Quantum Field Theory and the Structure of the Standard Model

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Outline

Exercises

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   - ▷ Basics: Poincaré symmetry
   - ▷ Particle physics with quantum fields
   - ▷ Global and gauge symmetries
     - – Internal symmetries and the gauge principle
     - – Quantization of gauge theories
     - – Spontaneous Symmetry Breaking
2. The Standard Model
   - ▷ Gauge group and field representations
   - ▷ Electroweak interactions
     - – One generation of quarks *or* leptons
     - – Electroweak SSB: Higgs sector, gauge boson and fermion masses
     - – Additional generations: fermion mixings (quarks *vs* leptons)
   - ▷ Strong interactions
   - ⋆ Anomalies?
   - ▷ Electroweak phenomenology

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1. Particles, fields and symmetries

Why Quantum Field Theory to describe Particle Physics?

- QFT is the (only) way to reconcile Quantum Mechanics and Special Relativity

[−] Wave equations (relativistic or not) cannot account for changing # of particles. And the relativistic versions suffer pathologies:

- ⁎ negative probability densities
- ⁎ negative-energy solutions
- ⁎ violation of causality

[+] Quantum fields:

- ⁎ provide a natural framework (*Fock space* of multiparticle states)
- ⁎ make sense of negative-energy solutions (*antiparticles*)
- ⁎ solve causality problem (*Feynman propagator*)
- ⁎ explains spin-statistics connection (*theorem*)
- ⁎ arguably, solve the wave-particle duality puzzle (no particles, *only fields*)

Basics: Poincaré symmetry

Guided by symmetry

- **Relativistic fields** are *irreps* of Poincaré group (rotations, boosts, translations)

  scalar $\phi(x)$, vector $V_\mu(x)$, tensor $h_{\mu\nu}(x)$, . . .
  Weyl $\psi_L(x)$, $\psi_R(x)$; Dirac $\psi(x)$, . . .

- **Lagrangian** densities: **local** $\mathcal{L}(x) = \mathcal{L}(\phi, \partial_\mu\phi)$ (maybe several "ϕ_i", ψ, V_μ, ...)

  invariant under Poincaré transformations

  – e.g. for a free Dirac field $\psi(x)$:

$$\boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu\partial_\mu \,, \quad \overline{\psi} \equiv \psi^\dagger\gamma^0$$

  ⋆ Field dynamics

  ⋆ Noether's theorem: (continuous) symmetry implies conservation laws
  (*energy, momentum, angular momentum*)

Lagrangians Dynamics (classical)

- Principle of least action: $\delta S = 0$ where $S = \int \mathrm{d}^4x\, \mathcal{L}(x)$

  $\Rightarrow$ Field EoM (E-L equations)

$$\delta S = \int \mathrm{d}^4x \sum_i \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i) \right)$$

$$= \int \mathrm{d}^4x \sum_i \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \right) \delta\phi_i = 0 \quad , \quad \forall\phi_i$$

(integrating by parts and assuming fields vanish at boundary)

  – e.g. EoM of a free Dirac field is the Dirac equation

$$\boxed{(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0}$$

$$\rightsquigarrow \quad \psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1,2} \left(a_{\boldsymbol{p},s}u^{(s)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + b^*_{\boldsymbol{p},s}v^{(s)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} \right)$$

with $p^2 = E^2_{\boldsymbol{p}} - \boldsymbol{p}^2 = m^2, \quad (p\!\!\!/ - m)u(\boldsymbol{p}) = 0, \quad (p\!\!\!/ + m)v(\boldsymbol{p}) = 0.$

Lagrangians | Quantization («particle» concept emerges)

- Impose canonical quantization rules:

commutation/anticommutation of fields with conjugate momenta $\Pi_i(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)}$

$$[\phi(t,\boldsymbol{x}),\Pi_\phi(t,\boldsymbol{y})] = \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{y}), \quad \{\psi(t,\boldsymbol{x}),\Pi_\psi(t,\boldsymbol{y})\} = \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

so that the Hamiltonian is bounded from below.

– e.g for a free fermion field, *anticommutation* is enforced! implying

$$\{a_{\boldsymbol{p},r}, a^\dagger_{\boldsymbol{k},s}\} = \{b_{\boldsymbol{p},r}, b^\dagger_{\boldsymbol{k},s}\} = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})\delta_{rs}\ , \quad \{a_{\boldsymbol{p},r}, a_{\boldsymbol{k},s}\} = \cdots = 0$$

– After *normal ordering* :: (all creation to left of annihilation opts) to subtract zero-point energy,

$$H = \int \mathrm{d}^3x :\mathcal{H}(x): = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\boldsymbol{p}} \sum_{s=1,2} (a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} + b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s})$$

⇒ Fields become operators that annihilate/create **particles/antiparticles**

$$|0\rangle \ \text{(vacuum)}\ , \quad a^\dagger_{\boldsymbol{p},s}|0\rangle \ \text{(1 particle)}\ , \quad b^\dagger_{\boldsymbol{p},s}|0\rangle \ \text{(1 antiparticle)}\ , \quad \ldots$$

⇒ Multiparticle states symmetric/antisymmetric under exchange (spin-statistics!)

