0$$

$$\Rightarrow \qquad \phi(x) \equiv v + \eta(x)\,, \quad \langle 0|\,\eta\,|0\rangle = 0$$

- At the quantum level, the same system is described by $\eta(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \quad \text{not invariant under} \quad \eta \mapsto -\eta$$

$$(m_\eta = \sqrt{2\lambda}\, v)$$

⇒ Lesson:

$\mathcal{L}(\phi)$ has the symmetry but the parameters can be such that the ground state of the Hamiltonian is not symmetric (Spontaneous Symmetry Breaking)

⇒ Note:

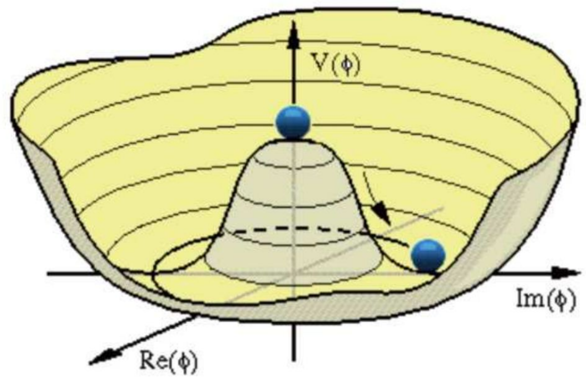
One may argue that $\mathcal{L}(\eta)$ exhibits an explicit breaking of the symmetry. However this is not the case since the coefficients of terms η^2, η^3 and η^4 are determined by just two parameters, λ and v (remnant of the original symmetry)

Spontaneous Symmetry Breaking — continuous symmetry

- Consider a complex scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad \text{invariant under U(1):} \quad \phi \mapsto e^{-iq\theta}\phi$$

$$\lambda > 0,\ \mu^2 < 0: \quad \langle 0|\,\phi\,|0\rangle \equiv \frac{v}{\sqrt{2}}, \quad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$



Take $v \in \mathbb{R}^+$. In terms of quantum fields:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + i\chi(x)], \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\chi\,|0\rangle = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \lambda v^2\eta^2 - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4$$

Note: if $ve^{i\alpha}$ (complex) replace η by $(\eta\cos\alpha - \chi\sin\alpha)$ and χ by $(\eta\sin\alpha + \chi\cos\alpha)$

$\Rightarrow$ The actual quantum Lagrangian $\mathcal{L}(\eta,\chi)$ is not invariant under U(1)

U(1) broken $\Rightarrow$ one scalar field remains massless: $m_\chi = 0$, $m_\eta = \sqrt{2\lambda}\,v$

Spontaneous Symmetry Breaking | continuous symmetry

- Another example: consider a real scalar SU(2) triplet $\Phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^\mathsf{T})(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi^\mathsf{T}\Phi - \frac{\lambda}{4}(\Phi^\mathsf{T}\Phi)^2 \quad \text{inv. under SU(2):} \quad \Phi \mapsto \mathrm{e}^{-\mathrm{i}T_a\theta^a}\Phi$$

that for $\lambda > 0$, $\mu^2 < 0$ acquires a VEV $\langle 0|\, \Phi^\mathsf{T}\Phi\,|0\rangle = v^2$ $\quad (\mu^2 = -\lambda v^2)$

Assume $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ v + \varphi_3(x) \end{pmatrix}$ and define $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi_3)(\partial^\mu \varphi_3) - \lambda v^2 \varphi_3^2 - \lambda v(2\varphi^\dagger\varphi + \varphi_3^2)\varphi_3 - \frac{\lambda}{4}(2\varphi^\dagger\varphi + \varphi_3^2)^2 + \frac{1}{4}\lambda v^4$$

$\Rightarrow$ Not symmetric under SU(2) but invariant under U(1):

$$\varphi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta}\varphi \quad (Q = \text{arbitrary}) \qquad\qquad \varphi_3 \mapsto \varphi_3 \quad (Q = 0)$$

SU(2) broken to U(1) $\Rightarrow 3 - 1 = 2$ broken generators

$\Rightarrow$ 2 (real) scalar fields (= 1 complex) remain massless: $m_\varphi = 0$, $m_{\varphi_3} = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

⇒ Goldstone's theorem: [Nambu '60; Goldstone '61]

The number of massless particles (Nambu-Goldstone bosons) is equal to the number of spontaneously broken generators of the symmetry

$$\text{Hamiltonian symmetric under group } G \quad \Rightarrow \quad [T_a, H] = 0\,, \quad a = 1, \dots, N$$

$$\text{By definition: } H|0\rangle = 0 \quad \Rightarrow \quad H(T_a|0\rangle) = T_a H|0\rangle = 0$$

– If $|0\rangle$ is such that $T_a|0\rangle = 0$ for all generators

⇒ non-degenerate minimum: *the* vacuum

– If $|0\rangle$ is such that $T_{a'}|0\rangle \neq 0$ for some (broken) generators a'

⇒ degenerate minimum: chose one (*true* vacuum) and $e^{-iT_{a'}\theta^{a'}}|0\rangle \neq |0\rangle$

⇒ excitations (particles) from $|0\rangle$ to $e^{-iT_{a'}\theta^{a'}}|0\rangle$ cost no energy: massless!

Spontaneous Symmetry Breaking | gauge symmetry

- Consider a U(1) gauge invariant Lagrangian for a complex scalar field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \,, \quad D_\mu = \partial_\mu + \mathrm{i}eQA_\mu$$

inv. under $\phi(x) \mapsto \phi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\phi(x) \,, \quad A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$

If $\lambda > 0$, $\mu^2 < 0$, the $\mathcal{L}$ in terms of quantum fields η and χ with null VEVs:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)] \,, \quad \mu^2 = -\lambda v^2$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) \\ &\boxed{-\lambda v^2\eta^2} - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4 \\ &\boxed{+\, eQvA_\mu\partial^\mu\chi} + eQA_\mu(\eta\partial^\mu\chi - \chi\partial^\mu\eta) \\ &\boxed{+\, \frac{1}{2}(eQv)^2A_\mu A^\mu} + \frac{1}{2}(eQ)^2A_\mu A^\mu(\eta^2 + 2v\eta + \chi^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$
$m_\chi = 0$

(ii) $M_A = |eQv|$ (!)

(iii) Term $A_\mu\partial^\mu\chi$ (?)

(iv) Add $\mathcal{L}_{\mathrm{GF}}$

Spontaneous Symmetry Breaking | gauge symmetry

- Removing the cross term and the (new) gauge fixing Lagrangian:

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A \chi)^2$$

$$\Rightarrow \quad \mathcal{L} + \mathcal{L}_{\rm GF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overbrace{M_A \partial_\mu (A^\mu \chi)}^{\substack{M_A[\partial_\mu A^\mu \chi + A_\mu \partial^\mu \chi] \\ \text{total deriv.}}}$$

$$+ \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}\xi M_A^2 \chi^2 + \dots$$

and the propagators of A_μ and χ are: (6)

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\varepsilon}$$

$\Rightarrow$ χ has a gauge-dependent mass: actually it is not a physical field!

Spontaneous Symmetry Breaking | gauge symmetry

- A more transparent parameterization of the quantum field ϕ is

$$\phi(x) \equiv e^{iQ\zeta(x)/v}\frac{1}{\sqrt{2}}[v+\eta(x)]\,, \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\zeta\,|0\rangle = 0$$

$$\phi(x) \mapsto e^{-iQ\zeta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}[v+\eta(x)] \quad \Rightarrow \quad \zeta \text{ gauged away!}$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) \\ &-\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \\ &+\frac{1}{2}(eQv)^2A_\mu A^\mu + \frac{1}{2}(eQ)^2A_\mu A^\mu(2v\eta+\eta^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$

(ii) $M_A = |eQv|$

(iii) No need for $\mathcal{L}_{\rm GF}$

$\Rightarrow$ This is the unitary gauge ($\xi \to \infty$): just physical fields

$$\widetilde{D}_{\mu\nu}(k) \to \frac{i}{k^2 - M_A^2 + i\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_A^2}\right] \quad \text{and} \quad \widetilde{D}(k) \to 0$$

Spontaneous Symmetry Breaking | gauge symmetry

⇒ Brout-Englert-Higgs mechanism:

[Anderson '62]
[Higgs '64; Englert, Brout '64; Guralnik, Hagen, Kibble '64]

The gauge bosons associated with the spontaneously broken generators become massive, the corresponding would-be Goldstone bosons are unphysical and can be absorbed, the remaining massive scalars (Higgs bosons) are physical (the smoking gun!)

- The would-be Goldstone bosons are 'eaten up' by the gauge bosons ('get fat') and disappear (gauge away) in the unitary gauge ($\xi \to \infty$)

  ⇒ Degrees of freedom are preserved

Before SSB:	2 (massless gauge boson)	+ 1 (Goldstone boson)
After SSB:	3 (massive gauge boson)	+ 0 (absorbed would-be Goldstone)

- For loops calculations, 't Hooft-Feynman gauge ($\xi = 1$) is more convenient:

  ⇒ Gauge boson propagators are simpler, but

  ⇒ Goldstone bosons must be included in internal lines

Spontaneous Symmetry Breaking | gauge symmetry

- Comments:
  - After SSB the FP ghost fields (unphysical) acquire a gauge-dependent mass, due to interactions with the scalar field(s):

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 - \xi_a M_{W^a}^2 + \mathrm{i}\varepsilon}$$

  - Gauge theories with SSB are renormalizable ['t Hooft, Veltman '72]

    UV divergences appearing at loop level can be removed by renormalization of parameters and fields of the classical Lagrangian ⇒ predictive!

2. The Standard Model

Gauge group and field representations

[Glashow '61; Weinberg '67; Salam '68]
[D. Gross, F. Wilczek; D. Politzer '73]

- The Standard Model is a gauge theory based on the local symmetry group:

$$\underbrace{\mathrm{SU}(3)_c}_{\text{strong}} \otimes \underbrace{\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y}_{\text{electroweak}} \to \mathrm{SU}(3)_c \otimes \underbrace{\mathrm{U}(1)_Q}_{\text{em}}$$

with the electroweak symmetry spontaneously broken to the electromagnetic $\mathrm{U}(1)_Q$ symmetry by the Brout-Englert-Higgs mechanism

- The particle (field) content: (ingredients: 12 *flavors* + 12 gauge bosons + H)

		Fermions	I	II	III	Q
spin $\frac{1}{2}$	Quarks	f	uuu	ccc	ttt	$\frac{2}{3}$
		f'	ddd	sss	bbb	$-\frac{1}{3}$
	Leptons	f	ν_e	ν_μ	ν_τ	0
		f'	e	μ	τ	-1

	Bosons	
spin 1	8 gluons	strong interaction
	$W^\pm, Z$	weak interaction
	γ	em interaction
spin 0	Higgs	origin of mass

$$Q_f = Q_{f'} + 1$$

Gauge group and field representations

- The fields lay in the following representations (color, weak isospin, hypercharge):

Multiplets	$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$	I	II	III	$Q = T_3 + Y$
Quarks	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ $-\frac{1}{3} = -\frac{1}{2} + \frac{1}{6}$
	$(\mathbf{3}, \mathbf{1}, \frac{2}{3})$	u_R	c_R	t_R	$\frac{2}{3} = 0 + \frac{2}{3}$
	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$	d_R	s_R	b_R	$-\frac{1}{3} = 0 - \frac{1}{3}$
Leptons	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$	$0 = \frac{1}{2} - \frac{1}{2}$ $-1 = -\frac{1}{2} - \frac{1}{2}$
	$(\mathbf{1}, \mathbf{1}, -1)$	e_R	μ_R	τ_R	$-1 = 0 - 1$
	$(\mathbf{1}, \mathbf{1}, 0)$	ν_{e_R}	ν_{μ_R}	ν_{τ_R}	$0 = 0 + 0$
Higgs	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$	(3 families of quarks & leptons)			UNIVERSAL

$\Rightarrow$ Electroweak (QFD): $SU(2)_L \otimes U(1)_Y$ Strong (QCD): $SU(3)_c$

Electroweak interactions

The EWSM with one family (of quarks or leptons)

- Consider two massless fermion fields $f(x)$ and $f'(x)$ with electric charges $Q_f = Q_{f'} + 1$ in three irreps of $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$:

$$\mathcal{L}_F^0 = \mathrm{i}\overline{f}\partial\!\!\!/ f + \mathrm{i}\overline{f}'\partial\!\!\!/ f' \qquad f_{R,L} = \frac{1}{2}(1 \pm \gamma_5)f\ , \quad f'_{R,L} = \frac{1}{2}(1 \pm \gamma_5)f'$$

$$= \mathrm{i}\overline{\Psi}_1\partial\!\!\!/\Psi_1 + \mathrm{i}\overline{\psi}_2\partial\!\!\!/\psi_2 + \mathrm{i}\overline{\psi}_3\partial\!\!\!/\psi_3 \quad ; \quad \Psi_1 = \underbrace{\begin{pmatrix} f_L \\ f'_L \end{pmatrix}}_{(\mathbf{2},y_1)}, \quad \psi_2 = \underbrace{f_R}_{(\mathbf{1},y_2)}, \quad \psi_3 = \underbrace{f'_R}_{(\mathbf{1},y_3)}$$

- To get a Langrangian invariant under gauge transformations:

$$\Psi_1(x) \mapsto U_L(x)\mathrm{e}^{-\mathrm{i}y_1\beta(x)}\Psi_1(x), \quad U_L(x) = \mathrm{e}^{-\mathrm{i}T_i\alpha^i(x)}, \quad T_i = \frac{\sigma_i}{2} \quad \text{(weak isospin gen.)}$$

$$\psi_2(x) \mapsto \mathrm{e}^{-\mathrm{i}y_2\beta(x)}\psi_2(x)$$

$$\psi_3(x) \mapsto \mathrm{e}^{-\mathrm{i}y_3\beta(x)}\psi_3(x)$$

The EWSM with one family | gauge invariance

⇒ Introduce gauge fields $W^i_\mu(x)$ $(i=1,2,3)$ and $B_\mu(x)$ through covariant derivatives:

$$\left.\begin{aligned} D_\mu \Psi_1 &= (\partial_\mu - \mathrm{i}g\widetilde{W}_\mu + \mathrm{i}g' y_1 B_\mu)\Psi_1\,, \quad \widetilde{W}_\mu \equiv \frac{\sigma_i}{2}W^i_\mu \\ D_\mu \psi_2 &= (\partial_\mu + \mathrm{i}g' y_2 B_\mu)\psi_2 \\ D_\mu \psi_3 &= (\partial_\mu + \mathrm{i}g' y_3 B_\mu)\psi_3 \end{aligned}\right\} \Rightarrow \boxed{\mathcal{L}_F} \quad (\not{P}, \not{C})$$

where two couplings g and g' have been introduced and

$$\widetilde{W}_\mu(x) \mapsto U_L(x)\widetilde{W}_\mu(x)U_L^\dagger(x) - \frac{\mathrm{i}}{g}(\partial_\mu U_L(x))U_L^\dagger(x)$$

$$B_\mu(x) \mapsto B_\mu(x) + \frac{1}{g'}\partial_\mu\beta(x)$$

⇒ Add Yang-Mills: gauge invariant kinetic terms for the gauge fields

$$\boxed{\mathcal{L}_{\mathrm{YM}}} = -\frac{1}{4}W^i_{\mu\nu}W^{i,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,, \quad W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon_{ijk}W^j_\mu W^k_\nu$$

(include self-interactions of the SU(2) gauge fields) and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

The EWSM with one family | mass terms forbidden

⇒ Note that mass terms are not invariant under $SU(2)_L \otimes U(1)_Y$, since LH and RH components do not transform the same:

$$m\overline{f}f = m(\overline{f_L}f_R + \overline{f_R}f_L)$$

⇒ Mass terms for the gauge bosons are not allowed either

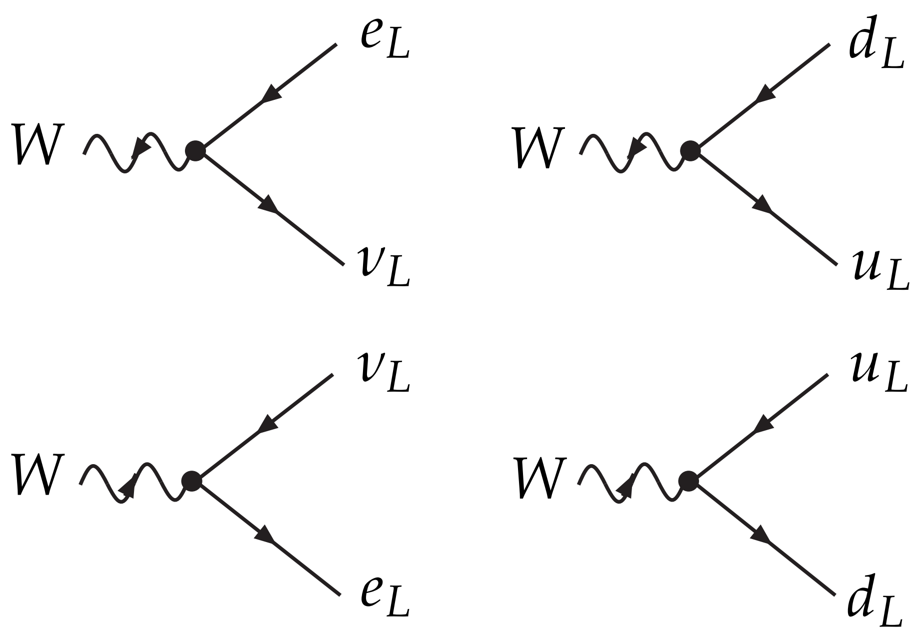
⇒ Next the different types of interactions are analyzed, and later the EWSB will be discussed

The EWSM with one family — charged current interactions

- $$\mathcal{L}_F \supset g\overline{\Psi}_1\gamma^\mu\widetilde{W}_\mu\Psi_1\,, \quad \widetilde{W}_\mu = \frac{1}{2}\begin{pmatrix} W^3_\mu & \sqrt{2}W^\dagger_\mu \\ \sqrt{2}W_\mu & -W^3_\mu \end{pmatrix}$$

$\Rightarrow$ charged current interactions of LH fermions with complex vector boson field W_μ:

$$\mathcal{L}_{\rm CC} = \frac{g}{2\sqrt{2}}\bar{f}\gamma^\mu(1-\gamma_5)f'W^\dagger_\mu + \text{h.c.}\,, \quad W_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu + \mathrm{i}W^2_\mu)$$



- The diagonal part of $g\overline{\Psi}_1\gamma^\mu\widetilde{W}_\mu\Psi_1$ and the remaining terms

$$\mathcal{L}_F \supset \frac{1}{2}g\overline{\Psi}_1\gamma^\mu\sigma_3 W_\mu^3\Psi_1 - g' B_\mu(y_1\overline{\Psi}_1\gamma^\mu\Psi_1 + y_2\overline{\psi}_2\gamma^\mu\psi_2 + y_3\overline{\psi}_3\gamma^\mu\psi_3)$$

$\Rightarrow$ neutral current interactions with neutral vector boson fields W_μ^3 and B_μ

We would like to identify B_μ with the photon field A_μ but that requires:

$$y_1 = y_2 = y_3 \quad \text{and} \quad g' y_j = eQ_j \quad \Rightarrow \quad \text{impossible!}$$

$\Rightarrow$ Since they are both neutral, try a combination:

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \equiv \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad s_W \equiv \sin\theta_W\ , \quad c_W \equiv \cos\theta_W$$

$\theta_W =$ weak mixing angle

$$\mathcal{L}_{\text{NC}} = \sum_{j=1}^{3} \overline{\psi}_j\gamma^\mu \left\{ -\left[gT_3 s_W + g' y_j c_W\right] A_\mu + \left[gT_3 c_W - g' y_j s_W\right] Z_\mu \right\} \psi_j$$

with $T_3 = \dfrac{\sigma_3}{2}$ (0) the third weak isospin component of the doublet (singlet)

- To make A_μ the photon field:

$$(1)\ \boxed{e = g s_W = g' c_W} \qquad (2)\ \boxed{Q = T_3 + Y}$$

where the electric charge operator is: $Q_1 = \begin{pmatrix} Q_f & 0 \\ 0 & Q_{f'} \end{pmatrix}, \quad Q_2 = Q_f\,, \quad Q_3 = Q_{f'}$

⇒ (1) Electroweak unification: g of SU(2) and g' of U(1) related to $e = \dfrac{gg'}{\sqrt{g^2 + g'^2}}$

⇒ (2) The hyperchages are fixed in terms of electric charges and weak isospin:

$$y_1 = Q_f - \frac{1}{2} = Q_{f'} + \frac{1}{2}\,, \qquad y_2 = Q_f\,, \qquad y_3 = Q_{f'}$$

$$\mathcal{L}_{\rm QED} = -e\, Q_f \overline{f} \gamma^\mu f\, A_\mu \quad + (f \to f')$$

⇒ RH neutrinos are sterile: $y_2 = Q_f = 0$

- The Z_μ is the neutral weak boson field:

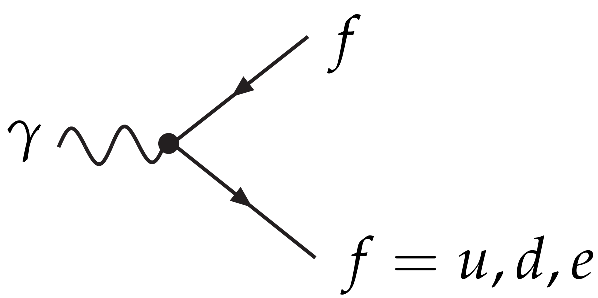
$$\mathcal{L}_{\rm NC}^{Z} = e\,\overline{f}\gamma^\mu(v_f - a_f\gamma_5)f\;Z_\mu \quad + (f \to f')$$

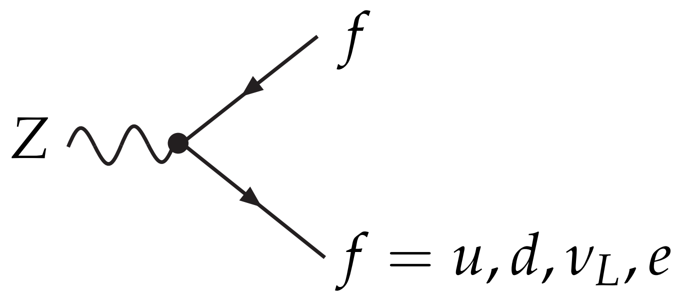
with

$$v_f = \frac{T_3^{f_L} - 2Q_f s_W^2}{2s_W c_W}\;, \quad a_f = \frac{T_3^{f_L}}{2s_W c_W}$$

- The complete neutral current Lagrangian reads:

$$\mathcal{L}_{\rm NC} = \mathcal{L}_{\rm QED} + \mathcal{L}_{\rm NC}^{Z}$$



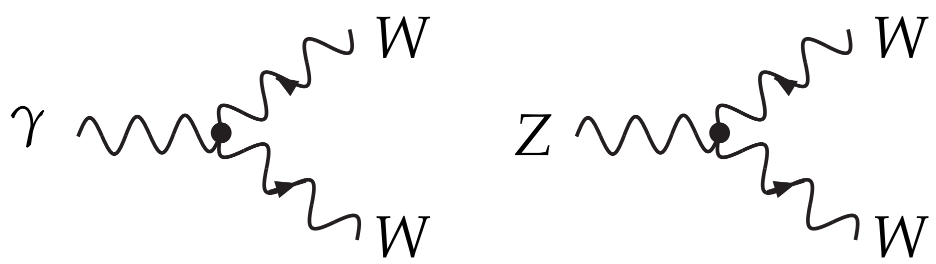


- Cubic:

$$\mathcal{L}_{\rm YM} \supset \mathcal{L}_3 = -\frac{\mathrm{i}ec_W}{s_W}\left\{W^{\mu\nu}W^\dagger_\mu Z_\nu - W^\dagger_{\mu\nu}W^\mu Z^\nu - W^\dagger_\mu W_\nu Z^{\mu\nu}\right\}$$
$$+\,\mathrm{i}e\left\{W^{\mu\nu}W^\dagger_\mu A_\nu - W^\dagger_{\mu\nu}W^\mu A^\nu - W^\dagger_\mu W_\nu F^{\mu\nu}\right\}$$

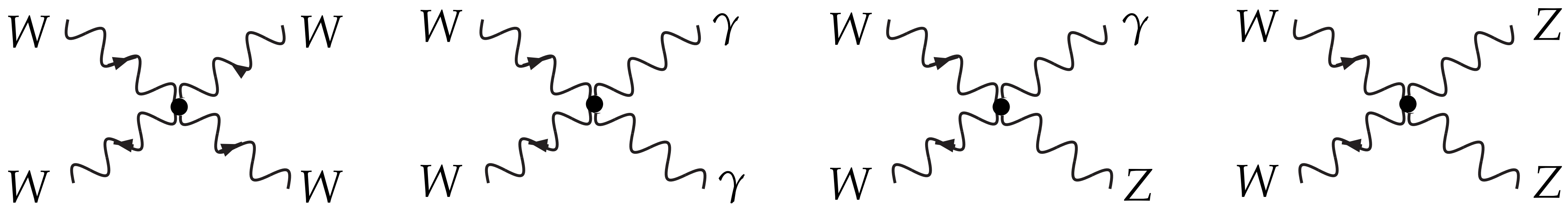
with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$



- Quartic:

$$\begin{aligned}\mathcal{L}_{\rm YM} \supset \mathcal{L}_4 = &-\frac{e^2}{2s_W^2}\left\{\left(W_\mu^\dagger W^\mu\right)^2 - W_\mu^\dagger W^{\mu\dagger} W_\nu W^\nu\right\}\\ &-\frac{e^2c_W^2}{s_W^2}\left\{W_\mu^\dagger W^\mu Z_\nu Z^\nu - W_\mu^\dagger Z^\mu W_\nu Z^\nu\right\}\\ &+\frac{e^2c_W}{s_W}\left\{2W_\mu^\dagger W^\mu Z_\nu A^\nu - W_\mu^\dagger Z^\mu W_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu Z^\nu\right\}\\ &-e^2\left\{W_\mu^\dagger W^\mu A_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu A^\nu\right\}\end{aligned}$$



Note: even number of W and no vertex with just γ or Z

Electroweak symmetry breaking | setup

- Out of the 4 gauge bosons of $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ with generators $T_1,\ T_2,\ T_3,\ Y$ we need all to be broken except the combination $Q = T_3 + Y$ so that A_μ remains massless and the other three gauge bosons get massive after SSB

  $\Rightarrow$ Introduce a complex SU(2) Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \quad \langle 0 | \Phi | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

  with gauge invariant Lagrangian ($\mu^2 = -\lambda v^2$):

$$\boxed{\mathcal{L}_\Phi} = (D_\mu \Phi)^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 , \qquad D_\mu \Phi = (\partial_\mu - \mathrm{i} g \widetilde{W}_\mu + \mathrm{i} g' y_\Phi B_\mu) \Phi$$

$$\text{take } y_\Phi = \frac{1}{2} \quad \Rightarrow \quad (T_3 + Y) |0\rangle = 0 \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0 \quad \checkmark$$

$$\{T_1, T_2, T_3 - Y\} |0\rangle \neq 0$$

Electroweak symmetry breaking — gauge boson masses

- Quantum fields in the unitary gauge:

$$\Phi(x) \equiv \exp\left\{\mathrm{i}\frac{\sigma_i}{2v}\theta^i(x)\right\}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H(x)\end{pmatrix}$$

$$\Phi(x) \mapsto \exp\left\{-\mathrm{i}\frac{\sigma_i}{2v}\theta^i(x)\right\}\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H(x)\end{pmatrix} \quad \Rightarrow$$

1 physical Higgs field $H(x)$

3 would-be Goldstones $\theta^i(x)$ gauged away

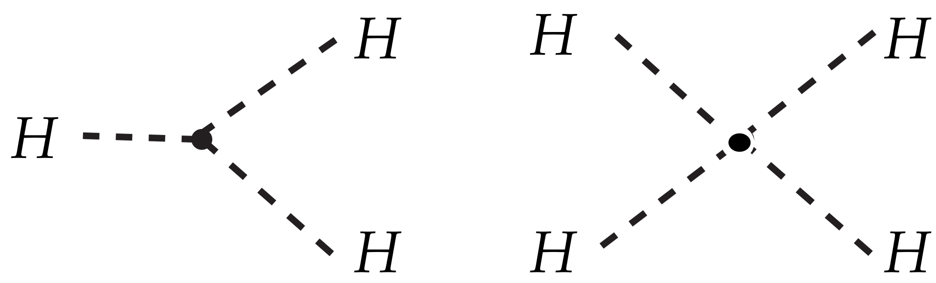
– The 3 dof apparently lost become the longitudinal polarizations of $\mathrm{W}^\pm$ and Z that get massive after SSB:

$$\mathcal{L}_\Phi \supset \mathcal{L}_M = \underbrace{\frac{g^2v^2}{4}}_{M_W^2} W^\dagger_\mu W^\mu + \underbrace{\frac{g^2v^2}{8c_W^2}}_{\frac{1}{2}M_Z^2} Z_\mu Z^\mu \quad \Rightarrow \quad \underbrace{M_W = M_Z c_W}_{\text{custodial symmetry}} = \frac{1}{2}gv$$

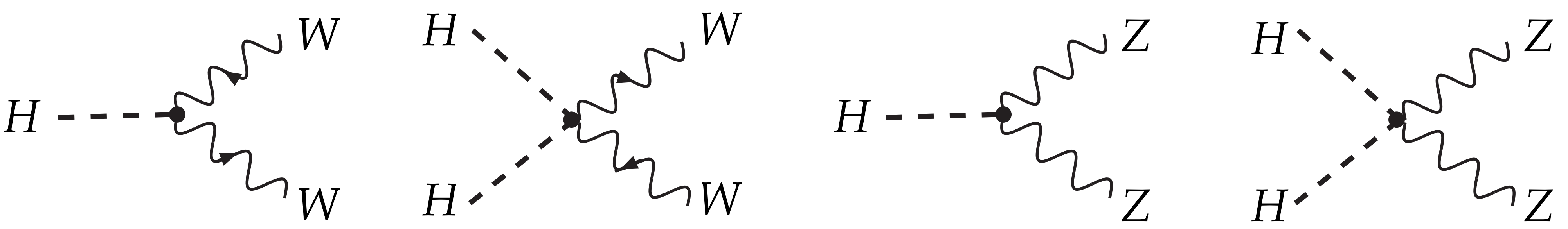
Electroweak symmetry breaking | Higgs sector

⇒ In the unitary gauge (just physical fields): $\mathcal{L}_\Phi = \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4$

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}M_H^2 H^2 - \frac{M_H^2}{2v}H^3 - \frac{M_H^2}{8v^2}H^4 \,, \quad M_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}\, v$$



$$\mathcal{L}_M + \mathcal{L}_{HV^2} = M_W^2 W_\mu^\dagger W^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\}$$



Electroweak symmetry breaking | Higgs sector

- Quantum fields in the R_ξ gauges:

$$\Phi(x) = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \equiv \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}[v + H(x) + \mathrm{i}\chi(x)] \end{pmatrix} , \quad \phi^-(x) = [\phi^+(x)]^*$$

$$\begin{aligned} \mathcal{L}_\Phi = {} & \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4 \\ & + (\partial_\mu\phi^+)(\partial^\mu\phi^-) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) \\ & + \mathrm{i}M_W\,(W_\mu\partial^\mu\phi^+ - W_\mu^\dagger\partial^\mu\phi^-) + \quad M_Z\,Z_\mu\partial^\mu\chi \\ & + \text{trilinear interactions [SSS, SSV, SVV]} \\ & + \text{quadrilinear interactions [SSSS, SSVV]} \end{aligned}$$

Electroweak symmetry breaking | gauge fixing

- To remove the cross terms $W_\mu \partial^\mu \phi^+$, $W^\dagger_\mu \partial^\mu \phi^-$, $Z_\mu \partial^\mu \chi$ and define propagators add:

$$\boxed{\mathcal{L}_{\rm GF}} = -\frac{1}{2\xi_\gamma}(\partial_\mu A^\mu)^2 - \frac{1}{2\xi_Z}(\partial_\mu Z^\mu - \xi_Z M_Z \chi)^2 - \frac{1}{\xi_W}|\partial_\mu W^\mu + \mathrm{i}\xi_W M_W \phi^-|^2$$

$\Rightarrow$ Massive propagators for gauge and (unphysical) would-be Goldstone fields:

$$\widetilde{D}^\gamma_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_\gamma)\frac{k_\mu k_\nu}{k^2}\right]$$

$$\widetilde{D}^Z_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_Z^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_Z)\frac{k_\mu k_\nu}{k^2 - \xi_Z M_Z^2}\right] \quad ; \quad \widetilde{D}^\chi(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\varepsilon}$$

$$\widetilde{D}^W_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_W^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_W)\frac{k_\mu k_\nu}{k^2 - \xi_W M_W^2}\right] \quad ; \quad \widetilde{D}^\phi(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\varepsilon}$$

('t Hooft-Feynman gauge: $\xi_\gamma = \xi_Z = \xi_W = 1$)

- The SM is a non-Abelian theory ⇒ add Faddeev-Popov ghosts $c_i(x)$ $(i = 1, 2, 3)$

$$c_1 \equiv \frac{1}{\sqrt{2}}(u_+ + u_-)\,, \quad c_2 \equiv \frac{\mathrm{i}}{\sqrt{2}}(u_+ - u_-)\,, \quad c_3 \equiv c_W\, u_Z - s_W\, u_\gamma$$

$$\boxed{\mathcal{L}_{\rm FP}} = \underbrace{(\partial^\mu \bar{c}_i)(\partial_\mu c_i - g\epsilon_{ijk} c_j W^k_\mu)}_{\text{U kinetic + [UUV]}} + \underbrace{\text{interactions with } \Phi}_{\text{U masses + [SUU]}}$$

⇒ Massive propagators for (unphysical) FP ghost fields:

$$\widetilde{D}^{u_\gamma}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\,, \quad \widetilde{D}^{u_Z}(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\varepsilon}\,, \quad \widetilde{D}^{u_\pm}(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\varepsilon}$$

('t Hooft-Feynman gauge: $\xi_Z = \xi_W = 1$)

$$
\begin{aligned}
\mathcal{L}_{\rm FP} = \quad & (\partial_\mu \overline{u}_\gamma)(\partial^\mu u_\gamma) + (\partial_\mu \overline{u}_Z)(\partial^\mu u_Z) + (\partial_\mu \overline{u}_+)(\partial^\mu u_+) + (\partial_\mu \overline{u}_-)(\partial^\mu u_-) \\
[\mathrm{UUV}] & \left\{ \begin{array}{l}
+ \mathrm{i}e[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]A_\mu - \dfrac{\mathrm{i}e c_W}{s_W}[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]Z_\mu \\
- \mathrm{i}e[(\partial^\mu \overline{u}_+)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_-]W^\dagger_\mu + \dfrac{\mathrm{i}e c_W}{s_W}[(\partial^\mu \overline{u}_+)u_Z - (\partial^\mu \overline{u}_Z)u_-]W^\dagger_\mu \\
+ \mathrm{i}e[(\partial^\mu \overline{u}_-)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_+]W_\mu - \dfrac{\mathrm{i}e c_W}{s_W}[(\partial^\mu \overline{u}_-)u_Z - (\partial^\mu \overline{u}_Z)u_+]W_\mu
\end{array} \right. \\
& - \xi_Z M_Z^2\, \overline{u}_Z u_Z - \xi_W M_W^2\, \overline{u}_+ u_+ - \xi_W M_W^2\, \overline{u}_- u_- \\
[\mathrm{SUU}] & \left\{ \begin{array}{l}
- e\xi_Z M_Z\, \overline{u}_Z \left[\dfrac{1}{2 s_W c_W} H u_Z - \dfrac{1}{2 s_W}\left(\phi^+ u_- + \phi^- u_+\right) \right] \\
- e\xi_W M_W\, \overline{u}_+ \left[\dfrac{1}{2 s_W}(H + \mathrm{i}\chi)u_+ - \phi^+ \left(u_\gamma - \dfrac{c_W^2 - s_W^2}{2 s_W c_W} u_Z \right) \right] \\
- e\xi_W M_W\, \overline{u}_- \left[\dfrac{1}{2 s_W}(H - \mathrm{i}\chi)u_- - \phi^- \left(u_\gamma - \dfrac{c_W^2 - s_W^2}{2 s_W c_W} u_Z \right) \right]
\end{array} \right.
\end{aligned}
$$

- We need masses for quarks and leptons without breaking gauge symmetry

$\Rightarrow$ Introduce Yukawa interactions:

$$\mathcal{L}_\mathrm{Y} = -\lambda_d \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R - \lambda_u \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} u_R$$

$$- \lambda_\ell \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \ell_R - \lambda_\nu \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \nu_R \quad + \mathrm{h.c.}$$

where $\widetilde{\Phi} \equiv i\sigma_2\Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ 7

$\Rightarrow$ After EW SSB, fermions acquire masses ($\overline{f}f = \overline{f_L}f_R + \overline{f_R}f_L$):

$$\mathcal{L}_\mathrm{Y} \supset -\frac{1}{\sqrt{2}}(v+H)\left\{\lambda_d\,\overline{d}d + \lambda_u\,\overline{u}u + \lambda_\ell\,\overline{\ell}\ell + \lambda_\nu\,\overline{\nu}\nu\right\} \quad \Rightarrow \quad m_f = \lambda_f\frac{v}{\sqrt{2}}$$

Additional generations — Yukawa matrices

- There are 3 generations of quarks and leptons in Nature. They are identical copies with the same properties under $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ differing only in their masses

  $\Rightarrow$ Take a general case of n generations and let u_i^I, d_i^I, ν_i^I, ℓ_i^I be the members of family i ($i = 1, \ldots, n$). Superindex I (interaction basis) was omitted so far

  $\Rightarrow$ General gauge invariant Yukawa Lagrangian:

$$\boxed{\mathcal{L}_\mathrm{Y}} = -\sum_{ij} \left\{ \begin{pmatrix} \overline{u}_{iL}^I & \overline{d}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(d)} \, d_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(u)} \, u_{jR}^I \right] + \begin{pmatrix} \overline{\nu}_{iL}^I & \overline{\ell}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(\ell)} \, \ell_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(\nu)} \, \nu_{jR}^I \right] \right\} + \text{h.c.}$$

  where $\lambda_{ij}^{(d)}$, $\lambda_{ij}^{(u)}$, $\lambda_{ij}^{(\ell)}$, $\lambda_{ij}^{(\nu)}$ are arbitrary Yukawa matrices

Additional generations | mass matrices

- After EW SSB, in n-dimensional matrix form:

$$\mathcal{L}_{\mathrm{Y}} \supset -\left(1+\frac{H}{v}\right)\left\{\overline{\mathbf{d}}^I_L\,\mathbf{M}_d\,\mathbf{d}^I_R + \overline{\mathbf{u}}^I_L\,\mathbf{M}_u\,\mathbf{u}^I_R + \overline{\boldsymbol{\ell}}^I_L\,\mathbf{M}_\ell\,\boldsymbol{\ell}^I_R + \overline{\boldsymbol{\nu}}^I_L\,\mathbf{M}_\nu\,\boldsymbol{\nu}^I_R + \text{h.c.}\right\}$$

with mass matrices

$$(\mathbf{M}_d)_{ij} \equiv \lambda^{(d)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_u)_{ij} \equiv \lambda^{(u)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_\ell)_{ij} \equiv \lambda^{(\ell)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_\nu)_{ij} \equiv \lambda^{(\nu)}_{ij}\frac{v}{\sqrt{2}}$$

⇒ Diagonalization determines mass eigenstates d_j, u_j, ℓ_j, ν_j in terms of interaction states d^I_j, u^I_j, ℓ^I_j, ν^I_j, respectively

⇒ Each $\mathbf{M}_f$ can be written as

$$\mathbf{M}_f = \mathbf{H}_f\,\mathcal{U}_f = \mathbf{V}^\dagger_f\,\mathcal{M}_f\,\mathbf{V}_f\,\mathcal{U}_f \quad \Longleftrightarrow \quad \mathbf{M}_f\mathbf{M}^\dagger_f = \mathbf{H}^2_f = \mathbf{V}^\dagger_f\,\mathcal{M}^2_f\,\mathbf{V}_f$$

with $\mathbf{H}_f \equiv \sqrt{\mathbf{M}_f\mathbf{M}^\dagger_f}$ a Hermitian positive definite matrix and $\mathcal{U}_f$ unitary

– Every $\mathbf{H}_f$ can be diagonalized by a unitary matrix $\mathbf{V}_f$

– The resulting $\mathcal{M}_f$ is diagonal and positive definite

Additional generations | fermion masses and mixings

- In terms of diagonal mass matrices (mass eigenstate basis):

$$\mathcal{M}_d = \text{diag}(m_d, m_s, m_b, \ldots)\,, \quad \mathcal{M}_u = \text{diag}(m_u, m_c, m_t, \ldots)$$
$$\mathcal{M}_\ell = \text{diag}(m_e, m_\mu, m_\tau, \ldots)\,, \quad \mathcal{M}_\nu = \text{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}, \ldots)$$

$$\mathcal{L}_\text{Y} \supset -\left(1 + \frac{H}{v}\right) \left\{ \overline{\mathbf{d}}\, \mathcal{M}_d\, \mathbf{d} + \overline{\mathbf{u}}\, \mathcal{M}_u\, \mathbf{u} + \overline{\boldsymbol{\ell}}\, \mathcal{M}_\ell\, \boldsymbol{\ell} + \overline{\boldsymbol{\nu}}\, \mathcal{M}_\nu\, \boldsymbol{\nu} \right\}$$

where fermion couplings to Higgs are proportional to masses and

$$\mathbf{d}_L \equiv \mathbf{V}_d\, \mathbf{d}^I_L \qquad \mathbf{u}_L \equiv \mathbf{V}_u\, \mathbf{u}^I_L \qquad \boldsymbol{\ell}_L \equiv \mathbf{V}_\ell\, \boldsymbol{\ell}^I_L \qquad \boldsymbol{\nu}_L \equiv \mathbf{V}_\nu\, \boldsymbol{\nu}^I_L$$
$$\mathbf{d}_R \equiv \mathbf{V}_d \mathcal{U}_d\, \mathbf{d}^I_R \qquad \mathbf{u}_R \equiv \mathbf{V}_u \mathcal{U}_u\, \mathbf{u}^I_R \qquad \boldsymbol{\ell}_R \equiv \mathbf{V}_\ell \mathcal{U}_\ell\, \boldsymbol{\ell}^I_R \qquad \boldsymbol{\nu}_R \equiv \mathbf{V}_\nu \mathcal{U}_\nu\, \boldsymbol{\nu}^I_R$$

$\Rightarrow$ Neutral Currents preserve chirality
$\overline{\mathbf{f}}^I_L \gamma^\mu \mathbf{f}^I_L = \overline{\mathbf{f}}_L \gamma^\mu \mathbf{f}_L$ and $\overline{\mathbf{f}}^I_R \gamma^\mu \mathbf{f}^I_R = \overline{\mathbf{f}}_R \gamma^\mu \mathbf{f}_R$
$\Big\} \Rightarrow \mathcal{L}_\text{NC}$ does not change family

$\Rightarrow$ GIM mechanism [Glashow, Iliopoulos, Maiani '70]

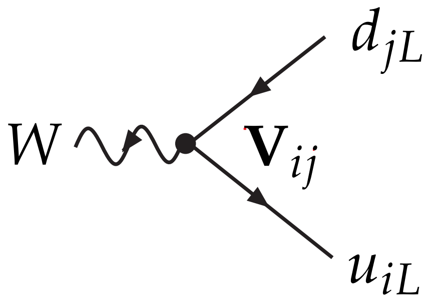
Additional generations | quark sector

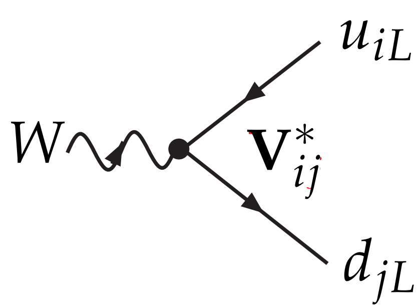
- However, in Charged Currents (also chirality preserving and only LH):

$$\overline{\mathbf{u}}^I_L \gamma^\mu \, \mathbf{d}^I_L = \overline{\mathbf{u}}_L \gamma^\mu \, \mathbf{V}_u \, \mathbf{V}_d^\dagger \mathbf{d}_L = \overline{\mathbf{u}}_L \gamma^\mu \, \mathbf{V} \mathbf{d}_L$$

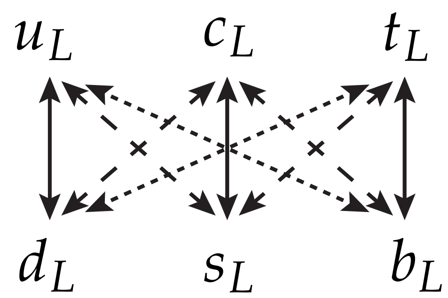
with $\mathbf{V} \equiv \mathbf{V}_u \, \mathbf{V}_d^\dagger$ the (unitary) CKM mixing matrix [Cabibbo '63; Kobayashi, Maskawa '73]

$$\Rightarrow \qquad \mathcal{L}_{\rm CC} = \frac{g}{2\sqrt{2}} \sum_{ij} \overline{u}_i \gamma^\mu (1-\gamma_5) \; \mathbf{V}_{ij} \; d_j \; W_\mu^\dagger + \text{h.c.}$$





$\mathcal{L}_{\rm CC}$ changes family !!



