Exercise 1: Covariant derivative

Defining the covariant derivative and the gauge fields by

$$D_\mu \equiv \partial_\mu - \mathrm{i}g\widetilde{W}_\mu\ , \quad \widetilde{W}_\mu \equiv T_a W^a_\mu$$

prove that the term $\overline{\Psi}\not{D}\Psi$ is invariant under gauge transformations:

$$\Psi \mapsto U\Psi\ , \quad U = \exp\{-\mathrm{i}T_a\theta^a(x)\}$$
$$\widetilde{W}_\mu \mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger$$

Exercise 2: Non abelian gauge transformations

The Yang-Mills Lagrangian is

$$\mathcal{L}_{\rm YM} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\}$$

where

$$\widetilde{W}_{\mu\nu} \equiv T_a W^a_{\mu\nu} = D_\mu\widetilde{W}_\nu - D_\nu\widetilde{W}_\mu = \partial_\mu\widetilde{W}_\nu - \partial_\nu\widetilde{W}_\mu - \mathrm{i}g[\widetilde{W}_\mu, \widetilde{W}_\nu]\ , \quad \widetilde{W}_\mu \equiv T_a W^a_\mu$$

and T_a are the N generators of a Lie group with algebra $[T_a, T_b] = \mathrm{i}f_{abc}T_c$.

i) Check that under a gauge transformation of the fields:

$$\widetilde{W}_\mu \mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger\ , \quad U = \exp\{-\mathrm{i}T_a\theta^a\}$$

the $\widetilde{W}_{\mu\nu}$ transforms as

$$\widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

and therefore $\mathcal{L}_{\rm YM}$ is gauge invariant.

ii) Check that one may write

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}$$

that contains kinetic terms and cubic and quartic interactions among the gauge fields.

iii) Check that

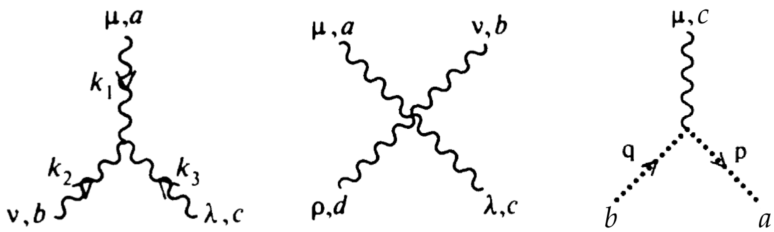
$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + gf_{abc}W^b_\mu W^c_\nu$$

iv) Check that under infinitesimal gauge transformations:

$$W^a_\mu \mapsto W^a_\mu - f_{abc}W^b_\mu\theta^c - \frac{1}{g}\partial_\mu\theta^a$$

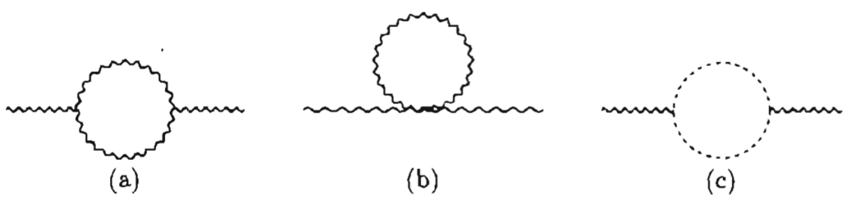
Exercise 3: Feynman rules of general non-Abelian gauge theories

Obtain the Feynman rules for cubic and quartic self-interactions among gauge fields in a general non-Abelian gauge theory, as well as those for the interactions of Faddeev-Popov ghosts with gauge fields:



Exercise 4: Faddeev-Popov ghosts and gauge invariance

Consider the 1-loop self-energy diagrams for non-Abelian gauge theories in the figure. Calculate the diagrams in the 't Hooft-Feynman gauge and show that the sum does not have the tensor structure $g_{\mu\nu}k^2 - k_\mu k_\nu$ required by the gauge invariance of the theory unless diagram (c) involving ghost fields is included.



Hint: Take Feynman rules from previous excercise and use dimensional regularization. It is convenient to use the Passarino-Veltman tensor decomposition of loop integrals:

$$\frac{\mathrm{i}}{16\pi^2}\{B_0,\, B_\mu,\, B_{\mu\nu}\} = \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1,\, q_\mu,\, q_\mu q_\nu\}}{q^2(q+k)^2}$$

$$\text{where} \quad B_0 = \Delta_\epsilon + \text{finite}$$

$$B_\mu = k_\mu B_1\,, \quad B_1 = -\frac{\Delta_\epsilon}{2} + \text{finite}$$

$$B_{\mu\nu} = g_{\mu\nu}B_{00} + k_\mu k_\nu B_{11}\,, \quad B_{00} = -\frac{k^2}{12}\Delta_\epsilon + \text{finite}\,, \quad B_{11} = \frac{\Delta_\epsilon}{3} + \text{finite}$$

with $\Delta_\epsilon = 2/\epsilon - \gamma + \ln 4\pi$ and $D = 4 - \epsilon$. You may check that the ultraviolet divergent part has the expected structure or find the final result in terms of scalar integrals, that for massless fields read:

$$B_1 = -\frac{1}{2}B_0\,, \quad B_{00} = -\frac{k^2}{4(D-1)}B_0\,, \quad B_{11} = \frac{D}{4(D-1)}B_0\,.$$

Do not forget a symmetry factor (1/2) in front of (a) and (b), and a factor (-1) in (c).