One-particle representations

- One-particle states are unitary irreps of the Poincaré group, so that

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\mathcal{P}^\dagger\mathcal{P}|\psi_2\rangle \quad \text{(invariant matrix elements)}$$

  $\mathcal{P}$ are represented by unitary operators in this space, and the generators $\underbrace{J^i \text{ (rotations)}, K^i \text{ (boosts)}}_{J_{\mu\nu}=-J_{\nu\mu}}$, P^μ (translations) by Hermitian operators.

$$(J^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i})$$

- Rotations form a compact subgroup (its finite dimensional irreps are unitary). But Lorentz group and Poincaré group are non-compact. Therefore:

  The *unitary* representations of the Poincaré group are *infinite-dimensional*.

- Poincaré group has two Casimir operators (commute with all generators)

$$\boxed{m^2 = P_\mu P^\mu, \quad W_\mu W^\mu} \qquad (W_\mu = \text{Pauli-Lubanski vector})$$

  whose eigenvalues label the irreps. Lorentz invariant (choose convenient frame).

One-particle representations | Wigner classification

- Two cases, characterized by mass m and spin j

  – $m \neq 0$: choose $P^\mu = (m, 0, 0, 0) \Rightarrow W_\mu W^\mu = -m^2 j(j+1)$
  $\Rightarrow$ **massive particles of spin j** have $2j+1$ dof ($j_3 = -j, -j+1, \ldots, j$)
  because $SU(2)$ is the *little group* (transformations leaving P^μ invariant)

  – $m = 0$: choose $P^\mu = (\omega, 0, 0, \omega) \Rightarrow W_\mu W^\mu = -\omega^2[(J^1 + K^2)^2 + (J^2 - K^1)^2]$
  $\Rightarrow$ **massless particles of spin j** have 2 dof (helicity $h = \pm j$)
  because now $SO(2)$ is the *little group* (rotations in plane $\perp$ to P^μ)

- Note: To construct a **unitary field theory with** V_μ (contains both spin 0 and 1) one has to choose carefully the Lagrangian so that the *physical theory* **never excites**:

  – the spin-0 component (if massive)

  – neither the longitudinal spin-1 component (if massless) $\Leftrightarrow$ gauge invariance

Particle physics

Particle physics — S-matrix elements

- Observables (cross sections, decays widths) expressed in terms of S-matrix elements ($m \to n$ processes)

$$ {}_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n | \boldsymbol{k}_1 \boldsymbol{k}_2 \cdots \boldsymbol{k}_m \rangle_{\text{in}} $$

(scalar fields/particles to simplify)

- Only free fields are related to particles/antiparticles ($a^\dagger_{\boldsymbol{p}}$, $b^\dagger_{\boldsymbol{p}}$).

We expect

$$ \phi(x) \xrightarrow[t\to-\infty]{} Z_\phi^{1/2} \phi_{\text{in}}(x) \; , \quad \phi(x) \xrightarrow[t\to+\infty]{} Z_\phi^{1/2} \phi_{\text{out}}(x) \; , $$

$\phi(x)$:	interacting fields
$\phi_{\text{in}}(x)$, $\phi_{\text{out}}(x)$:	free fields (before, after interaction)
Z_ϕ:	*wave function* renormalization

- LSZ reduction formula relates S-matrix elements with the (Fourier transform of) vacuum expectation values of *time-ordered* field products (correlators):

$$\left(\prod_{i=1}^{m}\frac{\mathrm{i}\sqrt{Z_\phi}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{\mathrm{i}\sqrt{Z_\phi}}{p_j^2-m^2}\right)\ {}_{\text{out}}\langle \boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_n|\boldsymbol{k}_1\boldsymbol{k}_2\cdots\boldsymbol{k}_m\rangle_{\text{in}}$$

$$=\int\left(\prod_{i=1}^{m}\mathrm{d}^4x_i\ \mathrm{e}^{-\mathrm{i}k_ix_i}\right)\int\left(\prod_{j=1}^{n}\mathrm{d}^4y_j\ \mathrm{e}^{+\mathrm{i}p_jy_j}\right)\langle 0|\,T\{\underbrace{\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)}_{\text{interacting fields}}\}\,|0\rangle$$

The correlator = the Green's function of $m+n$ points $G(\boldsymbol{p}_1\cdots\boldsymbol{p}_n;\boldsymbol{k}_1\cdots\boldsymbol{k}_m)$

▷ Physical particles (asymptotic states) are on-shell ($p^2-m^2=0$).

For on-shell incoming and outgoing particles, the rhs of LSZ formula (correlator) will have **poles** that cancel those in the prefactor of the lhs, yielding a regular S-matrix element [*residues* of the correlator].