$\Rightarrow$ If u_i or d_j had degenerate masses one could choose $\mathbf{V}_u = \mathbf{V}_d$ (field redefinition) and quark families would not mix. But they are *not degenerate*, so they mix!

$\Rightarrow$ $\mathbf{V}_u$ and $\mathbf{V}_d$ are not observable. Just masses and CKM mixings are observable

Additional generations | quark sector

- How many physical parameters in this sector?

  – Quark masses and CKM mixings determined by mass (or Yukawa) matrices

  – A general $n \times n$ unitary matrix, like the CKM, is given by

  $$n^2 \text{ real parameters} = n(n-1)/2 \text{ moduli} + n(n+1)/2 \text{ phases}$$

  Some phases are unphysical since they can be absorbed by field redefinitions:

  $$u_i \to \mathrm{e}^{\mathrm{i}\phi_i} u_i \,, \quad d_j \to \mathrm{e}^{\mathrm{i}\theta_j} d_j \quad \Rightarrow \quad \mathbf{V}_{ij} \to \mathbf{V}_{ij}\, \mathrm{e}^{\mathrm{i}(\theta_j - \phi_i)}$$

  Therefore $2n-1$ unphysical phases and the physical parameters are:

  $$\boxed{(n-1)^2 = n(n-1)/2 \text{ moduli } + (n-1)(n-2)/2 \text{ phases}}$$

Additional generations | quark sector

⇒ Case of $n = 2$ generations: 1 parameter, the Cabibbo angle θ_C:

$$\mathbf{V} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}$$

⇒ Case of $n = 3$ generations: 3 angles + 1 phase. In the standard parameterization:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{ud} & \mathbf{V}_{us} & \mathbf{V}_{ub} \\ \mathbf{V}_{cd} & \mathbf{V}_{cs} & \mathbf{V}_{cb} \\ \mathbf{V}_{td} & \mathbf{V}_{ts} & \mathbf{V}_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}\mathrm{e}^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \Rightarrow$$

δ only source of CP violation in the SM !

with $c_{ij} \equiv \cos\theta_{ij} \geq 0, \quad s_{ij} \equiv \sin\theta_{ij} \geq 0 \quad (i < j = 1,2,3)$ and $0 \leq \delta \leq 2\pi$

Additional generations — lepton sector

- If neutrinos were massless we could redefine the (LH) fields ⇒ no lepton mixing.

  However they *are* massive (though *very light* masses) ⇐ neutrino oscillations!

  – $\boxed{\nu\text{SM}}$ (introduce ν_R and get masses from *tiny* Yukawa couplings like quarks)

  Alternatively ...

- Neutrinos are special: [see next section]

  they *may* be their own antiparticle (Majorana) since they are neutral fermions

  ⇒ New mechanisms for generation of masses and mixings

  - Mass terms are different to Dirac case
  - Neutrinos and antineutrinos *may* mix
  - Intergenerational mixings are richer (*more* CP *phases*)

  – *If* they are Majorana $\boxed{\text{physics Beyond } \nu\text{SM}}$ (*seesaw* mechanism?)

Additional generations | lepton sector

- What we know about neutrinos:
  - From Z lineshape: $n = 3$ generations of *active* ν_L $[\nu_i\ (i = 1, \ldots, n)]$
    (but we do not know (*yet*) if neutrinos are Dirac or Majorana fermions)
  - From oscillations: active neutrinos are very light, non degenerate and mix
    PMNS matrix $\mathbf{U}$ [Pontecorvo '57; Maki, Nakagawa, Sakata '62; Pontecorvo '68]

$$|\nu_\alpha\rangle = \sum_i \mathbf{U}_{\alpha i}|\nu_i\rangle \quad \Longleftrightarrow \quad |\nu_i\rangle = \sum_\alpha \mathbf{U}^*_{\alpha i}|\nu_\alpha\rangle$$

mass eigenstates ν_i $(i = 1, 2, 3)$ / interaction states ν_α $(\alpha = e, \mu, \tau)$

$\Rightarrow$ *If* neutrinos *were* Majorana $\mathbf{U}$ *seems* unitary (for negligible light-heavy mixings) and analogous to $\mathbf{V}_u$, $\mathbf{V}_d$, $\mathbf{V}_\ell$ defined for quarks and charged leptons except for:

  - ν fields have both chiralities: $\nu_i = \nu_{iL} + \eta_i\,\nu^c_{iL}$
  - *If* ν's are Majorana, $\mathbf{U}$ *may* contain two additional physical (Majorana) phases that *cannot be absorbed* since then field phases are fixed by $\nu_i = \eta_i \nu^c_i$

Additional generations — lepton sector

⇒ Standard parameterization of the PMNS matrix:

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{e1} & \mathbf{U}_{e2} & \mathbf{U}_{e3} \\ \mathbf{U}_{\mu 1} & \mathbf{U}_{\mu 2} & \mathbf{U}_{\mu 3} \\ \mathbf{U}_{\tau 1} & \mathbf{U}_{\tau 2} & \mathbf{U}_{\tau 3} \end{pmatrix}$$

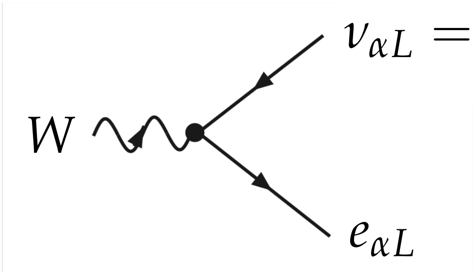
$$= \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}\,\mathrm{e}^{-i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}\,\mathrm{e}^{i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}\,\mathrm{e}^{i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}\,\mathrm{e}^{i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}\,\mathrm{e}^{i\delta} & c_{23}\,c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{i\alpha_{21}/2} & 0 \\ 0 & 0 & \mathrm{e}^{i\alpha_{31}/2} \end{pmatrix}$$

(different values than in CKM) (Majorana phases)

[$\theta_{13} \equiv \theta_{\odot}$, $\theta_{23} \equiv \theta_{\text{atm}}$, θ_{13} and δ measured in oscillations]

Additional generations | lepton sector

- **U** introduces family mixings in $\mathcal{L}_{\rm CC}$ (like CKM), but in this case:



$$\nu_{\alpha L} = \sum_i \mathbf{U}_{\alpha i} \nu_{iL}$$

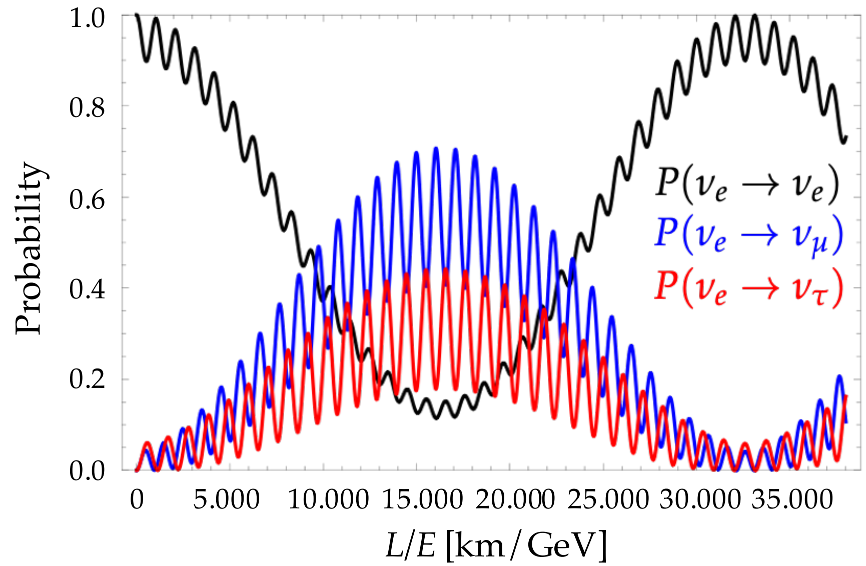
ν_α are *coherent* superpositions of mass eigenstates ν_i (produced/detected in association with ℓ_α)

ℓ_α (e, μ, τ) are mass eigenstates (do *not* oscillate) because $\Delta m^2_{ij} \ll \Delta m^2_{\mu e}$ [0706.1216]

$$|\nu_\alpha; t\rangle = \sum_i \mathbf{U}_{\alpha i} \mathrm{e}^{-\mathrm{i}E_i t} |\nu_i\rangle \,, \quad E_i = E + \frac{m_i^2}{2E}$$

$$\langle \nu_\beta | \nu_\alpha; t\rangle = \sum_i \mathbf{U}^*_{\beta i} \mathbf{U}_{\alpha i} \mathrm{e}^{-\mathrm{i}E_i t}$$

$$P(\nu_\alpha \to \nu_\beta) = \sum_{ij} \mathbf{U}^*_{\beta i} \mathbf{U}_{\alpha i} \mathbf{U}_{\beta j} \mathbf{U}^*_{\alpha j} \exp\left(-\mathrm{i}\frac{\Delta m^2_{ij}}{2E} t\right)$$



[see section on Neutrino phenomenology]

Neutrinos are special

Dirac vs Majorana fermions

- A Dirac fermion field is a spinor with 4 independent components: 2 LH+2 RH (left/right-handed particles and antiparticles)

$$\psi_L = P_L\psi\,, \quad \psi_R = P_R\psi\,, \quad \psi_L^c \equiv (\psi_L)^c = P_R\psi^c\,, \quad \psi_R^c \equiv (\psi_R)^c = P_L\psi^c$$

where $\psi^c \equiv C\overline{\psi}^{\mathsf{T}} = -\mathrm{i}\gamma^2\psi^*$ (charge conjugate), $C = -\mathrm{i}\gamma^2\gamma^0$, $P_{R,L} = \frac{1}{2}(1\pm\gamma_5)$

- A Majorana fermion field has just 2 independent components since $\psi^c \equiv \eta^*\psi$:

$$\psi_L = \eta\psi_R^c\,, \quad \psi_R = \eta\psi_L^c$$

where $\eta = -\mathrm{i}\eta_{\mathrm{CP}}$ (CP parity) with $|\eta|^2 = 1$. Only possible if neutral

[*Useful relations*: $\quad C^\dagger = C^T = C^{-1} = -C, \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T, \quad \overline{\psi^c} = \psi^T C$]

General mass terms

- Lorentz invariant terms:

$$\overline{\psi_R}\psi_L = \overline{\psi_L^c}\psi_R^c \quad \stackrel{\text{hc}}{\longleftrightarrow} \quad \overline{\psi_L}\psi_R = \overline{\psi_R^c}\psi_L^c \quad (\Delta F = 0)$$

$$\left.\begin{array}{rcl} \overline{\psi_L^c}\psi_L & = & \overline{\psi_L}\psi_L^c \\ \overline{\psi_R}\psi_R^c & = & \overline{\psi_R^c}\psi_R \end{array}\right\} \qquad (|\Delta F| = 2)$$

$$\Rightarrow \quad -\mathcal{L}_m = \underbrace{m_D\,\overline{\psi_R}\psi_L}_{\text{Dirac term}} + \underbrace{\frac{1}{2}m_L\,\overline{\psi_L^c}\psi_L + \frac{1}{2}m_R\,\overline{\psi_R}\psi_R^c}_{\text{Majorana terms}} \quad + \text{h.c.}$$

  - A Dirac fermion can only have a Dirac mass term (fermion number preserving)
  - Majorana fermions may have Majorana mass terms

$\Rightarrow$ In the SM:
- $*$ m_D from Yukawa coupling after EW SSB $\qquad (m_D = \lambda_\nu\, v/\sqrt{2})$
- $*$ m_L forbidden by gauge symmetry
- $*$ m_R compatible with gauge symmetry! (ν_R are sterile)

General mass terms (a more transparent parameterization)

- Rewrite previous mass terms introducing an array of two Majorana fermions:

$$\chi_L^0 = \begin{pmatrix} \psi_L \\ \psi_R^c \end{pmatrix}, \quad \chi^0 = \chi^{0c} = \chi_L^0 + \chi_L^{0c} \equiv \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \end{pmatrix}, \quad \begin{array}{l} \chi_1^0 = \chi_1^{0c} = \chi_{1L}^0 + \chi_{1L}^{0c} \equiv \psi_L + \psi_L^c \\ \chi_2^0 = \chi_2^{0c} = \chi_{2L}^0 + \chi_{2L}^{0c} \equiv \psi_R^c + \psi_R \end{array}$$

$$\Rightarrow \quad -\mathcal{L}_m = \frac{1}{2}\overline{\chi_L^{0c}}\,\mathbf{M}\,\chi_L^0 + \text{h.c.} \quad \text{with} \quad \mathbf{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

M is a square symmetric matrix ⇒ diagonalizable by a unitary matrix $\widetilde{\mathcal{U}}$:

$$\widetilde{\mathcal{U}}^{\mathsf{T}}\mathbf{M}\,\widetilde{\mathcal{U}} = \mathcal{M} = \text{diag}(m_1', m_2'), \quad \chi_L^0 = \widetilde{\mathcal{U}}\chi_L \quad (\chi_L^{0c} = \widetilde{\mathcal{U}}^*\chi_L^c)$$

To get positive eigenvalues $m_i = \eta_i m_i'$ (physical masses) replace $\chi_{iL} = \sqrt{\eta_i}\,\xi_{iL}$

$$\chi_L^0 = \mathcal{U}\xi_L, \quad \mathcal{U} = \widetilde{\mathcal{U}}\text{diag}(\sqrt{\eta_1}, \sqrt{\eta_2}), \quad \begin{array}{l} \xi_1 = \xi_{1L} + \xi_{1L}^c \\ \xi_2 = \xi_{2L} + \xi_{2L}^c \end{array} \quad \text{(physical fields)}$$

General mass terms

♣ Case of only Dirac term $\qquad \boxed{\chi_L^0 = (\nu_L, \nu_R^c)}$

$(m_L = m_R = 0)$

$$\mathbf{M} = \begin{pmatrix} 0 & m_D \\ m_D & 0 \end{pmatrix} \quad \Rightarrow \quad \widetilde{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \quad m_1' = -m_D , \quad m_2' = m_D$$

Eigenstates $\Rightarrow$ Physical states

$$\chi_{1L} = \frac{1}{\sqrt{2}}(\nu_L - \nu_R^c) \qquad \xi_{1L} = -\mathrm{i}\chi_{1L} \quad [\eta_1 = -1]$$

$$\chi_{2L} = \frac{1}{\sqrt{2}}(\nu_L + \nu_R^c) \qquad \xi_{2L} = \chi_{2L} \quad [\eta_2 = +1]$$

with masses $m_1 = m_2 = m_D$

$$\Rightarrow \quad -\mathcal{L}_m = m_D(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R) = \frac{1}{2} m_D(\overline{\xi_{1L}^c}\xi_{1L} + \overline{\xi_{2L}^c}\xi_{2L}) + \text{h.c.}$$

One Dirac fermion = two Majorana of equal mass and opposite CP parities

General mass terms

♣ Case of seesaw (type I) $\quad \boxed{\chi_L^0 = (\nu_L, N_R^c)}$

$(m_D \ll m_R)$ [Yanagida '79; Gell-Mann, Ramond, Slansky '79; Mohapatra, Senjanovic '80]

$$\mathbf{M} = \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \quad \Rightarrow \quad \widetilde{\mathcal{U}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$m_1 \equiv m_\nu \simeq \frac{m_D^2}{m_R} \ll m_2 \equiv m_N \simeq m_R$$

$$\theta \simeq \frac{m_D}{m_R} \simeq \sqrt{\frac{m_\nu}{m_N}} \ll 1$$

$$\begin{aligned} \chi_{1L} &\approx \nu_L - \frac{m_D}{m_R} N_R^c \approx \nu_L \quad \Rightarrow \quad \xi_{1L} \approx -\mathrm{i}\nu_L \\ \chi_{2L} &\approx \frac{m_D}{m_R} \nu_L + N_R^c \approx N_R^c \quad \Rightarrow \quad \xi_{2L} \approx N_R^c \end{aligned} \quad \Rightarrow \quad -\mathcal{L}_m = \underbrace{\frac{1}{2} m_\nu \, \overline{\nu_L^c} \nu_L}_{\text{gauge invariant??}} + \frac{1}{2} m_N \, \overline{N_R^c} N_R + \text{h.c.}$$

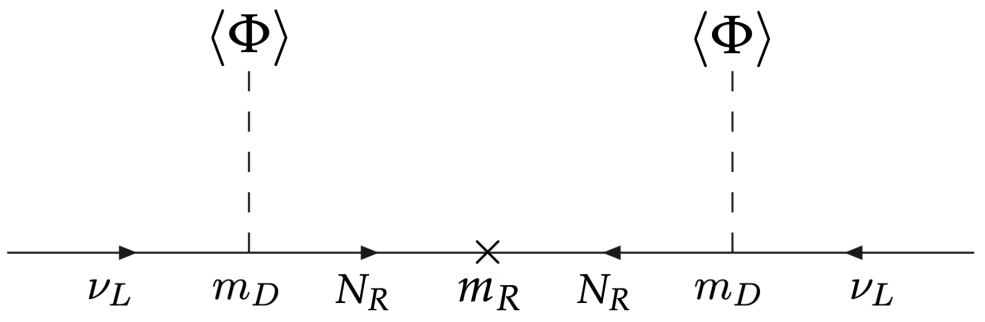
General mass terms

♣ Case of seesaw (type I) $\chi_L^0 = (\nu_L, N_R^c)$

$\frac{1}{2} m_\nu\, \overline{\nu_L^c} \nu_L$ comes after EW SSB from a dim-5 effective interaction, that is gauge-invariant but lepton-number violating (Weinberg operator):

$$\mathcal{L}_{\text{Weinberg}} = -\frac{1}{2}\frac{\lambda_\nu^2}{m_R}(\overline{L}\widetilde{\Phi})(\widetilde{\Phi}^T L^c) + \text{h.c.}$$

with $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$



Perhaps the observed neutrino ν_L is the LH component of a light Majorana ν (then $\overline{\nu}$ = RH) and light because of a very heavy Majorana neutrino N

e.g. $m_D = \lambda_\nu \frac{v}{\sqrt{2}} \sim 100$ GeV , $m_R \sim m_N \sim 10^{14}$ GeV $\Rightarrow$ $m_\nu \sim 0.1$ eV ✓

General mass terms

♣ Case of seesaw (type I): $\boxed{\chi_L^0 = (\nu_{\alpha L}, N_{Rj}^c)}$ **several generations**

$$\alpha = e, \mu, \tau \text{ (active)} \qquad j = 1, \ldots, n_R \geq 2 \text{ (sterile)}$$

$$\mathbf{M} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \qquad \text{with blocks} \quad \begin{cases} 0 : 3 \times 3 & M_D : 3 \times n_R \\ M_D^T : n_R \times 3 & M_R : n_R \times n_R \end{cases}$$

For $M_D \ll M_R$, and taking M_R diagonal to simplify:

$$\mathcal{U}^T \, \mathbf{M} \, \mathcal{U} \approx \begin{pmatrix} \mathbf{U}^T M_D M_R^{-1} M_D^T \mathbf{U} & 0 \\ 0 & M_R \end{pmatrix} \equiv \begin{pmatrix} M_\nu^{\text{diag}} & 0 \\ 0 & M_N^{\text{diag}} \end{pmatrix}$$

The 3×3 block $\mathbf{U}$ is *approximately unitary* because it is contained in $\mathcal{U}$:

$$\mathcal{U} \approx \begin{pmatrix} \mathbf{U} & \mathcal{O}(m_D/m_R) \\ \mathcal{O}(m_D/m_R) & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \nu_\alpha = \nu_{\alpha L} + \nu_{\alpha L}^c \quad \text{with} \quad \nu_{\alpha L} = \mathbf{U}_{\alpha i} \nu_{iL}$$

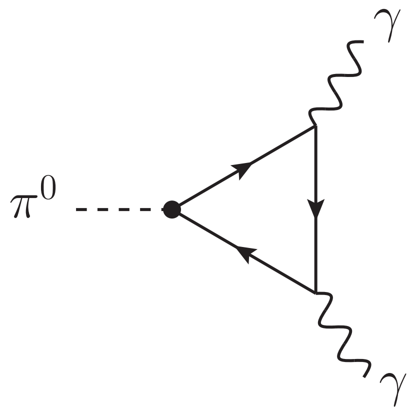
$$(\chi_L^0 = \mathcal{U} \xi_L)$$

Anomalies?

About anomalies

- Anomaly: a symmetry of the classical Lagrangian broken by quantum corrections.
- Anomalies appear when *both* axial ($\psi\gamma^\mu\gamma_5\psi$) and vector ($\psi\gamma^\mu\psi$) currents involved.

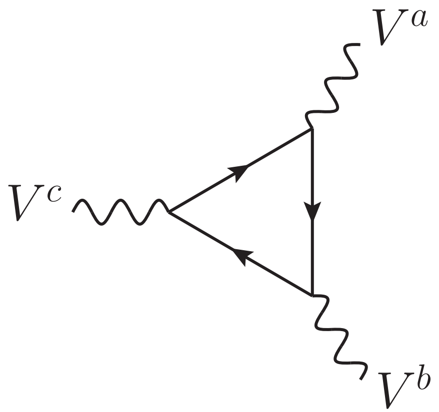
– Anomalies of global symmetries are welcome. For example:
$\pi^0 \to \gamma\gamma$ thanks to coupling of an axial current $j_A^\mu = (\overline{u}\gamma^\mu\gamma_5 u - \overline{d}\gamma^\mu\gamma_5 d)$ to two electromagnetic (vector) currents, breaking the conservation of the axial current ($\partial^\mu j_A^\mu \neq 0$) at 1 loop, even in the limit of massless quarks.



– However, gauge anomalies are a disaster:
they break Ward-Takahashi identities spoiling renormalizability.

Gauge anomalies

- The gauge anomalies are generated by **triangle diagrams** connecting three gauge bosons V^a, V^b, V^c, each coupled to fermions by $(\overline{\Psi}_L\gamma^\mu T^a_L\Psi_L + \overline{\Psi}_R\gamma^\mu T^a_R\Psi_R)V^a_\mu$ with T^a_L (T^a_R) the associated generators:



+ crossed

$$\mathcal{A}^{abc} = \mathrm{Tr}\left(\{T^a_L, T^b_L\}T^c_L\right) - \mathrm{Tr}\left(\{T^a_R, T^b_R\}T^c_R\right)$$

[traces include summation over *all* fermions]

Gauge symmetry is preserved at quantum level if *every* $\mathcal{A}^{abc} = 0$.

- In $SU(3)_c \times SU(2)_L \times U(1)_Y$ we have $T^a \in \{\frac{1}{2}\lambda^i, \frac{1}{2}\sigma^i, Y\}$ with

$$\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}$$
$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}\,\mathbb{1}$$
$$\mathrm{Tr}(\lambda^i) = \mathrm{Tr}(\sigma^i) = 0$$

Gauge anomalies | cancel!

- Since $SU(3)_c$ is non-chiral (not anomalous), the only non trivial combinations are

$$SU(3)^2U(1): \quad \mathrm{Tr}(\{\lambda^i,\lambda^j\}Y\}) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{quarks}} (Y_L - Y_R) = 0 \quad \checkmark$$

$$SU(2)^2U(1): \quad \mathrm{Tr}(\{\sigma^i,\sigma^j\}Y) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{leptons}} Y_L + N_c \sum_{\text{quarks}} Y_L = 0 \quad \checkmark$$

$$U(1)^3: \quad \mathrm{Tr}(Y^3) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{leptons}} (Y_L^3 - Y_R^3) + N_c \sum_{\text{quarks}} (Y_L^3 - Y_R^3) = 0 \quad \checkmark$$

where

	ν_e	e	u	d
Y_L	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
Y_R	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$

and anomalies cancel if $N_c = 3$.

- In particular the second constraint is equivalent to

$$Q_\nu + Q_e + N_c(Q_u + Q_d) = -1 + \frac{1}{3}N_c = 0 \quad \Rightarrow \quad N_c = 3 \quad (!!)$$

$\Rightarrow$ The electroweak SM needs leptons + quarks in every generation !!

$\Rightarrow$ The electroweak SM needs the QCD part !!

3. Electroweak Pheno

The full Lagrangian — fields and interactions

$$\boxed{\mathcal{L} = \mathcal{L}_F + \mathcal{L}_{\rm YM} + \mathcal{L}_\Phi + \mathcal{L}_{\rm Y} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}}$$

$$\begin{aligned} \mathcal{L}_F &\supset \mathcal{L}_{\rm CC} + \mathcal{L}_{\rm NC} \\ \mathcal{L}_{\rm YM} &\supset \mathcal{L}_{VVV} + \mathcal{L}_{VVVV} \\ \mathcal{L}_\Phi &\supset \text{gauge boson masses} \\ \mathcal{L}_Y &\supset \text{fermion masses and mixings} \end{aligned}$$

- Fields:
  - [F] fermions
  - [S] scalars (Higgs and unphysical Goldstones)
  - [V] vector bosons
  - [U] unphysical ghosts

- Interactions: [FFV] [FFS] [SSV] [SVV] [SSVV]
  [VVV] [VVVV] [SSS] [SSSS]
  [SUU] [UUVV]

Full SM Lagrangian | types of interactions

- Lorentz structure of generic interactions (normalized to e):

$$\mathcal{L}_{\rm FFV} = e\,\overline{\psi}_i\gamma^\mu(g_V - g_A\gamma_5)\psi_j\,V_\mu = e\,\overline{\psi}_i\gamma^\mu(g_L P_L + g_R P_R)\psi_j\,V_\mu$$
$$\mathcal{L}_{\rm FFS} = e\,\overline{\psi}_i(g_S - g_P\gamma_5)\psi_j\,\phi = e\,\overline{\psi}_i(c_L P_L + c_R P_R)\psi_j\,\phi$$
$$\mathcal{L}_{\rm VVV} = -\mathrm{i}e\,c_{VVV}\left(W^{\mu\nu}W^\dagger_\mu V_\nu - W^\dagger_{\mu\nu}W^\mu V^\nu - W^\dagger_\mu W_\nu V^{\mu\nu}\right)$$
$$\mathcal{L}_{\rm VVVV} = e^2\,c_{VVVV}\left(2W^\dagger_\mu W^\mu V_\nu V'^\nu - W^\dagger_\mu V^\mu W_\nu V'^\nu - W^\dagger_\mu V'^\mu W_\nu V^\nu\right)$$
$$\mathcal{L}_{\rm SSV} = -\mathrm{i}e\,c_{SSV}\,\phi\overleftrightarrow{\partial_\mu}\phi'\,V^\mu$$
$$\mathcal{L}_{\rm SVV} = e\,c_{SVV}\,\phi\,V^\mu V'_\mu$$
$$\mathcal{L}_{\rm SSVV} = e^2\,c_{SSVV}\,\phi\phi' V^\mu V'_\mu$$
$$\mathcal{L}_{\rm SSS} = e\,c_{SSS}\,\phi\phi'\phi''$$
$$\mathcal{L}_{\rm SSSS} = e^2\,c_{SSSS}\,\phi\phi'\phi''\phi''',$$

where $\quad \phi\overleftrightarrow{\partial_\mu}\phi' \equiv \phi_i\partial_\mu\phi' - (\partial_\mu\phi_i)\phi' \quad$ and $\quad V_\mu \in \{A_\mu, Z_\mu, W_\mu, W^\dagger_\mu\}$.

Full SM Lagrangian **Feynman Rules**

8

- Feynman rules for generic vertices normalized to e (all momenta incoming):

$(\mathrm{i}\mathcal{L})$

$$
\begin{aligned}
[\mathrm{FFV}_\mu] &= \mathrm{i}e\gamma^\mu(g_L P_L + g_R P_R) \\
[\mathrm{FFS}] &= \mathrm{i}e(c_L P_L + c_R P_R) \\
[\mathrm{V}_\mu(k_1)\mathrm{V}_\nu(k_2)\mathrm{V}_\rho(k_3)] &= \mathrm{i}e\, c_{VVV}\left[g_{\mu\nu}(k_2-k_1)_\rho + g_{\nu\rho}(k_3-k_2)_\mu + g_{\mu\rho}(k_1-k_3)_\nu\right] \\
[\mathrm{V}_\mu\mathrm{V}_\nu\mathrm{V}_\rho\mathrm{V}_\sigma] &= \mathrm{i}e^2\, c_{VVVV}\left[2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right] \\
[\mathrm{S}(p)\mathrm{S}(p')\mathrm{V}_\mu] &= \mathrm{i}e\, c_{SSV}\,(p_\mu - p'_\mu) \\
[\mathrm{SV}_\mu\mathrm{V}_\nu] &= \mathrm{i}e\, c_{SVV} g_{\mu\nu} \\
[\mathrm{SSV}_\mu\mathrm{V}_\nu] &= \mathrm{i}e^2\, c_{SSVV}\, g_{\mu\nu} \\
[\mathrm{SSS}] &= \mathrm{i}e\, c_{SSS} \\
[\mathrm{SSSS}] &= \mathrm{i}e^2\, c_{SSSS}
\end{aligned}
$$

Note: $g_{L,R} = g_V \pm g_A$ $\quad\partial_\mu \to -\mathrm{i}p_\mu$ $\quad$ Attention to symmetry factors!

$c_{L,R} = g_S \pm g_P$ $\quad$ *e.g.* $2\times HZZ$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

FFV	$\bar{f}_i f_j \gamma$	$\bar{f}_i f_j Z$	$\bar{u}_i d_j W^+$	$\bar{d}_j u_i W^-$	$\bar{\nu}_i \ell_j W^+$	$\bar{\ell}_j \nu_i W^-$
g_L	$-Q_f \delta_{ij}$	$g_+^f \delta_{ij}$	$\frac{1}{\sqrt{2} s_W} \mathbf{V}_{ij}$	$\frac{1}{\sqrt{2} s_W} \mathbf{V}_{ij}^*$	$\frac{1}{\sqrt{2} s_W} \delta_{ij}$	$\frac{1}{\sqrt{2} s_W} \delta_{ij}$
g_R	$-Q_f \delta_{ij}$	$g_-^f \delta_{ij}$	0	0	0	0

$$g_\pm^f \equiv v_f \pm a_f \qquad v_f = \frac{T_3^{f_L} - 2Q_f s_W^2}{2 s_W c_W} \qquad a_f = \frac{T_3^{f_L}}{2 s_W c_W}$$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

FFS	$\bar{f}_i f_j H$	$\bar{f}_i f_j \chi$	$\bar{u}_i d_j \phi^+$	$\bar{d}_j u_i \phi^-$
c_L	$-\dfrac{1}{2s_W}\dfrac{m_{f_i}}{M_W}\delta_{ij}$	$-\dfrac{\mathrm{i}}{2s_W}2T_3^{f_L}\dfrac{m_{f_i}}{M_W}\delta_{ij}$	$+\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{u_i}}{M_W}\mathbf{V}_{ij}$	$-\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{d_j}}{M_W}\mathbf{V}_{ij}^*$
c_R	$-\dfrac{1}{2s_W}\dfrac{m_{f_i}}{M_W}\delta_{ij}$	$+\dfrac{\mathrm{i}}{2s_W}2T_3^{f_L}\dfrac{m_{f_i}}{M_W}\delta_{ij}$	$-\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{d_j}}{M_W}\mathbf{V}_{ij}$	$+\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{u_j}}{M_W}\mathbf{V}_{ij}^*$

$(f = u, d, \ell)$

FFS	$\bar{\nu}_i \ell_j \phi^+$	$\bar{\ell}_j \nu_i \phi^-$
c_L	$+\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{\nu_i}}{M_W}\delta_{ij}$	$-\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{\ell_j}}{M_W}\delta_{ij}$
c_R	$-\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{\ell_j}}{M_W}\delta_{ij}$	$+\dfrac{1}{\sqrt{2}s_W}\dfrac{m_{\nu_i}}{M_W}\delta_{ij}$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

SVV	HZZ	HW^+W^-	$\phi^\pm W^\mp \gamma$	$\phi^\pm W^\mp Z$
c_{SVV}	$M_W/(s_W c_W^2)$	M_W/s_W	$-M_W$	$-M_W s_W/c_W$

SSV	$\chi H Z$	$\phi^\pm \phi^\mp \gamma$	$\phi^\pm \phi^\mp Z$	$\phi^\mp H W^\pm$	$\phi^\mp \chi W^\pm$
c_{SSV}	$-\dfrac{\mathrm{i}}{2s_W c_W}$	∓ 1	$\pm\dfrac{c_W^2 - s_W^2}{2s_W c_W}$	$\mp\dfrac{1}{2s_W}$	$-\dfrac{\mathrm{i}}{2s_W}$

VVV	$W^+W^-\gamma$	W^+W^-Z
c_{VVV}	-1	c_W/s_W

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

VVVV	$W^+W^+W^-W^-$	W^+W^-ZZ	$W^+W^-\gamma Z$	$W^+W^-\gamma\gamma$
c_{VVVV}	$\frac{1}{s_W^2}$	$-\frac{c_W^2}{s_W^2}$	$\frac{c_W}{s_W}$	-1

SSVV	HHW^-W^+	$HHZZ$
c_{SSVV}	$\frac{1}{2s_W^2}$	$\frac{1}{2s_W^2c_W^2}$

SSS	HHH
c_{SSS}	$-\frac{3M_H^2}{2M_W s_W}$

SSSS	$HHHH$
c_{SSSS}	$-\frac{3M_H^2}{4M_W^2 s_W^2}$

– Would-be Goldstone bosons in [SSVV], [SSS] and [SSSS] omitted

– Faddeev-Popov ghosts in [SUU] and [UUVV] omitted

– All Feynman rules from FeynArts (same conventions; $\chi, \phi^\pm \to G^0, G^\pm$):

http://www.ugr.es/local/jillana/SM/FeynmanRulesSM.pdf

Input parameters

- Parameters:

17 +9 =	1	1	1	1	9 +3	4	6
formal:	g	g'	v	λ	λ_f	$\mathbf{V}_{\text{CKM}}$	$\mathbf{U}_{\text{PMNS}}$
practical:	α	M_W	M_Z	M_H	m_f		

where $g = \dfrac{e}{s_W}$ $\quad g' = \dfrac{e}{c_W}$ and

$$\underbrace{\alpha = \frac{e^2}{4\pi} \qquad M_W = \frac{1}{2} g v \qquad M_Z = \frac{M_W}{c_W}}_{g,\ g',\ v} \qquad M_H = \sqrt{2\lambda}\, v \qquad m_f = \frac{v}{\sqrt{2}} \lambda_f$$

⇒ Many (more) experiments

⇒ After Higgs discovery, for the first time *all* parameters measured!

Input parameters

- Experimental values [Particle Data Group '20]

  - Fine structure constant:

    $\alpha^{-1} = 137.035\,999\,150\,(33)$ Harvard cyclotron (g_e) [1712.06060]

    $\alpha^{-1} = 137.035\,999\,046\,(27)$ atom interferometry (Cesium) [1812.04130]

    $\alpha^{-1} = 137.035\,999\,206\,(11)$ atom interferometry (Rubidium) [Nature 588, 61(2020)]

  - The SM predicts $M_W < M_Z$ in agreement with measurements:

    $M_Z = (91.1876 \pm 0.0021)$ GeV LEP1/SLD

    $M_W = (80.379 \pm 0.012)$ GeV LEP2/Tevatron/LHC

  - Top quark mass:

    $m_t = (172.25 \pm 0.30)$ GeV Tevatron/LHC

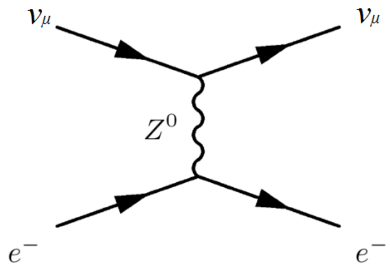
  - Higgs boson mass:

    $M_H = (125.46 \pm 0.17)$ GeV LHC

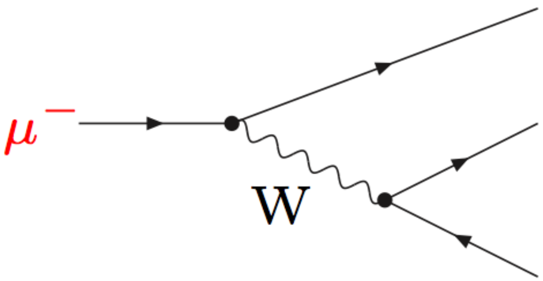
  - ...