Exercise 5: Propagator of a massive vector boson field

Consider the Proca Lagrangian of a massive vector boson field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu \ , \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Show that the propagator of A_μ is

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right]$$

Exercise 6: Propagator of a massive gauge field

Consider the U(1) gauge invariant Lagrangian $\mathcal{L}$ with gauge fixing $\mathcal{L}_{\text{GF}}$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \\ \mathcal{L}_{\text{GF}} &= -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A \chi)^2 \ , \quad \text{with} \quad D_\mu = \partial_\mu + \mathrm{i}eA_\mu \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned}$$

where $M_A = ev$ after spontaneous symmetry breaking ($\mu^2 < 0$, $\lambda > 0$) when the complex scalar field ϕ acquires a VEV and is parameterized by

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi(x) + \mathrm{i}\chi(x)] \ , \quad \mu^2 = -\lambda v^2.$$

Show that the propagators of φ, χ and the gauge field A_μ are respectively

$$\begin{aligned}\widetilde{D}^\varphi(k) &= \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}\varepsilon} \quad \text{with } M_\varphi^2 = -2\mu^2 = 2\lambda v^2 \\ \widetilde{D}^\chi(k) &= \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\varepsilon} \ , \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]\end{aligned}$$

Exercise 7: The conjugate Higgs doublet

Show that $\widetilde{\Phi} \equiv i\sigma_2\Phi^* = \begin{pmatrix}\phi^{0*} \\ -\phi^-\end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix}$, with $\phi^- = (\phi^+)^*$. What are the weak isospins, hypercharges and electric charges of ϕ^0, ϕ^{0*}, ϕ^+, ϕ^-? *Hint*: Use the property of Pauli matrices: $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$.

Exercise 8: Lagrangian and Feynman rules of the Standard Model

Try to reproduce the Lagrangian and the corresponding Feynman rules of as many Standard Model interactions as you can. Of particular interest/difficulty are [VVV] and [VVVV].

Exercise 9: $e^+e^- \to f\bar{f}$

Check that the differential and total cross-sections in the CM system for $e^+e^- \to f\bar{f}$ ($f \neq e$) in the SM at tree level are:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = N_c^f \frac{\alpha^2}{4s}\beta_f \left\{ \left[1+\cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2-1)\ G_2(s) + 2\beta_f \cos\theta\ G_3(s)\right\}$$

$$\sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]\ , \quad \beta_f = \sqrt{1-4m_f^2/s}$$

with

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}\chi_Z(s) + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$

$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$

$$G_3(s) = 2Q_e Q_f a_e a_f \mathrm{Re}\chi_Z(s) + 4v_e v_f a_e a_f |\chi_Z(s)|^2$$

$$\chi_Z(s) \equiv \frac{s}{s-M_Z^2+\mathrm{i}M_Z\Gamma_Z}\ , \quad N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

Exercise 10: Z pole observables at tree level

Show that

(a) $$\Gamma(f\bar{f}) \equiv \Gamma(Z\to f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2+a_f^2)\ , \quad N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

(b) $$\sigma_{\text{had}} = 12\pi \frac{\Gamma(\mathrm{e^+e^-})\Gamma(\text{had})}{M_Z^2\Gamma_Z^2}$$

(c) $$A_{FB} = \frac{\sigma(\cos\theta>0)-\sigma(\cos\theta<0)}{\sigma(\cos\theta>0)+\sigma(\cos\theta<0)}\frac{3}{4}A_f\ , \quad \text{with } A_f = \frac{2v_f a_f}{v_f^2+a_f^2}$$

Exercise 11: Higgs partial decay widths at tree level

Show that

(a) $$\Gamma(H\to f\bar{f}) = N_c^f \frac{G_F M_H}{4\pi\sqrt{2}} m_f^2 \left(1-\frac{4m_f^2}{M_H^2}\right)^{3/2}\ , \quad N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

(b) $$\Gamma(H\to W^+W^-) = \frac{G_F M_H^3}{8\pi\sqrt{2}}\sqrt{1-\frac{4M_W^2}{M_H^2}}\left(1-\frac{4M_W^2}{M_H^2}+\frac{12M_W^4}{M_H^4}\right)$$

$$\Gamma(H\to ZZ) = \frac{G_F M_H^3}{16\pi\sqrt{2}}\sqrt{1-\frac{4M_Z^2}{M_H^2}}\left(1-\frac{4M_Z^2}{M_H^2}+\frac{12M_Z^4}{M_H^4}\right)$$