Particle physics | **Perturbation theory** | **Feynman diagrams**

- The correlators can be expressed in terms of free fields "ϕ_0" :

$$\langle 0|\, T\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\langle 0|\, T\left\{\phi_0(x_1)\cdots\phi_0(x_n)\exp\left[\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}[\phi_0(x)]\right]\right\}|0\rangle}{\langle 0|\, T\left\{\exp\left[\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}[\phi_0(x)]\right]\right\}|0\rangle}$$

- In perturbation theory one expands the exponential and computes every correlator using *Wick's theorem* (all possible "contractions")

$$\textit{contraction} \equiv \overbrace{\phi(x)\phi}(y) = D_F(x-y) = \langle 0|\, T\{\phi(x)\phi(y)\}\,|0\rangle = \quad \textit{Feynman propagator}$$

- Feynman diagrams/rules provide a systematic procedure to organize/compute the perturbative series in terms of *propagators* (and vertices)

- **Note:** functional quantization (path integral) provides an *alternative* method

$$\langle 0|\, T\{\hat\phi(x_1)\cdots\hat\phi(x_n)\}\,|0\rangle = \frac{\int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_n)\; \mathrm{e}^{\mathrm{i}S}}{\int \mathcal{D}\phi\; \mathrm{e}^{\mathrm{i}S}} \qquad \text{perturbatively } \textit{or} \text{ not! } (\textit{lattice})$$

- **Causality** requires $[\phi(x), \phi^\dagger(y)] = 0$ if $(x-y)^2 < 0$ (*spacelike* interval)

▷ Recall that a (free) field is a combination of positive and negative energy waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}px} \right)$$

▷ From the commutation relations of creation and annihilation operators:

$$[\phi(x), \phi^\dagger(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{E_{\boldsymbol{p}}}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{E_{\boldsymbol{q}}}} \left(\mathrm{e}^{-\mathrm{i}(px-qy)} [a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] + \mathrm{e}^{\mathrm{i}(px-qy)} [b^\dagger_{\boldsymbol{p}}, b_{\boldsymbol{q}}] \right)$$

$$= \Delta(x-y) - \Delta(y-x)$$

where the first (second) contribution comes from particles (antiparticles) and

$$\Delta(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 E_{\boldsymbol{p}}} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

▷ If $(x-y)^2 < 0$ choose frame where $x - y \equiv (0, \boldsymbol{r})$. Then

$$\Delta(x-y) = \Delta(y-x) \propto \frac{m}{r}\mathrm{e}^{-mr} \neq 0, \quad \text{for } mr \gg 1$$

Therefore:

$$\text{If only particles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) \neq 0 \ (!!)$$

$$\text{If } \textit{both} \text{ particles and antiparticles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) - \Delta(y-x) = 0 \ (\checkmark)$$

- In fact the **Feynman propagator** contains *both* contributions:

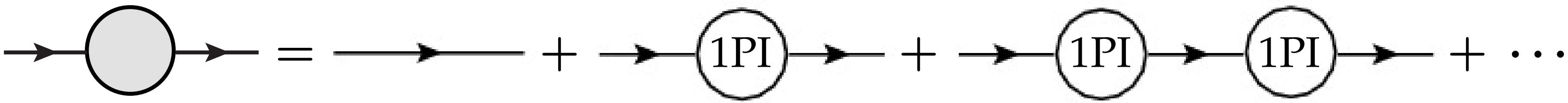
$$D_F(x-y) = \langle 0| T\{\phi(x)\phi^\dagger(y)\} |0\rangle = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)$$

  – Probability amplitude that particle created in y propagates to x, if $x^0 > y^0$
  – Probability amplitude that antiparticle created in x propagates to y, if $y^0 > x^0$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \quad \text{where} \quad \varepsilon \to 0^+ (\text{usually omitted})$$

Particle physics | **Perturbation theory** | **LSZ** (rewritten)

- Corrections to external legs (external propagators) can be resummed:

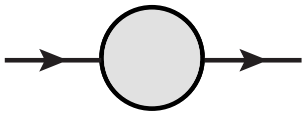


$$= \frac{\mathrm{i}}{p^2 - m_0^2} + \frac{\mathrm{i}}{p^2 - m_0^2}[-\mathrm{i}M^2(p^2)]\frac{\mathrm{i}}{p^2 - m_0^2} + \ldots$$

$$= \frac{\mathrm{i}}{p^2 - m_0^2 - M^2(p^2)} \qquad (m_0 = \text{mass in } \mathcal{L})$$

and Taylor expanding about $p^2 = m^2$ (*physical* mass):