Observables and experiments

Weak NC discovery (1973)



- Low energy observables ($Q^2 \ll M_Z^2$)
  - ν-nucleon (NuTeV) and νe (CERN) scattering
    asymmetries CC/NC and $\nu/\bar{\nu}$ $\Rightarrow$ $\boxed{s_W^2}$
  - Parity and Atomic Parity violation (SLAC, CERN, Jefferson Lab, Mainz)
    LR asymmetries $\mathrm{e}_{R,L} N \to \mathrm{e}X$ and Z effects on atomic transitions $\Rightarrow$ $\boxed{s_W^2}$
  - muon decay: $\mu \to e\,\overline{\nu}_e \nu_\mu$ (PSI)
    lifetime



$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2)$$

$$f(x) \equiv 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x = 0.99981295 \qquad \Rightarrow \boxed{G_F}$$

$$\mathrm{i}\mathcal{M} = \left(\frac{\mathrm{i}e}{\sqrt{2}s_W}\right)^2 \bar{e}\gamma^\rho \nu_L \frac{-\mathrm{i}g_{\rho\delta}}{q^2 - M_W^2} \overline{\nu_L}\gamma^\delta \mu \equiv \overbrace{-\mathrm{i}\frac{4G_F}{\sqrt{2}}(\bar{e}\gamma^\rho \nu_L)(\overline{\nu_L}\gamma_\rho \mu)}^{\text{Fermi theory } (-q^2 \ll M_W^2)}\ ; \quad \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 M_W^2}$$

Observables and experiments

- Low energy observables

⇒ Fermi constant provides the Higgs VEV (electroweak scale):

$$v = \left(\sqrt{2}G_F\right)^{-1/2} \approx 246\,\text{GeV}$$

and constrains the product $M_W^2 s_W^2$, which implies

$$M_Z^2 > M_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F s_W^2} > \frac{\pi\alpha}{\sqrt{2}G_F} \approx (37.4\ \text{GeV})^2$$

⇒ Consistency checks: e.g. from muon lifetime:

$$G_F = 1.166\,378\,7(6) \times 10^{-5}\ \text{GeV}^{-2}$$

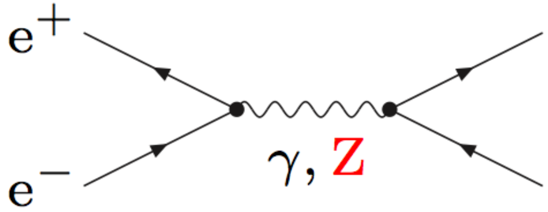
If one compares with (tree level result)

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 M_W^2} = \frac{\pi\alpha}{2(1 - M_W^2/M_Z^2)M_W^2} \approx 1.125 \times 10^{-5}$$

a discrepancy that disappears when *quantum corrections* are included

Observables and experiments

- $e^+e^- \to \bar{f}f$ (PEP, PETRA, TRISTAN, ..., LEP1, SLD)



$$\frac{d\sigma}{d\Omega} = N_c^f \frac{\alpha^2}{4s}\beta_f \left\{ \left[1+\cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2-1)\, G_2(s) + 2\beta_f \cos\theta\, G_3(s) \right\}$$

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_eQ_f v_e v_f \mathrm{Re}\chi_Z(s) + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$

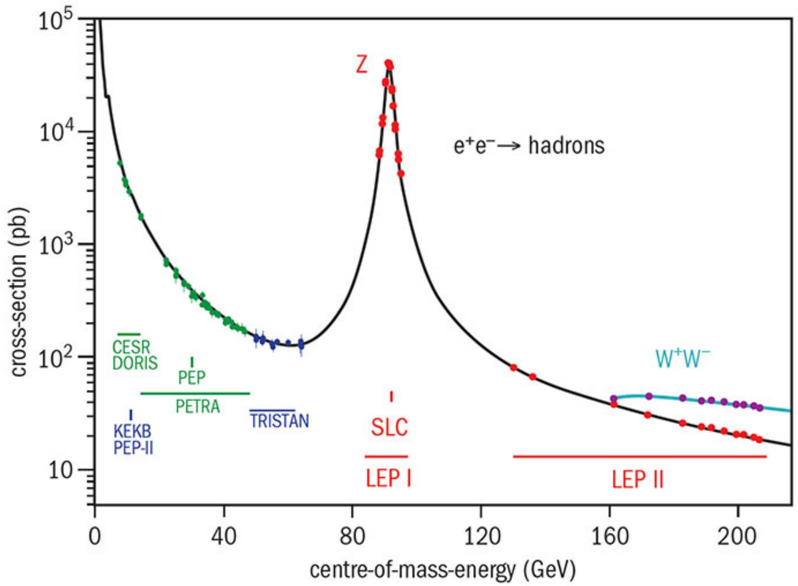
$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$

$$G_3(s) = 2Q_eQ_f a_e a_f \mathrm{Re}\chi_Z(s) + 4v_e v_f a_e a_f |\chi_Z(s)|^2 \quad \Rightarrow \quad \boxed{A_{FB}(s)}$$

$$\chi_Z(s) \equiv \frac{s}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}, \quad N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

$$\sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]$$

$$\beta_f = \sqrt{1-4m_f^2/s}$$

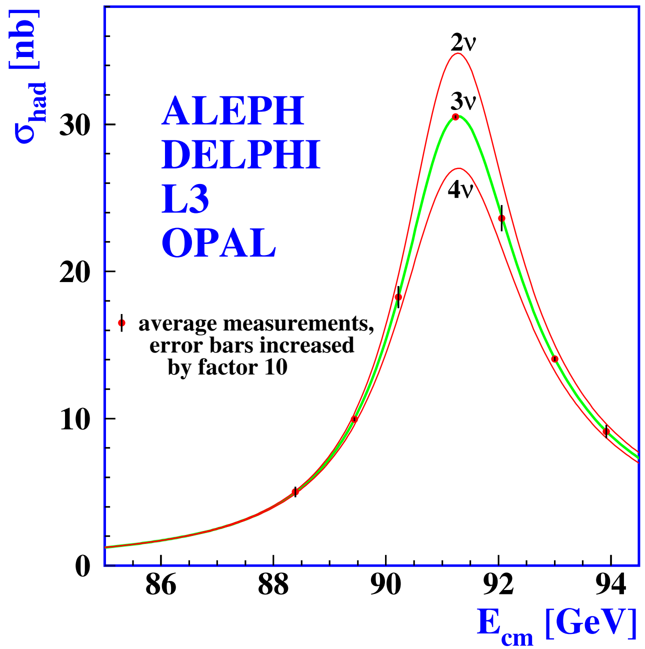


Observables and experiments

- Z pole observables (LEP1/SLD)

  $M_Z,\ \Gamma_Z,\ \sigma_{\rm had},\ A_{FB},\ A_{LR},\ R_b,\ R_c,\ R_\ell \quad \Rightarrow \boxed{M_Z,\ s_W^2}$

  from $e^+e^- \to \bar{f}f$ at the Z pole ($\gamma - Z$ interference vanishes). Neglecting m_f:



$$\sigma^0_{\rm had} = 12\pi \frac{\Gamma(e^+e^-)\Gamma({\rm had})}{M_Z^2\Gamma_Z^2}$$

9

$$R_b = \frac{\Gamma(b\bar{b})}{\Gamma({\rm had})} \quad R_c = \frac{\Gamma(c\bar{c})}{\Gamma({\rm had})} \quad R_\ell = \frac{\Gamma({\rm had})}{\Gamma(\ell^+\ell^-)}$$

$$\left[\Gamma(Z \to f\bar{f}) \equiv \Gamma(f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)\right]$$

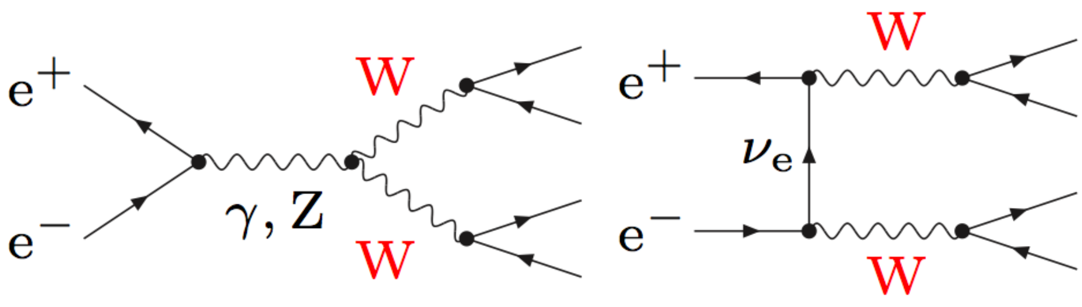
$$\Gamma_Z \simeq 2.5\ {\rm GeV} \Rightarrow N_\nu \simeq 3\ ✓$$

Forward-Backward and (if polarized e^-) Left-Right asymmetries due to Z:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}A_f\frac{A_e + P_e}{1 + P_e A_e} \qquad A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = A_e P_e \quad \text{with } A_f \equiv \frac{2v_f a_f}{v_f^2 + a_f^2}$$

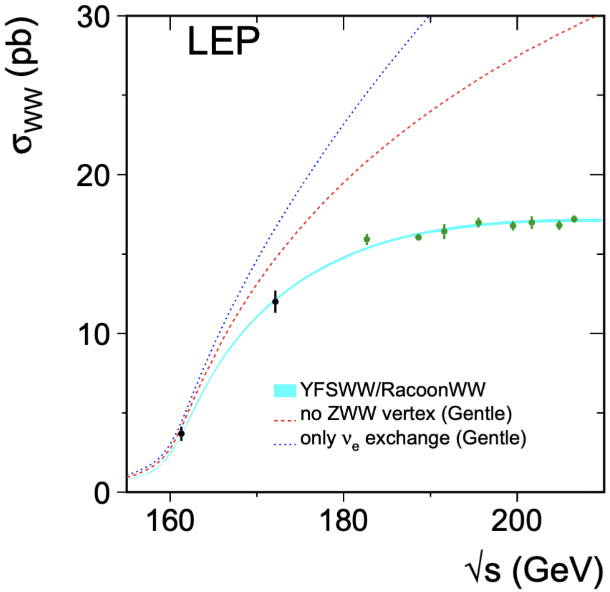
Observables and experiments

- W-pair production (LEP2)
  $e^+e^- \to WW \to 4\,f\;(+\gamma)$



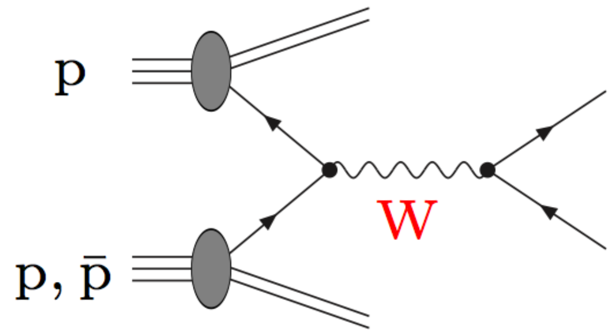
$\Rightarrow \boxed{M_W}$

$\Downarrow$



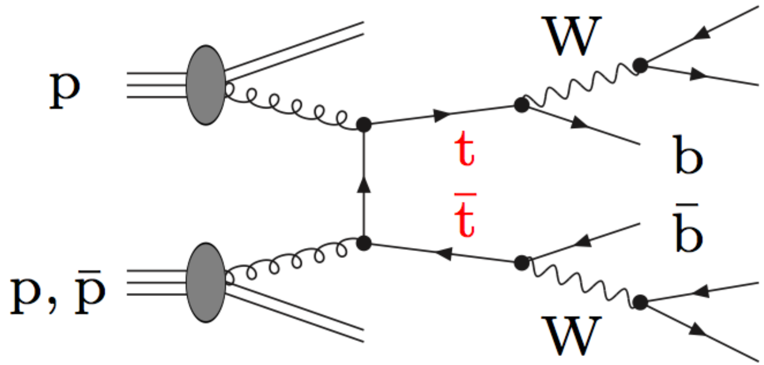
[1302.3415]

- W production (Tevatron/LHC)
  $pp/p\bar{p} \to W \to \ell\nu_\ell\;(+\gamma)$



$\Rightarrow \boxed{M_W}$

- Top-quark production (Tevatron/LHC)
  $pp/p\bar{p} \to t\bar{t} \to 6\,f$



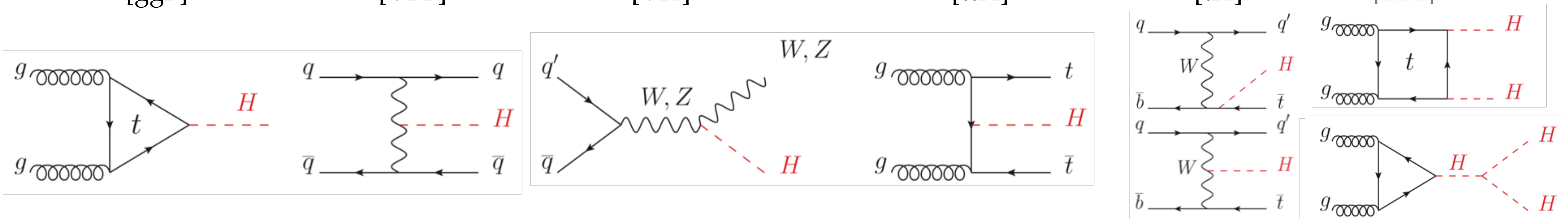
$\Rightarrow \boxed{m_t}$

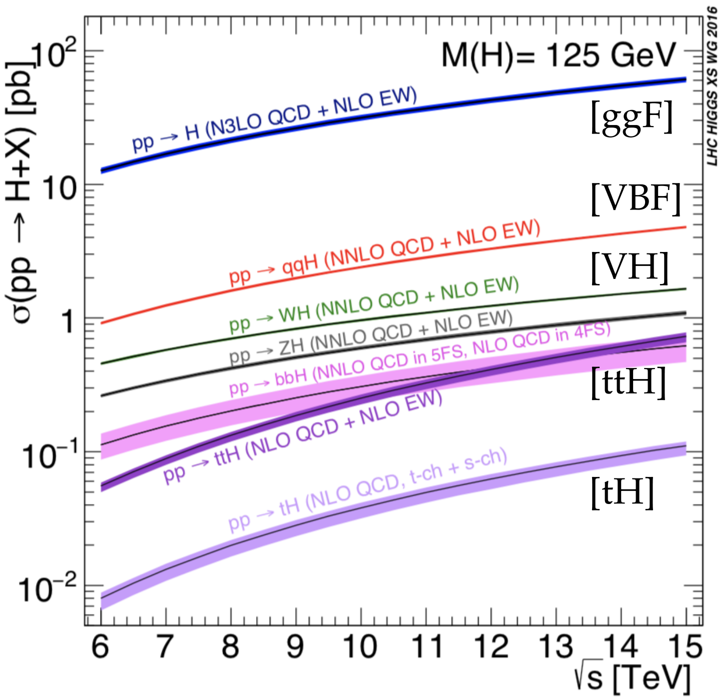
Observables and experiments

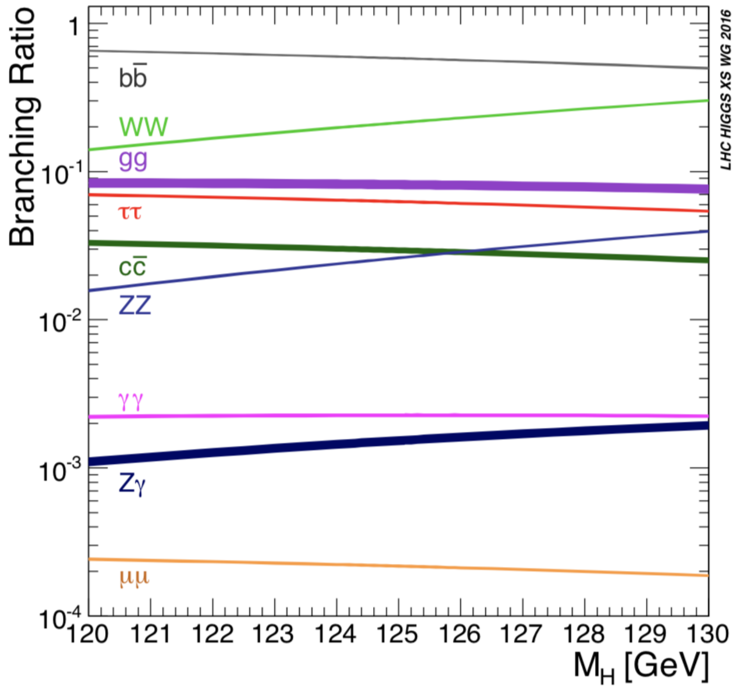
- Higgs (LHC) [PDG '20]

  Single and Double H production and decay to different channels ⇒ $\boxed{M_H}$

[ggF] [VBF] [VH] [ttH] [tH] [HH]







10

Observables and experiments

- Higgs (LHC)

[PDG '20]

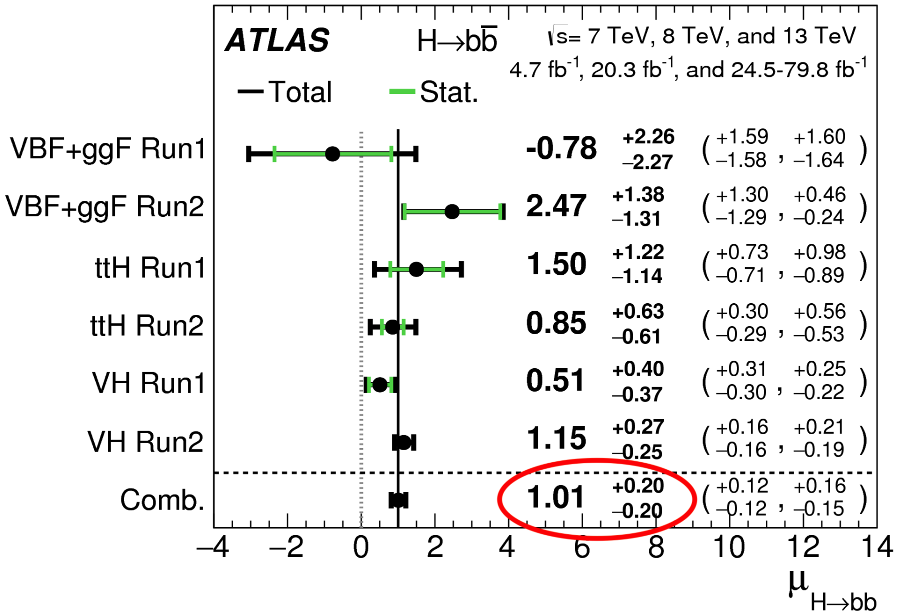
Signal strength $\mu = \dfrac{(\sigma \cdot \mathrm{BR})_{\mathrm{obs}}}{(\sigma \cdot \mathrm{BR})_{\mathrm{SM}}}$	Run 1	Run 2
ATLAS	1.17 ± 0.27	1.02 ± 0.14
CMS	$1.18^{+0.26}_{-0.23}$	$1.18^{+0.17}_{-0.14}$

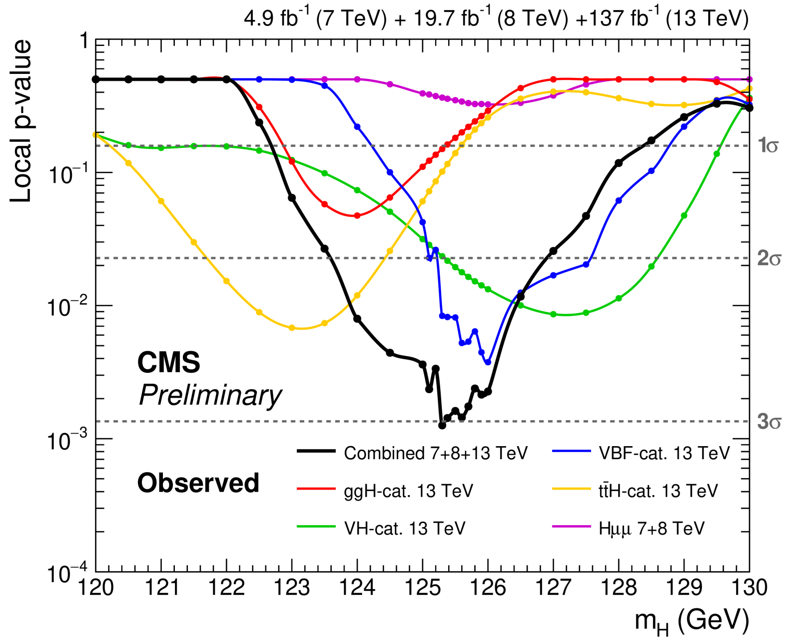
Per channel:

$\gamma\gamma, ZZ, W^+W^-, \tau^+\tau^- > 5\sigma$

$b\bar{b}$ $> 5\sigma$ [Jul '18!]

$\mu^+\mu^-$ $\sim 3\sigma$ [Jul '20!!]

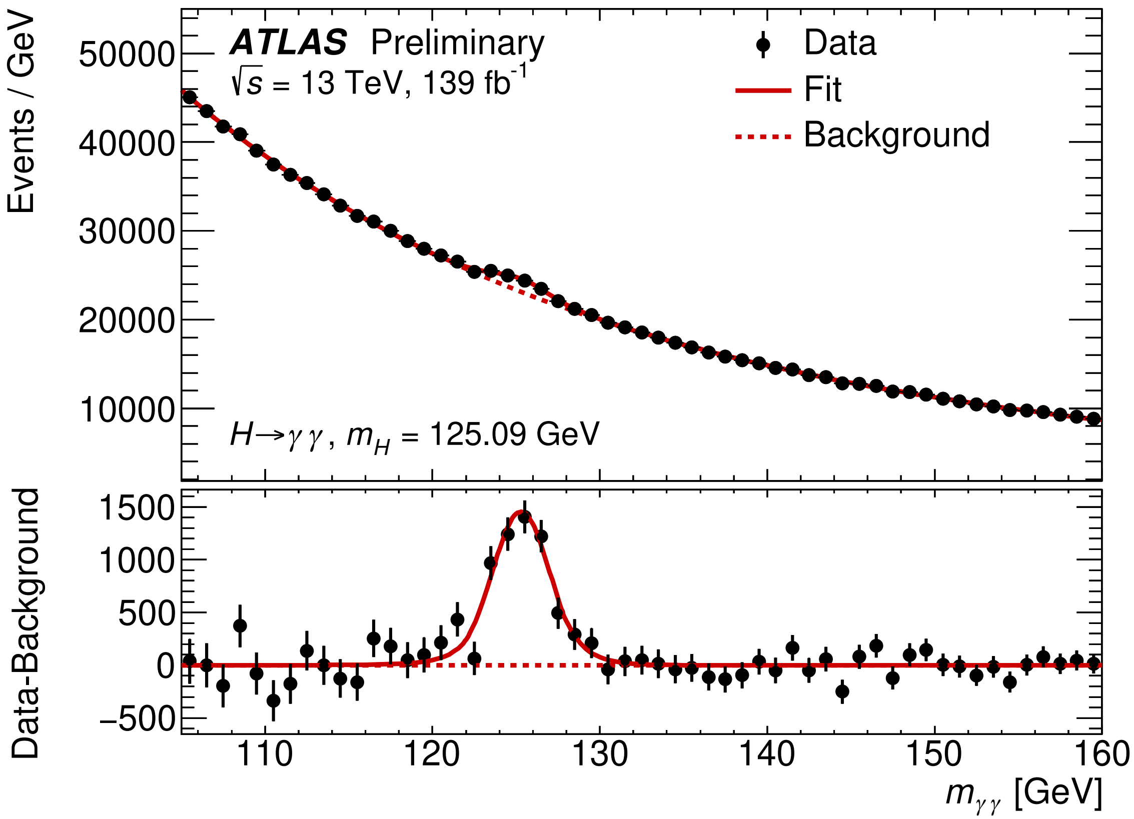




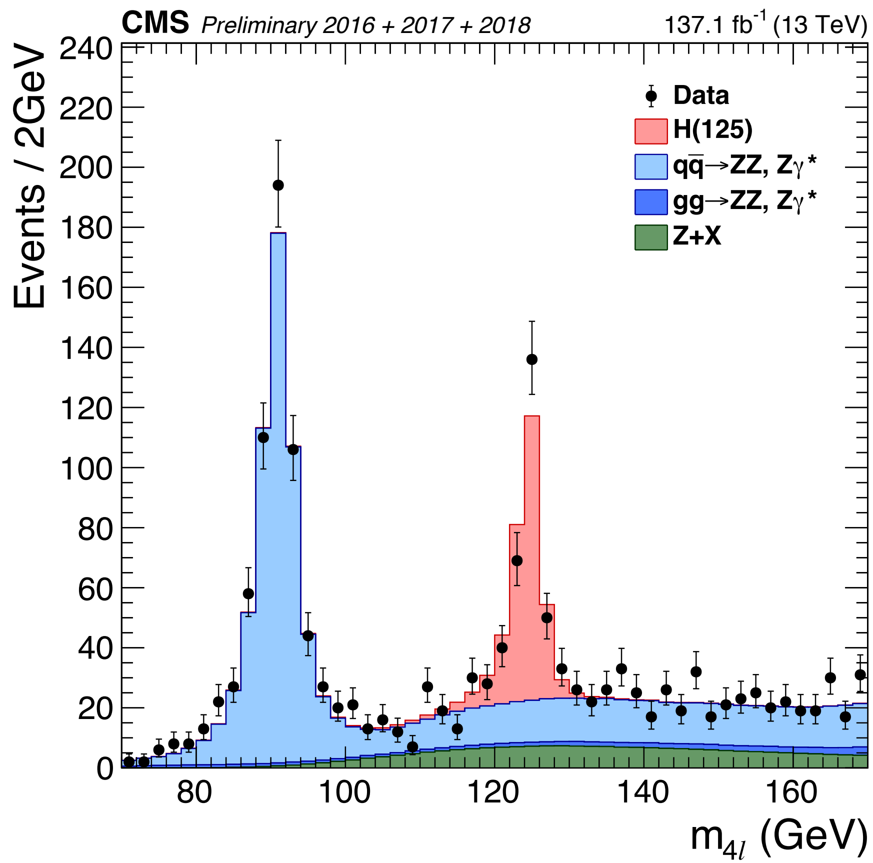
Observables and experiments

- Higgs mass (LHC) [PDG '20]

$H \to \gamma\gamma$

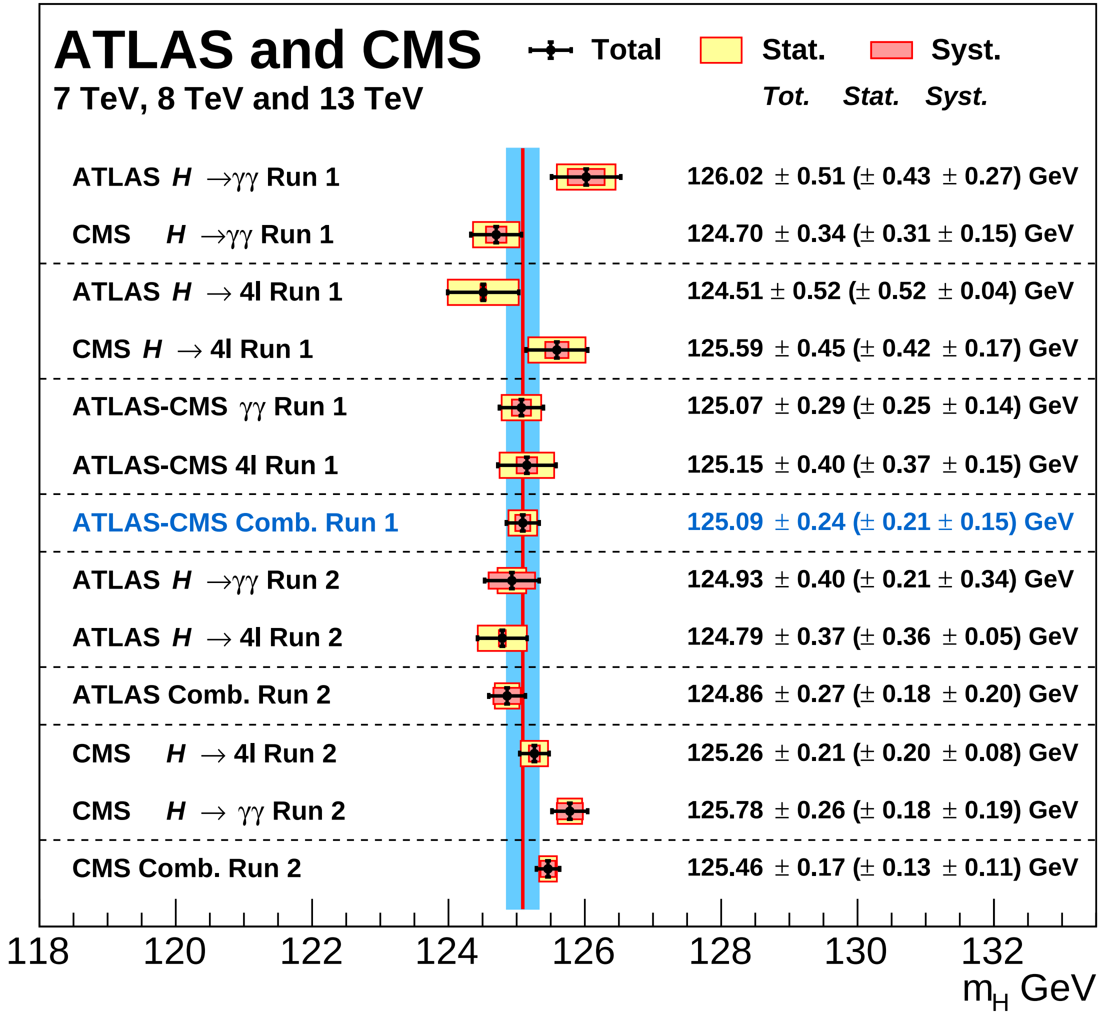


$H \to ZZ^* \to 4\ell$



Observables and experiments

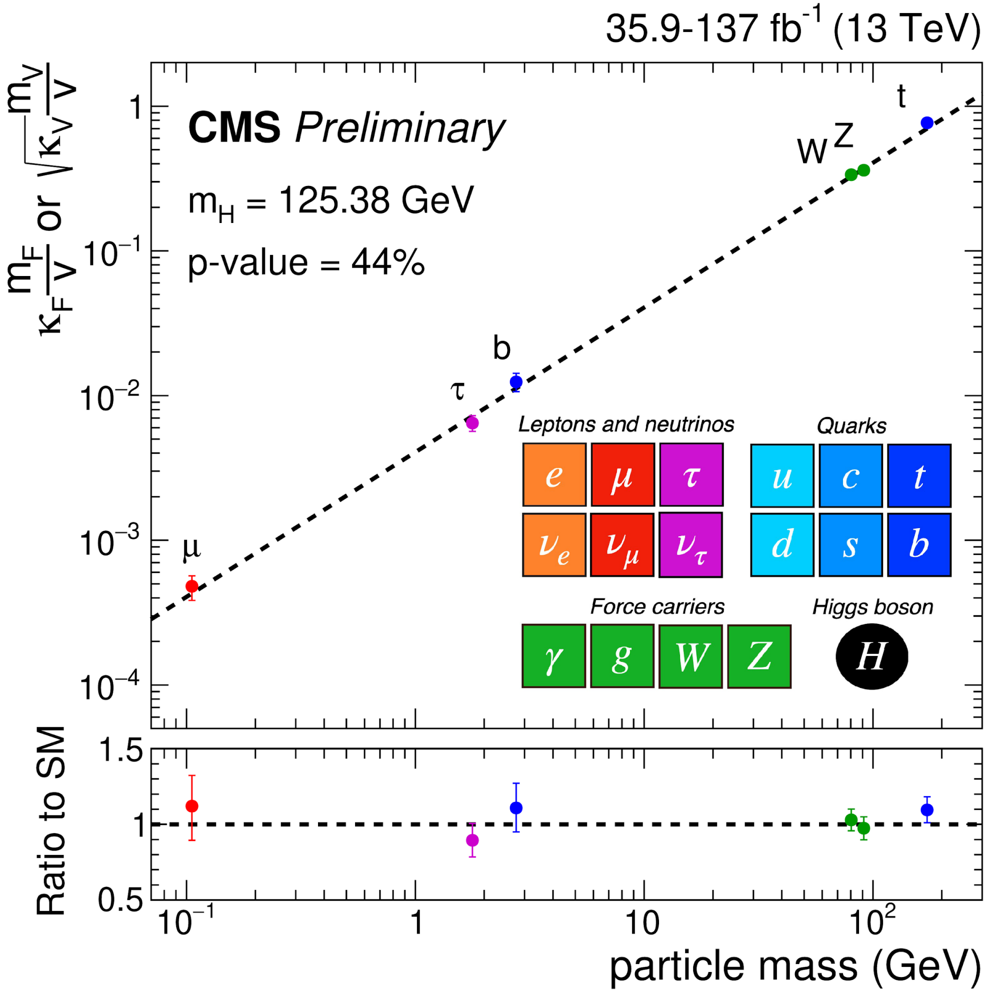
- Higgs mass (LHC) [PDG '20]



Observables and experiments

- Higgs couplings (LHC)

[2009.04363]



H self-couplings not yet observed

$\Leftarrow \kappa_F$ or $\kappa_V = \dfrac{\text{obs}}{\text{SM}}$

proben over more that 3 orders of mangitude!

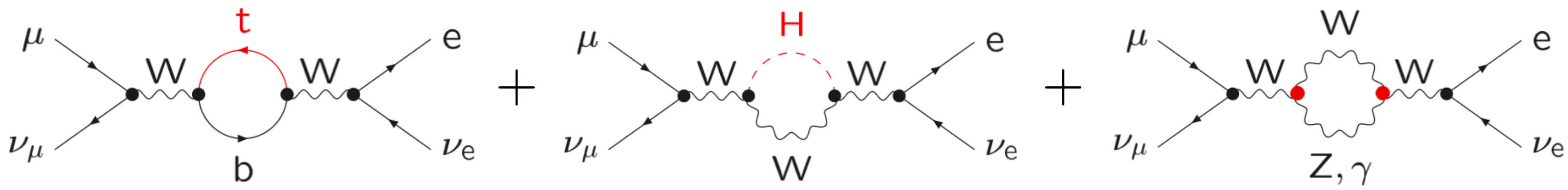
Precise determination of parameters

- Experimental precision requires accurate predictions ⇒ quantum corrections (complication: loop calculations involve renormalization)

– Correction to G_F from muon lifetime:

$$\frac{G_F}{\sqrt{2}} \to \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2(1-M_W^2/M_Z^2)M_W^2}[1+\Delta r(m_t, M_H)]$$

when loop corrections are included:



$+ \cdots + 2$ loops

Since muon lifetime is measured more precisely than M_W, it is traded for G_F:

$$\Rightarrow M_W^2(\alpha, G_F, M_Z, m_t, M_H) = \frac{M_Z^2}{2}\left(1+\sqrt{1-\frac{4\pi\alpha}{\sqrt{2}G_F M_Z^2}[1+\Delta r(m_t, M_H)]}\right)$$

(correlation between M_W, m_t and M_H, given α, G_F and M_Z)

Precise determination of parameters

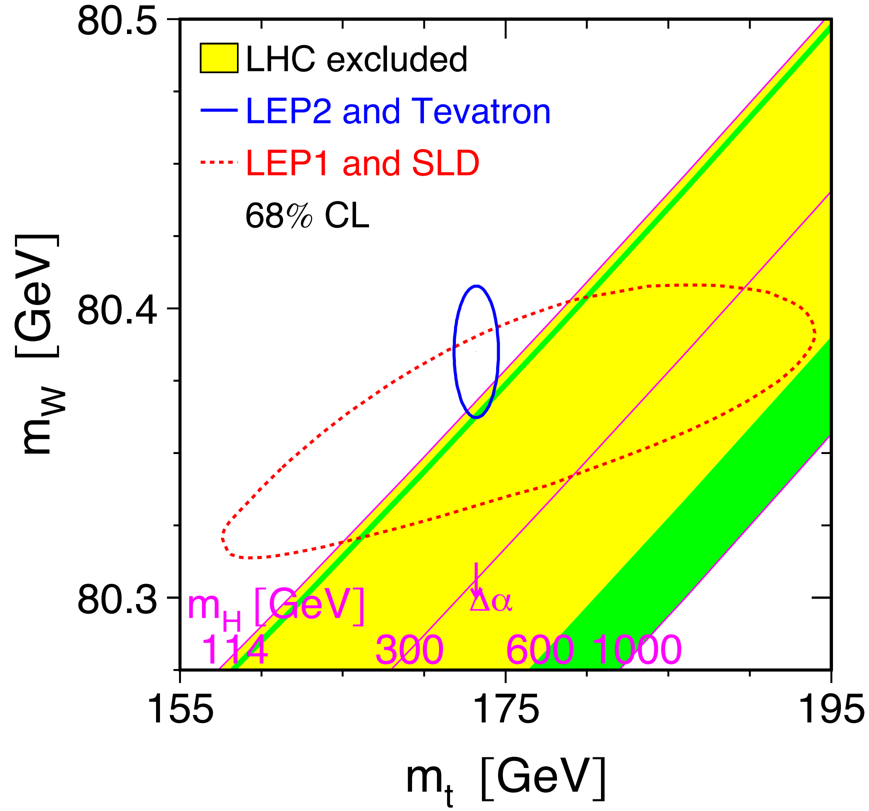
Indirect constraints from LEP1/SLD $M_H(M_W, m_t)$

Direct measurements from LEP2/Tevatron

Allowed regions for M_H allowed by direct searches

LHC excluded

[LEPEWWG 2013]



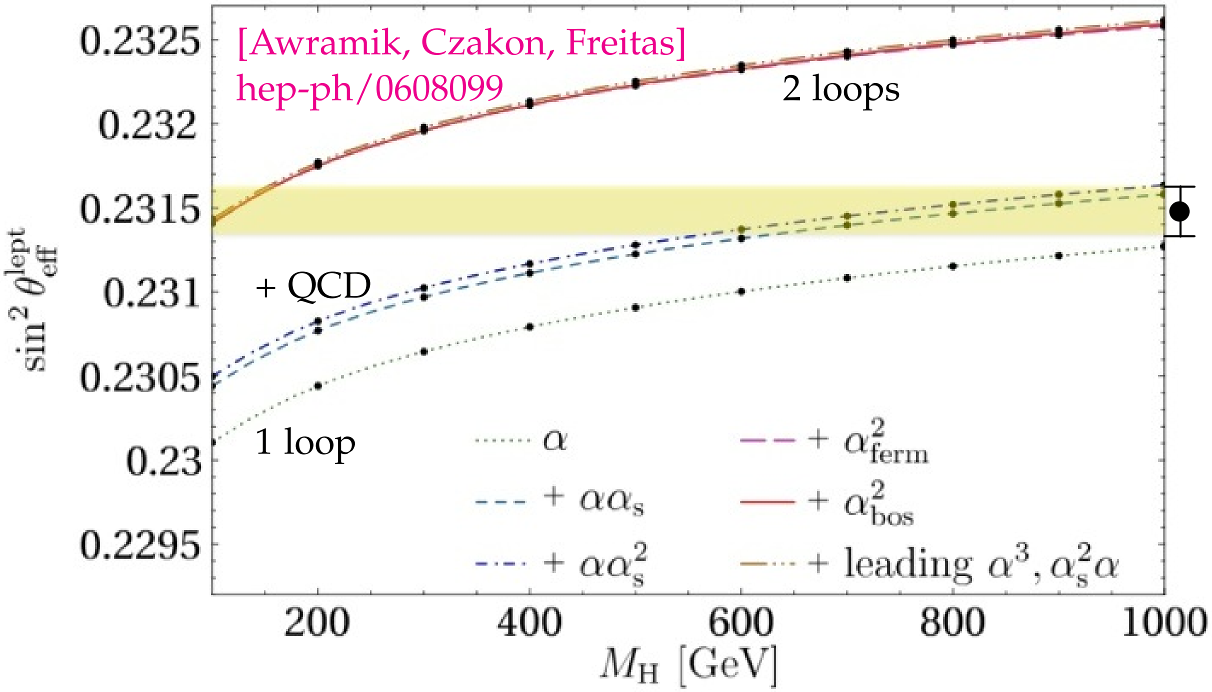
Precise determination of parameters

– Corrections to vector and axial couplings from Z pole observables:

$$v_f \to g_V^f = v_f + \Delta g_V^f \qquad a_f \to g_A^f = a_f + \Delta g_A^f$$

$$\Rightarrow \sin^2\theta_{\rm eff}^f \equiv \frac{1}{4|Q_f|}\left|1 - {\rm Re}(g_V^f/g_A^f)\right| \equiv \overbrace{(1 - M_W^2/M_Z^2)}^{s_W^2}\ \kappa_Z^f$$

(Two) loop calculations are crucial and point to a light Higgs:



$s_W^2 = 0.22290 \pm 0.00029$ (tree)

$\sin^2\theta_{\rm eff}^{\rm lept} = 0.23148 \pm 0.000017$ (exp)

Precise determination of parameters

- In addition, experiments and observables testing the flavor structure of the SM: flavor conserving: dipole moments, … flavor changing: $b \to s\gamma$, …

  $\Rightarrow$ very sensitive to new physics through loop corrections

  Extremely precise measurements are:

  – electron magnetic moment:

$$\left.\begin{array}{rl} \text{exp:} & g_e/2 = 1.001\,159\,652\,182\,032\,(720) \\ \text{theo:} & \text{QED (5 loops!)} \end{array}\right\} \Rightarrow \alpha^{-1} = 137.035\,999\,150\,(33)$$

  – muon anomalous magnetic moment: $a_\mu = (g_\mu - 2)/2$

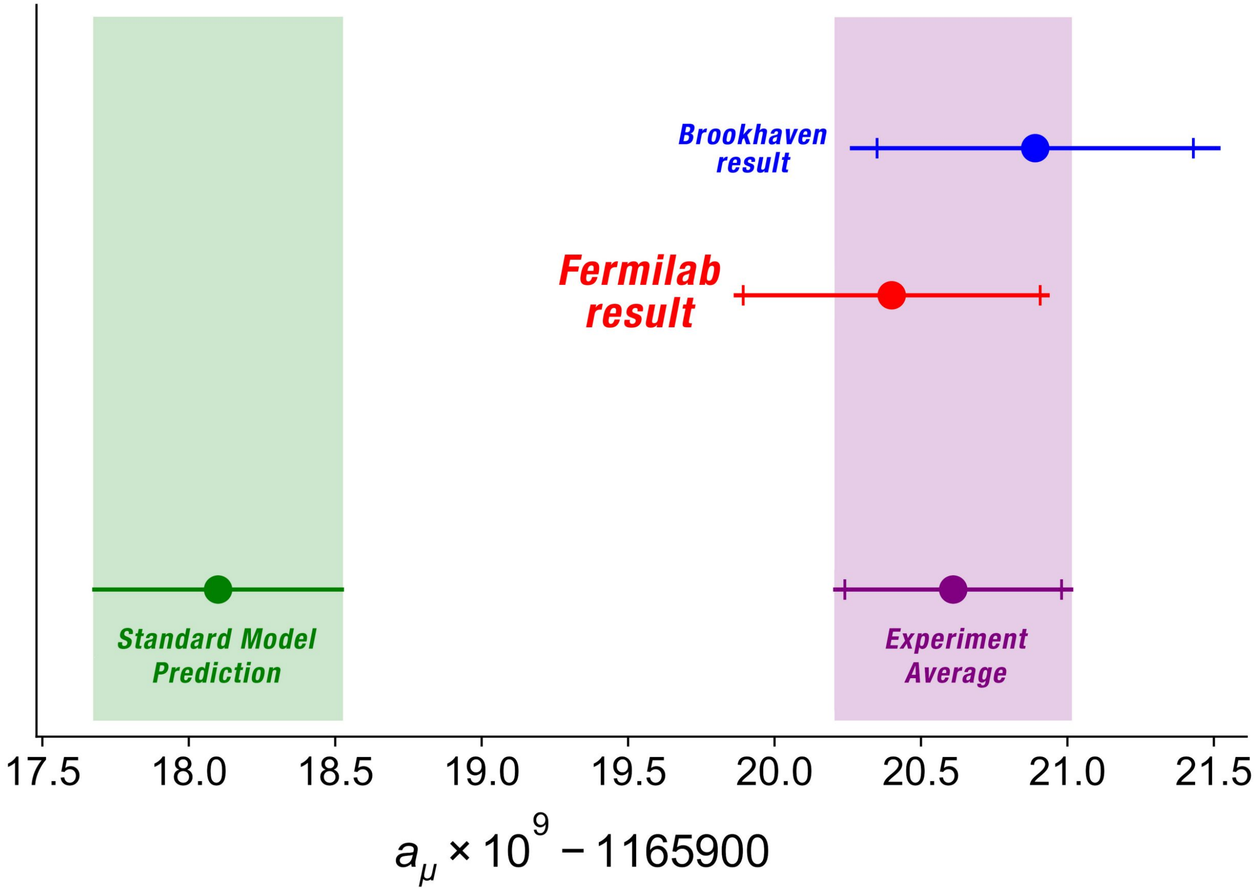
$a_\mu^{\text{exp}} = 116\,592\,089\,(63) \times 10^{-11}$		[Brookhaven '06]
$a_\mu^{\text{QED}} = 116\,584\,719$	$\times 10^{-11}$	[QED: 5 loops]
$a_\mu^{\text{EW}} = 154$	$(1) \times 10^{-11}$	[W, Z, H: 2 loops]
$a_\mu^{\text{had}} = 6\,937$	$(43) \times 10^{-11}$	[$e^+e^- \to$ had]
$\mathbf{a_\mu^{\text{SM}} = 116\,591\,810\,(43) \times 10^{-11}}$		**[Theory Initiative '20]**

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 279\,(76) \times 10^{-11}$$

$$3.7\sigma\ !$$

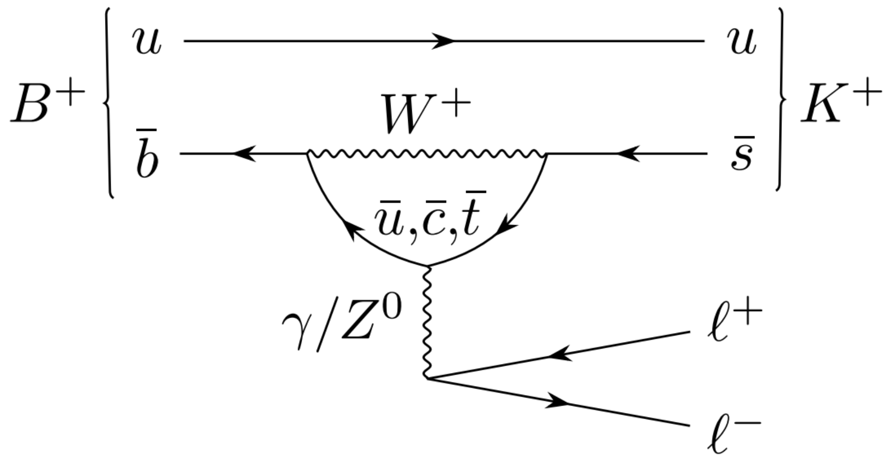
- New Muon $g-2$ Experiment at Fermilab. First results! [2104.03281]

$$a_\mu^{\text{exp}} = 116\,592\,061\,(41) \times 10^{-11}$$

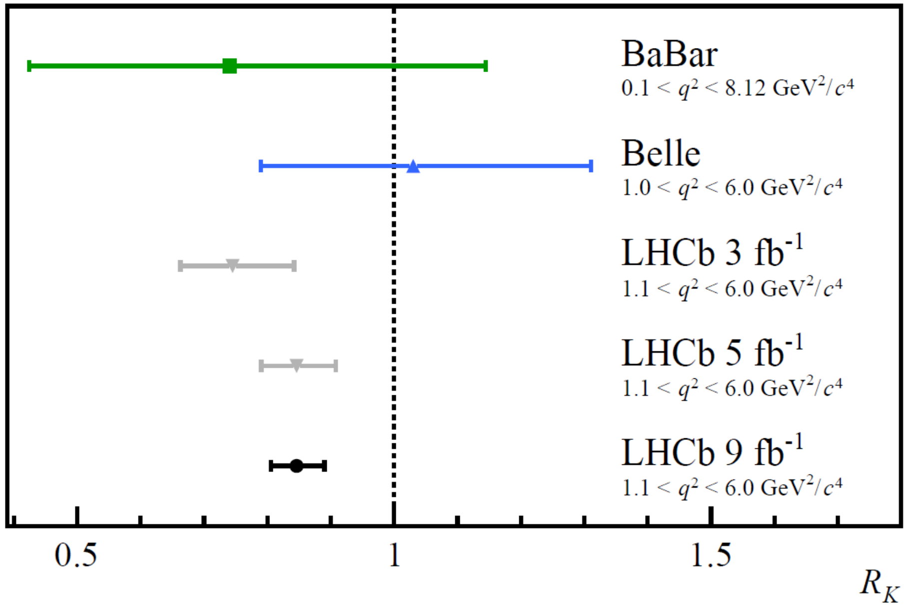


$\Rightarrow 4.2\sigma$!!

- Test of lepton universality in b decays at LHCb [2103.11769]



$$R_K = \frac{\mathcal{B}(B^+ \to K^+\mu^+\mu^-)}{\mathcal{B}(B^+ \to K^+e^+e^-)} = 0.846^{+0.044}_{-0.041} \quad \Rightarrow$$



Global fits

- Fit parameters from a list of observables: find the $\chi^2_{\rm min}$ varying some of them
  [$n_{\rm dof}$ = # of observables minus # of parameters]

  http://gfitter.desy.de [1803.01853]

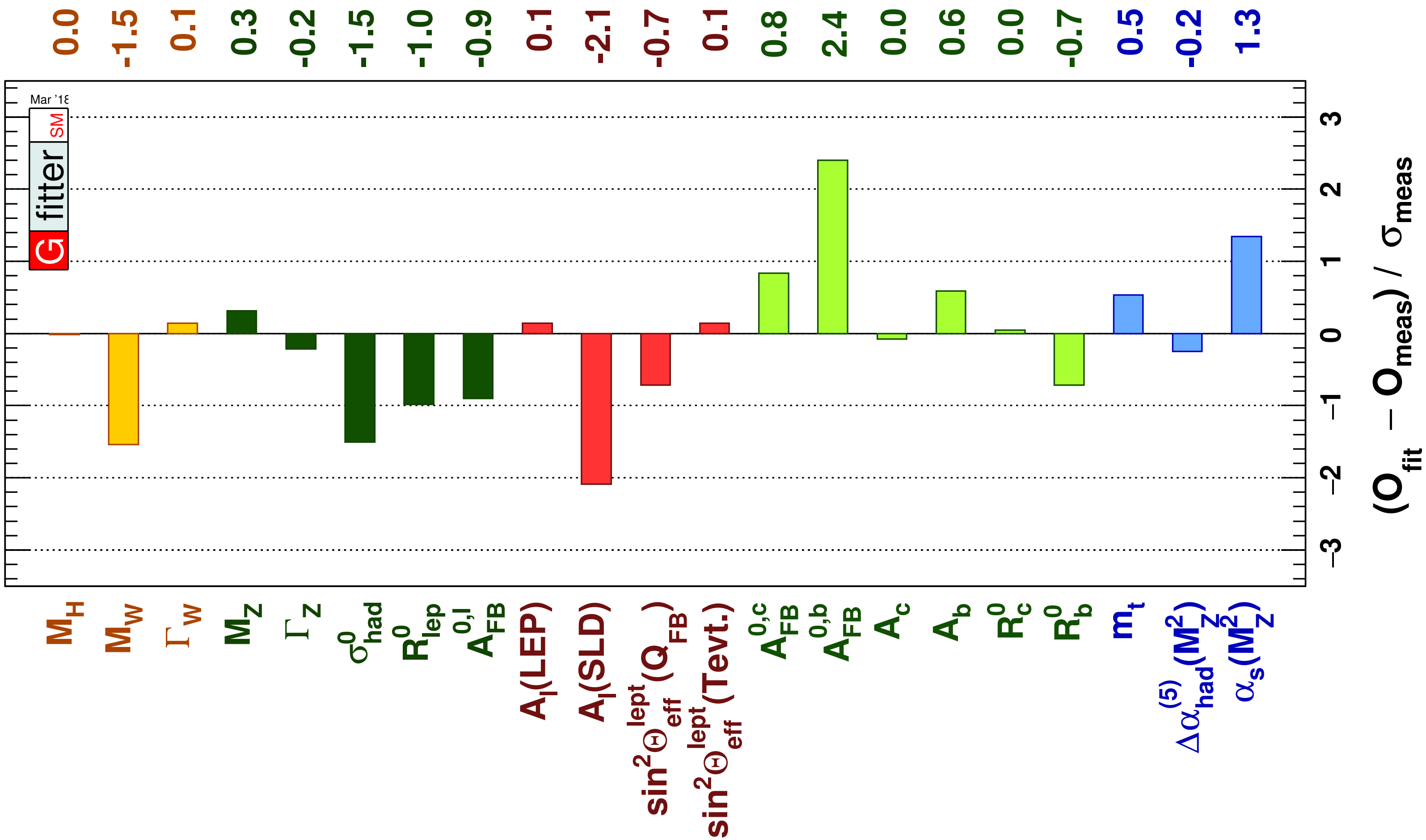
$n_{\rm dof}$	$\chi^2_{\rm min}$	p-value
15	18.6	0.23

(goodness of fit)

Parameter	Input value	Free in fit
M_H [GeV]	125.1 ± 0.2	Yes
M_W [GeV]	80.379 ± 0.013	–
Γ_W [GeV]	2.085 ± 0.042	–
M_Z [GeV]	91.1875 ± 0.0021	Yes
Γ_Z [GeV]	2.4952 ± 0.0023	–
$\sigma^0_{\rm had}$ [nb]	41.540 ± 0.037	–
R^0_ℓ	20.767 ± 0.025	–
$A^{0,\ell}_{\rm FB}$	0.0171 ± 0.0010	–
A_ℓ $^{(\star)}$	0.1499 ± 0.0018	–
$\sin^2\theta^\ell_{\rm eff}(Q_{\rm FB})$	0.2324 ± 0.0012	–
$\sin^2\theta^\ell_{\rm eff}$(Tevt.)	0.23148 ± 0.00033	–
A_c	0.670 ± 0.027	–
A_b	0.923 ± 0.020	–
$A^{0,c}_{\rm FB}$	0.0707 ± 0.0035	–
$A^{0,b}_{\rm FB}$	0.0992 ± 0.0016	–
R^0_c	0.1721 ± 0.0030	–
R^0_b	0.21629 ± 0.00066	–
$\overline{m}_c$ [GeV]	$1.27^{+0.07}_{-0.11}$	Yes
$\overline{m}_b$ [GeV]	$4.20^{+0.17}_{-0.07}$	Yes
m_t [GeV]$^{(\nabla)}$	172.47 ± 0.68	Yes
$\Delta\alpha^{(5)}_{\rm had}(M^2_Z)$ $^{(\dagger\triangle)}$	2760 ± 9	Yes
$\alpha_s(M^2_Z)$	–	Yes

Global fits (Comparisons)

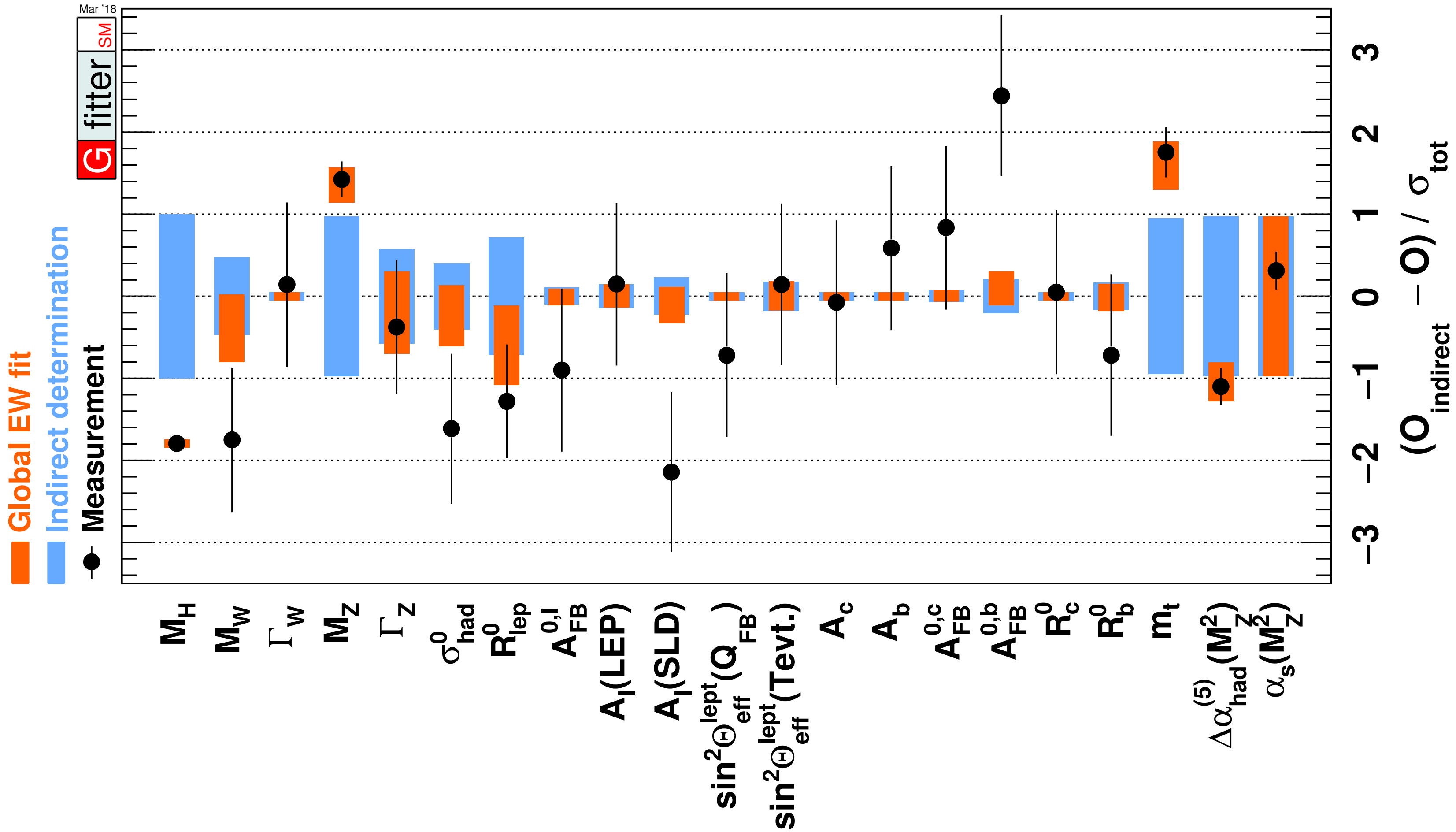
- Compare direct measurements of the observables with fit values:



$\Rightarrow$ some tensions (none above 3σ): $A_\ell(\text{SLD})$, $A^b_{FB}(\text{LEP})$, R_b, ...

Global fits (Comparisons)

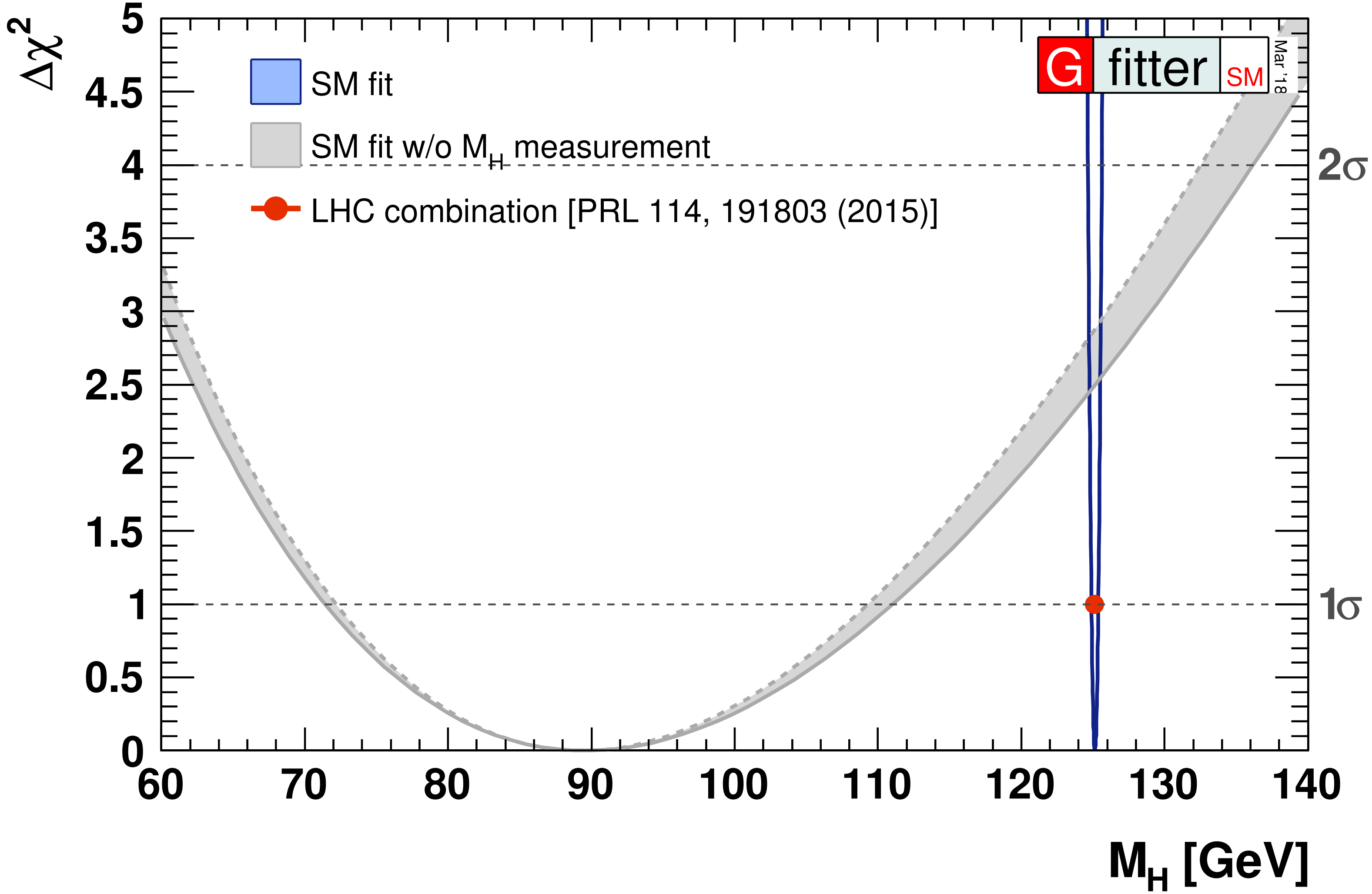
- Compare indirect determinations with fit values (error bars are direct measmts.):



[indirect determination means fit without using constraint from given direct measurement]

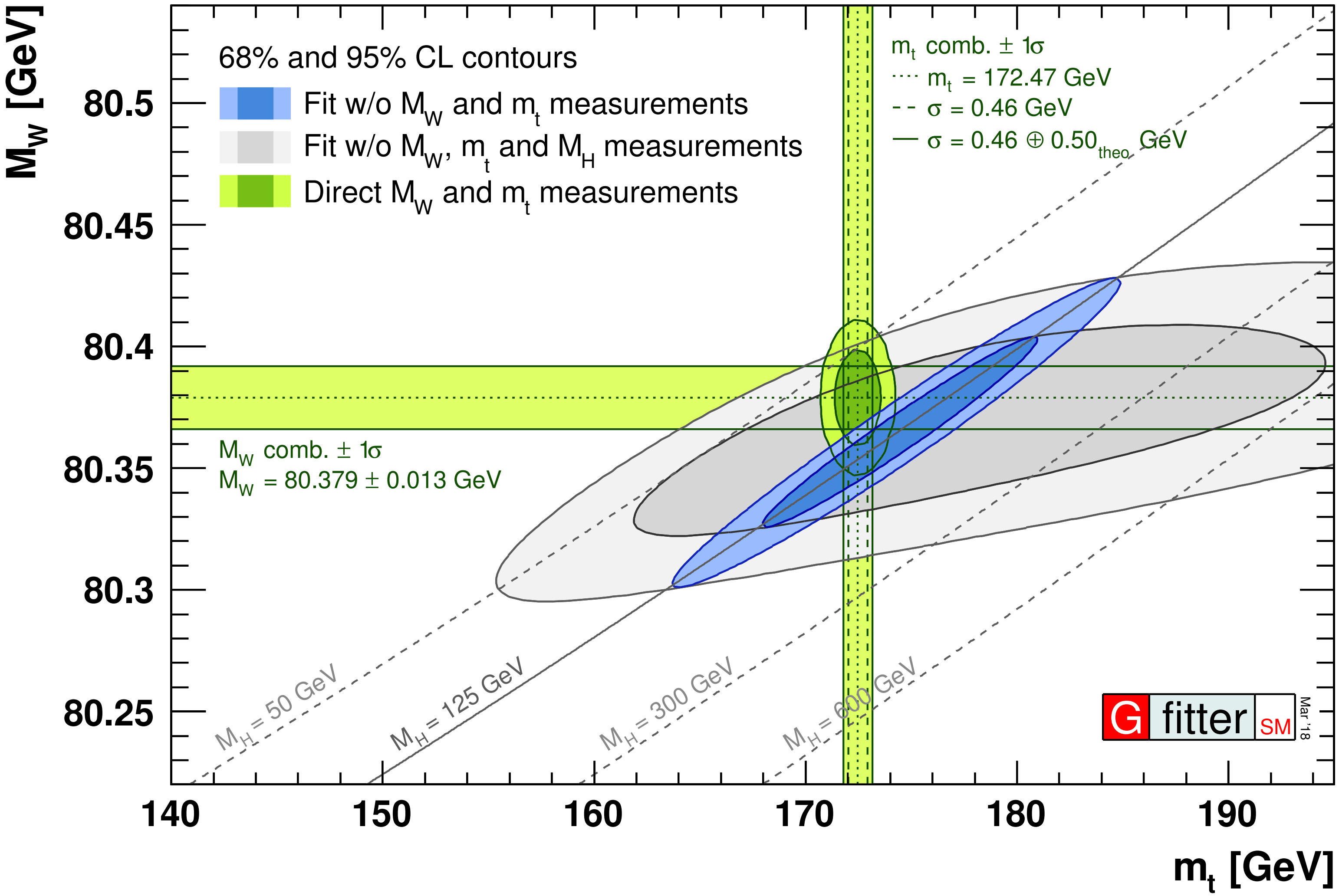
Global fits (Conclusions)

⇒ Fits prefer a somewhat lighter Higgs:



Global fits (Conclusions)

⇒ In general, impressive consistency of the SM, e.g.:



Neutrinos

Neutrinos | **Dirac vs Majorana**

- General mass terms compatible with $SU_L(2) \times U_Y(1)$ gauge symmetry (1 family):

$$-\mathcal{L}_m = \frac{1}{2} \begin{pmatrix} \overline{\nu_L^c} & \overline{\nu_R} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.}$$

  - 1 Dirac = light ν $\quad [m_R = 0]$

$$-\mathcal{L}_m = m_D \overline{\nu_R} \nu_L + \text{h.c.}$$
$$m_\nu = m_D \quad (4 \text{ dof})$$

  - 2 Majorana = light ν + heavy N $\quad [m_D \ll m_R]$ $\quad$ (seesaw) $\quad$ LNV

$$\begin{aligned} -\mathcal{L}_m &= m_D \overline{\nu_R} \nu_L + \frac{1}{2} m_R \overline{\nu_R} \nu_R^c + \text{h.c.} \\ &= \frac{1}{2} m_\nu \overline{\nu_L^c} \nu_L + \frac{1}{2} m_N \overline{N_R} N_R^c + \text{h.c.} \end{aligned}$$
$$m_\nu = \frac{m_D^2}{m_R} \quad (2 \text{ dof}) \,, \quad m_N = m_R \quad (2 \text{ dof})$$

Neutrinos | Dirac vs Majorana

- Neutrinos at low energies (3 families of light neutrinos):

  – 3 light Dirac: $\nu_\alpha = \nu_{\alpha L} + \nu_{\alpha R}$ (like quarks)

  $$\nu_{\alpha L} = U_{\alpha i}\nu_{iL}\ , \quad \nu_{\alpha R} = V_{\alpha i}\nu_{iR}\ , \quad V^\dagger M_\nu U = \mathcal{M}_{\rm diag}$$

  – 3 light Majorana: $\nu_\alpha = \nu_{\alpha L} + \nu^c_{\alpha L}$ $(\nu_{\alpha R} = \nu^c_{\alpha L})$

  $$\nu_{\alpha L} = U_{\alpha i}\nu_{iL}\ , \quad \nu_{\alpha R} = U^*_{\alpha i}\nu_{iR}\ , \quad U^T M_\nu U = \mathcal{M}_{\rm diag} \quad \Leftarrow M_\nu = M_D M_R^{-1} M_D^T$$

  ▷ In both cases, mixing matrix shows up in Charged Currents:

  $$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \sum_{\alpha i} \overline{\ell_{L\alpha}} \gamma^\mu U_{\alpha i}\nu_{iL} W_\mu + \text{h.c.}$$

  (basis where charged leptons are diagonal)

  ▷ In Majorana case there are also Neutral Current interactions with the Z:

  $$\mathcal{L}^Z_{\rm NC} \supset \frac{g}{4c_W} \sum_{ij} \overline{\nu}_j \gamma^\mu (P_L C_{ij} - P_R C^*_{ij}) \nu_j Z_\mu, \quad C_{ij} = \sum_{\alpha=1}^{3} U^*_{\alpha i} U_{\alpha j}$$

Neutrinos — Dirac vs Majorana

- About (light) neutrino flavour mixing (3 families):

  – Dirac: All phases absorbed by field redefinitions but one phase remains.
  In the standard parameterization

$$U_{\text{Dirac}} = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{i\delta} & c_{23}\, c_{13} \end{pmatrix}$$

  – Majorana: Since $\nu_{\alpha R} = \nu^c_{\alpha L}$ two extra phases remain

$$U_{\text{Majorana}} = U_{\text{Dirac}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}} & 0 \\ 0 & 0 & e^{i\alpha_{31}} \end{pmatrix}$$

- **The solar neutrino problem**
  - The Sun produces ν_e's whose flux can be calculated using solar models
  - The flux of ν_e measured on the Earth in all expts reduced by a factor 0.3–0.5

  $\Rightarrow$ Explained by oscillations $\nu_e \to \nu_{\mu,\tau}$

- **The atmospheric neutrino problem**
  - Cosmic rays produce π's in the atmosphere that should give a flux of ν_μ's and ν_e's in (2:1)

$$\pi \to \bar{\nu}_\mu \mu \to \bar{\nu}_\mu \nu_\mu \bar{\nu}_e \, e$$

  - The observed flux of ν_μ is largely reduced

  $\Rightarrow$ Explained by oscillations $\nu_\mu \to \nu_\tau$

Neutrinos | Oscillations in vacuum

- Case of two family mixing:

  Define $|\nu_e\rangle$ and $|\nu_\mu\rangle$ the flavour eigenstates producing e and μ respectively in a CC

  Assume:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

  where $|\nu_1\rangle$ and $|\nu_2\rangle$ are the energy eigenstates. Their time evolution is given by

$$\begin{aligned} |\nu_e, t\rangle &= \cos\theta\, \mathrm{e}^{-\mathrm{i}E_1 t}\, |\nu_1\rangle + \sin\theta\, \mathrm{e}^{-\mathrm{i}E_2 t}\, |\nu_2\rangle \\ |\nu_\mu, t\rangle &= -\sin\theta\, \mathrm{e}^{-\mathrm{i}E_1 t}\, |\nu_1\rangle + \cos\theta\, \mathrm{e}^{-\mathrm{i}E_2 t}\, |\nu_2\rangle \end{aligned}$$

  Then the probability of oscillation from ν_e to ν_μ after a time t is

$$P(\nu_e \to \nu_\mu; t) = |\langle \nu_\mu | \nu_e, t\rangle|^2 = \sin^2(2\theta) \sin^2\left(\frac{\Delta E}{2} t\right) , \quad \Delta E = E_2 - E_1$$

Neutrinos | Oscillations in vacuum

- Case of two family mixing:

  For ultrarelativistic neutrinos with definite momentum ($p \gg m_i$)

$$E_i = \sqrt{m_i^2 + p^2} \approx p + \frac{m_i^2}{2p}\,, \quad L \approx t\,, \quad p \approx E$$

$$P(\nu_e \to \nu_\mu; L) = \sin^2(2\theta)\sin^2\left(\pi\frac{L}{L_{\rm osc}}\right)$$

  where the oscillation length is

$$L_{\rm osc} = \pi\frac{4E}{\Delta m^2} = \frac{\pi}{1.27}\frac{E/\text{GeV}}{\Delta m^2/\text{eV}^2}\ \text{km}\,, \quad \Delta m^2 = |m_2^2 - m_1^2|$$

  - Oscillations are only sensitive to (squared) mass differences
  - If $L \gg L_{\rm osc}$ (too fast oscillations) then average: $\langle P(\nu_e \to \nu_\mu; t)\rangle = \frac{1}{2}\sin^2(2\theta)$
  - For subtleties on the theory of neutrino oscillations see E. Akhmedov: http://arxiv.org/pdf/0706.1216.pdf, http://arxiv.org/pdf/0905.1903.pdf

Neutrinos | Oscillations in vacuum

- Case of three family mixing: $|\nu_\alpha\rangle = U_{\alpha i}\,|\nu_i\rangle$ (see next slide):

$$P(\nu_\alpha \to \nu_\beta; L) = |\,\langle \nu_\beta | \nu_\alpha, L\rangle\,|^2 = \left| \sum_{ij} \langle \nu_j |\, U^*_{\beta j} \mathrm{e}^{-\mathrm{i}E_i L} U_{\alpha i}\,|\nu_i\rangle \right|^2 = \left| \sum_i U^*_{\beta i} U_{\alpha i} \mathrm{e}^{-\mathrm{i}m_i^2 L/(2E)} \right|^2$$

$$= \sum_{ij} U_{\beta j} U^*_{\alpha j} U_{\beta i} U^*_{\alpha i} \mathrm{e}^{-\mathrm{i}\Delta m^2_{ij} L/(2E)}$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \mathrm{Re}(U_{\beta j} U^*_{\alpha j} U^*_{\beta i} U_{\alpha i}) \sin^2 \frac{\Delta m^2_{ij} L}{4E}$$

$$+ 2 \sum_{i>j} \mathrm{Im}(U_{\beta j} U^*_{\alpha j} U^*_{\beta i} U_{\alpha i}) \sin \frac{\Delta m^2_{ij} L}{2E}\,, \qquad \Delta m^2_{ij} = m_i^2 - m_j^2$$

  - Majorana phases are irrelevant
  - $P(\nu_\beta \to \nu_\alpha; U) = P(\nu_\alpha \to \nu_\beta; U^*)$ and CPT $\Rightarrow P(\bar\nu_\alpha \to \bar\nu_\beta) = P(\nu_\beta \to \nu_\alpha)$
    Therefore $P(\bar\nu_\alpha \to \bar\nu_\beta; U) = P(\nu_\alpha \to \nu_\beta; U^*)$
    $\Rightarrow$ if CP conserved ($U = U^*$) then $P(\bar\nu_\alpha \to \bar\nu_\beta) = P(\nu_\alpha \to \nu_\beta)$

Neutrinos — Oscillations in vacuum

▷ Use:

$$\langle \nu_j | \nu_i \rangle = \delta_{ij}, \quad \Delta m_{ij}^2 \equiv m_i^2 - m_j^2, \quad \sum_{ij} = \sum_{i=j} + \sum_{i>j} + \sum_{i<j}$$

▷ Use the unitarity of U in:

$$\begin{aligned}\sum_{i=j} (U_{\beta j} U_{\alpha j}^* U_{\beta i}^* U_{\alpha i}) &= \sum_j (U_{\beta j} U_{\alpha j}^*) \sum_i (U_{\beta i}^* U_{\alpha i}) - \sum_{i \neq j} (U_{\beta j} U_{\alpha j}^* U_{\beta i}^* U_{\alpha i}) \\ &= \delta_{\alpha\beta} - 2 \sum_{i>j} \mathrm{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta i}^* U_{\alpha i}) \ .\end{aligned}$$

▷ Combine:

$$\left(\sum_{i>j} + \sum_{i<j} \right) [\cdots] = \sum_{i>j} \left[2\mathrm{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta i}^* U_{\alpha i}) \cos \frac{\Delta m_{ij}^2 L}{2E} + 2\mathrm{Im}(U_{\beta j} U_{\alpha j}^* U_{\beta i}^* U_{\alpha i}) \sin \frac{\Delta m_{ij}^2 L}{2E} \right]$$

▷ And substitute: $\quad 1 - \cos \dfrac{\Delta m_{ij}^2 L}{2E} = 2 \sin^2 \dfrac{\Delta m_{ij}^2 L}{4E}$

- Comments:
  - Nature seems to have chosen 3 flavours with two very different squared mass differences: $\Delta m^2_\odot \equiv \Delta m^2_{21} \ll \Delta m^2_{\rm atm} \equiv \Delta m^2_{31} \simeq \Delta m^2_{32}$. Then

$$P(\nu_\alpha \to \nu_\beta \neq \nu_\alpha) \approx P^{\rm short}_{\alpha\beta} + P^{\rm long}_{\alpha\beta}$$

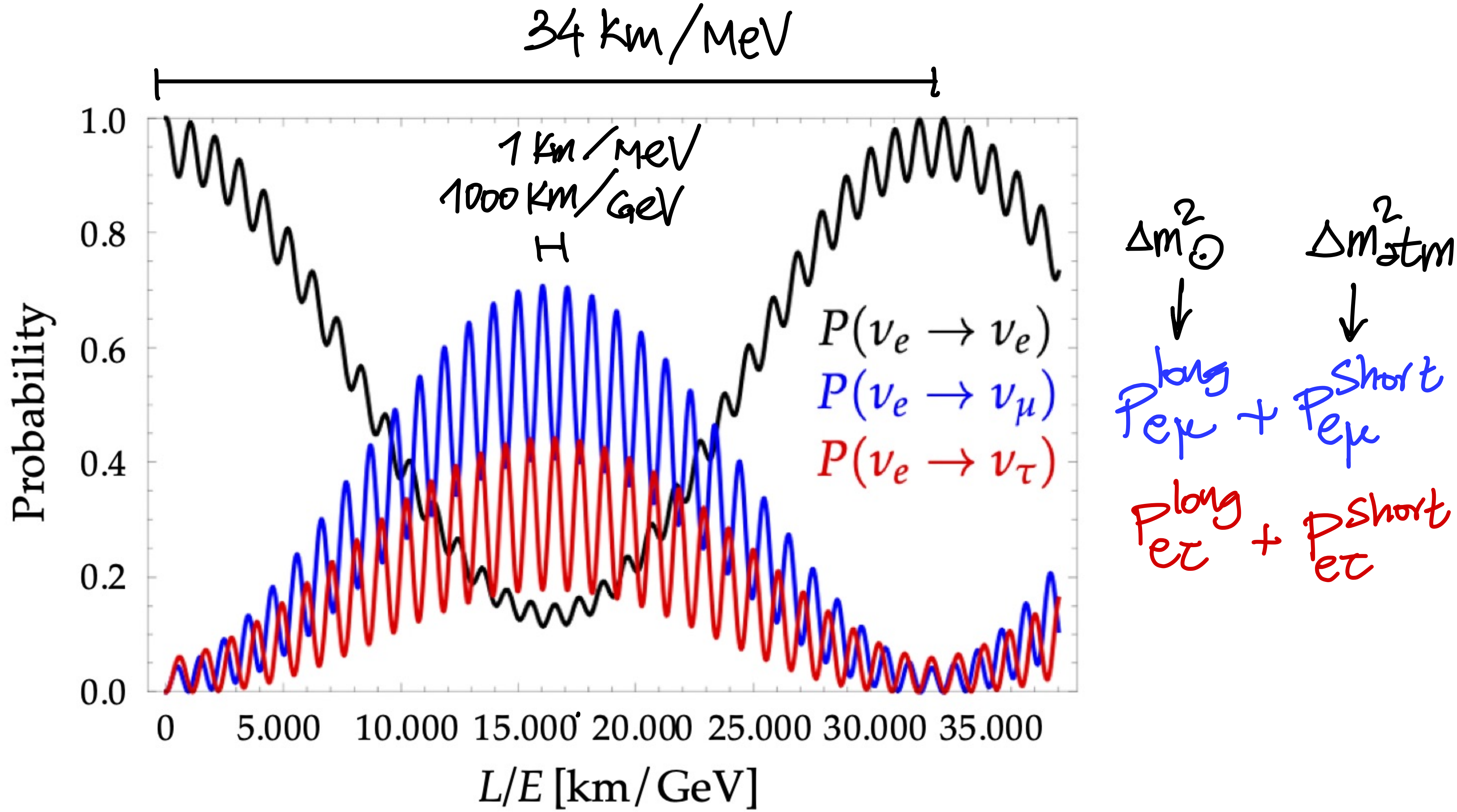
  with

$$P^{\rm short}_{\alpha\beta} = 4U^2_{\beta 3}U^2_{\alpha 3}\sin^2[1.27\Delta m^2_{\rm atm} L/E]$$
$$P^{\rm long}_{\alpha\beta} = -4U_{\beta 2}U_{\alpha 2}U_{\beta 1}U_{\alpha 1}\sin^2[1.27\Delta m^2_\odot L/E]$$

  where we have neglected CPV effects and used that unitarity of U:

$$U_{\beta 1}U_{\alpha 1} + U_{\beta 2}U_{\alpha 2} = -U_{\beta 3}U_{\alpha 3}$$

  - Selecting the appropriate L/E the oscillation is sensitive to the short or the long component only (decoupled) and can be effectively treated as a two-flavour mixing


34 km/MeV
1 km/MeV
1000 km/GeV
1.0
0.8
0.6
0.4
0.2
0.0
Probability
0
5.000
10.000
15.000
20.000
25.000
30.000
35.000
L/E [km/GeV]
P(νe → νe)
P(νe → νμ)
P(νe → ντ)
Δm²☉
Δm²atm
P^long_eμ + P^short_eμ
P^long_eτ + P^short_eτ

– Values of Δm^2 that can be *explored* in different experiments.
SBL (LBL) stands for Short (Long) Baseline

Experiment	L [m]	E [MeV]	Δm^2 [eV^2]
Reactors SBL	10^2	1	10^{-2}
Reactors LBL	10^3	1	10^{-3}
	10^5	1	10^{-5}
Accelerators SBL	10^3	10^3	1
Accelerators LBL	10^6	10^3	10^{-3}
Atmospheric	10^4 – 10^7	10^3	10^{-1} – 10^{-4}
Solar *	10^{11}	1	10^{-11}

* Since $\Delta m^2_\odot \sim 10^{-5} \gg 10^{-11}$ eV^2 $\Rightarrow L \gg L_{\rm osc} \sim 100$ km and $P \sim \frac{1}{2}\sin^2(2\theta)$

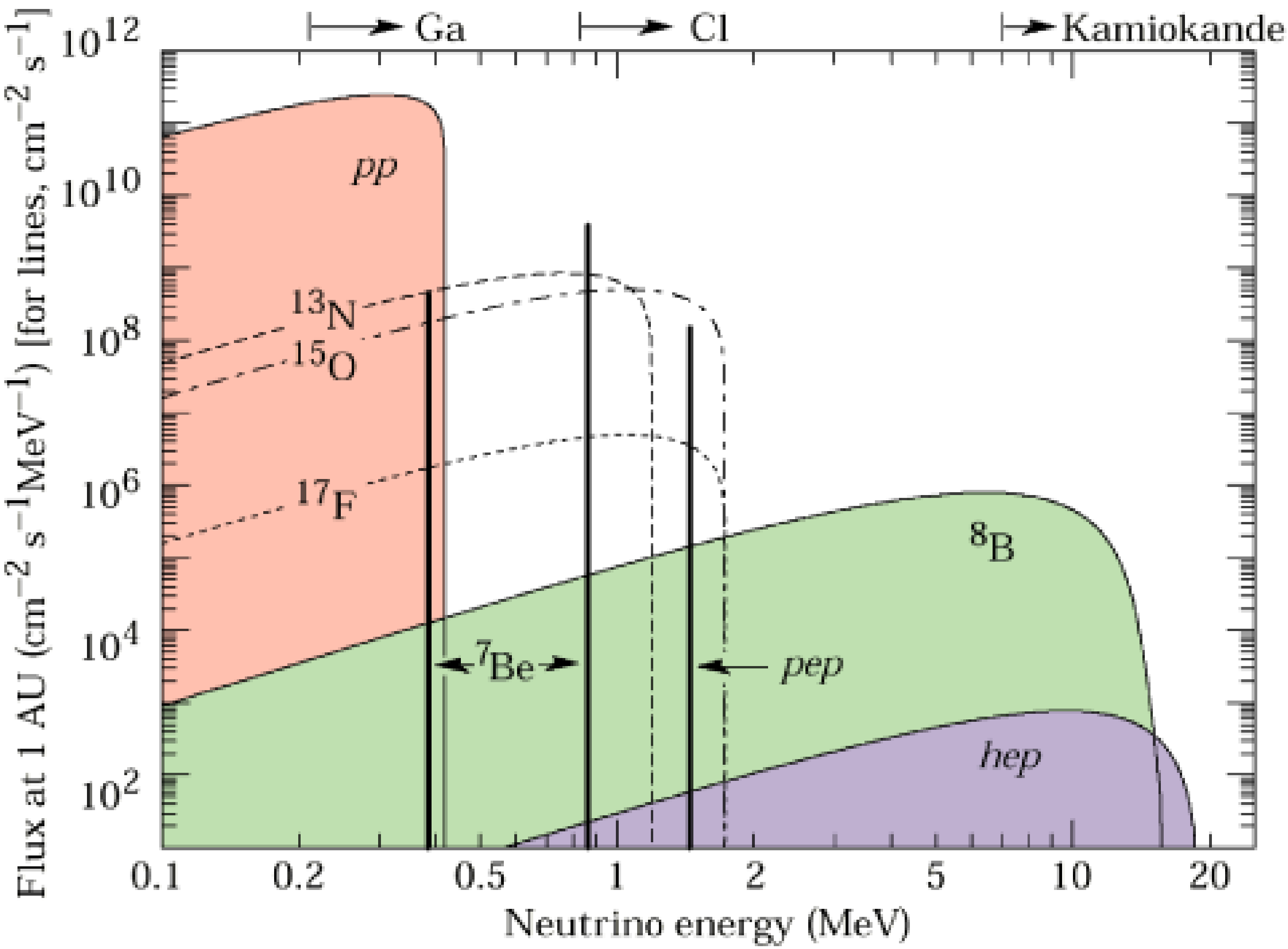
Mikheyev-Smirnov-Wolfenstein (MSW) effect