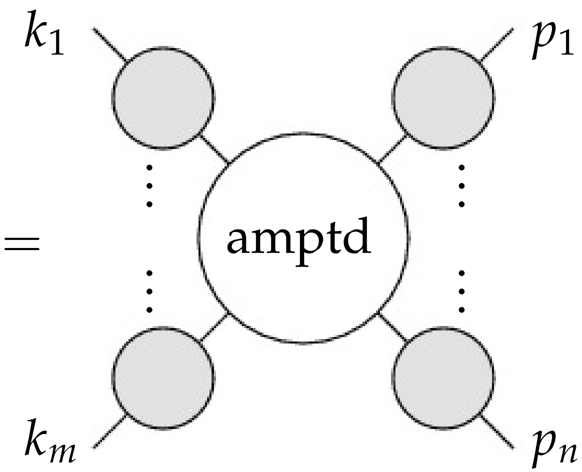
$$p^2 - m_0^2 - M^2(p^2) = (p^2 - m^2)\left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)$$

$\Rightarrow$  $= \dfrac{\mathrm{i}Z_\phi}{p^2 - m^2} + \text{regular near } p^2 = m^2$

$$\text{with} \quad m^2 = m_0^2 + M^2(m^2)\,, \quad Z_\phi = \left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)^{-1}$$

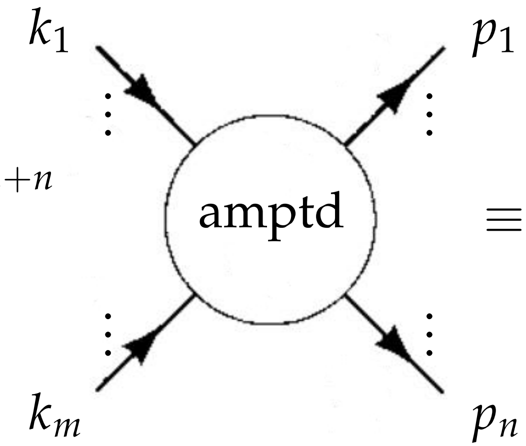
▷ Then we may factor out external legs from *amputated* diagrams:

$$G(\boldsymbol{p}_1 \cdots \boldsymbol{p}_n; \boldsymbol{k}_1 \cdots \boldsymbol{k}_m) =$$



and express the LSZ formula in a simpler form:

$${}_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n | \boldsymbol{k}_1 \boldsymbol{k}_2 \cdots \boldsymbol{k}_m \rangle_{\text{in}} = \left(\sqrt{Z}\right)^{m+n}$$



$$\equiv (2\pi)^4 \delta^4\left(\sum_i p_i - \sum_j k_j\right) \mathrm{i}\mathcal{M}$$

- Feynman rules require integration over loop momenta resulting *sometimes* in divergent expressions.

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\mathcal{M}^{(1)}}_{\text{divergent?}} + \ldots$$

(the loop expansion is also an expansion in powers of $\hbar$: *quantum* corrections)

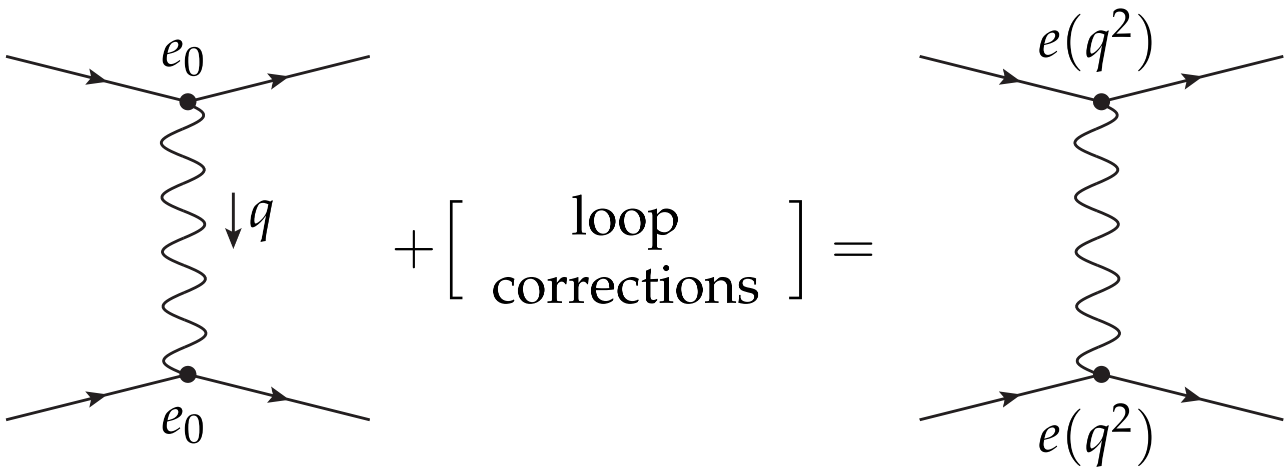
- **Regularization** and **renormalization** needed to make sense of these divergences.

▷ One assumes that fields and parameters in the Lagrangian (*bare*) must be redefined order by order in terms of new ones (*renormalized*) so that physical predictions are finite

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\widehat{\mathcal{M}}^{(1)}}_{\text{finite}} + \ldots$$

▷ As a consequence, renormalized *coupling constants run* (depend on a scale)

e.g.