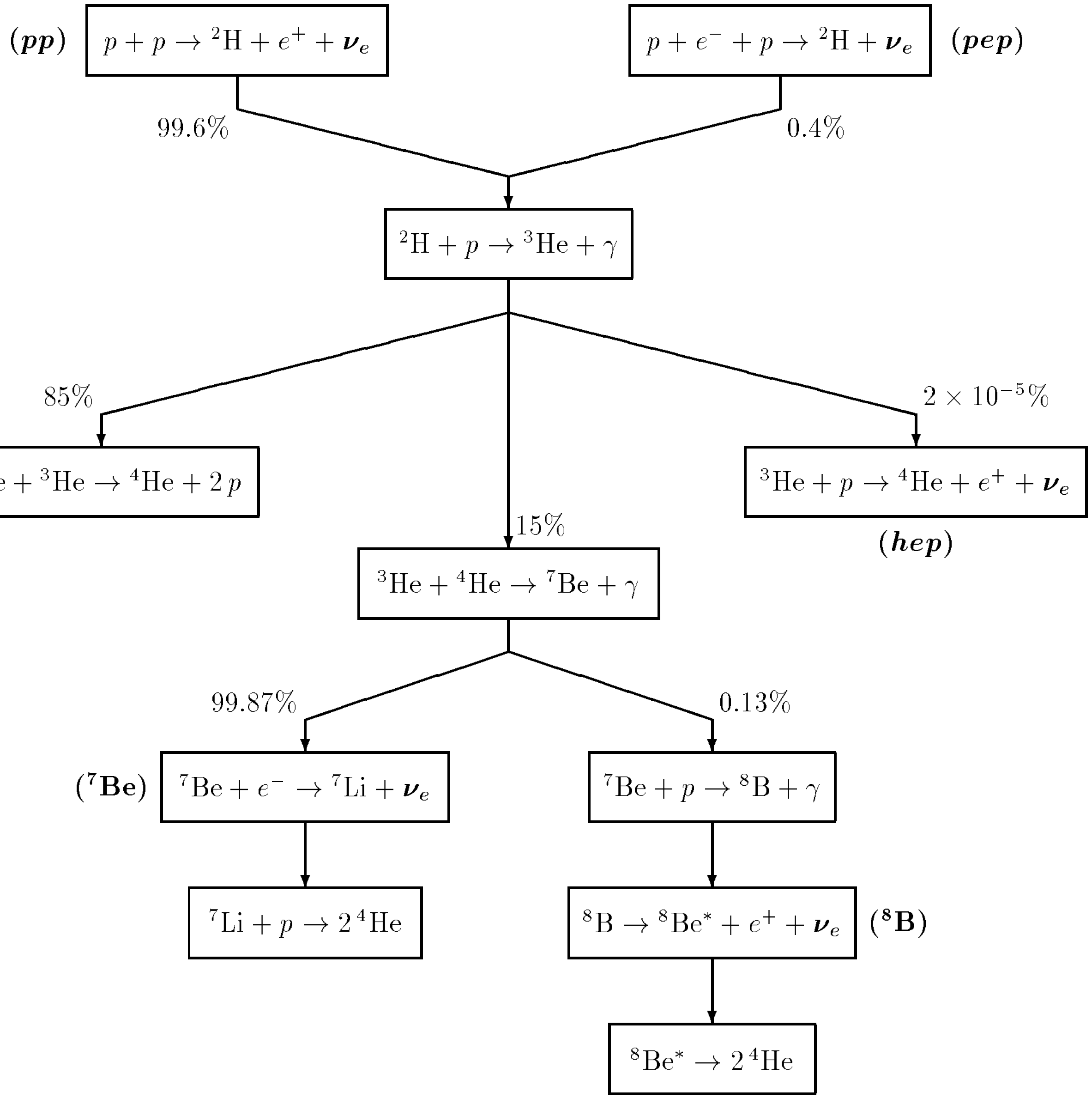
– Electron neutrinos travelling through dense matter (Sun, Earth, supernovas) suffer Charged Current interactions enhancing their oscillation probability

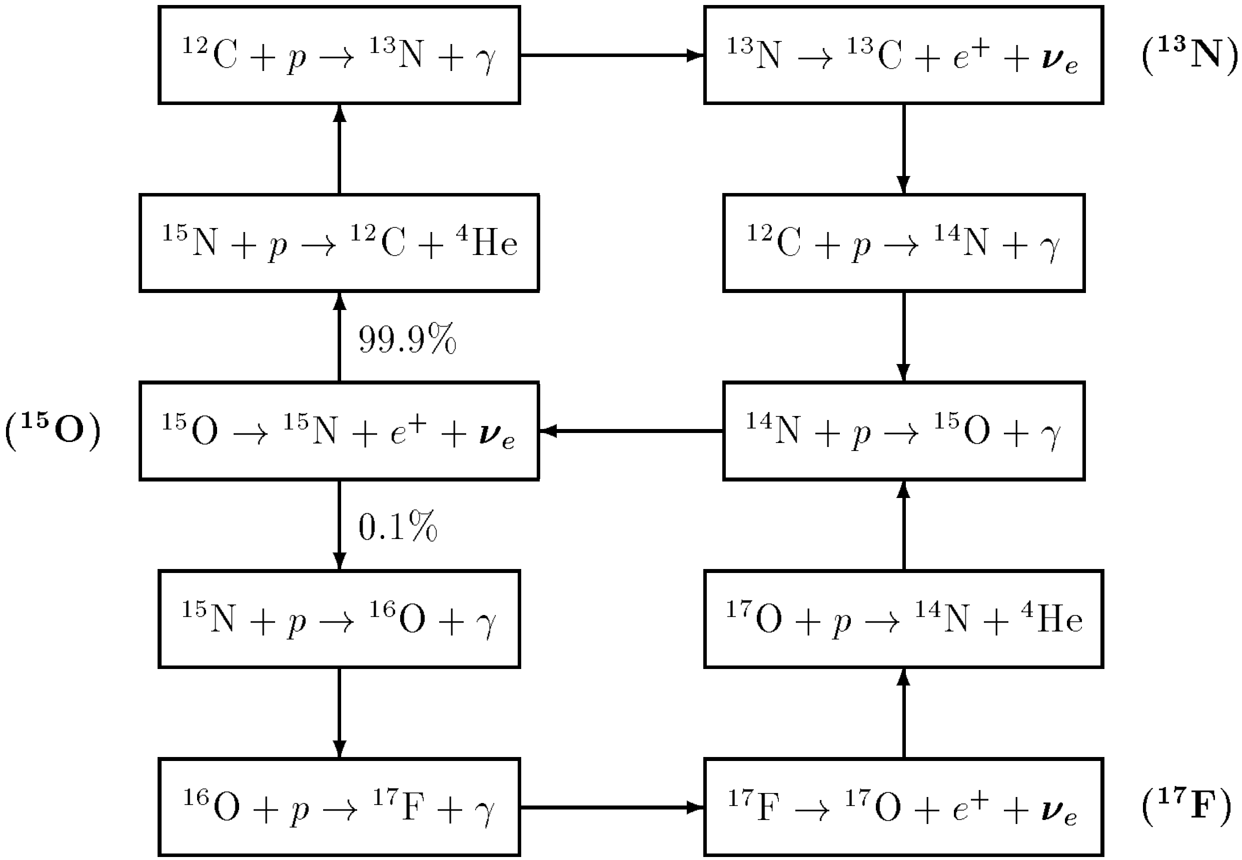
– This effect is important for high energy solar neutrinos, above 2 MeV, and it transforms most of the ν_e into ν_μ when leaving the Sun

We skip their description here

- Solar neutrino fluxes [at $L = 1$ AU $\sim 10^{11}$ km] $E \sim$ MeV



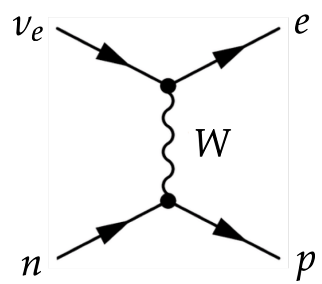

(**pp**)
$p + p \to {}^2\mathrm{H} + e^+ + \nu_e$
$p + e^- + p \to {}^2\mathrm{H} + \nu_e$
(**pep**)
99.6%
0.4%
${}^2\mathrm{H} + p \to {}^3\mathrm{He} + \gamma$
85%
2×10^{-5}%
${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + 2\,p$
${}^3\mathrm{He} + p \to {}^4\mathrm{He} + e^+ + \nu_e$
15%
(**hep**)
${}^3\mathrm{He} + {}^4\mathrm{He} \to {}^7\mathrm{Be} + \gamma$
99.87%
0.13%
(**⁷Be**)
${}^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu_e$
${}^7\mathrm{Be} + p \to {}^8\mathrm{B} + \gamma$
${}^7\mathrm{Li} + p \to 2\,{}^4\mathrm{He}$
${}^8\mathrm{B} \to {}^8\mathrm{Be}^* + e^+ + \nu_e$
(**⁸B**)
${}^8\mathrm{Be}^* \to 2\,{}^4\mathrm{He}$


$^{12}\mathrm{C} + p \to {}^{13}\mathrm{N} + \gamma$
$^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \boldsymbol{\nu_e}$
$(^{13}\mathbf{N})$
$^{15}\mathrm{N} + p \to {}^{12}\mathrm{C} + {}^{4}\mathrm{He}$
$^{12}\mathrm{C} + p \to {}^{14}\mathrm{N} + \gamma$
99.9%
$(^{15}\mathbf{O})$
$^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \boldsymbol{\nu_e}$
$^{14}\mathrm{N} + p \to {}^{15}\mathrm{O} + \gamma$
0.1%
$^{15}\mathrm{N} + p \to {}^{16}\mathrm{O} + \gamma$
$^{17}\mathrm{O} + p \to {}^{14}\mathrm{N} + {}^{4}\mathrm{He}$
$^{16}\mathrm{O} + p \to {}^{17}\mathrm{F} + \gamma$
$^{17}\mathrm{F} \to {}^{17}\mathrm{O} + e^+ + \boldsymbol{\nu_e}$
$(^{17}\mathbf{F})$

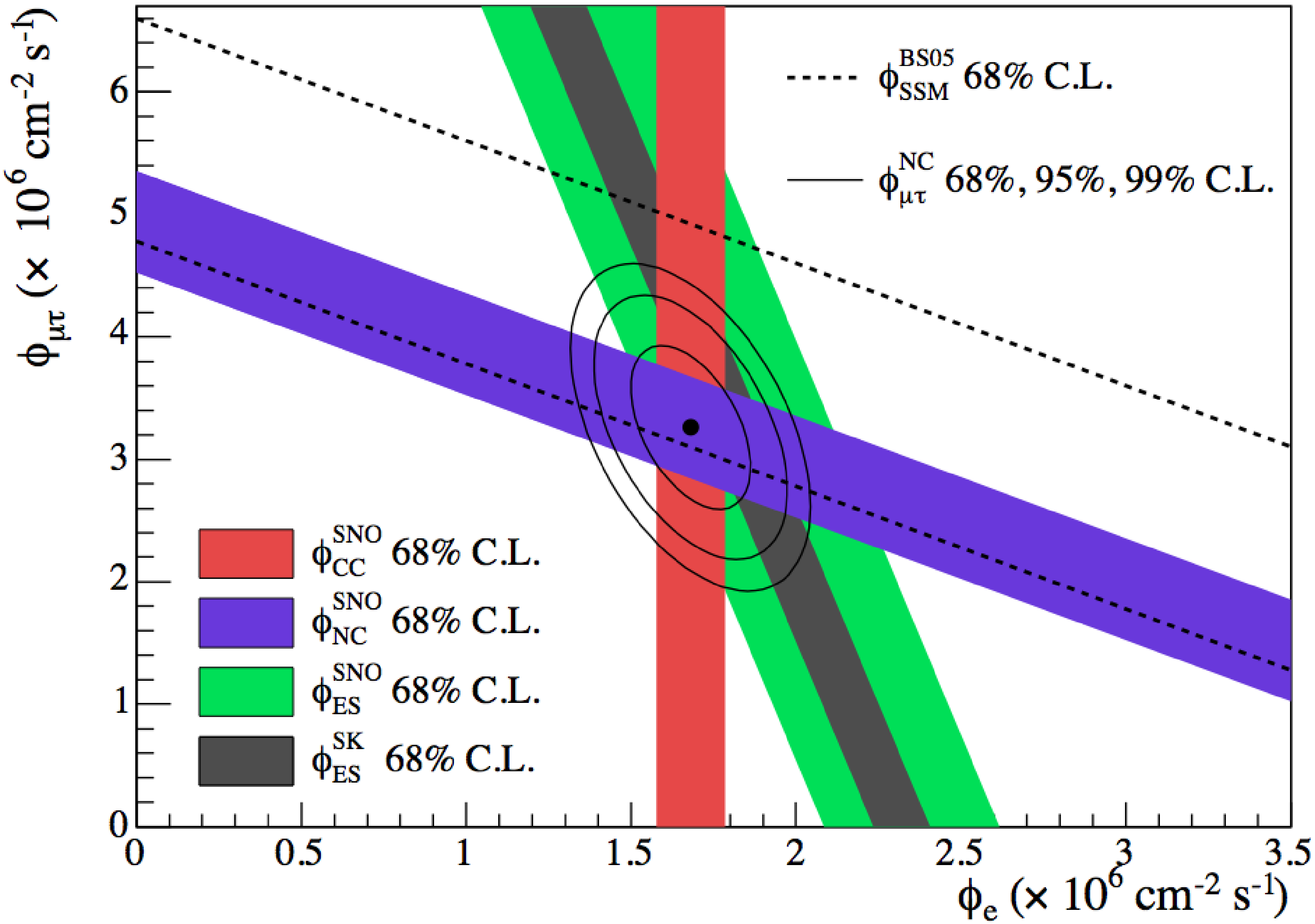
- Experiments

Experiment	Reaction	Energy threshold [MeV]
Homestake	$\nu_e\,{}^{37}\text{Cl} \to e\,{}^{37}\text{Ar}$	0.814
SAGE, Gallex/GNO	$\nu_e\,{}^{71}\text{Ga} \to e\,{}^{71}\text{Ge}$	0.233
SuperKamiokande	$\nu_{e,x}\,e \to \nu_{e,x}\,e$	5.5
SNO	ES: $\nu_{e,x}\,e \to \nu_{e,x}\,e$ CC: $\nu_e\,D \to ppe$ NC: $\nu_x\,D \to \nu_x\,pn$	5.5

	Cl	Ar	Ga	Ge
Z	17	18	31	32

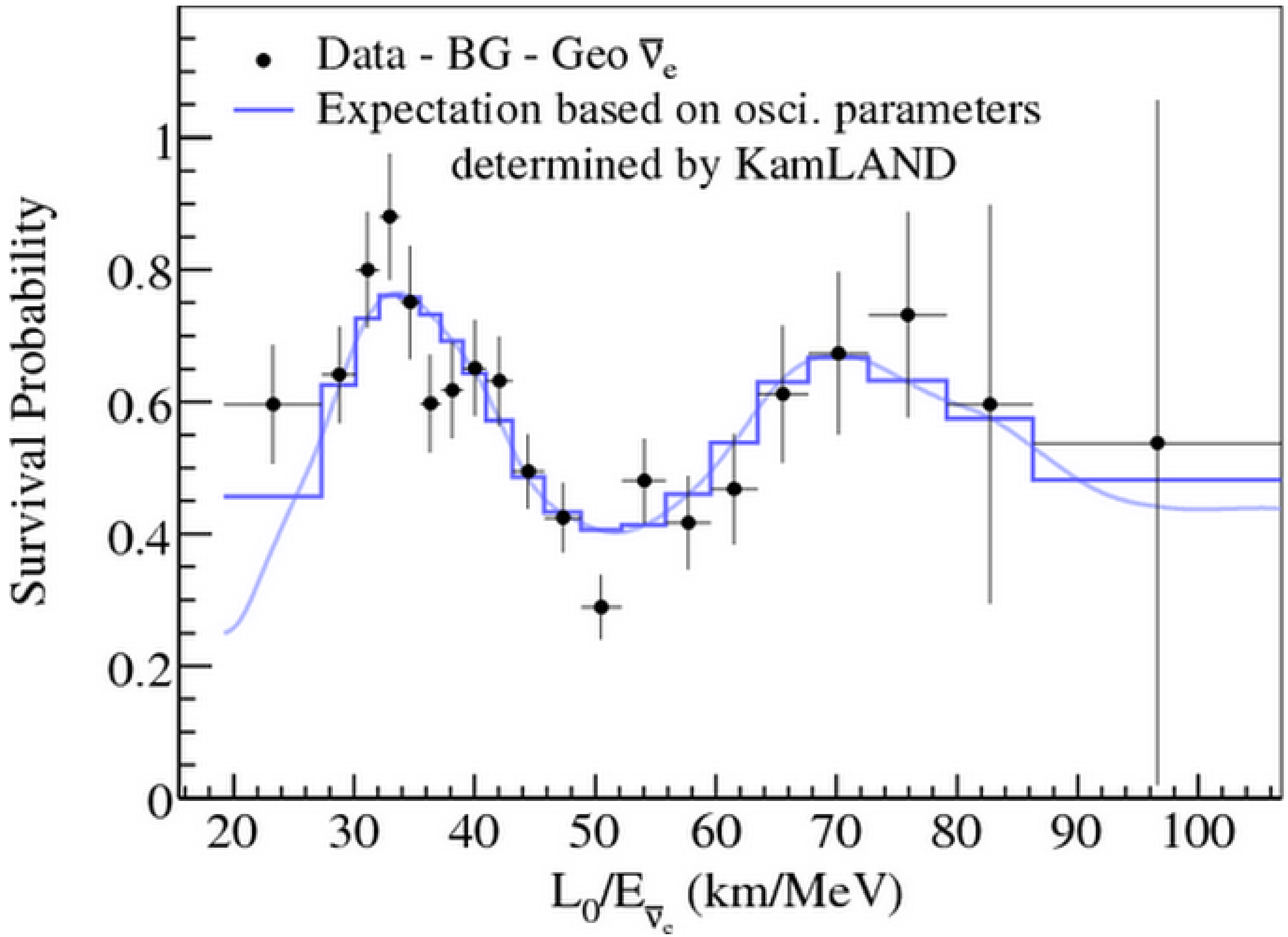


- SuperKamiokande + SNO

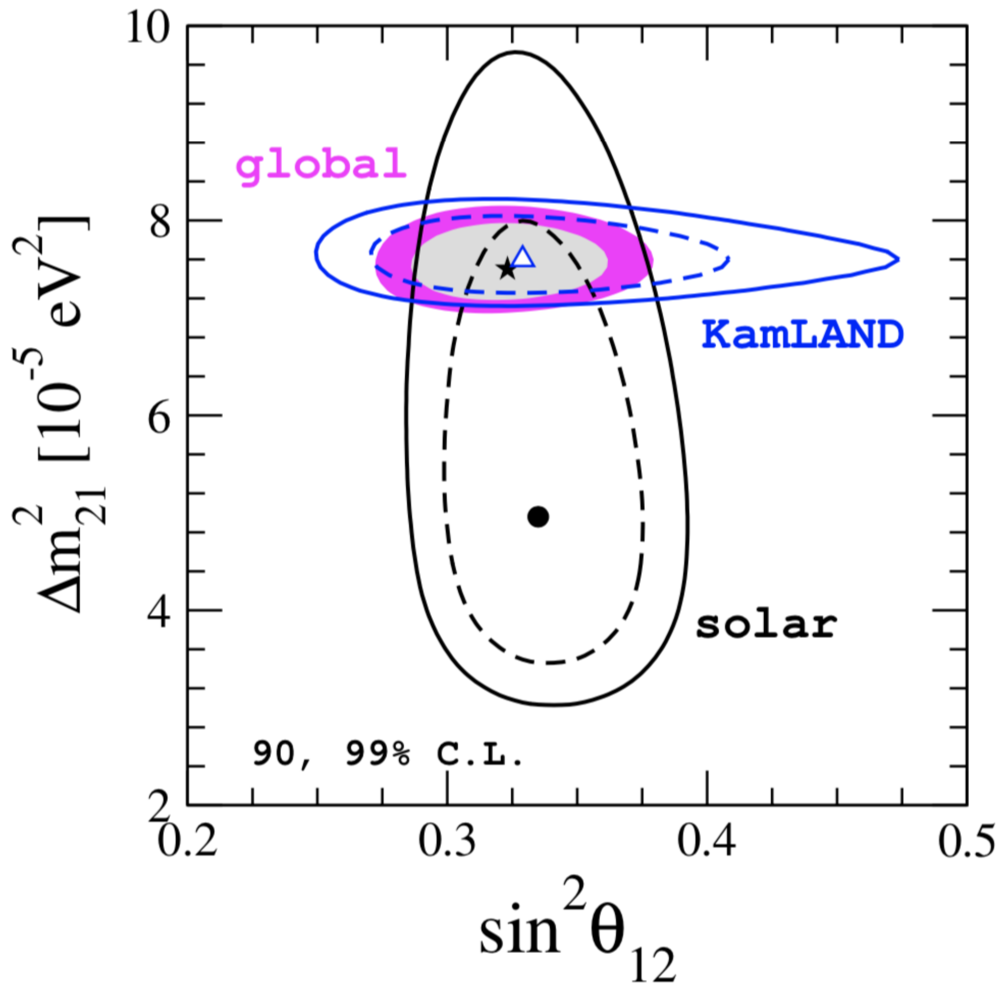


- Test with reactor neutrinos ($E \sim$ MeV, $\phi \sim 10^{20}\,\mathrm{s}^{-1}\,\mathrm{GW}^{-1}$): KamLAND

  A very LBL reactor $L \sim 180$ km measuring $P(\bar{\nu}_e \to \bar{\nu}_e)$ as a function of the energy



- Global fit including KamLAND [deSalas '19]



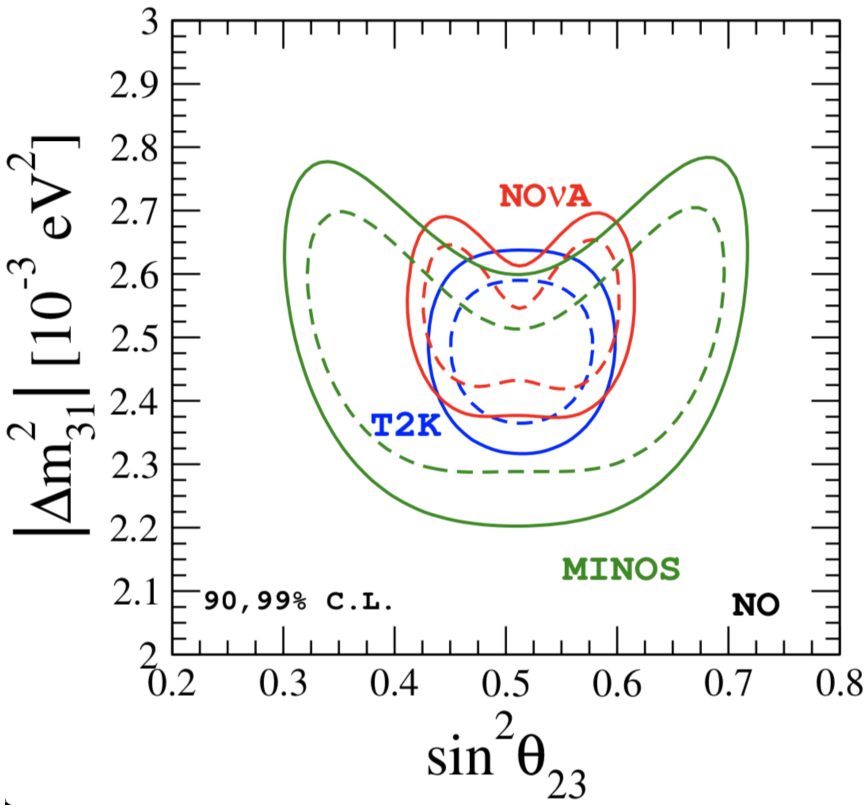
- Atmospheric fluxes of ν_μ and ν_e in (2:1) from $\pi \to \bar\nu_\mu \mu \to \bar\nu_\mu \nu_\mu \bar\nu_e\, e$
  $L \sim 10$ km [downgoing] – 10^4 km [upgoing]
  $E \sim 10^2 - 10^4$ MeV
  $\phi \sim 100$ m^{-2} s^{-1}
  (much less abundant but much more energetic than solar)

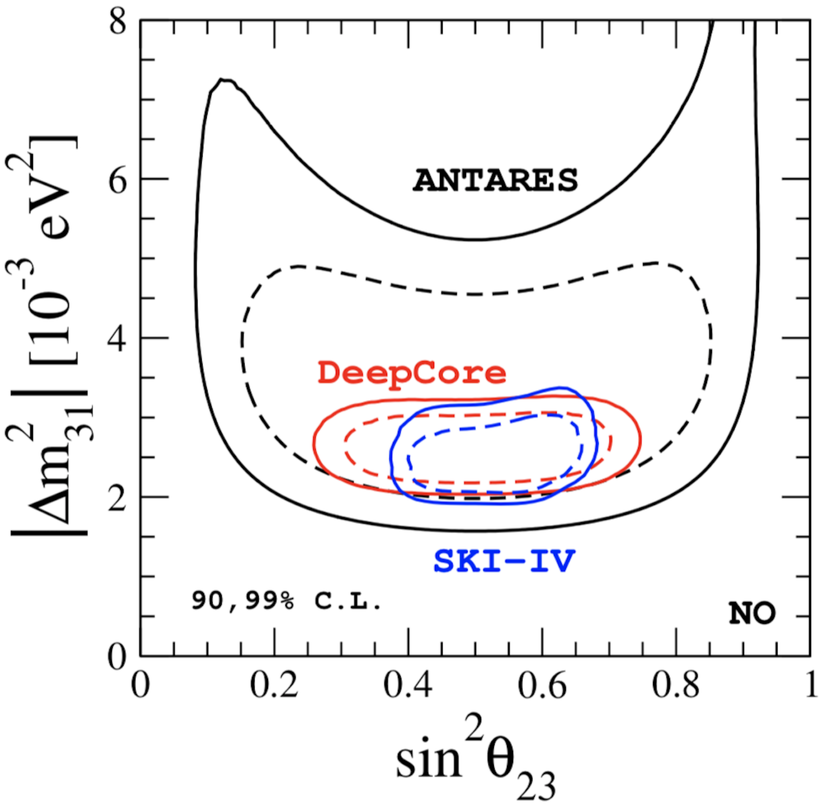
- Experiment: SuperKamiokande (atmospheric neutrinos)
  Reaction: $\nu_i N \to \ell_i N'$ detecting ℓ_i by Cherenkov $\Rightarrow$ direction, flavour (not charge)
  Result: ν_e flux unchanged and $\nu_\mu \to \nu_x$

- Tests with (LBL) accelerator neutrinos (ν_μ disappearance):

Opera	$L = 732$ km	$E \sim 10$ GeV	(CERN $\to$ Gran Sasso)
K2K	$L = 250$ km	$E \sim 1$ GeV	(KEK $\to$ Kamioka)
T2K	$L = 295$ km	$E \sim 1$ GeV	(JPARC $\to$ Kamioka)
MINOS	$L = 735$ km	$E \sim 3$ GeV	(Fermilab $\to$ Soudan (MN))
NOνA	$L = 810$ km	$E \sim 2$ GeV	(Fermilab $\to$ Ash River (MN) [new])

- Global fit [atm (SK), K2K] and comparison with T2K and MINOS (IceCube/DeepCore and ANTARES less constraining) [deSalas '19]

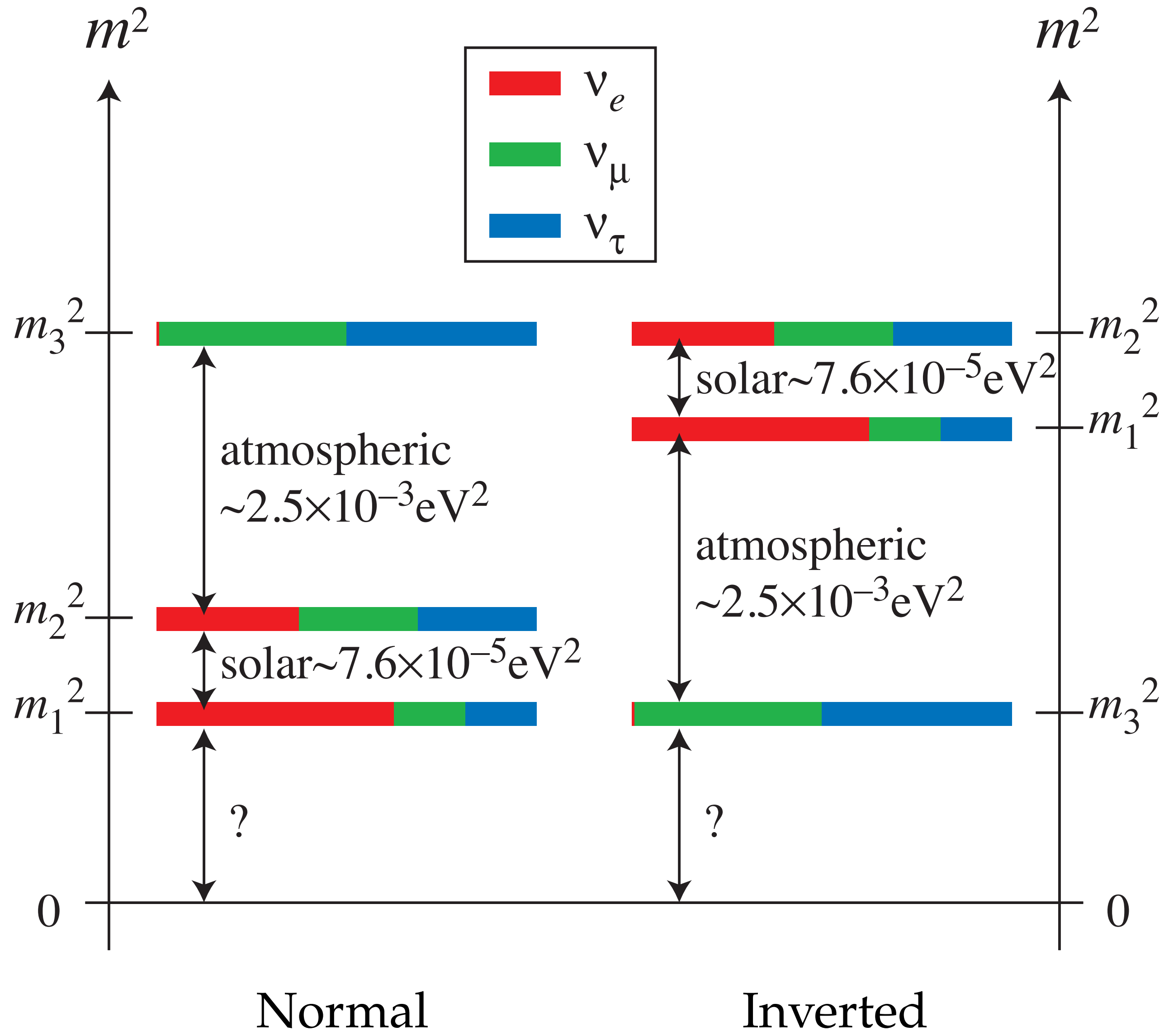




Two solutions:

– Normal ordering (NO): $\Delta m^2_{31} > 0$

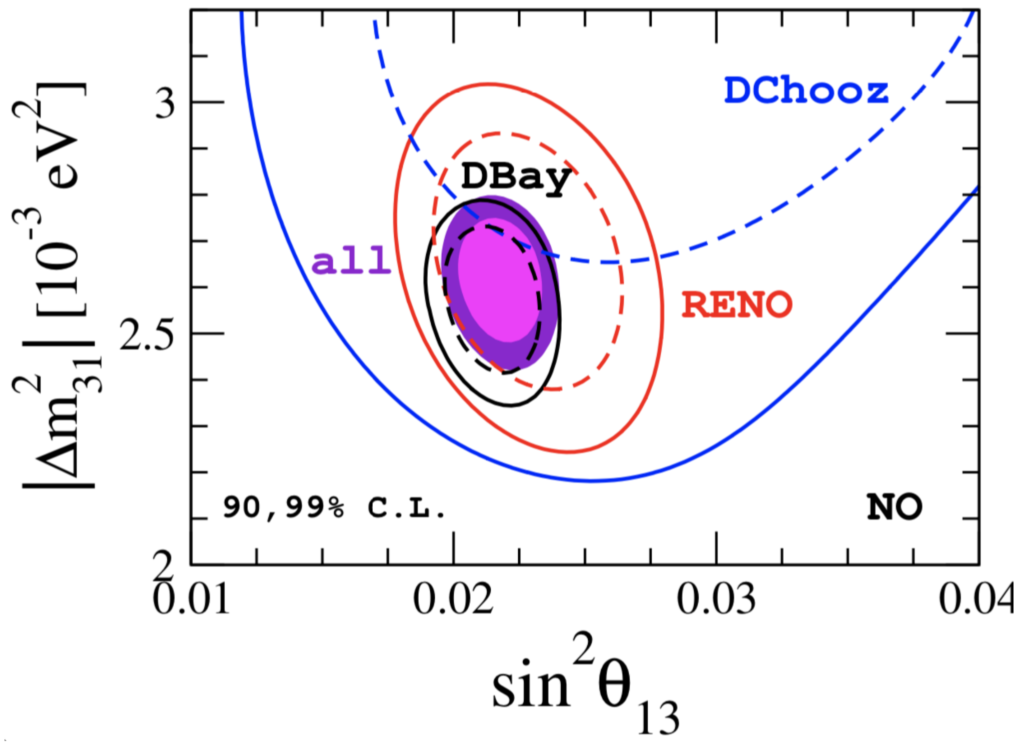
– Inverted ordering (IO): $\Delta m^2_{31} < 0$ (not shown here)



m^2
m^2
ν_e
ν_μ
ν_τ
$m_3{}^2$
$m_2{}^2$
$m_1{}^2$
atmospheric
~2.5×10^{-3}eV^2
solar~7.6×10^{-5}eV^2
?
0
$m_2{}^2$
$m_1{}^2$
$m_3{}^2$
solar~7.6×10^{-5}eV^2
atmospheric
~2.5×10^{-3}eV^2
?
0
Normal
Inverted

Degenerate?

- Measurements of $\theta_{13} \neq 0$ since 2012 by:
  - Reactor experiments (Daya Bay, RENO and Double Chooz)
  - Accelerator experiments (MINOS and T2K) [deSalas '19]



⇒ From global fits some indirect information on δ

Parameters

www.nu-fit.org

[1811.05487]

NuFIT 4.1 (2019)

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.2$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
without SK atmospheric data	$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$
	$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$
	$\sin^2\theta_{23}$	$0.558^{+0.020}_{-0.033}$	$0.427 \to 0.609$	$0.563^{+0.019}_{-0.026}$	$0.430 \to 0.612$
	$\theta_{23}/^\circ$	$48.3^{+1.1}_{-1.9}$	$40.8 \to 51.3$	$48.6^{+1.1}_{-1.5}$	$41.0 \to 51.5$
	$\sin^2\theta_{13}$	$0.02241^{+0.00066}_{-0.00065}$	$0.02046 \to 0.02440$	$0.02261^{+0.00067}_{-0.00064}$	$0.02066 \to 0.02461$
	$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \to 8.99$	$8.65^{+0.13}_{-0.12}$	$8.26 \to 9.02$
	$\delta_{\rm CP}/^\circ$	222^{+38}_{-28}	$141 \to 370$	285^{+24}_{-26}	$205 \to 354$
	$\dfrac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
	$\dfrac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.523^{+0.032}_{-0.030}$	$+2.432 \to +2.618$	$-2.509^{+0.032}_{-0.030}$	$-2.603 \to -2.416$

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 10.4$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$
	$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.75}$	$31.61 \to 36.27$
	$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$	$0.433 \to 0.609$	$0.565^{+0.017}_{-0.022}$	$0.436 \to 0.610$
	$\theta_{23}/^\circ$	$48.6^{+1.0}_{-1.4}$	$41.1 \to 51.3$	$48.8^{+1.0}_{-1.2}$	$41.4 \to 51.3$
	$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$	$0.02044 \to 0.02435$	$0.02259^{+0.00065}_{-0.00065}$	$0.02064 \to 0.02457$
	$\theta_{13}/^\circ$	$8.60^{+0.13}_{-0.13}$	$8.22 \to 8.98$	$8.64^{+0.12}_{-0.13}$	$8.26 \to 9.02$
	$\delta_{\rm CP}/^\circ$	221^{+39}_{-28}	$144 \to 357$	282^{+23}_{-25}	$205 \to 348$
	$\dfrac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
	$\dfrac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.528^{+0.029}_{-0.031}$	$+2.436 \to +2.618$	$-2.510^{+0.030}_{-0.031}$	$-2.601 \to -2.419$

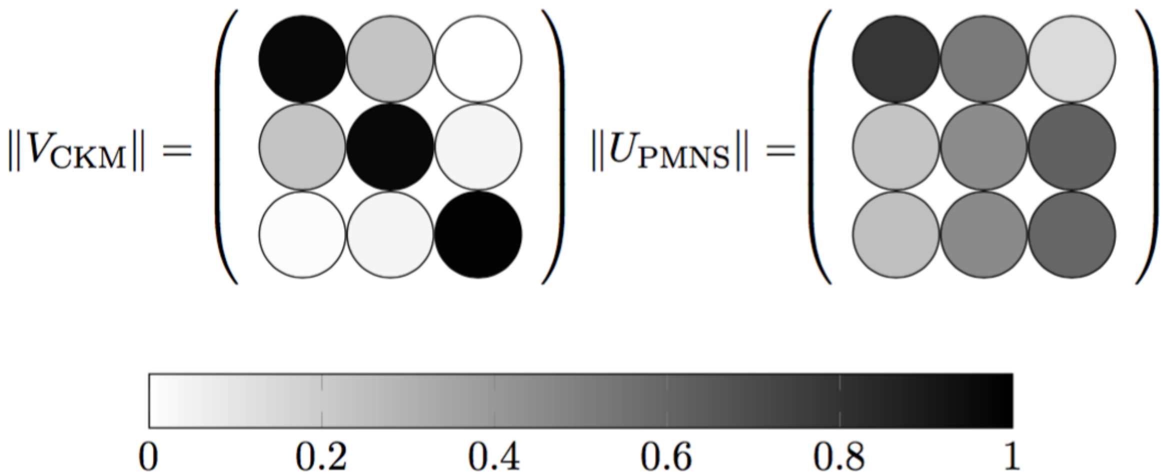
Mixing matrix www.nu-fit.org [1811.05487]

NuFIT 4.1 (2019)

$$|U|_{3\sigma}^{\text{w/o SK-atm}} = \begin{pmatrix} 0.797 \to 0.842 & 0.518 \to 0.585 & 0.143 \to 0.156 \\ 0.244 \to 0.496 & 0.467 \to 0.678 & 0.646 \to 0.772 \\ 0.287 \to 0.525 & 0.488 \to 0.693 & 0.618 \to 0.749 \end{pmatrix}$$

$$|U|_{3\sigma}^{\text{with SK-atm}} = \begin{pmatrix} 0.797 \to 0.842 & 0.518 \to 0.585 & 0.143 \to 0.156 \\ 0.243 \to 0.490 & 0.473 \to 0.674 & 0.651 \to 0.772 \\ 0.295 \to 0.525 & 0.493 \to 0.688 & 0.618 \to 0.744 \end{pmatrix}$$

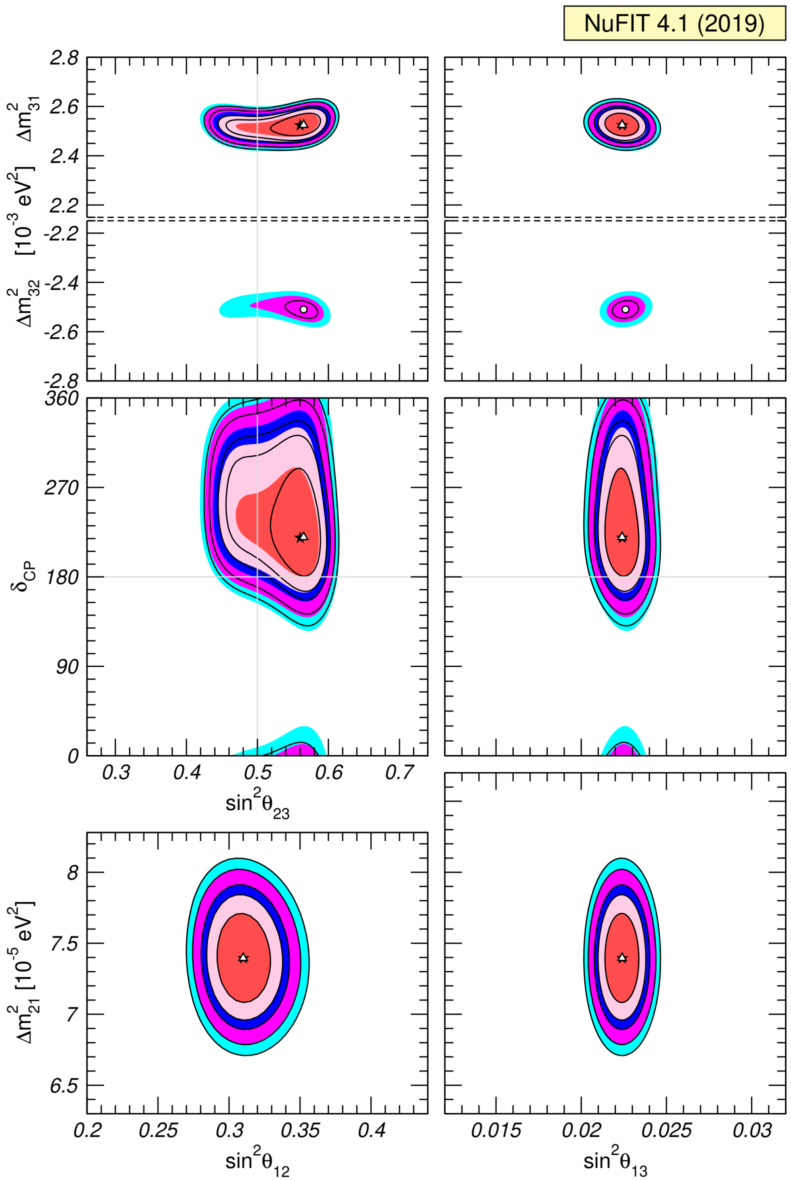
Comparison CKM vs PMNS:

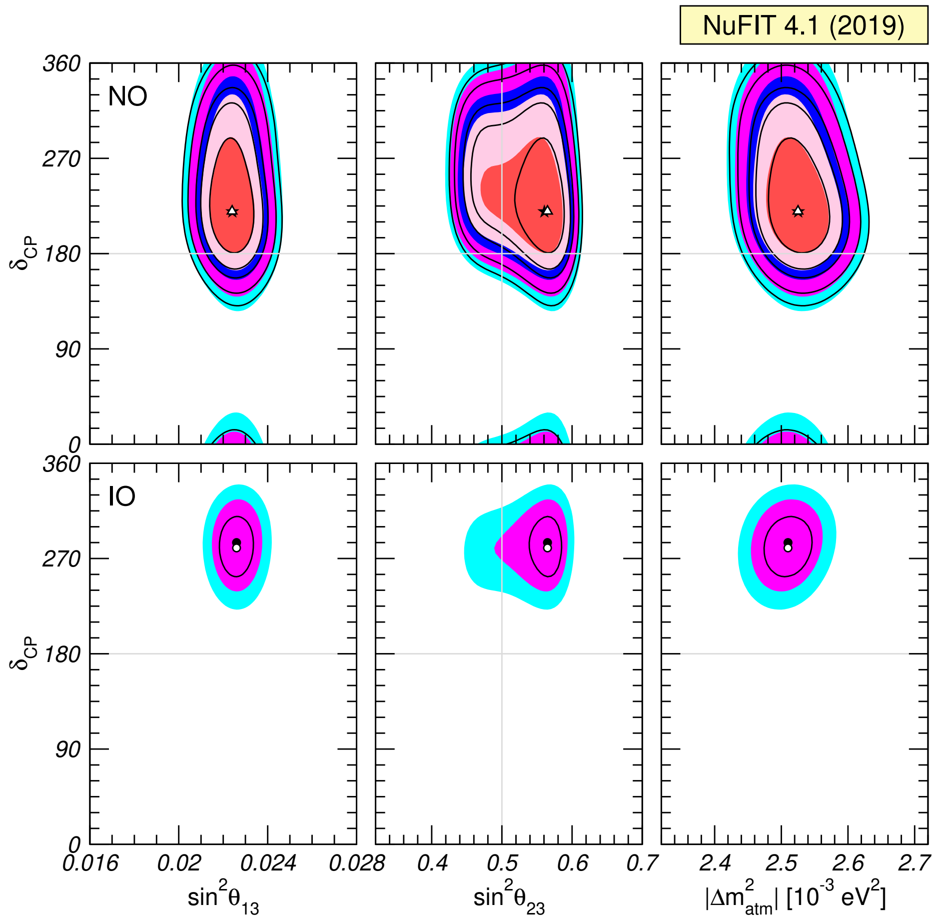


Two-dimensional allowed regions

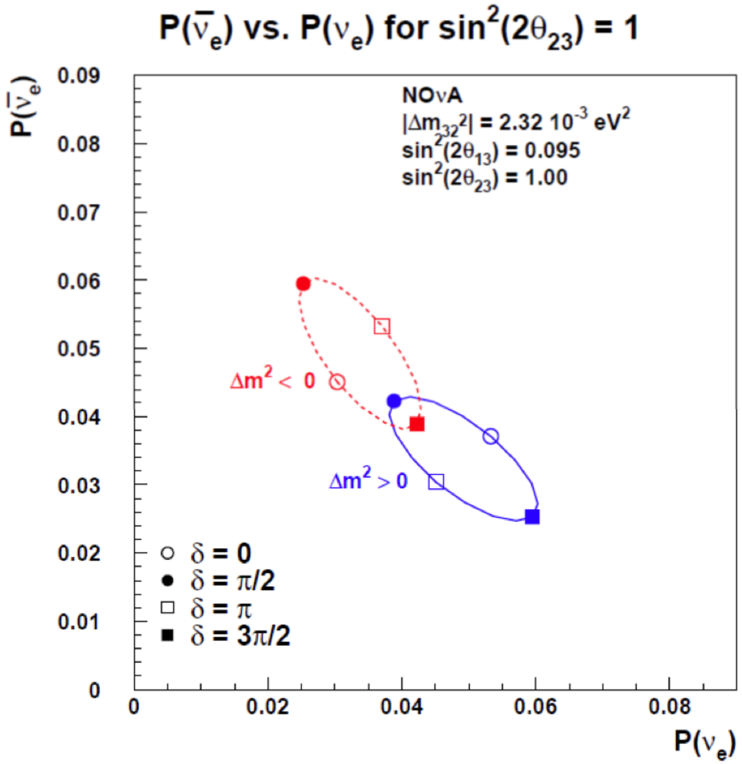
www.nu-fit.org

[1811.05487]





- Nоνa (from 2016)



  – Measurement of $\text{sign}(\Delta m^2_{31})$ (strong dependence on θ_{23})

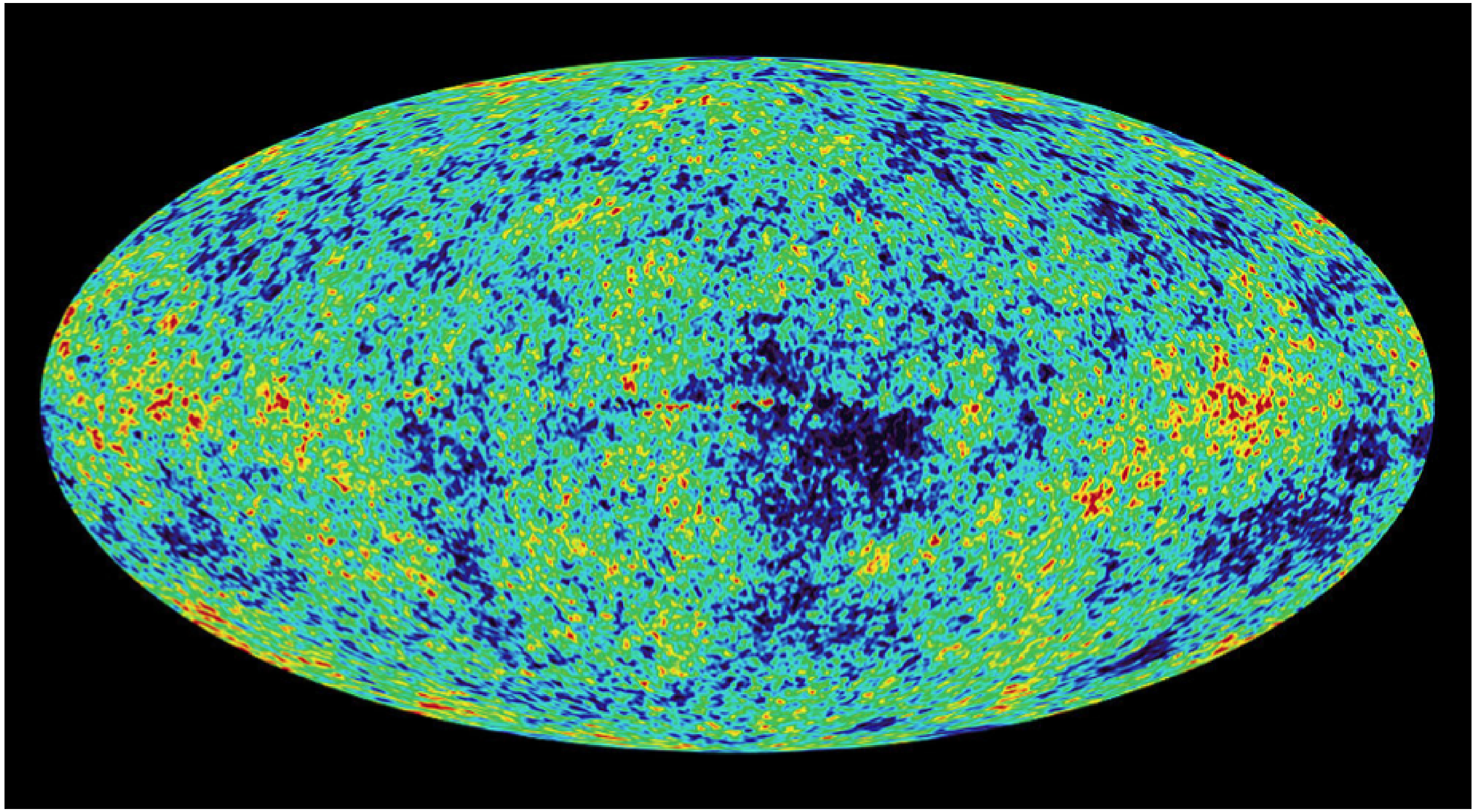
  – Measurement of δ from direct CP asymmetry:

$$\mathcal{A}_{\text{CP}} = \frac{P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)}{P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)}$$

Neutrinos | Absolute mass scale

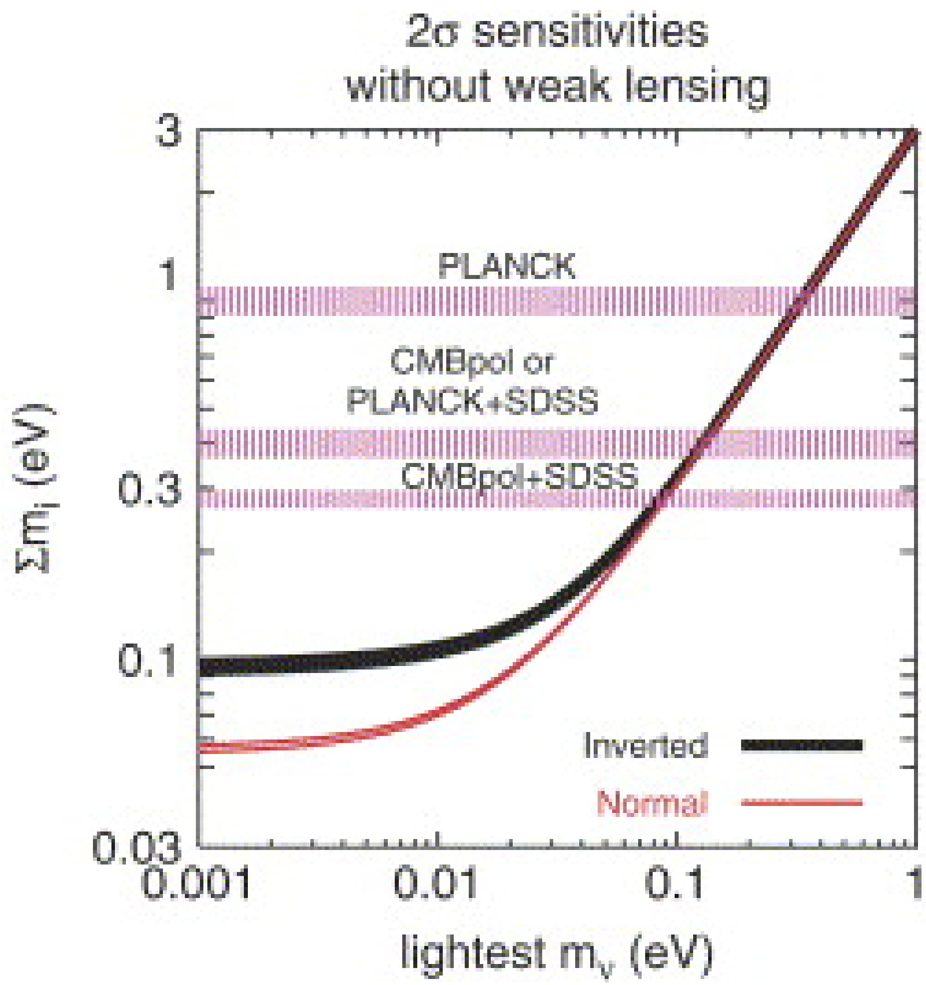
- Cosmology: if $m_\nu \neq 0$ neutrinos contribute to the mass density of the universe

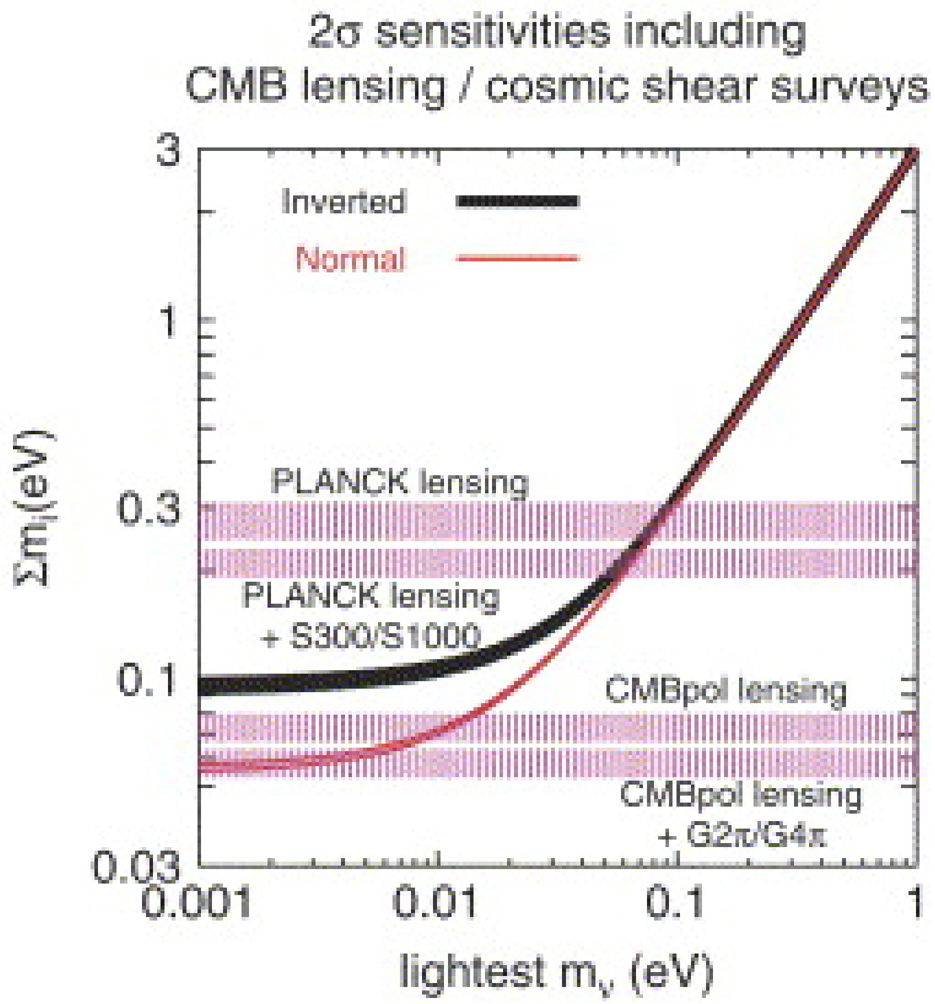
From CMB and LSS (hypothesis dependent)



$$\sum_i m_{\nu_i} < 0.23 - 0.59 \text{ eV [PLANCK]}$$

- Cosmology: if $m_\nu \neq 0$ neutrinos contribute to the mass density of the universe



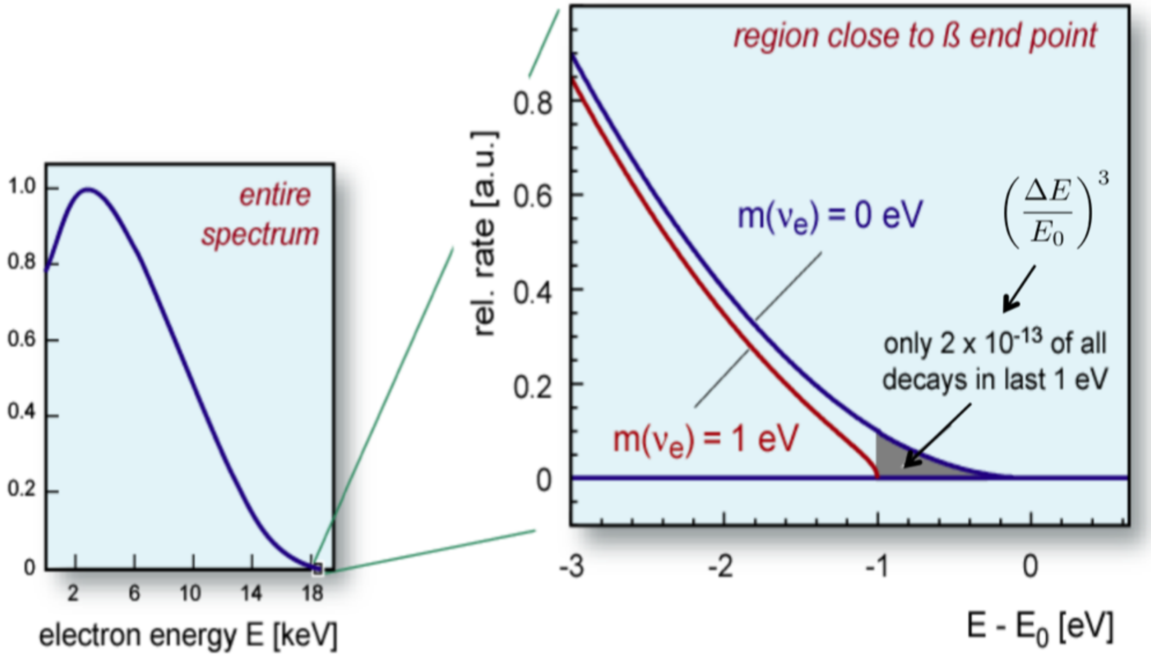


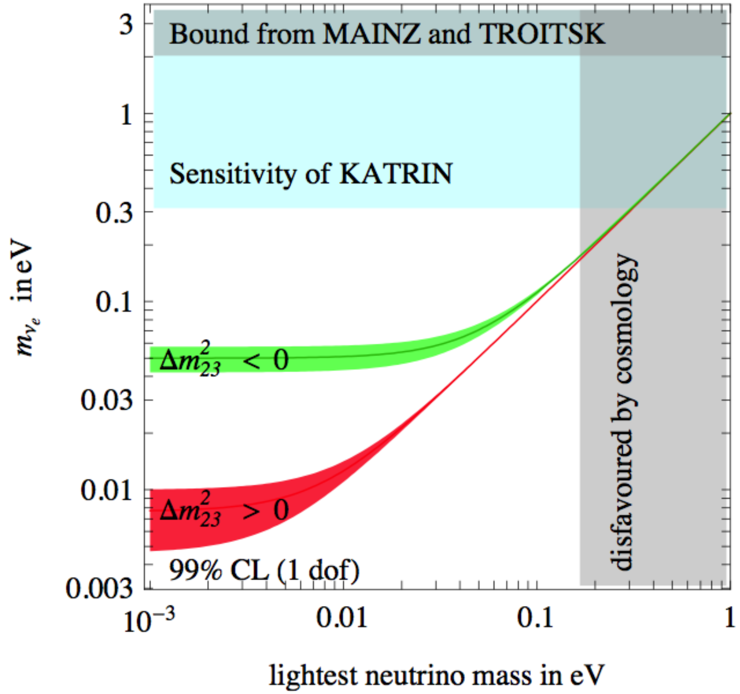
Neutrinos — Absolute mass scale

- Beta decay of tritium: $^3\text{H} \to ^3\text{He} + e^- + \bar{\nu}_e$ (end point of electron spectrum)

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \sum_i |U_{ei}|^2 \Gamma(m^2_{\nu_i}, E) \approx \Gamma(\langle m_{\nu_e}\rangle, E)$$

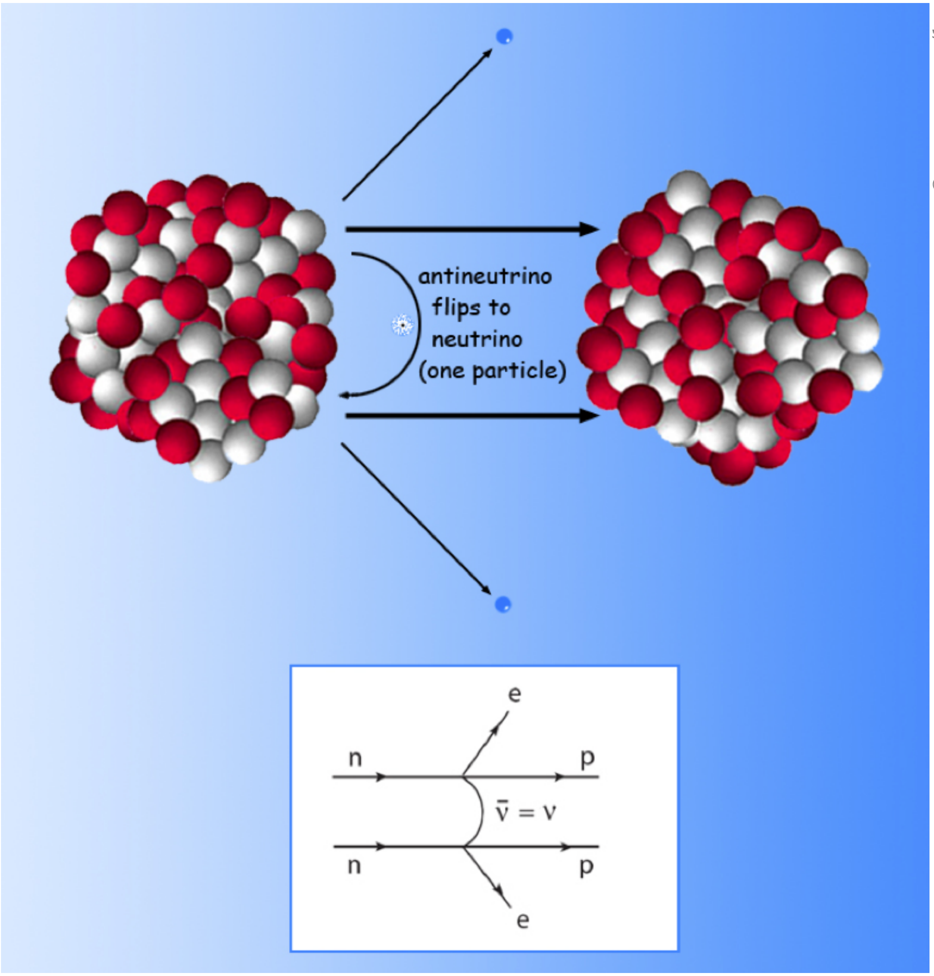
$$m^2_\beta \equiv \langle m_{\nu_e}\rangle^2 = \sum_i |U_{ei}|^2 m^2_{\nu_i} = c^2_{13}(m^2_1 c^2_{12} + m^2_2 s^2_{12}) + m^2_3 s^2_{13}$$





Neutrinos — Dirac or Majorana?

- Neutrinoless double-beta decay ($0\nu\beta\beta$)
  - $2\nu\beta\beta$ observed with $T_{2\nu\beta\beta} \sim 10^{20}$ years
  - $0\nu\beta\beta$ requires LNV (Majorana ν's)

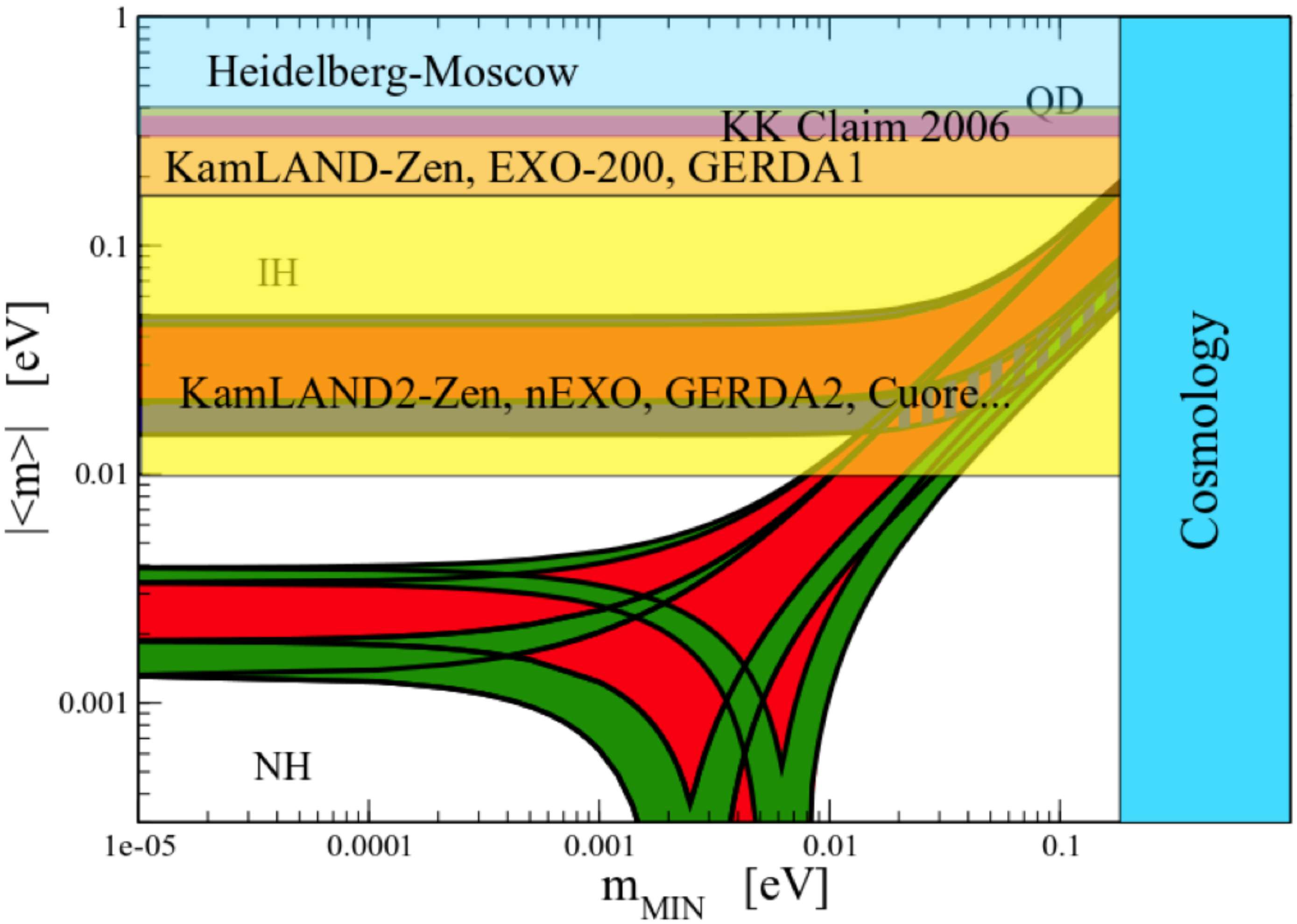


Suppressed by $m_{\beta\beta}^2$ but enhanced by phase space

$$m_{\beta\beta} = \left|\sum_i U_{ei}^2 m_i\right| = \left|c_{13}^2(m_1 c_{12}^2 + m_2 s_{12}^2 \mathrm{e}^{2\mathrm{i}\alpha}) + m_3 s_{13}^2 \mathrm{e}^{2\mathrm{i}(\beta-\delta)}\right|$$

Neutrinos — Dirac or Majorana?

- Neutrinoless double-beta decay ($0\nu\beta\beta$)



Neutrinos | Summary of parameters

N_ν (active and light)	3	LEP
$\Delta m^2_{21} \sim 7.5 \times 10^{-5}$ eV^2	$\theta_{12} \sim 35^\circ$	Solar, KamLAND
$\Delta m^2_{31} \sim \pm 2.5 \times 10^{-3}$ eV^2	$\theta_{23} \sim 45^\circ$	Atmospheric, K2K, MINOS
	$\theta_{13} \sim 9^\circ$	T2K, MINOS, Double Chooz, Daya Bay, RENO
$m_{\beta\beta} \equiv \lvert \sum_i U^2_{ei} m_{\nu_i} \rvert$	$\lesssim 0.2$ eV	KamLAND-Zen, EXO, HM, IGEX, . . .
$m_\beta \equiv \sqrt{\sum_i \lvert U_{ei}\rvert^2 m^2_{\nu_i}}$	$\lesssim 2.3$ eV	Mainz, Troitsk
$\sum_i m_{\nu_i}$	$\lesssim 1$ eV	Cosmology
sign(Δm^2_{31})	?	Noνa, NF, BB, SB, . . .
CP, δ	?	Noνa, NF, BB, SB, . . .
Dirac or Majorana?	?	HM?, $0\nu\beta\beta$

Summary and conclusions

Summary and conclusions

- The SM is a gauge theory with spontaneous symmetry breaking (renormalizable)
- Confirmed by many low and high energy experiments with remarkable accuracy, at the level of quantum corrections, with (almost) no significant deviations
- In spite of its tremendous success, it leaves fundamental questions unanswered:

  why 3 generations? why the observed pattern of fermion masses and mixings?

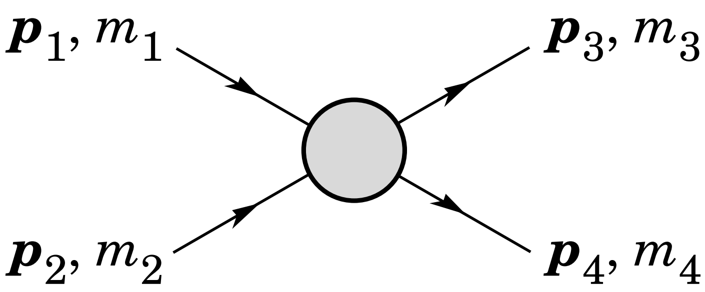
- And there are several hints for physics beyond:
  - phenomenological:
    - $(g_\mu - 2)$
    - neutrino masses
    - flavor anomalies
    - baryon asymmetry
    - dark matter
    - dark energy
  - conceptual:
    - gravity is not included
    - hierarchy problem
    - cosmological constant

    The SM is an Effective Theory valid up to electroweak scale?

4. Tools

Kinematics

$2 \to 2$ Kinematics



$$p_1 + p_2 = p_3 + p_4$$

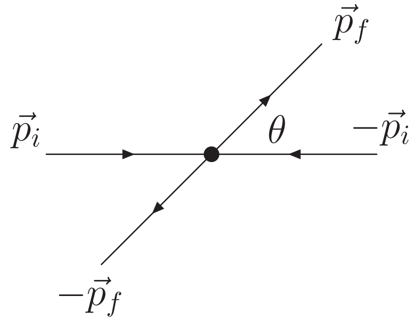
Mandelstam variables

(Lorentz invariant)

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 = m_1^2 + m_2^2 + 2(p_1 \cdot p_2) \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_1^2 + m_3^2 - 2(p_1 \cdot p_3) \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_1^2 + m_4^2 - 2(p_1 \cdot p_4) \\
s + t + u &= \sum_{i=1}^{4} m_i^2 .
\end{aligned}$$

Center of mass frame (CM)

Consider particular case: $\quad m_1 = m_2 \equiv m_i, \quad m_3 = m_4 \equiv m_f$



$$
\begin{aligned}
p_1 &= (E, 0, 0, |\boldsymbol{p_i}|) \\
p_2 &= (E, 0, 0, -|\boldsymbol{p_i}|) \\
p_3 &= (E, |\boldsymbol{p_f}| \sin\theta, 0, |\boldsymbol{p_f}| \cos\theta) \\
p_4 &= (E, -|\boldsymbol{p_f}| \sin\theta, 0, -|\boldsymbol{p_f}| \cos\theta)
\end{aligned}
$$

$$
\begin{aligned}
s &= 4E^2 = E_{\text{CM}}^2 \\
t &= -\frac{s}{2}(1 - \beta_i \beta_f \cos\theta) + m_i^2 + m_f^2 \\
u &= -\frac{s}{2}(1 + \beta_i \beta_f \cos\theta) + m_i^2 + m_f^2
\end{aligned}
$$

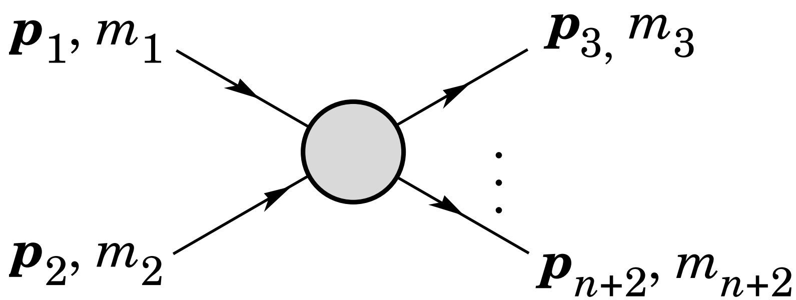
$$(s \geq \text{máx}\{4m_i^2, 4m_f^2\};\ t, u \leq -|m_i^2 - m_f^2|)$$

where $E^2 - |\boldsymbol{p}_{i,f}|^2 = m_{i,f}^2, \quad \beta_{i,f} = |\boldsymbol{p}_{i,f}|/E = \sqrt{1 - 4m_{i,f}^2/s}.$

Scalar products:

$$
\begin{aligned}
m_i^2 + (p_1 \cdot p_2) = m_f^2 + (p_3 \cdot p_4) &= 2E^2 = \frac{s}{2} \\
(p_1 \cdot p_3) = (p_2 \cdot p_4) &= E^2(1 - \beta_i \beta_f \cos\theta) = \frac{m_i^2 + m_f^2 - t}{2} \\
(p_1 \cdot p_4) = (p_2 \cdot p_3) &= E^2(1 + \beta_i \beta_f \cos\theta) = \frac{m_i^2 + m_f^2 - u}{2}
\end{aligned}
$$

Cross-section

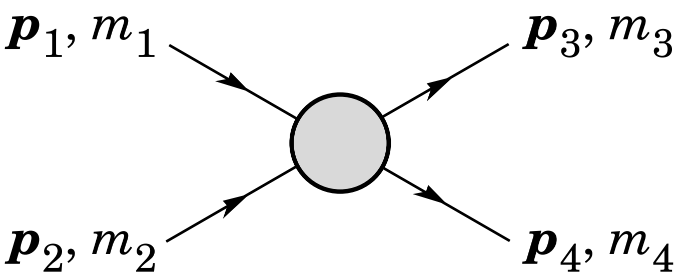


$$\mathrm{d}\sigma(i \to f) = \frac{1}{4\left\{(p_1 p_2)^2 - m_1^2 m_2^2\right\}^{1/2}} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_i - p_f) \prod_{j=3}^{n+2} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}$$

- ▷ Sum over initial polarizations and/or average over final polarizations if the initial state is unpolarized and/or the final state polarization is not measured
- ▷ Divide the total cross-section by a symmetry factor $S = \prod_i n_i!$ if there are n_i identical particles of species i in the final state

Cross-section | case $2 \to 2$ in CM frame

$$\boldsymbol{q} = \boldsymbol{p}_1 = -\boldsymbol{p}_2$$



$$\boldsymbol{p} = \boldsymbol{p}_3 = -\boldsymbol{p}_4$$

$$\Rightarrow \int \mathrm{d}\Phi_2 \equiv (2\pi)^4 \int \delta^4(p_1 + p_2 - p_3 - p_4) \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4} = \int \frac{|\boldsymbol{p}| \mathrm{d}\Omega}{16\pi^2 E_{\mathrm{CM}}}$$

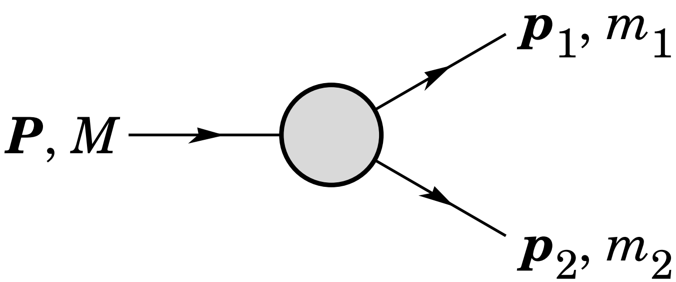
and if $m_1 = m_2 \quad \Rightarrow \quad 4\{(p_1 p_2)^2 - m_1^2 m_2^2\}^{1/2} = 4E_{\mathrm{CM}}|\boldsymbol{q}|$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(1,2 \to 3,4) = \frac{1}{64\pi^2 E_{\mathrm{CM}}^2} \frac{|\boldsymbol{p}|}{|\boldsymbol{q}|} |\mathcal{M}|^2}$$

Decay width

$$\mathrm{d}\Gamma(i\to f) = \frac{1}{2M}|\mathcal{M}|^2(2\pi)^4\delta^4(P-p_f)\prod_{j=1}^{n}\frac{\mathrm{d}^3p_j}{(2\pi)^3 2E_j}$$

case $1 \to 2$



$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}(i\to 1,2) = \frac{1}{32\pi^2}\frac{|\boldsymbol{p}|}{M^2}|\mathcal{M}|^2$$

▷ Note that masses M, m_1 and m_2 fix final energies and momenta:

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} \qquad E_2 = \frac{M^2 - m_1^2 + m_2^2}{2M}$$

$$|\boldsymbol{p}| = |\boldsymbol{p}_1| = |\boldsymbol{p}_2| = \frac{\left\{[M^2-(m_1+m_2)^2][M^2-(m_1-m_2)^2]\right\}^{1/2}}{2M}$$

General Feynman Rules

Reglas generales

Para el cálculo de funciones de Green o de amplitudes invariantes de scattering $\mathcal{M}_{fi}$.

1. Dibujar todos los diagramas conectados y topológicamente distintos en el orden deseado de teoría de perturbaciones.

   En cada diagrama:

2. Asociar momentos externos a todas las líneas externas y L momentos internos a los L loops. Determinar los momentos de las líneas internas de modo que el cuadri-momento se conserve en cada vértice.

3. Asignar un propagador a cada línea interna:

[bosón escalar] $\dfrac{\mathrm{i}}{p^2 - m^2}$

[fermión spin 1/2] $\dfrac{\mathrm{i}(\not{p} + m)}{p^2 - m^2}$

μ  ν

[bosón vectorial] $\dfrac{\mathrm{i}}{p^2 - m^2}\left[-g_{\mu\nu} + (1-\xi)\dfrac{p_\mu p_\nu}{(p^2 - \xi m^2)}\right]$ (R_ξ)

($\xi = 1$: gauge 't Hooft-Feynman; $\xi = 0$: gauge de Landau; $\xi = \infty$: gauge unitario)

Reglas generales

4. A cada vértice asignar un peso compuesto por los siguientes factores:
   (a) La constante de acoplamiento que aparezca en $\mathrm{i}\mathcal{L}_{\text{int}}$.
   (b) Por cada derivada de un campo ϕ cualquiera $\partial_\mu \phi$ asociar $(-\mathrm{i}p_\mu)$ donde p es el correspondiente momento entrante.
   (c) Un factor proviniente de la degeneración de partículas idénticas en cada vértice. (Por ejemplo, $\times 2$ para ZZH, $\times 4$ para $ZZHH$.)

5. Por cada momento interno q no fijado por la conservación de momento en cada vértice (loops), introducir un factor

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$$

   e integrar, (si es necesario) después de regularizar.

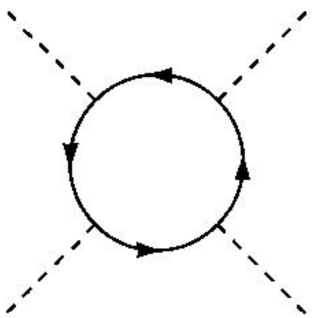
Reglas generales

6. Multiplicar la contribución de cada diagrama por:

(a) Un factor (-1) entre diagramas que difieren entre sí solo por el intercambio de dos fermiones externos idénticos. (Por ejemplo, los dos diagramas del scattering de Møller, $e^-e^- \to e^-e^-$, o los dos del scattering de Bhabha, $e^+e^- \to e^+e^-$, a nivel árbol.)

(b) Un factor de simetría $1/S$ donde S el número de permutaciones de líneas internas y vértices que deja invariante el diagrama si las líneas externas permanecen fijadas.

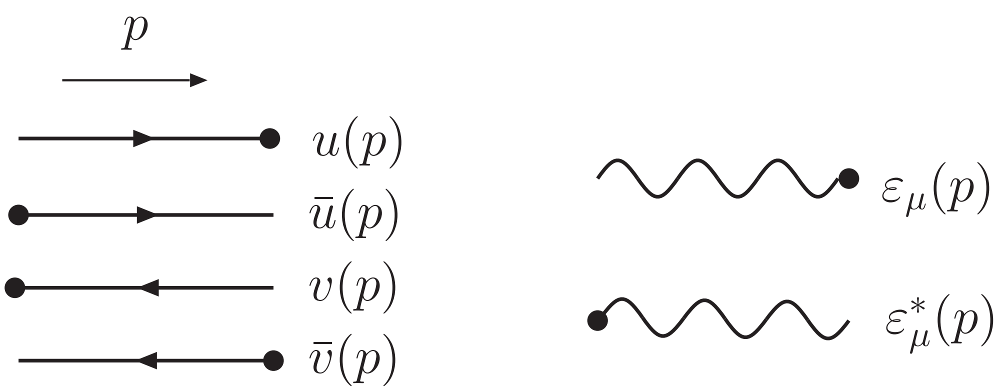
(c) Un factor (-1) por cada loop fermiónico, pues:



$$= \bar\psi\psi\bar\psi\psi\bar\psi\psi\bar\psi\psi = -\mathrm{Tr}\left[\psi\bar\psi\;\psi\bar\psi\;\psi\bar\psi\;\psi\bar\psi\right] = -\mathrm{Tr}(\widetilde{S}_F\widetilde{S}_F\widetilde{S}_F\widetilde{S}_F)$$

Reglas generales

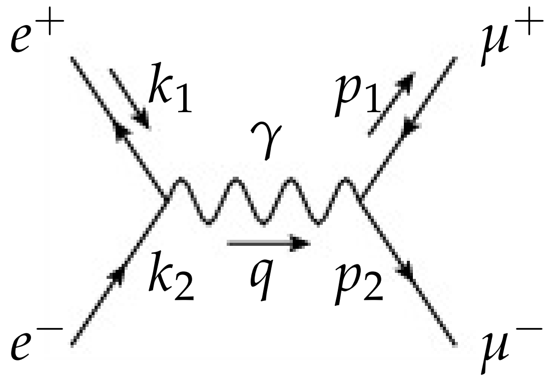
7. Para obtener $\mathrm{i}\mathcal{M}_{fi}$, poner las líneas externas sobre su capa de masas, es decir $p_i^2 = m_i^2$. Poner por cada línea fermiónica externa un espinor: $u(p)$ [o $v(p)$] para fermiones [o antifermiones] entrantes con momento p; $\bar{u}(p)$ [o $\bar{v}(p)$] para fermiones [o antifermiones] salientes con momento p. Poner vectores de polarización $\varepsilon_\mu(p,\lambda)$ [o $\varepsilon_\mu^*(p,\lambda)$] para bosones vectoriales entrantes [o salientes] con momento p.



Calculation of a simple process

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Consideremos la aniquilación de un electrón y un positrón para dar un muón y un antimuón. En QED este proceso viene descrito a orden más bajo de TP (nivel árbol) por el diagrama de la figura.



▷ El muón tiene la misma carga del electrón, $Q_\mu = Q_e = -1$, y una masa M unas 200 veces mayor que la masa m del electrón.

▷ Vamos a hallar paso a paso y en detalle la sección eficaz de este proceso.