$e_0 =$ *bare* coupling $\qquad e(q^2) =$ *renormalized* coupling

$q^2 =$ *renormalization scale* (at which e is "measured")

Note: e is not an observable

Global and gauge symmetries

Internal symmetries | **free Lagrangian**

- In addition to **spacetime** (Poincaré) symmetries, the free Lagrangian

$$\text{(Dirac)} \qquad \boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \, , \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

$\Rightarrow$ Invariant under **internal** global U(1) phase transformations:

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta}\psi(x) \, , \quad Q, \ \theta \text{ (constants)} \in \mathbb{R}$$

$\Rightarrow$ By Noether's theorem, divergentless current:

$$\mathcal{J}^\mu = Q\,\overline{\psi}\gamma^\mu\psi \, , \quad \partial_\mu \mathcal{J}^\mu = 0$$

and a conserved «charge»

$$\mathcal{Q} = \int \mathrm{d}^3x \, \mathcal{J}^0, \quad \partial_t \mathcal{Q} = 0$$

Internal symmetries | free Lagrangian

- For a free fermion quantum field:

⇒ The Noether charge is an operator:*

$$\mathcal{Q} = Q \int \mathrm{d}^3x \; :\overline{\psi}\gamma^0\psi: = Q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} - b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s} \right)$$

$$\mathcal{Q}\; a^\dagger_{\boldsymbol{k},s} |0\rangle = +Q\; a^\dagger_{\boldsymbol{k},s} |0\rangle \;\; \text{(particle)}\,, \qquad \mathcal{Q}\; b^\dagger_{\boldsymbol{k},s} |0\rangle = -Q\; b^\dagger_{\boldsymbol{k},s} |0\rangle \;\; \text{(antiparticle)}$$

* normal ordering prescription for fermionic operators

$$: a_{\boldsymbol{p},r} a^\dagger_{\boldsymbol{q},s} : \equiv -a^\dagger_{\boldsymbol{q},s} a_{\boldsymbol{p},r}\,, \qquad : b_{\boldsymbol{p},r} b^\dagger_{\boldsymbol{q},s} : \equiv -b^\dagger_{\boldsymbol{q},s} b_{\boldsymbol{p},r}$$

The gauge principle | gauge symmetry dictates interactions

- To make $\mathcal{L}_0$ invariant under local ≡ gauge transformations of U(1):

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\psi(x) \,, \quad \theta = \theta(x) \in \mathbb{R}$$

perform the minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eQA_\mu \quad \text{(covariant derivative)}$$

where a gauge field $A_\mu(x)$ is introduced transforming as:

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \quad \Leftarrow \quad \boxed{D_\mu\psi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta(x)}D_\mu\psi} \quad \overline{\psi}\not{D}\psi \text{ inv.}$$ (1)

⇒ The new Lagrangian contains **interactions** between ψ and A_μ:

$$\boxed{\mathcal{L}_{\text{int}} = -e\, Q\, \overline{\psi}\gamma^\mu\psi A_\mu} \quad \propto \begin{cases} \text{coupling} & e \\ \text{charge} & Q \end{cases}$$

$$(= -e\, \mathcal{J}^\mu A_\mu)$$

The gauge principle | gauge invariance dictates interactions

- Dynamics for the gauge field ⇒ add gauge invariant kinetic term:

$$\text{(Maxwell)} \qquad \boxed{\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad \Leftarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$

  – The full U(1) gauge invariant Lagrangian for a fermion field $\psi(x)$ reads:

$$\mathcal{L}_{\text{sym}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_1) \quad \text{(QED)}$$

  – The same applies to a complex scalar field $\phi(x)$:

$$\mathcal{L}_{\text{sym}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{(sQED)}$$

- A general gauge symmetry group G is an *compact* N-dimensional Lie group

$$\mathrm{g} \in G\,, \quad \mathrm{g}(\boldsymbol{\theta}) = \mathrm{e}^{-\mathrm{i}T_a\theta^a}\,, \quad a = 1,\ldots,N$$

$$\theta^a = \theta^a(x) \in \mathbb{R}\,, \quad T_a = \text{Hermitian generators}\,, \quad [T_a, T_b] = \mathrm{i}f_{abc}T_c \quad \text{(Lie algebra)}$$

structure constants: $f_{abc} = 0$ Abelian

$f_{abc} \neq 0$ non-Abelian

$\Rightarrow$ *Unitary* finite-dimensional irreducible representations:

$g(\boldsymbol{\theta})$ represented by $U(\boldsymbol{\theta})$

$$d \times d \text{ matrices}: \quad U(\boldsymbol{\theta}) \text{ [given by } \{T_a\} \text{ algebra representation]}$$

$$d\text{-multiplet}: \quad \Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x)\,, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

- Examples:

G	N	Abelian	
U(1)	1	Yes	
SU(n)	n^2-1	No	($n \times n$ unitary matrices with det $=1$)