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- En primer lugar, asignamos momentos a todas las partículas del diagrama y usamos la conservación del cuadrimomento en cada vértice, lo que fija el cuadrimomento del fotón virtual que se propaga entre los dos vértices de interacción,

$$q = k_1 + k_2 = p_1 + p_2 \; .$$

▷ Las patas externas son fermiones, cuyos espines etiquetamos mediante índices r_1, r_2, s_1, s_2 que toman dos valores posibles $\{1, 2\}$.

- Aplicando las reglas de Feynman, recorriendo cada línea fermiónica en sentido contrario al flujo fermiónico, el elemento de matriz invariante viene dado por

$$\mathrm{i}\mathcal{M} = \overline{u}^{(s_2)}(\boldsymbol{p}_2)(\mathrm{i}e\gamma^\beta)v^{(s_1)}(\boldsymbol{p}_1)\frac{(-\mathrm{i})}{q^2}\left[g_{\alpha\beta} - (1-\xi)\frac{q_\alpha q_\beta}{q^2}\right]\overline{v}^{(r_1)}(\boldsymbol{k}_1)(\mathrm{i}e\gamma^\alpha)u^{(r_2)}(\boldsymbol{k}_2) \; .$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Nótese que como los fermiones externos están sobre su capa de masas satisfacen las respectivas ecuaciones de Dirac,

$$\not{k}_1 v^{(r_1)}(\boldsymbol{k}_1) = -m v^{(r_1)}(\boldsymbol{k}_1) \, , \quad \not{k}_2 u^{(r_2)}(\boldsymbol{k}_2) = m u^{(r_2)}(\boldsymbol{k}_2) \, ,$$

  así que la amplitud no depende del parámetro ξ, como deber ser, ya que

$$q_\alpha \overline{v}^{(r_1)}(\boldsymbol{k}_1)\gamma^\alpha u^{(r_2)}(\boldsymbol{k}_2) = \overline{v}^{(r_1)}(\boldsymbol{k}_1)(\not{k}_1 + \not{k}_2)u^{(r_2)}(\boldsymbol{k}_2) = 0 \, .$$

- Podríamos haber trabajado desde el principio en el gauge de 't Hooft-Feynman ($\xi = 1$). Por tanto,

$$\mathcal{M} = \frac{e^2}{q^2}\overline{u}^{(s_2)}(\boldsymbol{p}_2)\gamma^\alpha v^{(s_1)}(\boldsymbol{p}_1)\; \overline{v}^{(r_1)}(\boldsymbol{k}_1)\gamma_\alpha u^{(r_2)}(\boldsymbol{k}_2) \, .$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Para hallar $|\mathcal{M}|^2$, nótese que

$$(\overline{u}\gamma^\alpha v)^* = v^\dagger\gamma^{\alpha\dagger}\gamma^{0\dagger}u = v^\dagger\gamma^0\gamma^0\gamma_\alpha\gamma^0 u = \overline{v}\gamma^\alpha u \,,$$

donde se ha usado

$$\overline{u} = u^\dagger\gamma^0 \,, \quad \gamma^{\alpha\dagger} = \gamma_\alpha \,, \quad \gamma^0\gamma_\alpha\gamma^0 = \gamma^\alpha \,.$$

Se trata además de un número complejo que podemos multiplicar en cualquier orden. Lo mismo ocurre con la otra línea fermiónica.

▷ Conviene escribir,

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\overline{u}^{(s_2)}(\boldsymbol{p}_2)\gamma^\alpha v^{(s_1)}(\boldsymbol{p}_1)\,\overline{v}^{(s_1)}(\boldsymbol{p}_1)\gamma^\beta u^{(s_2)}(\boldsymbol{p}_2) \times \overline{v}^{(r_1)}(\boldsymbol{k}_1)\gamma_\alpha u^{(r_2)}(\boldsymbol{k}_2)\,\overline{u}^{(r_2)}(\boldsymbol{k}_2)\gamma_\beta v^{(r_1)}(\boldsymbol{k}_1) \quad (1)$$

Podemos ahora hacer uso de las propiedades de espinores y matrices de Dirac, que conducen a multitud de identidades (Diracología).

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

▷ En particular, puede verse que los dos estados de espín a lo largo del eje z satisfacen

$$u^{(1)}(\boldsymbol{p})\overline{u}^{(1)}(\boldsymbol{p}) = (\not{p} + m)\frac{1+\gamma_5 \not{n}}{2}\,,$$
$$u^{(2)}(\boldsymbol{p})\overline{u}^{(2)}(\boldsymbol{p}) = (\not{p} + m)\frac{1-\gamma_5 \not{n}}{2}\,,$$
$$v^{(1)}(\boldsymbol{p})\overline{v}^{(1)}(\boldsymbol{p}) = (\not{p} - m)\frac{1+\gamma_5 \not{n}}{2}\,,$$
$$v^{(2)}(\boldsymbol{p})\overline{v}^{(2)}(\boldsymbol{p}) = (\not{p} - m)\frac{1-\gamma_5 \not{n}}{2}\,,$$

donde $n^\mu = (0,0,0,1)$ en el sistema de referencia en el que $p^\mu = (m,0,0,0)$. En general,

$$u(\boldsymbol{p},n)\overline{u}(\boldsymbol{p},n) = (\not{p} + m)\frac{1+\gamma_5 \not{n}}{2}\,, \quad v(\boldsymbol{p},n)\overline{v}(\boldsymbol{p},n) = (\not{p} - m)\frac{1+\gamma_5 \not{n}}{2}$$

proyectan sobre polarizaciones bien definidas a lo largo de una dirección n^μ, que cumple $n^2 = -1$ y $p_\mu n^\mu = 0$.

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Si elegimos, por simplicidad, el eje z como dirección del movimiento, $p^\mu = (E, 0, 0, |\boldsymbol{p}|)$, los operadores anteriores proyectan sobre los dos estados de helicidad de partícula y antipartícula, respectivamente, si tomamos $n^\mu = \pm(|\boldsymbol{p}|/m, 0, 0, E/m)$.

- En particular, en el límite ultrarrelativista ($E \gg m$) los proyectores sobre quiralidades *right* y *left* de partícula y antipartícula son:

$$u^{(1)}(\boldsymbol{p})\overline{u}^{(1)}(\boldsymbol{p}) = (\not{p}+m)\frac{1+\gamma_5\not{n}}{2} \to u_R(p)\overline{u}_R(\boldsymbol{p}) = (\not{p}+m)\frac{1+\gamma_5}{2}\,,$$

$$u^{(2)}(\boldsymbol{p})\overline{u}^{(2)}(\boldsymbol{p}) = (\not{p}+m)\frac{1-\gamma_5\not{n}}{2} \to u_L(p)\overline{u}_L(\boldsymbol{p}) = (\not{p}+m)\frac{1-\gamma_5}{2}\,,$$

$$v^{(1)}(\boldsymbol{p})\overline{v}^{(1)}(\boldsymbol{p}) = (\not{p}-m)\frac{1+\gamma_5\not{n}}{2} \to v_L(p)\overline{v}_L(\boldsymbol{p}) = (\not{p}-m)\frac{1-\gamma_5}{2}\,,$$

$$v^{(2)}(\boldsymbol{p})\overline{v}^{(2)}(\boldsymbol{p}) = (\not{p}-m)\frac{1-\gamma_5\not{n}}{2} \to v_R(\boldsymbol{p})\overline{v}_R(\boldsymbol{p}) = (\not{p}-m)\frac{1+\gamma_5}{2}\,.$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

▷ Otra propiedad que se demuestra fácilmente de lo anterior es

$$\overline{u}(\boldsymbol{p},n)\Gamma u(\boldsymbol{p},n) = \mathrm{Tr}\left[\Gamma(\not{p}+m)\frac{1+\gamma_5\not{n}}{2}\right] , \quad \overline{v}(\boldsymbol{p},n)\Gamma v(\boldsymbol{p},n) = \mathrm{Tr}\left[\Gamma(\not{p}-m)\frac{1+\gamma_5\not{n}}{2}\right]$$

donde Γ es una matriz 4×4 arbitraria.

▷ Por otro lado, si los fermiones no están polarizados el cálculo se simplifica notablemente pues podemos aplicar directamente las relaciones de completitud,

$$\sum_s u^{(s)}(\boldsymbol{p})\overline{u}^{(s)}(\boldsymbol{p}) = \not{p}+m , \quad \sum_s v^{(s)}(\boldsymbol{p})\overline{v}^{(s)}(\boldsymbol{p}) = \not{p}-m ,$$

que conducen a

$$\sum_s \overline{u}^{(s)}(\boldsymbol{p})\Gamma u^{(s)}(\boldsymbol{p}) = \mathrm{Tr}\left[\Gamma(\not{p}+m)\right] , \quad \sum_s \overline{v}^{(s)}(\boldsymbol{p})\Gamma v^{(s)}(\boldsymbol{p}) = \mathrm{Tr}\left[\Gamma(\not{p}-m)\right] .$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Volvamos a nuestro cálculo y supongamos por simplicidad que tanto los fermiones iniciales como los finales no están polarizados. Tenemos entonces que promediar sobre espines iniciales y sumar sobre espines finales:

$$\widetilde{\sum_{r_i}}\sum_{s_i}|\mathcal{M}|^2 = \frac{1}{4}\sum_{r_i}\sum_{s_i}|\mathcal{M}|^2$$
$$= \frac{e^4}{4q^4}\mathrm{Tr}[\gamma^\alpha(\not{p}_1 - M)\gamma^\beta(\not{p}_2 + M)]\mathrm{Tr}[\gamma_\alpha(\not{k}_2 + m)\gamma_\beta(\not{k}_1 - m)] \, ,$$

  que aparece como el producto de las trazas de las dos cadenas fermiónicas.

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- Para hallar las trazas volvemos a recurrir a la Diracología. Necesitamos en particular,

$$\begin{aligned} \mathrm{Tr}[\#\ \text{impar}\ \gamma'\text{s}] &= 0 \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \end{aligned}$$

de donde

$$\begin{aligned} \mathrm{Tr}[\gamma^\alpha(\not{p}_1 - M)\gamma^\beta(\not{p}_2 + M)] &= \mathrm{Tr}[\gamma^\alpha\not{p}_1\gamma^\beta\not{p}_2] - M^2\mathrm{Tr}[\gamma^\alpha\gamma^\beta] \\ &= 4(p_1^\alpha p_2^\beta - (p_1p_2)g^{\alpha\beta} + p_1^\beta p_2^\alpha) - 4M^2g^{\alpha\beta} \\ \mathrm{Tr}[\gamma_\alpha(\not{k}_2 + m)\gamma_\beta(\not{k}_1 - m)] &= \mathrm{Tr}[\gamma_\alpha\not{k}_1\gamma_\beta\not{k}_2] - m^2\mathrm{Tr}[\gamma_\alpha\gamma_\beta] \\ &= 4(k_{1\alpha}k_{2\beta} - (k_1k_2)g_{\alpha\beta} + k_{1\beta}k_{2\alpha}) - 4m^2g_{\alpha\beta} \end{aligned}$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

y por tanto,

$$
\begin{aligned}
\widetilde{\sum_{r_i}}\sum_{s_i}|\mathcal{M}|^2 = \frac{16e^4}{4q^4}[&(p_1k_1)(p_2k_2) - (p_1p_2)(k_1k_2) + (p_1k_2)(p_2k_1) - m^2(p_1p_2) \\
&- (p_1p_2)(k_1k_2) + 4(p_1p_2)(k_1k_2) - (p_1p_2)(k_1k_2) + 4m^2(p_1p_2) \\
&+ (p_1k_2)(p_2k_1) - (p_1p_2)(k_1k_2) + (p_1k_1)(p_2k_2) - m^2(p_1p_2) \\
&- M^2(k_1k_2) + 4M^2(k_1k_2) - M^2(k_1k_2) + 4M^2m^2] \\
= \frac{8e^4}{q^4}[&(p_1k_1)(p_2k_2) + (p_1k_2)(p_2k_1) + m^2(p_1p_2) + M^2(k_1k_2) + 2M^2m^2] \,. \qquad (2)
\end{aligned}
$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

- El siguiente paso es elegir un sistema de referencia. Supongamos el sistema centro de masas y sea θ el ángulo que forma el μ^+ saliente con el e^+ incidente,

$$
\begin{aligned}
k_1^\mu &= E(1,0,0,\beta_i) \ , \\
k_2^\mu &= E(1,0,0,-\beta_i) \ , \quad \beta_i = \sqrt{1-m^2/E^2} \ , \\
p_1^\mu &= E(1,\beta_f \sin\theta, 0, \beta_f\cos\theta) \ , \\
p_2^\mu &= E(1,-\beta_f \sin\theta, 0, -\beta_f\cos\theta) \ , \quad \beta_f = \sqrt{1-M^2/E^2} \ .
\end{aligned}
$$

Entonces,

$$
\begin{aligned}
q^2 = (k_1+k_2)^2 = (p_1+p_2)^2 &= E_{\rm CM}^2 = 4E^2 \ , \\
(p_1k_1) = (p_2k_2) &= E^2(1-\beta_i\beta_f\cos\theta) \ , \\
(p_1k_2) = (p_2k_1) &= E^2(1+\beta_i\beta_f\cos\theta) \ , \\
(p_1p_2) = E^2(1+\beta_f^2) &= E^2(2-M^2/E^2) \ , \\
(k_1k_2) = E^2(1+\beta_i^2) &= E^2(2-m^2/E^2)
\end{aligned}
$$

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

y la expresión anterior queda

$$\widetilde{\sum_{r_i}}\sum_{s_i} |\mathcal{M}|^2 = \frac{e^4}{2E^4}[2E^4(1+\beta_i^2\beta_f^2\cos^2\theta) + 2E^2(m^2+M^2)]$$

$$= e^4\left[1 + 4\frac{m^2+M^2}{E_{\mathrm{CM}}^2} + \left(1-\frac{4m^2}{E_{\mathrm{CM}}^2}\right)\left(1-\frac{4M^2}{E_{\mathrm{CM}}^2}\right)\cos^2\theta\right] .$$

- La sección eficaz diferencial del proceso se obtiene a partir de

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\mathrm{CM}}^2}\frac{|\boldsymbol{p}|}{|\boldsymbol{k}|}|\mathcal{M}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\mathrm{CM}}^2}\sqrt{\frac{E_{\mathrm{CM}}^2-4M^2}{E_{\mathrm{CM}}^2-4m^2}}\left[1 + 4\frac{m^2+M^2}{E_{\mathrm{CM}}^2} + \left(1-\frac{4m^2}{E_{\mathrm{CM}}^2}\right)\left(1-\frac{4M^2}{E_{\mathrm{CM}}^2}\right)\cos^2\theta\right]$$

donde se ha sustituido la constante de estructura fina $\alpha = e^2/(4\pi)$.

Un proceso sencillo: $e^+e^- \to \mu^+\mu^-$

▷ Nótese que $E_{\rm CM} > 2M > 2m$, la energía umbral del proceso. La sección eficaz total es

$$\sigma = \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \int \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \, .$$

▷ En el límite ultrarrelativista ($E_{\rm CM} \gg M > m$),

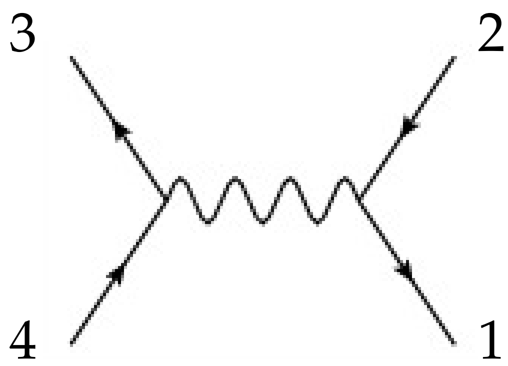
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \to \frac{\alpha^2}{4E_{\rm CM}^2}(1+\cos^2\theta)$$

$$\sigma \to \frac{4\pi\alpha^2}{3E_{\rm CM}^2} \, .$$

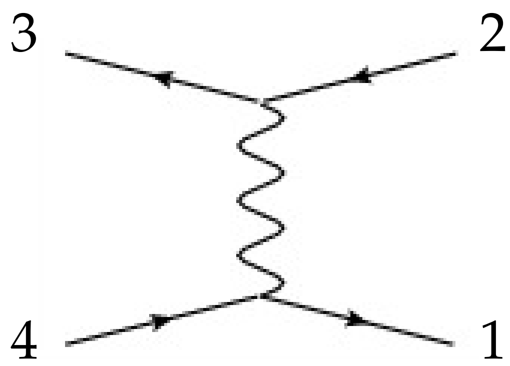
Comentarios | Sobre los signos relativos entre diagramas

- En QED se trabaja con campos espinoriales y ya hemos visto que hay que tener cuidado porque las contracciones de Wick de estos campos pueden dar lugar a signos relativos entre los distintos diagramas que contribuyen a la amplitud de un proceso. Recordemos que hay que mirar si la reordenación de los espinores corresponde a una permutación par o impar. Veamos unos cuantos ejemplos.

– Scattering de Bhabha: $e^+e^- \to e^+e^-$



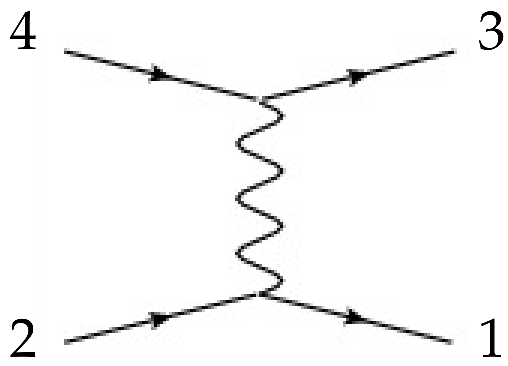
$\overline{u}_1 v_2 \overline{v}_3 u_4$



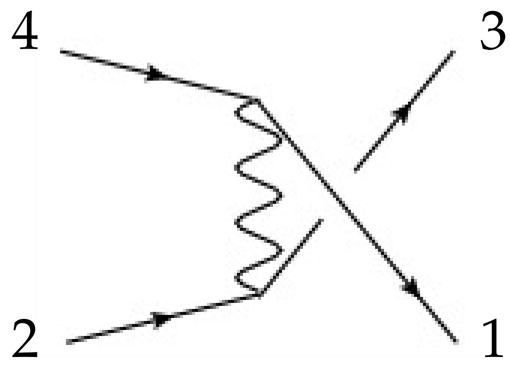
$-\overline{u}_1 u_4 \overline{v}_3 v_2$

Comentarios — Sobre los signos relativos entre diagramas

– Scattering de Møller: $e^-e^- \to e^-e^-$

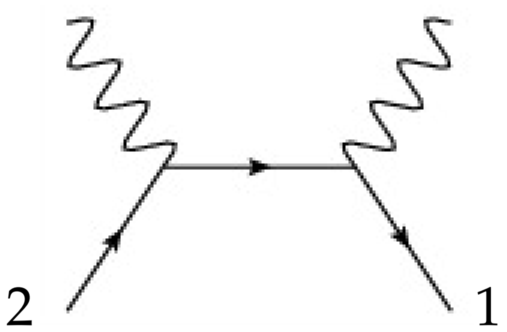


$\overline{u}_1 u_2 \overline{u}_3 u_4$

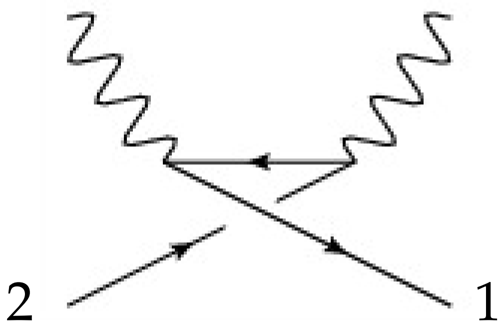


$-\overline{u}_1 u_4 \overline{u}_3 u_2$

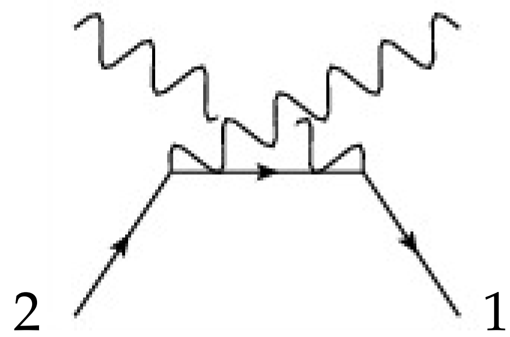
– Scattering de Compton: $e\gamma \to e\gamma$ (¡no hay cambio de signo!)



$\overline{u}_1 u_2$



$=$



$+\overline{u}_1 u_2$

Comentarios | Sobre partículas idénticas

- Recordemos también que si hay dos partículas idénticas en el estado final (por ejemplo, $\gamma\gamma$, e^+e^+, e^-e^-) la sección eficaz total es

$$\sigma = \frac{1}{2}\int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,.$$

– Caso del fotón.
  Tiene dos estados de polarización (transversos). Supongamos el sistema de referencia en el que $k^\mu = (\omega, 0, 0, \omega)$ (nuestras conclusiones serán independientes de esta elección gracias a la covariancia Lorentz). Entonces, pueden ser

$$\begin{aligned} &\text{lineales:} && \epsilon^\mu(\boldsymbol{k}, 1) = (0, 1, 0, 0)\ , && \epsilon^\mu(\boldsymbol{k}, 2) = (0, 0, 1, 0) \\ &\text{elípticas:} && \epsilon^\mu(\boldsymbol{k}, L) = (0, \cos\theta, \mathrm{i}\sin\theta, 0)\ , && \epsilon^\mu(\boldsymbol{k}, R) = (0, \cos\theta, -\mathrm{i}\sin\theta, 0)\ . \end{aligned}$$

  En cualquier caso, si sumamos sobre los dos estados de polarización,

$$\sum_\lambda \epsilon^*_\mu(\boldsymbol{k}, \lambda)\epsilon_\nu(\boldsymbol{k}, \lambda) = -g_{\mu\nu} + Q_{\mu\nu}\ , \quad Q_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ .$$

  Veamos que, debido a la invariancia gauge, en la práctica podemos ignorar el término $Q_{\mu\nu}$.

Comentarios | Sobre las polarizaciones de los bosones vectoriales

– Caso del fotón.
En efecto, la amplitud de un proceso arbitrario de QED que involucre un fotón externo con momento k (tomamos un fotón saliente) puede escribirse con toda generalidad como

$$\mathcal{M}(\boldsymbol{k},\lambda) = \epsilon^*_\mu(\boldsymbol{k},\lambda)\mathcal{M}^\mu(\boldsymbol{k})$$

y cualquier observable, en este sistema de referencia, será proporcional a

$$\begin{aligned}\sum_\lambda |\mathcal{M}(\boldsymbol{k},\lambda)|^2 &= \sum_{\lambda=1,2} \epsilon^*_\mu(\boldsymbol{k},\lambda)\epsilon_\nu(\boldsymbol{k},\lambda)\mathcal{M}^\mu(\boldsymbol{k})\mathcal{M}^{\nu*}(\boldsymbol{k}) \\ &= |\mathcal{M}^1(\boldsymbol{k})|^2 + |\mathcal{M}^2(\boldsymbol{k})|^2 \ . \end{aligned} \tag{3}$$

– Caso del fotón.

Ahora bien, sabemos que el campo del fotón se acopla a una corriente conservada mediante una interacción $\int \mathrm{d}^4x\ j^\mu(x) A_\mu(x)$, con $\partial_\mu j^\mu(x) = 0$, así que

$$\mathcal{M}^\mu(\boldsymbol{k}) = \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} \langle f|\, j^\mu(x)\,|i\rangle$$

donde los estados inicial y final incluyen todas las partículas externas excepto el fotón en cuestión.

Como la simetría gauge se debe preservar también a nivel cuántico, de la conservación de la corriente y la expresión anterior deducimos[a]

$$k_\mu \mathcal{M}^\mu(\boldsymbol{k}) = \mathrm{i} \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} \langle f|\, \partial_\mu j^\mu(x)\,|i\rangle = 0$$

$$\Rightarrow k_\mu \mathcal{M}^\mu(\boldsymbol{k}) = \omega \mathcal{M}^0(\boldsymbol{k}) - \omega \mathcal{M}^3(\boldsymbol{k}) = 0 \Rightarrow \mathcal{M}^0(\boldsymbol{k}) = \mathcal{M}^3(\boldsymbol{k})\ .$$

[a] $0 = \int \mathrm{d}^4x\ \partial_\mu \left[\mathrm{e}^{\mathrm{i}kx} \langle f|\, j^\mu(x)\,|i\rangle\right] = \mathrm{i}k_\mu \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} \langle f|\, j^\mu(x)\,|i\rangle + \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} \langle f|\, \partial_\mu j^\mu(x)\,|i\rangle.$

Comentarios | Sobre las polarizaciones de los bosones vectoriales

– Caso del fotón.

Así que podemos reescribir $\sum_\lambda |\mathcal{M}(\boldsymbol{k},\lambda)|^2$ como

$$\sum_{\lambda=1,2} \epsilon^*_\mu(\boldsymbol{k},\lambda)\epsilon_\nu(\boldsymbol{k},\lambda)\mathcal{M}^\mu(\boldsymbol{k})\mathcal{M}^{\nu *}(\boldsymbol{k})$$

$$= |\mathcal{M}^1(\boldsymbol{k})|^2 + |\mathcal{M}^2(\boldsymbol{k})|^2 + |\mathcal{M}^3(\boldsymbol{k})|^2 - |\mathcal{M}^0(\boldsymbol{k})|^2$$

que equivale a reemplazar

$$\sum_\lambda \epsilon^*_\mu(\boldsymbol{k},\lambda)\epsilon_\nu(\boldsymbol{k},\lambda) \to -g_{\mu\nu} \ .$$

– Caso de un bosón vectorial masivo.

Tiene tres estados de polarización (uno longitudinal y dos transversos). En este caso podemos elegir el sistema de referencia en reposo, $k^\mu = (M, 0, 0, 0)$ y los estados de polarización

$$\epsilon^\mu(\boldsymbol{k}, 1) = (0, 1, 0, 0)\ , \quad \epsilon^\mu(\boldsymbol{k}, 2) = (0, 0, 1, 0)\ , \quad \epsilon^\mu(\boldsymbol{k}, 3) = (0, 0, 0, 1)\ .$$

Si sumamos sobre polarizaciones,

$$\sum_\lambda \epsilon^*_\mu(\boldsymbol{k}, \lambda)\epsilon_\nu(\boldsymbol{k}, \lambda) = -g_{\mu\nu} + Q_{\mu\nu}\ , \quad Q_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ ,$$

en el sistema de referencia en reposo.

Comentarios | Sobre las polarizaciones de los bosones vectoriales

– Caso de un bosón vectorial masivo.
Podemos obtener la expresión válida para $k^\mu = (k^0, \boldsymbol{k})$ con $M^2 = (k^0)^2 - \boldsymbol{k}^2$ haciendo un boost con $\gamma = k^0/M$, $\gamma\boldsymbol{\beta} = \boldsymbol{k}/M$,

$$\Lambda^\mu{}_{\mu'} = \begin{pmatrix} \gamma & \gamma\beta_1 & \gamma\beta_2 & \gamma\beta_3 \\ \gamma\beta_1 & & & \\ \gamma\beta_2 & & \delta_{ij} + (\gamma - 1)\dfrac{\beta_i\beta_j}{\beta^2} & \\ \gamma\beta_3 & & & \end{pmatrix}$$

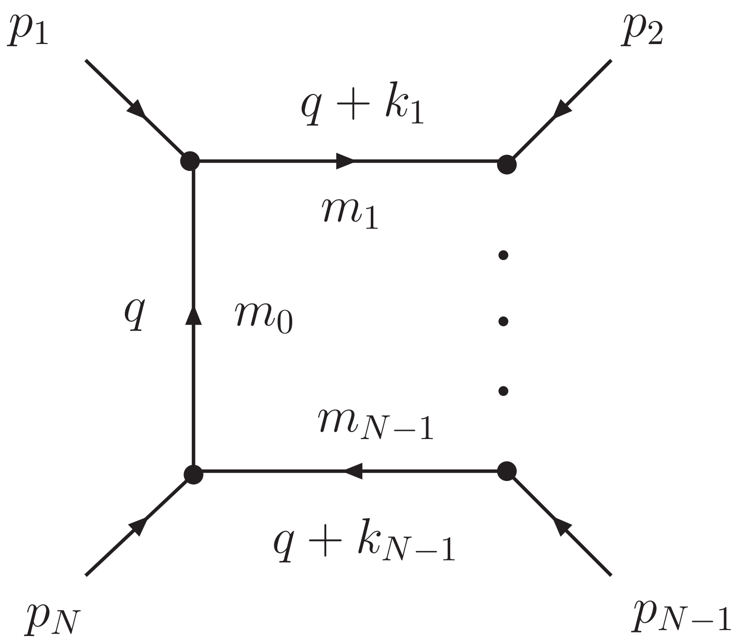
que conduce a

$$\sum_\lambda \epsilon^*_\mu(\boldsymbol{k},\lambda)\epsilon_\nu(\boldsymbol{k},\lambda) = -g_{\mu\nu} + \Lambda^0{}_\mu\Lambda^0{}_\nu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} .$$

Loop calculations

Structure of one-loop amplitudes

- Consider the following generic one-loop diagram with N external legs:



$$k_1 = p_1, \quad k_2 = p_1 + p_2, \quad \ldots \quad k_{N-1} = \sum_{i=1}^{N-1} p_i$$

- It contains general integrals of the kind:

$$\frac{\mathrm{i}}{16\pi^2} T^N_{\mu_1\ldots\mu_P} \equiv \mu^{4-D} \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{q_{\mu_1} \cdots q_{\mu_P}}{[q^2 - m_0^2][(q+k_1)^2 - m_1^2] \cdots [(q+k_{N-1})^2 - m_{N-1}^2]}$$

Structure of one-loop amplitudes

- ▷ D dimensional integration in dimensional regularization
- ▷ Integrals are symmetric under permutations of Lorentz indices
- ▷ Scale μ introduced to keep the proper mass dimensions
- ▷ P is the number of q's in the numerator and determines the tensor structure of the integral (scalar if $P = 0$, vector if $P = 1$, etc.). Note that $P \leq N$
- ▷ Notation: A for T^1, B for T^2, etc. For example, the scalar integrals A_0, B_0, etc.
- ▷ The tensor integrals can be decomposed as a linear combination of the Lorentz covariant tensors that can be built with $g_{\mu\nu}$ and a set of linearly independent momenta [Passarino, Veltman '79]
- ▷ The choice of basis is not unique

  Here we use the basis formed by $g_{\mu\nu}$ and the momenta k_i, where the the tensor coefficients are totally symmetric in their indices [Denner '93]

  This is the basis used by the computer package LoopTools [www.feynarts.de/looptools]

Structure of one-loop amplitudes

- We focus here on:

$$
\begin{aligned}
B_\mu &= k_{1\mu} B_1 \\
B_{\mu\nu} &= g_{\mu\nu} B_{00} + k_{1\mu} k_{1\nu} B_{11} \\
C_\mu &= k_{1\mu} C_1 + k_{2\mu} C_2 \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + \sum_{i,j=1}^{2} k_{i\mu} k_{j\nu} C_{ij} \\
C_{\mu\nu\rho} &= \ldots
\end{aligned}
$$

- We will see that the scalar integrals A_0 and B_0 and the tensor integral coefficients B_1, B_{00}, B_{11} and C_{00} are divergent in $D = 4$ dimensions (ultraviolet divergence, equivalent to take cutoff $\Lambda \to \infty$ in q)

- It is possible to express every tensor coefficient in terms of scalar integrals (scalar reduction) [Denner '93]

Explicit calculation

- Basic ingredients:

– Euler Gamma function:

$$\Gamma(x+1) \quad = \quad x\Gamma(x)$$

Taylor expansion around poles at $x = 0, -1, -2, \ldots$:

$$\begin{aligned} x = 0: \quad \Gamma(x) &= \frac{1}{x} - \gamma + \mathcal{O}(x) \\ x = -1: \quad \Gamma(x) &= -\frac{1}{(x+1)} + \gamma - 1 + \cdots + \mathcal{O}(x+1) \end{aligned}$$

where $\gamma \approx 0.5772\ldots$ is Euler-Mascheroni constant

– Feynman parameters:

$$\frac{1}{a_1 a_2 \cdots a_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{[x_1 a_1 + x_2 a_2 + \cdots x_n a_n]^n}$$

Explicit calculation

– The following integrals (with $\varepsilon \to 0^+$) will be needed:

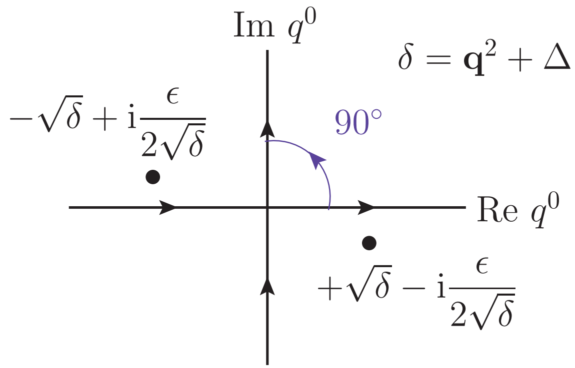
$$\int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2 - \Delta + \mathrm{i}\varepsilon)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-D/2}$$

$$\Rightarrow \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{q^2}{(q^2 - \Delta + \mathrm{i}\varepsilon)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n - D/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-D/2-1}$$

▷ Let's solve the first integral in Euclidean space: $q^0 = \mathrm{i}q_E^0$, $\boldsymbol{q} = \boldsymbol{q}_E$, $q^2 = -q_E^2$,

$$\int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2 - \Delta + \mathrm{i}\varepsilon)^n} = \mathrm{i}(-1)^n \int \frac{\mathrm{d}^D q_E}{(2\pi)^D} \frac{1}{(q_E^2 + \Delta)^n}$$

(equivalent to a Wick rotation of 90°). The second integral follows from this one



Explicit calculation

In D-dimensional spherical coordinates:

$$\int \frac{d^D q_E}{(2\pi)^D} \frac{1}{(q_E^2+\Delta)^n} = \int d\Omega_D \int_0^\infty dq_E q_E^{D-1} \frac{1}{(q_E^2+\Delta)^n} \equiv \mathcal{I}_A \times \mathcal{I}_B$$

where
$$\mathcal{I}_A = \int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

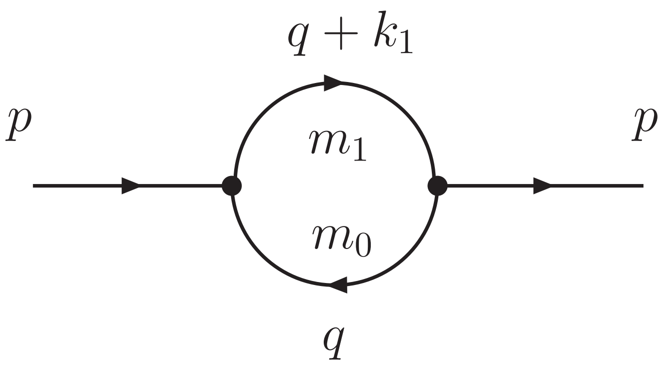
$$\text{since } (\sqrt{\pi})^D = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^D = \int d^D x\, e^{-\sum_{i=1}^D x_i^2} = \int d\Omega_D \int_0^\infty dx\, x^{D-1} e^{-x^2}$$
$$= \left(\int d\Omega_D\right) \frac{1}{2} \int_0^\infty dt\, t^{D/2-1} e^{-t} = \left(\int d\Omega_D\right) \frac{1}{2}\Gamma(D/2)$$

and, changing variables: $t = q_E^2$, $z = \Delta/(t+\Delta)$, we have

$$\mathcal{I}_B = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-D/2} \int_0^1 dz\, z^{n-D/2-1}(1-z)^{D/2-1} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-D/2} \frac{\Gamma(n-D/2)\Gamma(D/2)}{\Gamma(n)}$$

where Euler Beta function was used: $B(\alpha,\beta) = \int_0^1 dz\, z^{\alpha-1}(1-z)^{\beta-1} = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Explicit calculation | Two-point functions



$$\frac{\mathrm{i}}{16\pi^2}\{B_0,\ B^\mu,\ B^{\mu\nu}\}\,(\text{args}) = \mu^{4-D}\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\frac{\{1,\ q^\mu,\ q^\mu q^\nu\}}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]}$$

- ▷ $k_1 = p$
- ▷ The integrals depend on the masses m_0, m_1 and the invariant p^2:

$$(\text{args}) = (p^2; m_0^2, m_1^2)$$

Explicit calculation **Two-point functions**

- Using Feynman parameters,

$$\frac{1}{a_1 a_2} = \int_0^1 \mathrm{d}x \frac{1}{[a_1 x + a_2(1-x)]^2}$$

$$\Rightarrow \frac{\mathrm{i}}{16\pi^2}\{B_0,\ B^\mu,\ B^{\mu\nu}\} = \mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\{1,\ -A^\mu,\ q^\mu q^\nu + A^\mu A^\nu\}}{(q^2-\Delta_2)^2}$$

with

$$\Delta_2 = x^2 p^2 + x(m_1^2 - m_0^2 - p^2) + m_0^2$$

$$\begin{aligned} a_1 &= (q+p)^2 - m_1^2 \\ a_2 &= q^2 - m_0^2 \end{aligned}$$

and a loop momentum shift to obtain a perfect square in the denominator:

$$q^\mu \to q^\mu - A^\mu, \quad A^\mu = x p^\mu$$

Explicit calculation — Two-point functions

- Then, the scalar function is:

$$\begin{aligned}\frac{\mathrm{i}}{16\pi^2}B_0 &= \mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{1}{(q^2-\Delta_2)^2} \\ \Rightarrow \quad B_0 &= \Delta_\epsilon - \int_0^1 \mathrm{d}x\ \ln\frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]\end{aligned}$$

where $\Delta_\epsilon \equiv \dfrac{2}{\epsilon} - \gamma + \ln 4\pi$ and the Euler Gamma function was expanded around $x=0$ for $D=4-\epsilon$, using $x^\epsilon = \exp\{\epsilon \ln x\} = 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2)$:

$$\mu^{4-D}\frac{\mathrm{i}\Gamma(2-D/2)}{(4\pi)^{D/2}}\left(\frac{1}{\Delta_2}\right)^{2-D/2} = \frac{\mathrm{i}}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{\Delta_2}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

- Comparing with the definitions of the tensor coefficientes we have:

$$\begin{aligned}\frac{\mathrm{i}}{16\pi^2}B^\mu &= -\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{A^\mu}{(q^2-\Delta_2)^2} \\ \Rightarrow \quad B_1 &= -\frac{1}{2}\Delta_\epsilon + \int_0^1 \mathrm{d}x\ x\ln\frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]\end{aligned}$$

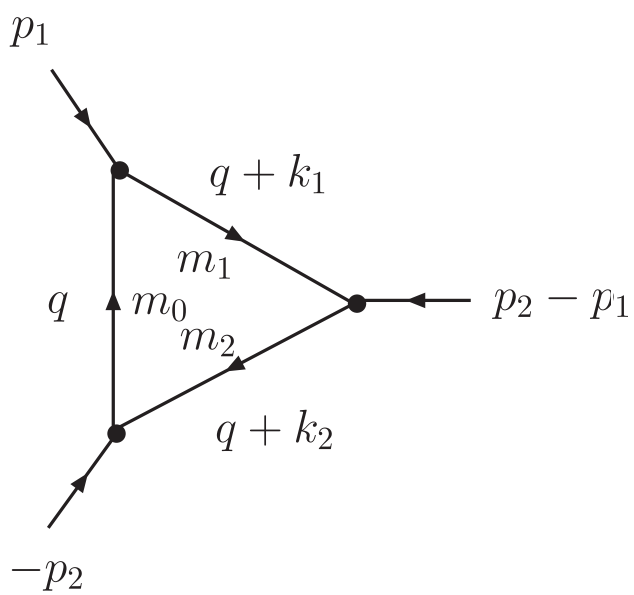
and

$$\begin{aligned} \frac{\mathrm{i}}{16\pi^2} B^{\mu\nu} &= \mu^{4-D} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(q^2/D) g^{\mu\nu} + A^\mu A^\nu}{(q^2 - \Delta_2)^2} \\ \Rightarrow \quad B_{00} &= -\frac{1}{12}(p^2 - 3m_0^2 - 3m_1^2)\Delta_\epsilon + \mathcal{O}(\epsilon) \qquad [D = 4-\epsilon] \\ B_{11} &= \frac{1}{3}\Delta_\epsilon - \int_0^1 \mathrm{d}x\; x^2 \ln\frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D = 4-\epsilon] \end{aligned}$$

where $q^\mu q^\nu$ have been replaced by $(q^2/D) g^{\mu\nu}$ in the integrand and the Euler Gamma function was expanded around $x = -1$ for $D = 4 - \epsilon$:

$$-\mu^{4-D} \frac{\mathrm{i}\Gamma(1-D/2)}{(4\pi)^{D/2} 2\Gamma(2)} \left(\frac{1}{\Delta_2}\right)^{1-D/2} = \frac{\mathrm{i}}{16\pi^2} \frac{1}{2} \left(\Delta_\epsilon - \ln\frac{\Delta_2}{\mu^2} + 1\right) \Delta_2 + \mathcal{O}(\epsilon)$$

Three-point functions



$$\frac{i}{16\pi^2}\{C_0,\ C^\mu,\ C^{\mu\nu}\}\,(\text{args}) = \mu^{4-D}\int\frac{d^Dq}{(2\pi)^D}\frac{\{1,\ q^\mu,\ q^\mu q^\nu\}}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]}$$

▷ It is convenient to choose the external momenta so that:

$$k_1 = p_1, \quad k_2 = p_2.$$

▷ The integrals depend on the masses m_0, m_1, m_2 and the invariants:

$$(\text{args}) = (p_1^2, Q^2, p_2^2; m_0^2, m_1^2, m_2^2), \quad Q^2 \equiv (p_2-p_1)^2.$$

Explicit calculation | Three-point functions

- Using Feynman parameters,

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{[a_1 x + a_2 y + a_3(1-x-y)]^3}$$

$$\Rightarrow \frac{\mathrm{i}}{16\pi^2}\{C_0,\ C^\mu,\ C^{\mu\nu}\} = 2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1,\ -A^\mu,\ q^\mu q^\nu + A^\mu A^\nu\}}{(q^2-\Delta_3)^3}$$

with

$$\Delta_3 = x^2 p_1^2 + y^2 p_2^2 + xy(p_1^2 + p_2^2 - Q^2) + x(m_1^2 - m_0^2 - p_1^2) + y(m_2^2 - m_0^2 - p_2^2) + m_0^2$$

$$\begin{aligned} a_1 &= (q+p_1)^2 - m_1^2 \\ a_2 &= (q+p_2)^2 - m_2^2 \\ a_3 &= q^2 - m_0^2 \end{aligned}$$

and a loop momentum shift to obtain a perfect square in the denominator:

$$q^\mu \to q^\mu - A^\mu, \quad A^\mu = x p_1^\mu + y p_2^\mu$$

- Then the scalar function is:

$$\frac{\mathrm{i}}{16\pi^2}C_0 = 2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2-\Delta_3)^3}$$

$$\Rightarrow \quad C_0 = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{1}{\Delta_3} \qquad [D=4]$$

- Comparing with the definitions of the tensor coefficientes we have:

$$\frac{\mathrm{i}}{16\pi^2}C^\mu = -2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{A^\mu}{(q^2-\Delta_3)^3}$$

$$\Rightarrow \quad C_1 = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{x}{\Delta_3} \qquad [D=4]$$

$$C_2 = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y}{\Delta_3} \qquad [D=4]$$

Explicit calculation | Three-point functions

$$\frac{\mathrm{i}}{16\pi^2}C^{\mu\nu} = 2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(q^2/D)g^{\mu\nu} + A^\mu A^\nu}{(q^2-\Delta_3)^3}$$

$$\Rightarrow \quad C_{11} = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{x^2}{\Delta_3} \qquad [D=4]$$

$$C_{22} = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y^2}{\Delta_3} \qquad [D=4]$$

$$C_{12} = -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{xy}{\Delta_3} \qquad [D=4]$$

$$C_{00} = \frac{1}{4}\Delta_\epsilon - \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \ln\frac{\Delta_3}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]$$

where $\Delta_\epsilon \equiv \frac{2}{\epsilon} - \gamma + \ln 4\pi$ and $q^\mu q^\nu$ was replaced by $(q^2/D)g^{\mu\nu}$ in the integrand

In C_{00} the Euler Gamma function was expanded around $x=0$ for $D=4-\epsilon$:

$$\mu^{4-D}\frac{\mathrm{i}\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(3)}\left(\frac{1}{\Delta_3}\right)^{2-D/2} = \frac{\mathrm{i}}{16\pi^2}\frac{1}{2}\left(\Delta_\epsilon - \ln\frac{\Delta_3}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

Note about Diracology in D dimensions

- Attention should be paid to the traces of Dirac matrices when working in D dimensions (dimensional regularization) since

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}_{4\times 4}, \quad g^{\mu\nu}g_{\mu\nu} = \mathrm{Tr}\{g^{\mu\nu}\} = D$$

Thus, the following identities involving contractions of Lorentz indices can be proven:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= D \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(D-2)\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-D)\gamma^\nu\gamma^\rho\gamma^\sigma
\end{aligned}$$

- Para el cálculo del momento dipolar magnético anómalo del electrón en QED, necesitaremos las siguientes funciones de tres puntos, evaluadas en:

$$\begin{aligned} p_1^2 = p_2^2 = m^2 \quad & \text{(electrones } \textit{on-shell}\text{)} \\ Q^2 = 0 \quad & \text{(fotón } \textit{on-shell}\text{)} \\ m_0 = 0 \quad & \text{(masa del fotón)} \\ m_1 = m_2 = m \quad & \text{(masa del electrón)} \\ \Rightarrow \quad & \Delta_3 = m^2(x+y)^2. \end{aligned}$$

  Las integrales básicas son entonces

$$\begin{aligned} C_0 &= \text{divergente en el infrarrojo (no se necesita)}, \\ C_1 = C_2 &= \frac{1}{2m^2}, \\ C_{11} = C_{22} = 2\,C_{12} &= -\frac{1}{6m^2}. \\ C_{00} &= \text{divergente en el ultravioleta (no se necesita)}, \end{aligned}$$

  donde $C \equiv C(m^2, 0, m^2; 0, m^2, m^2)$.