  - U(1): 1 generator (q), one-dimensional irreps only
  - SU(2): 3 generators

    $f_{abc} = \epsilon_{abc}$ (Levi-Civita symbol)
    - Fundamental irrep ($d=2$): $T_a = \frac{1}{2}\sigma_a$ (3 Pauli matrices)
    - Adjoint irrep ($d = N = 3$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$
  - SU(3): 8 generators

    $f^{123} = 1$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$, $f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367} = \frac{1}{2}$
    - Fundamental irrep ($d=3$): $T_a = \frac{1}{2}\lambda_a$ (8 Gell-Mann matrices)
    - Adjoint irrep ($d = N = 8$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$

(for SU(n): f_{abc} totally antisymmetric)

The gauge principle | non-Abelian gauge theories

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of G:

$$\mathcal{L}_0 = \overline{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi\ , \qquad \Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x)\ , \quad \boldsymbol{\theta} = \boldsymbol{\theta}(x) \in \mathbb{R}$$

substitute the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu\ , \quad \widetilde{W}_\mu \equiv T_a W^a_\mu$$

where a gauge field $W^a_\mu(x)$ per generator is introduced, transforming as:

$$\widetilde{W}_\mu(x) \mapsto \widetilde{W}'_\mu(x) = \underbrace{U\widetilde{W}_\mu(x)U^\dagger}_{\text{adjoint irrep}} - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \quad \Leftarrow \quad \boxed{D_\mu\Psi \mapsto UD_\mu\Psi} \quad \overline{\Psi}D\!\!\!\!/\,\Psi \text{ inv.}$$ (1)

$\Rightarrow$ The new Lagrangian contains interactions between Ψ and W^a_μ:

$$\boxed{\mathcal{L}_{\text{int}} = g\ \overline{\Psi}\gamma^\mu T_a\Psi W^a_\mu} \quad \propto \begin{cases} \text{coupling} & g \\ \text{charge} & T_a \end{cases}$$

$$(= g\ \mathcal{J}^\mu_a W^a_\mu)$$

The gauge principle | non-Abelian gauge theories

- Dynamics for the gauge fields ⇒ add gauge invariant kinetic terms:

$$\text{(Yang-Mills)} \qquad \boxed{\mathcal{L}_{\rm YM} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\} = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}}$$

$$\widetilde{W}_{\mu\nu} \equiv T_a W^a_{\mu\nu} \equiv D_\mu \widetilde{W}_\nu - D_\nu \widetilde{W}_\mu = \partial_\mu \widetilde{W}_\nu - \partial_\nu \widetilde{W}_\mu - ig[\widetilde{W}_\mu, \widetilde{W}_\nu] \;\Leftrightarrow\; \widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

$$\Rightarrow \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$$ (2)

⇒ $\mathcal{L}_{\rm YM}$ contains cubic and quartic **self-interactions** of the gauge fields W^a_μ:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$$

$$\mathcal{L}_{\rm cubic} = -\frac{1}{2} g f_{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) W^{b,\mu} W^{c,\nu}$$

$$\mathcal{L}_{\rm quartic} = -\frac{1}{4} g^2 f_{abe} f_{cde}\, W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu}$$

Quantization **propagators** (particle interpretation of field correlators)

- The (Feynman) propagator of a scalar field:

$$D_F(x-y) = \langle 0|\, T\{\phi(x)\phi^\dagger(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

(Feynman *prescription* $\varepsilon \to 0^+$)

is a Green's function of the Klein-Gordon operator:

$$(\Box_x + m^2)D_F(x-y) = -\mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{D}_F(p) = \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}$$

- The propagator of a fermion field:

$$S_F(x-y) = \langle 0|\, T\{\psi(x)\overline{\psi}(y)\}\,|0\rangle = (\mathrm{i}\not{\partial}_x + m)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

is a Green's function of the Dirac operator:

$$(\mathrm{i}\not{\partial}_x - m)S_F(x-y) = \mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{S}_F(p) = \frac{\mathrm{i}}{\not{p}-m+\mathrm{i}\varepsilon}$$

Quantization of gauge theories | propagators

- HOWEVER a gauge field propagator cannot be defined unless $\mathcal{L}$ is modified:

(e.g. modified Maxwell) $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

Euler-Lagrange: $$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu = 0$$

– In momentum space the propagator is the inverse of:

$$-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \quad \Rightarrow \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\Rightarrow$ Note that $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular!

$\Rightarrow$ One may argue that $\mathcal{L}$ above will not lead to Maxwell equations …

unless we fix a (Lorenz) gauge where: (remove redundancy)

$$\partial^\mu A_\mu = 0 \quad \Leftarrow \quad A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu\Lambda \ \text{ with } \ \partial^\mu\partial_\mu\Lambda \equiv -\partial^\mu A_\mu$$

Quantization of gauge theories — gauge fixing (Abelian case)

- The extra term is called Gauge Fixing:

$$\boxed{\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$\Rightarrow$ modified $\mathcal{L}$ equivalent to Maxwell Lagrangian just in the gauge $\partial^\mu A_\mu = 0$

$\Rightarrow$ the ξ-dependence always cancels out in physical amplitudes

- Several choices for the gauge fixing term (simplify calculations): R_ξ gauges

$$\text{('t Hooft-Feynman gauge)} \quad \xi = 1: \quad \widetilde{D}_{\mu\nu}(k) = -\frac{\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}$$

$$\text{(Landau gauge)} \quad \xi = 0: \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right]$$

Quantization of gauge theories | gauge fixing (non-Abelian case)

- For a non-Abelian gauge theory, the gauge fixing terms:

$$\mathcal{L}_{\text{GF}} = -\sum_a \frac{1}{2\xi_a}(\partial^\mu W_\mu^a)^2$$

allow to define the propagators:

$$\widetilde{D}^{ab}_{\mu\nu}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_a)\frac{k_\mu k_\nu}{k^2}\right]$$

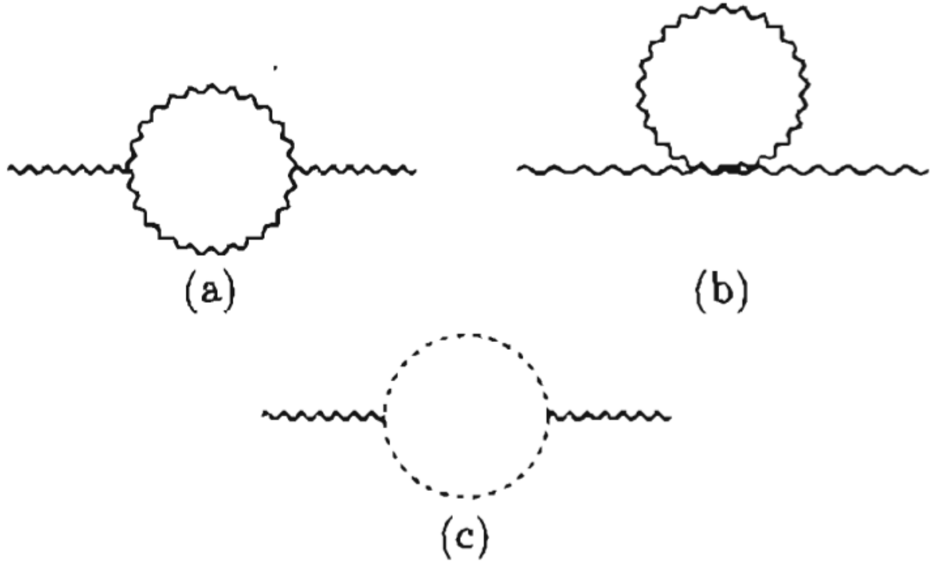
HOWEVER, unlike the Abelian case, this is not the end of the story …

- Add Faddeev-Popov *ghost* fields $c_a(x)$, $a = 1, \ldots, N$: ('t Hooft-Feynman gauge)

$$\boxed{\mathcal{L}_{\rm FP} = (\partial^\mu \bar{c}_a)(D_\mu^{\rm adj})_{ab} c_b = (\partial^\mu \bar{c}_a)(\partial_\mu c_a - g f_{abc} c_b W_\mu^c)} \quad \Leftarrow \quad D_\mu^{\rm adj} = \partial_\mu - \mathrm{i} g T_c^{\rm adj} W_\mu^c$$

Computational trick: *anticommuting* scalar fields, just in loops as virtual particles

$\Rightarrow$ Faddeev-Popov ghosts needed to preserve gauge symmetry: (3)



Self Energy

$$= \Pi_{\mu\nu} = \mathrm{i}(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$ (4)

Ward identity: $k^\mu \Pi_{\mu\nu} = 0$

with

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\varepsilon}$$

[(-1) sign for closed loops! (like fermions)]

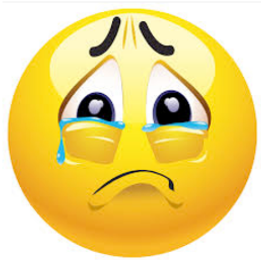
- Then the full quantum Lagrangian is

$$\boxed{\mathcal{L}_{\rm sym} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}}$$

⇒ Note that in the case of a massive vector field

$$\text{(Proca)} \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu$$

it is not gauge invariant!!!



What about the gauge principle???