- Para el cálculo de las contribuciones débiles (y de supersimetría) a los momentos dipolares magnéticos se necesitan las siguientes funciones de tres puntos, evaluadas en:

$$\begin{aligned} p_1^2 = p_2^2 &= 0 && \text{(se desprecian las masas de los fermiones externos)} \\ Q^2 &= 0 && \text{(fotón } \textit{on-shell}\text{)} \\ m_0 &= M_1 && \text{(masa de la partícula virtual no acoplada al fotón externo)} \\ m_1 = m_2 &= M_2 && \text{(masa de las otras partículas virtuales)} \\ &\Rightarrow && \Delta_3 = (M_2^2 - M_1^2)(x+y) + M_1^2. \end{aligned}$$

Casos sencillos | Integrales para $(g-2)_\ell$ en SM y MSSM

▷ Las integrales básicas son

$$\begin{aligned}
C_0 &= \frac{1}{M_1^2}\frac{1-x_{21}+\ln x_{21}}{(1-x_{21})^2},\\
C_1 = C_2 &= \frac{1}{M_1^2}\frac{-3+4x_{21}-x_{21}^2-2\ln x_{21}}{4(1-x_{21})^3},\\
C_{11} = C_{22} = 2\,C_{12} &= \frac{1}{M_1^2}\frac{11-18x_{21}+9x_{21}^2-2x_{21}^3+6\ln x_{21}}{18(1-x_{21})^4},\\
C_{00} &= \text{divergente en el ultravioleta (no se necesita).}
\end{aligned}$$

▷ O bien

$$\begin{aligned}
\overline{C}_0 &= \frac{1}{M_1^2}\frac{-1+x_{21}-x_{21}\ln x_{21}}{(1-x_{21})^2},\\
\overline{C}_1 = \overline{C}_2 &= \frac{1}{M_1^2}\frac{1-4x_{21}+3x_{21}^2-2x_{21}^2\ln x_{21}}{4(1-x_{21})^3},\\
\overline{C}_{11} = \overline{C}_{22} = 2\,\overline{C}_{12} &= \frac{1}{M_1^2}\frac{-2+9x_{21}-18x_{21}^2+11x_{21}^3-6x_{21}^3\ln x_{21}}{18(1-x_{21})^4},
\end{aligned}$$

donde $C \equiv C(0,0,0;M_1^2,M_2^2,M_2^2)$, $\overline{C} \equiv C(0,0,0;M_2^2,M_1^2,M_1^2)$ y $x_{21} \equiv M_2^2/M_1^2$.

- Para el cálculo de las contribuciones débiles (y de supersimetría) a las transiciones radiativas del tipo $\mu \to e\gamma$ se usan las funciones de tres puntos anteriores.

▷ Para comprobar que esta transición es puramente magnética, es decir no contribuyen las auto-energías de las patas externas, conviene conocer explícitamente C_{00} evaluada en la misma configuración anterior:

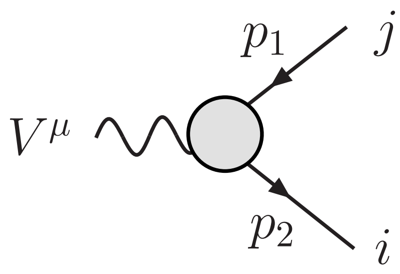
$$C_{00}(0,0,0;M_1^2,M_2^2,M_2^2) = -\frac{1}{2}B_1(0;M_1^2,M_2^2)$$

y las siguientes funciones de dos puntos evaluadas en $p^2=0$, $m_0=M_1$, $m_1=M_2$:

$$\begin{aligned}
B_0(0;M_1^2,M_2^2) &= \Delta_\epsilon + 1 - \frac{M_1^2 \ln \dfrac{M_1^2}{\mu^2} - M_2^2 \ln \dfrac{M_2^2}{\mu^2}}{M_1^2 - M_2^2} \\
B_1(0;M_1^2,M_2^2) &= -\frac{1}{2}\Delta_\epsilon + \frac{4M_1^4 - 3M_2^4 - M_1^2M_2^2 - 2\ln \dfrac{M_1^2}{M_2^2}}{4(M_1^2-M_2^2)^2} \\
&= -B_0(0;M_2^2,M_1^2) - B_1(0;M_2^2,M_1^2)
\end{aligned}$$

Factores de forma dipolares a un loop

El vértice vector-fermión más general



$\mathrm{i}\Gamma^\mu(p_1, p_2)$

- La estructura Lorentz más general del vértice vector-fermión contiene 24 términos independientes, que son combinaciones de los cuadrivectores $p \equiv p_1 + p_2$, $q \equiv p_2 - p_1$ y las 16 matrices de Dirac (indicadas abajo entre paréntesis):

$$
\begin{aligned}
(1) \quad &: \quad p^\mu,\ q^\mu, \\
(\gamma_5) \quad &: \quad \gamma_5 p^\mu,\ \gamma_5 q^\mu, \\
(\gamma^\alpha) \quad &: \quad \gamma^\mu,\ p^\mu \not{p},\ p^\mu \not{q},\ q^\mu \not{p},\ q^\mu \not{q},\ \epsilon^{\mu\nu\alpha\beta}\gamma_\nu p_\alpha q_\beta, \\
(\gamma_5\gamma^\alpha) \quad &: \quad \gamma_5\gamma^\mu,\ \gamma_5 p^\mu \not{p},\ \gamma_5 p^\mu \not{q},\ \gamma_5 q^\mu \not{p},\ \gamma_5 q^\mu \not{q},\ \gamma_5 \epsilon^{\mu\nu\alpha\beta}\gamma_\nu p_\alpha q_\beta, \\
(\sigma^{\alpha\beta}) \quad &: \quad \sigma^{\mu\nu} p_\nu,\ \sigma^{\mu\nu} q_\nu,\ p^\mu \sigma^{\alpha\beta} p_\alpha q_\beta,\ q^\mu \sigma^{\alpha\beta} p_\alpha q_\beta, \\
&\quad\ \ \epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} p_\nu,\ \epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} q_\nu,\ p^\mu \epsilon^{\alpha\beta\rho\sigma}\sigma_{\alpha\beta} p_\rho q_\sigma,\ q^\mu \epsilon^{\alpha\beta\rho\sigma}\sigma_{\alpha\beta} p_\rho q_\sigma.
\end{aligned}
$$

El vértice vector-fermión más general

- Con frecuencia el vértice se escribe de la siguiente forma:

$$\begin{aligned} \mathrm{i}\Gamma^\mu(p_1,p_2) &= \mathrm{i}e\,[\gamma^\mu(F_V - F_A\gamma_5) + (\mathrm{i}F_M + F_E\gamma_5)\sigma^{\mu\nu}q_\nu + (\mathrm{i}F_S + F_P\gamma_5)q^\mu \\ &\quad +(F_{MV} + \mathrm{i}F_{EV}\gamma_5)p^\mu + (F_{TS} + \mathrm{i}F_{TP}\gamma_5)\sigma^{\mu\nu}p_\nu + \ldots]\,. \end{aligned}$$

▷ Los factores de forma F_i son en general funciones de todos los escalares independientes (invariantes Lorentz) que se puedan construir con los vectores p_1 y p_2, es decir, $F_i(p_1^2, p_2^2, q^2)$. La constante e se ha introducido por conveniencia, de modo que los acoplamientos quedan normalizados a los de la electrodinámica cuántica (QED).

- Si ambos fermiones están *on-shell* (es decir, $p^2 = m^2$), la ecuación de Dirac nos permite eliminar los términos omitidos anteriormente y también F_{MV}, F_{EV} F_{TS} y F_{TP}, pues ya no son independientes y entonces

$$\textit{ij on-shell}:\ \mathrm{i}\Gamma^\mu(p_1,p_2) = \mathrm{i}e\big[\gamma^\mu(F_V - F_A\gamma_5) + (\mathrm{i}F_M + F_E\gamma_5)\sigma^{\mu\nu}q_\nu + (\mathrm{i}F_S + F_P\gamma_5)q^\mu\big]\,.$$

El vértice vector-fermión más general

▷ Basta usar la siguiente relación entre matrices de Dirac:

$$\gamma_5\gamma_\rho\epsilon^{\rho\mu\nu\sigma} = \frac{\mathrm{i}}{6}(\gamma^\mu\gamma^\nu\gamma^\sigma + \gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\sigma\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\sigma\gamma^\nu\gamma^\mu),$$

la ecuación de Dirac (ED): $\not{p}_1 u(p_1) = m_1 u(p_1)$, $\not{p}_2 u(p_2) = m_2 u(p_2)$, y las identidades de Gordon (que se deducen de la ED y $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$):

$$\bar{u}(p_2)\sigma^{\mu\nu}(p_2 \pm p_1)_\nu u(p_1) = \bar{u}(p_2)\left\{-\mathrm{i}(m_2 \mp m_1)\gamma^\mu + \mathrm{i}(p_2 \mp p_1)^\mu\right\} u(p_1),$$

$$\bar{u}(p_2)\gamma_5\sigma^{\mu\nu}(p_2 \pm p_1)_\nu u(p_1) = \bar{u}(p_2)\left\{-\mathrm{i}(m_2 \pm m_1)\gamma^\mu\gamma_5 + \mathrm{i}\gamma_5(p_2 \mp p_1)^\mu\right\} u(p_1).$$

- Si el bosón vectorial V también está *on-shell*, su polarización satisface $q^\mu\varepsilon_\mu = 0$ y por tanto los factores de forma F_S, F_P son irrelevantes y el vértice se reduce a:

$$\textit{Vij on-shell}: \quad \mathrm{i}\Gamma^\mu(p_1, p_2) = \mathrm{i}e\left[\gamma^\mu(F_V - F_A\gamma_5) + (\mathrm{i}F_M + F_E\gamma_5)\sigma^{\mu\nu}q_\nu\right].$$

Los factores de forma F_V, F_A y $F_{M,E}$ se denominan vectorial, axial y dipolares, respectivamente.

Una partícula masiva de spin 1 de momento p^μ tiene tres grados de libertad de polarización $\varepsilon^\mu(\lambda = 1,2,3)$ que verifican $p^\mu\varepsilon_\mu(\lambda) = 0$ y $\varepsilon^\mu(\lambda)\varepsilon_\mu(\lambda') = -\delta_{\lambda\lambda'}$. En reposo: $p^\mu = (M,0,0,0)$, $\varepsilon_1^\mu = (0,1,0,0)$, $\varepsilon_2^\mu = (0,0,1,0)$, $\varepsilon_3^\mu = (0,0,0,1)$.
En movimiento: $p^\mu = (E,0,0,p)$, $\varepsilon_x^\mu = (0,1,0,0)$, $\varepsilon_y^\mu = (0,0,1,0)$, $\varepsilon_L^\mu = (p/M,0,0,E/M)$ [circular: $\varepsilon_\pm^\mu = 1/\sqrt{2}(\epsilon_x^\mu \pm \mathrm{i}\epsilon_y^\mu)$].
Si tiene masa nula (ej. el fotón) no existe el s.r. en reposo y sólo existen las dos polarizaciones transversales.

El vértice vector-fermión más general

- Si $V = \gamma$ (fotón) la invariancia gauge U(1) impone la conservación de la corriente, $q_\mu \Gamma^\mu = 0$, y por tanto para fermiones *on-shell*:

$$\begin{array}{lrcl} [V=\gamma] & (m_i - m_j)F_V + \mathrm{i}q^2 F_S & = & 0, \\ ij \textit{ on-shell} & -(m_i + m_j)F_A + q^2 F_P & = & 0. \end{array}$$

- En consecuencia, si también el fotón está *on-shell* ($q^2 = 0$) y los fermiones son idénticos ($m = m_i = m_j$), necesariamente $F_A = 0$. El vértice electromagnético viene entonces descrito por tres constantes, relacionadas con la carga y los momentos dipolar magnético y dipolar eléctrico:

$$\boxed{\gamma ii \textit{ on-shell} \;:\quad \mathrm{i}\Gamma^\mu_{i=j} = \mathrm{i}e\left[\gamma^\mu F_V + (\mathrm{i}F_M + F_E\gamma_5)\sigma^{\mu\nu}q_\nu\right]}$$

El vértice vector-fermión más general

▷ Entonces, de acuerdo con nuestra convención para la derivada covariante,

$$\begin{aligned} eQ_f \equiv -eF_V(0) &= \text{carga eléctrica del fermión } f, \\ \mu \equiv -\left(\frac{e}{2m}F_V(0) + eF_M(0)\right) &= \text{momento dipolar magnético (MDM)}, \\ \mathrm{a} \equiv 2m\frac{F_M(0)}{F_V(0)} &= \text{momento dipolar magnético anómalo (AMDM)}, \\ \mathrm{d} = -eF_E(0) &= \text{momento dipolar eléctrico (EDM)}. \end{aligned}$$

- Así, a nivel árbol (electrodinámica clásica), un electrón tiene acoplamientos $F_V = 1$, $F_A = F_M = F_E = 0$, y por tanto carga $Q_e = -1$ y momento dipolar magnético

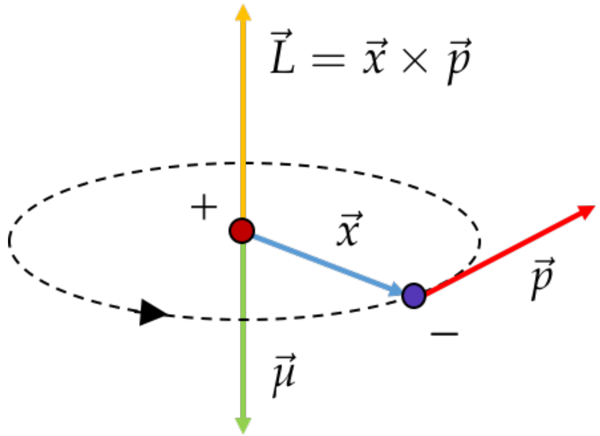
$$\mu = -\frac{\mathrm{g}}{2}\frac{e}{2m}, \quad \mathrm{g} \approx 2 \quad \Leftarrow \quad \text{interacción no relativista} \quad \boldsymbol{\mu}\cdot\mathbf{B} \equiv \mu\,\boldsymbol{\sigma}\cdot\mathbf{B}$$

donde $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ es el spin y g es la razón giromagnética o factor de Landé.

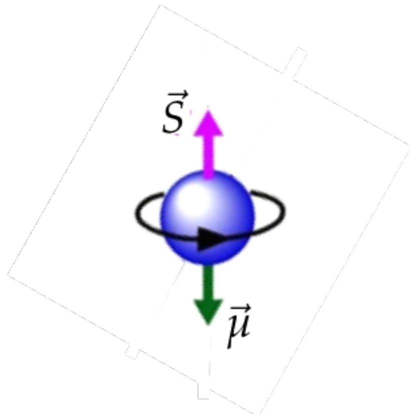
Nótese que el momento anómalo y la razón giromagnética se definen sólo para partículas cargadas.
Sin embargo una partícula neutra puede tener momento magnético dado por $\mu = -eF_M(0)$.

Momento dipolar magnético

$$\boldsymbol{\mu} = \frac{eQ}{2m}\boldsymbol{L}$$



$$\boldsymbol{\mu} = \mathrm{g}\frac{eQ}{2m}\boldsymbol{S} = \underbrace{\frac{\mathrm{g}}{2}\frac{eQ}{2m}}_{\mu}\boldsymbol{\sigma} \tag{5}$$



El vértice vector-fermión más general

- Las correcciones cuánticas inducen valores no nulos de AMDM y EDM. Las condiciones de renormalización fijan $F_V(0) = -Q_f$ (a todo orden de teoría de perturbaciones), pero aparece un AMDM que viene dado por

$$\mathrm{a} = \frac{\mathrm{g}-2}{2} = -2m\frac{F_M(0)}{Q_f} \Rightarrow \mu = \frac{\mathrm{g}}{2}\frac{eQ_f}{2m} = (1+\mathrm{a})\frac{eQ_f}{2m}.$$

- Por otro lado, las ecuaciones anteriores implican $F_V = F_A = 0$ para fermiones distintos. Es decir, procesos tales como $\mu \to e\gamma$ se deben sólo a transiciones dipolares,

$$\boxed{\gamma ij \textit{ on-shell}: \quad \mathrm{i}\Gamma^{\mu}_{i\neq j} = \mathrm{i}e(\mathrm{i}F_M + F_E\gamma_5)\sigma^{\mu\nu}q_\nu}$$

El vértice vector-fermión más general

- En general, todos los factores de forma son reales a nivel árbol para fermiones externos iguales, por la hermiticidad del lagrangiano, pero se pueden hacer complejos al introducir las correcciones cuánticas (regla de Cutkosky).

  La amplitud se hace compleja cuando sea posible cortar el diagrama en dos diagramas tales que ambos describan procesos físicos. Se trata de una aplicación del teorema óptico. Es fácil darse cuenta de que si $V = \gamma$ la amplitud ha de ser siempre real porque el fotón tiene masa nula.

- Los factores de forma que acompañan a los operadores de dimensión mayor que cuatro (todos menos F_V y F_A), ej. los dipolares, son nulos a nivel árbol en cualquier teoría renormalizable. Por tanto sus correcciones a un loop son finitas.

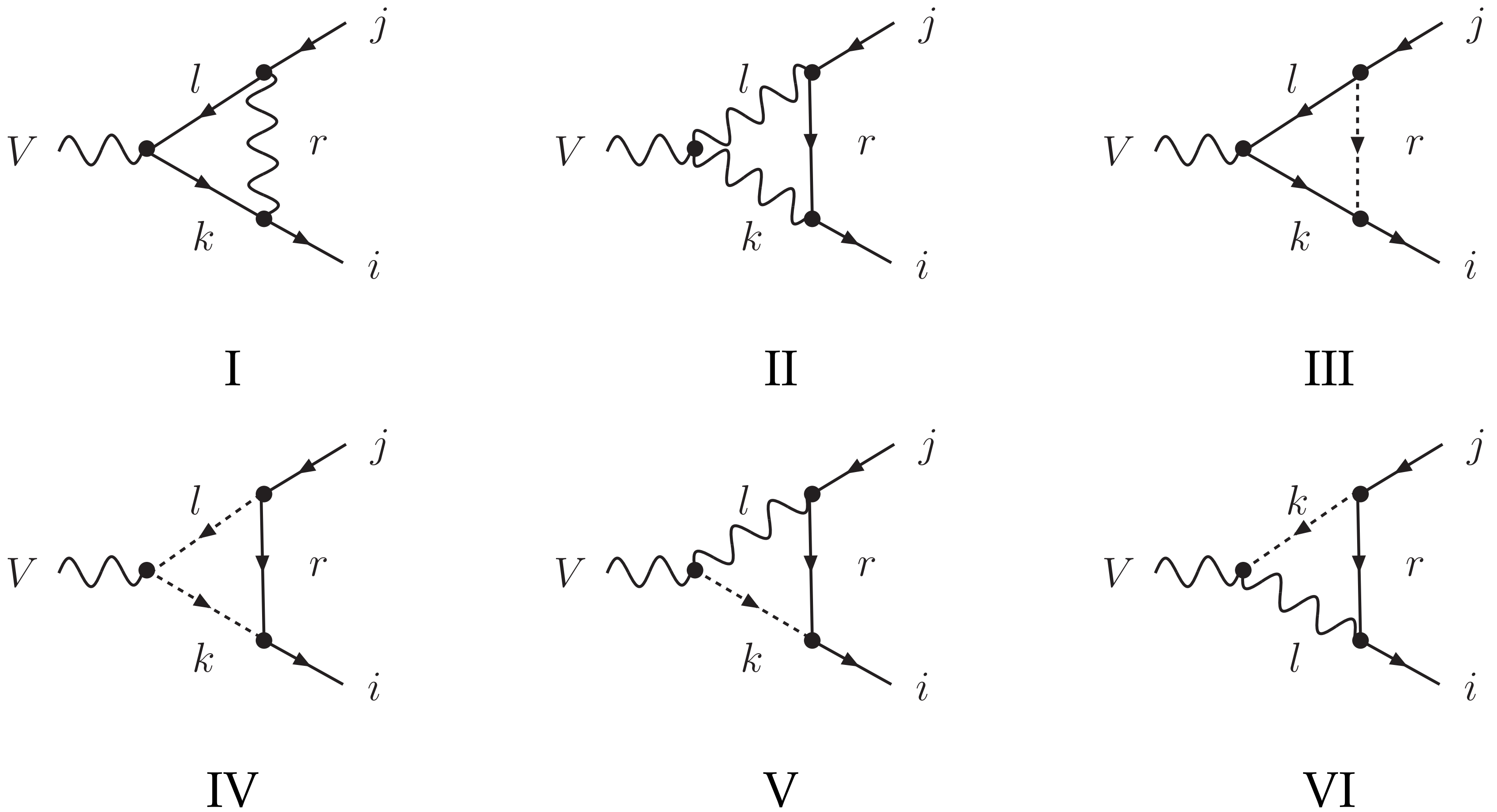
  Además acoplan fermiones de quiralidades contrarias, por lo que deben ser proporcionales a alguna masa fermiónica, ya sea interna o externa.

- Los factores de forma F_V, F_A y F_M multiplican sendos bilineales pares bajo CP, mientras que F_E acompaña a uno impar. Esto significa que si CP se conserva el momento dipolar F_E se anula si $i = j$, aunque esto no ocurre si los fermiones externos son distintos.

- Similarmente, si P se conserva (ej. en QED) F_A y F_E son nulos.

El vértice vector-fermión más general

- Los diagramas que contribuyen a un loop al vértice efectivo vector-fermión pueden agruparse en seis clases o topologías distintas:



El momento magnético anómalo

- Los momentos magnéticos anómalos de electrón y muón son observables medidos con gran precisión. Como se deben enteramente a correcciones cuánticas ponen especialmente a prueba la consistencia de la teoría.

- El momento magnético del electrón se ha medido en un ciclotrón en Harvard con una precisión impresionante: [Particle Data Group, Phys. Rev. **D86** (2012) 010001]

$$\frac{g_e}{2} = 1.001\,159\,652\,180\,76\,(27)$$

- El momento magnético del muón se obtiene a partir de la frecuencia de precesión del spin respecto a un campo magnético homogéneo en un anillo de almacenamiento de muones:

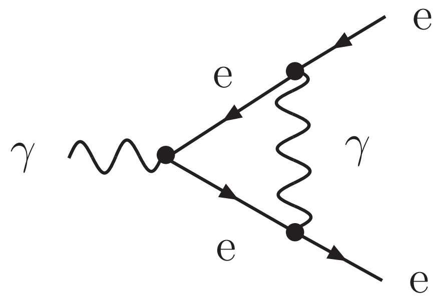
$$\omega_a = a_\mu \frac{eB}{2m}, \quad a_\mu = \frac{(g_\mu - 2)}{2} \; .$$

  Se ha medido con gran precisión en el experimento E821 de Brookhaven:

  [G.W. Bennett *et al.*, Phys. Rev. D **73** (2006) 072003]

$$\frac{g_\mu}{2} = 1.001\,165\,920\,91\,(63)$$

El momento magnético anómalo en QED



- En QED sólo existe un diagrama a un loop (clase I) y el único acoplamiento no nulo es del tipo [VFF] con $g_V = 1$, $g_A = 0$. La configuración de masas y momentos es también muy simple. El AMDM del electrón es entonces:

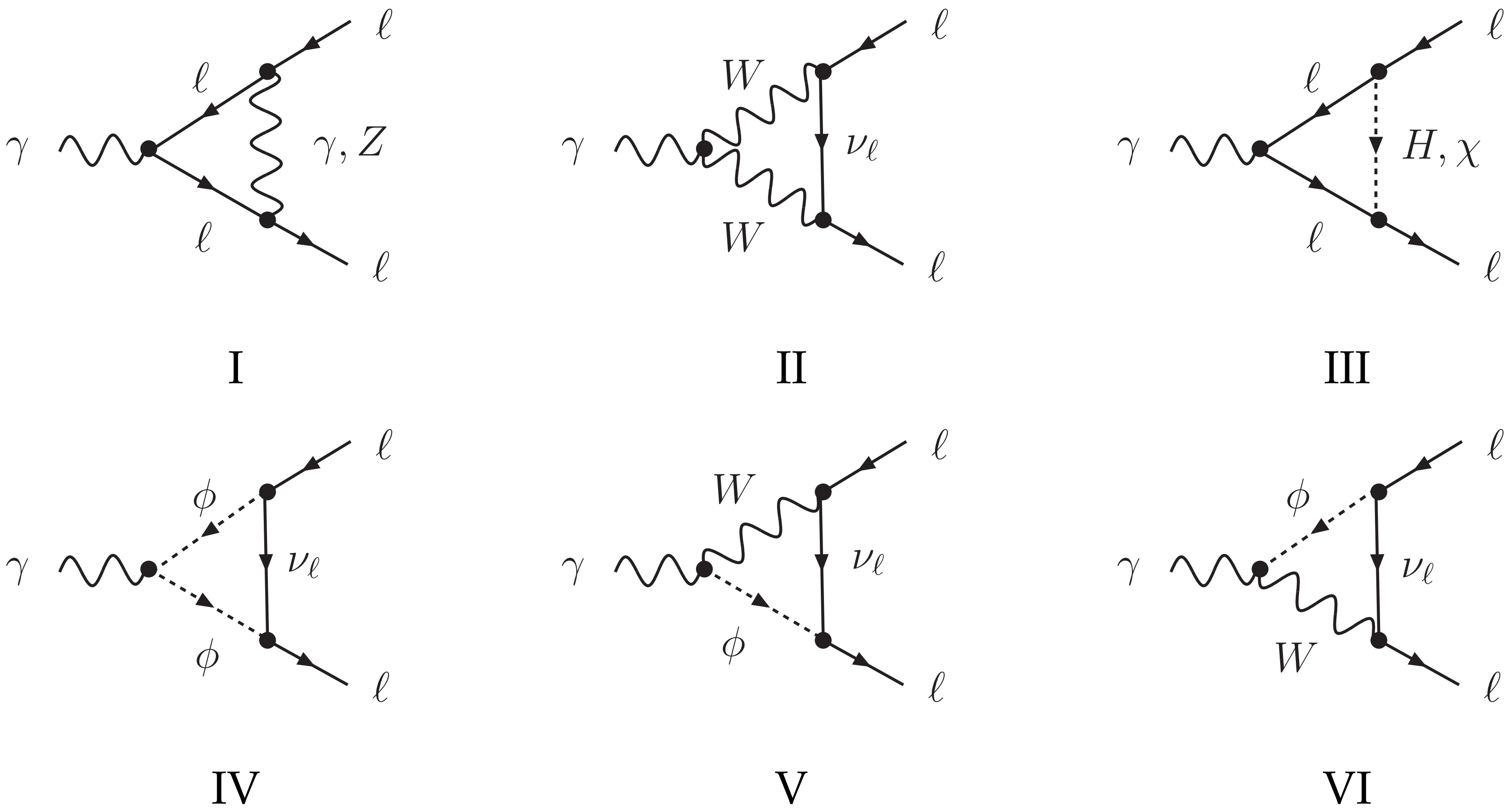
$$F_M(0) = \frac{\alpha}{4\pi} 2m(C_1 + C_2 + C_{11} + C_{22} + 2C_{12}) = \frac{\alpha}{4\pi}\frac{1}{m},$$

$$\mathrm{a} = \frac{\mathrm{g}-2}{2} = 2m\, F_M(0) = \frac{\alpha}{2\pi},$$

donde $\alpha = e^2/4\pi$ es la constante de estructura fina y hemos utilizado las funciones de tres puntos con argumentos $(m^2, 0, m^2; 0, m^2, m^2)$ evaluadas anteriormente.

El momento magnético anómalo en el SM

- Los diagramas a un loop en el gauge de ’t Hooft-Feynman (incluyendo QED) son:



- Para los nuevos, necesitaremos más paciencia, las reglas de Feynman del SM y las funciones de tres puntos con argumentos del tipo $(0, 0, 0; M_1^2, M_2^2, M_2^2)$ evaluadas anteriormente. (Se puede despreciar la masa del leptón ℓ.)

El momento magnético anómalo en el SM

- Sumando todas las contribuciones a un loop:

$$a_\ell = \frac{(g_\ell - 2)}{2} = \underbrace{\frac{\alpha}{2\pi}}_{\text{QED}} + \frac{G_F}{8\pi^2\sqrt{2}} m_\ell^2 \left\{ \underbrace{\frac{10}{3}}_{\text{W}} \underbrace{- \frac{1}{3}\left[5 - (1 - 4s_W^2)^2\right]}_{\text{Z}} + \underbrace{\mathcal{O}\left(\frac{m_\ell^2}{M_H^2} \ln \frac{M_H^2}{m_\ell^2}\right)}_{\text{H}} \right\}$$

donde

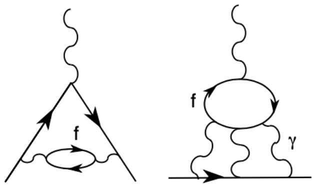
$$G_F = \frac{\pi\alpha}{\sqrt{2} s_W^2 M_W^2}(1 + \Delta r) = 1.1663787\,(6) \times 10^{-5}\ \text{GeV}^{-2} \Leftarrow \text{vida media del muon}$$

$$\alpha^{-1} = 137.035\,999\,139\,(31) \Leftarrow \text{comparando con } (g_e - 2) \text{ en QED a 5 loops}$$

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2} = 0.223, \quad m_e = 0.511\ \text{MeV}, \quad m_\mu = 0.106\ \text{GeV}, \quad m_\tau = 1.777\ \text{GeV}.$$

▷ Nótese que $(1 - 4s_W^2)^2 \simeq 0.012$ por lo que la Z contribuye aproximadamente la mitad que la W y con signo opuesto. La contribución del Higgs es despreciable.

- A continuación resumimos las predicciones del SM y las medidas actuales del momento anómalo del muón a_μ [Particle Data Group, Phys. Rev. **D98** (2018) 030001]

CÁLCULOS TEÓRICOS		Contribución a $a_\mu(\times 10^{11})$
QED	coeficiente de $\left(\frac{\alpha}{\pi}\right)^n$	
$n=1$	0.5	116 140 973
$n=2$	0.765 857 425 (17)	...
$n=3$	24.050 509 96 (32)	...
$n=4$	130.879 6 (63)	...
$n=5$	753. (10)	...
Total		116 584 719
Débil	1. loop	195
	2. loop	-41
Hadrónica		LO: 6 930 (34)
		N(N)LO: 7 (26)
TOTAL		$a_\mu^{\text{teo}} = 116\,591\,810\,(43)$

MEDIDA EXPERIMENTAL	
MEDIDA EXPERIMENTAL: E821 (Brookhaven '06)	$a_\mu^{\text{exp}} = 116\,592\,089\,(63)$
Muon $g-2$ (Fermilab '21)	$a_\mu^{\text{exp}} = 116\,592\,061\,(41)$

$\Rightarrow$ Discrepancia de 3.7σ (4.2σ)!!

El proceso raro $\mu \to e\gamma$

- Se trata de un proceso con violación de sabor leptónico (LFV), que en el SM con neutrinos sin masa está prohibido.

- Sin embargo, en el SM con neutrinos masivos o en otras extensiones del SM, como supersimetría, este proceso puede darse.

- La colaboración MEG lleva acabo un experimento en el PSI (Suiza) desde el 2004 con el haz de muones más intenso del mundo. No han observado ningún suceso, lo que pone una cota actualmente de $\mathcal{B}(\mu \to e\gamma) < 4.2 \times 10^{-13}$. El objetivo es llegar en unos años hasta 10^{-14}.

- Los experimentos BaBar en el PEP-II de SLAC (EE UU) y Belle en el KEKB (Japón) ponen cotas a desintegraciones similares del τ. Las más actuales son $\mathcal{B}(\tau \to \mu\gamma) < 4.4 \times 10^{-8}$ y $\mathcal{B}(\tau \to e\gamma) < 3.3 \times 10^{-8}$.

El proceso raro $\mu \to e\gamma$

- Recordemos que se trata de transiciones dipolares, lo que puede comprobarse explícitamente hallando las contribuciones a F_V y F_A, que son nulas.

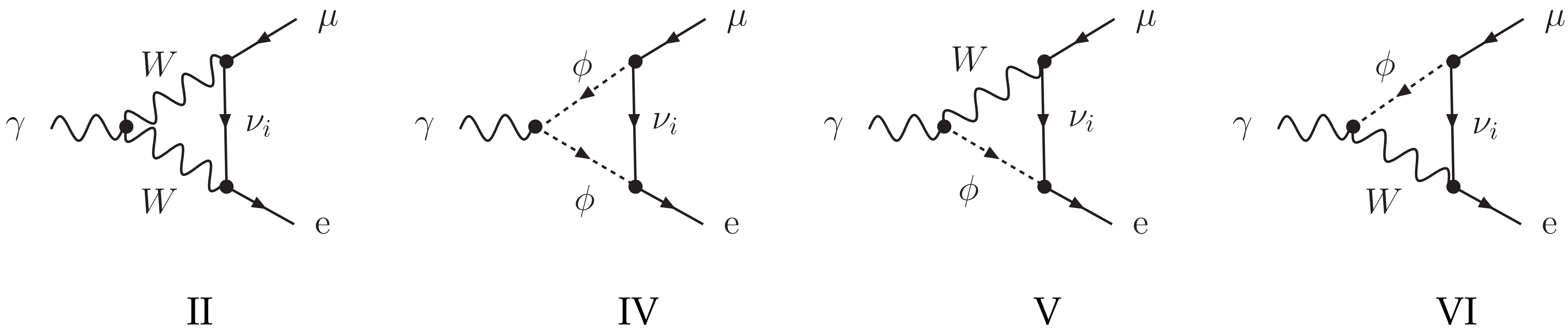
- Las anchuras y fracciones de desintegración relevantes son

$$
\begin{aligned}
\Gamma(\ell_j \to \ell_i \gamma) &= \frac{\alpha}{2} m_{\ell_j}^3 \left(|F_M|^2 + |F_E|^2 \right), \\
\Gamma(\ell_j \to \ell_i \nu_j \bar{\nu}_i) &= \frac{G_F^2 m_{\ell_j}^5}{192\pi^3}, \quad G_F = \frac{\pi \alpha_W}{\sqrt{2} M_W^2}, \\
\frac{\mathcal{B}(\ell_j \to \ell_i \gamma)}{\mathcal{B}(\ell_j \to \ell_i \nu_j \bar{\nu}_i)} &= \frac{\Gamma(\ell_j \to \ell_i \gamma)}{\Gamma(\ell_j \to \ell_i \nu_i \bar{\nu}_j)} \\
&= \frac{12\alpha}{\pi} \frac{M_W^4}{m_{\ell_j}^2} \left(\frac{4\pi}{\alpha_W} \right)^2 \left(|F_M|^2 + |F_E|^2 \right),
\end{aligned}
$$

donde $\mathcal{B}(\ell_j \to \ell_i \nu_j \bar{\nu}_i) = 1/0.17/0.17$ para $\ell_j \ell_i = \mu e/\tau\mu/\tau e$.

$\mu \to e\gamma$ en el SM con neutrinos masivos

- Los diagramas a un loop en el gauge de 't Hooft-Feynman son:



- A continuación listamos las contribuciones de cada clase de diagramas que se obtienen usando las funciones de tres puntos con argumentos $(0,0,0; m_{\nu_i}^2, M_W^2, M_W^2)$.

$\mu \to e\gamma$ en el SM con neutrinos masivos

- Definiendo $x_i \equiv m_{\nu_i}^2 / M_W^2$

$$\text{II:} \quad F_M = -\mathrm{i}F_E = -\frac{\alpha_W}{16\pi} m_\mu \sum_i U_{ei} U_{\mu i}^* \; [3\overline{C}_{11} - \overline{C}_1]$$

$$\text{IV:} \quad F_M = -\mathrm{i}F_E = -\frac{\alpha_W}{16\pi} m_\mu \sum_i U_{ei} U_{\mu i}^* \; x_i \; \left[\overline{C}_0 + 3\overline{C}_1 + \frac{3}{2}\overline{C}_{11}\right]$$

$$\text{V:} \quad F_M = -\mathrm{i}F_E = 0$$

$$\text{VI:} \quad F_M = -\mathrm{i}F_E = \frac{\alpha_W}{16\pi} m_\mu \sum_i U_{ei} U_{\mu i}^* \; \overline{C}_1$$

$$\text{Total:} \quad F_M = -\mathrm{i}F_E = \frac{\alpha_W}{16\pi} \frac{m_\mu}{M_W^2} \sum_i U_{ei} U_{\mu i}^* \; F_W(x_i)$$

$$\text{donde} \quad F_W(x) = \frac{10 - 33x + 45x^2 - 4x^3}{12(1-x)^3} + \frac{3x^3}{2(1-x)^4} \ln x \to \frac{5}{6} - \frac{x}{4} + \mathcal{O}(x^2)$$

- Por tanto, para neutrinos ligeros y usando la unitariedad de U,

$$\mathcal{B}(\mu \to e\gamma)|_{\text{SM}} = \frac{3\alpha}{2\pi} \left| \sum_i U_{ei} U_{\mu i}^* F_W(x_i) \right|^2 \simeq \frac{3\alpha}{32\pi} \left| \sum_i U_{ei} U_{\mu i}^* \; x_i \right|^2 \lesssim 10^{-54} ,$$

donde se han sustituido los ángulos de mezcla y Δm_{ij}^2 medidos en oscilaciones.

$\mu \to e\gamma$ en el SM con neutrinos masivos

- Nota importante: En las expresiones anteriores se ha despreciado m_e. Para recuperar la contribución de la W al momento magnético anómalo conviene reinsertar m_e:

$$
\begin{aligned}
F_M &= \frac{\alpha_W}{16\pi}\sum_i \left\{ m_\mu U_{ei}U^*_{\mu i} + m_e U^*_{ei}U_{\mu i} \right\} [\dots] \\
-iF_E &= \frac{\alpha_W}{16\pi}\sum_i \left\{ m_\mu U_{ei}U^*_{\mu i} - m_e U^*_{ei}U_{\mu i} \right\} [\dots]
\end{aligned}
$$

Entonces obtenemos que en efecto, para $\ell = \mu = e$, la contribución de la W a a_ℓ es

$$
a_\ell = 2m_\ell F_M(0) = \frac{G_F}{8\pi^2\sqrt{2}} 4m_\ell^2 F_W(0) = \frac{10}{3}\frac{G_F}{8\pi^2\sqrt{2}} m_\ell^2
$$

pues, despreciando las correcciones radiativas de G_F,

$$
\frac{G_F}{8\pi^2\sqrt{2}} = \frac{\alpha_W}{16\pi}\frac{1}{M_W^2},
$$

y $d_\ell = 0$ si no hay fases complejas en U o si los neutrinos no tienen masa.