– The propagator is:

5

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right)$$

Spontaneous Symmetry Breaking | discrete symmetry

- Consider a real scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad \text{invariant under} \quad \phi \mapsto -\phi$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

$\mu^2,\ \lambda \in \mathbb{R}$ (Real/Hermitian Hamiltonian) and $\lambda > 0$ (existence of a ground state)

(a) $\mu^2 > 0$: min of $V(\phi)$ at $\phi = 0$

(b) $\mu^2 < 0$: min of $V(\phi)$ at $\phi = v \equiv \pm\sqrt{\frac{-\mu^2}{\lambda}}$, in QFT $\langle 0|\,\phi\,|0\rangle = v \neq 0$ (VEV)

– A quantum field must have $v = 0$
$a\,|0\rangle = 0$

$$\Rightarrow \qquad \phi(x) \equiv v + \eta(x)\,, \quad \langle 0|\,\eta\,|0\rangle = 0$$

Spontaneous Symmetry Breaking | discrete symmetry

- At the quantum level, the same system is described by $\eta(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \quad \text{not invariant under} \quad \eta \mapsto -\eta$$

$$(m_\eta = \sqrt{2\lambda}\, v)$$

⇒ Lesson:

$\mathcal{L}(\phi)$ has the symmetry but the parameters can be such that the ground state of the Hamiltonian is not symmetric (Spontaneous Symmetry Breaking)

⇒ Note:

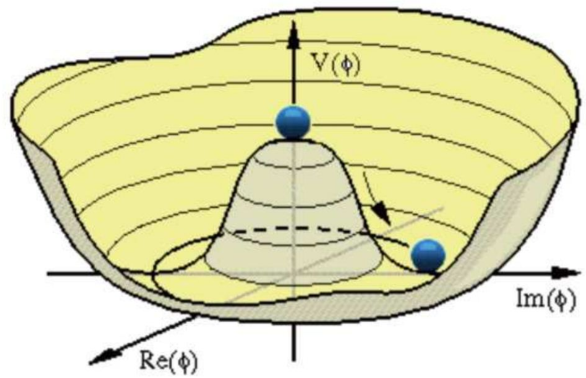
One may argue that $\mathcal{L}(\eta)$ exhibits an explicit breaking of the symmetry. However this is not the case since the coefficients of terms η^2, η^3 and η^4 are determined by just two parameters, λ and v (remnant of the original symmetry)

Spontaneous Symmetry Breaking — continuous symmetry

- Consider a complex scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \quad \text{invariant under U(1):} \quad \phi \mapsto e^{-iq\theta}\phi$$

$$\lambda > 0,\ \mu^2 < 0: \quad \langle 0|\,\phi\,|0\rangle \equiv \frac{v}{\sqrt{2}}, \quad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$



Take $v \in \mathbb{R}^+$. In terms of quantum fields:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + i\chi(x)], \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\chi\,|0\rangle = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \lambda v^2 \eta^2 - \lambda v \eta(\eta^2 + \chi^2) - \frac{\lambda}{4}(\eta^2 + \chi^2)^2 + \frac{1}{4}\lambda v^4$$

Note: if $v e^{i\alpha}$ (complex) replace η by $(\eta \cos\alpha - \chi \sin\alpha)$ and χ by $(\eta \sin\alpha + \chi \cos\alpha)$

$\Rightarrow$ The actual quantum Lagrangian $\mathcal{L}(\eta, \chi)$ is not invariant under U(1)

U(1) broken $\Rightarrow$ one scalar field remains massless: $m_\chi = 0$, $m_\eta = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

- Another example: consider a real scalar SU(2) triplet $\Phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^\mathsf{T})(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi^\mathsf{T}\Phi - \frac{\lambda}{4}(\Phi^\mathsf{T}\Phi)^2 \quad \text{inv. under SU(2):} \quad \Phi \mapsto \mathrm{e}^{-\mathrm{i}T_a\theta^a}\Phi$$

that for $\lambda > 0,\ \mu^2 < 0$ acquires a VEV $\langle 0|\, \Phi^\mathsf{T}\Phi \,|0\rangle = v^2 \qquad (\mu^2 = -\lambda v^2)$

Assume $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ v + \varphi_3(x) \end{pmatrix}$ and define $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi_3)(\partial^\mu \varphi_3) - \lambda v^2 \varphi_3^2 - \lambda v(2\varphi^\dagger\varphi + \varphi_3^2)\varphi_3 - \frac{\lambda}{4}(2\varphi^\dagger\varphi + \varphi_3^2)^2 + \frac{1}{4}\lambda v^4$$

$\Rightarrow$ Not symmetric under SU(2) but invariant under U(1):

$$\varphi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta}\varphi \quad (Q = \text{arbitrary}) \qquad\qquad \varphi_3 \mapsto \varphi_3 \quad (Q = 0)$$

SU(2) broken to U(1) $\Rightarrow 3 - 1 = 2$ broken generators

$\Rightarrow$ 2 (real) scalar fields (= 1 complex) remain massless: $m_\varphi = 0$, $m_{\varphi_3} = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

⇒ Goldstone's theorem: [Nambu '60; Goldstone '61]

The number of massless particles (Nambu-Goldstone bosons) is equal to the number of spontaneously broken generators of the symmetry

$$\text{Hamiltonian symmetric under group } G \quad \Rightarrow \quad [T_a, H] = 0\,, \quad a = 1, \ldots, N$$

$$\text{By definition: } H|0\rangle = 0 \quad \Rightarrow \quad H(T_a|0\rangle) = T_a H|0\rangle = 0$$

– If $|0\rangle$ is such that $T_a|0\rangle = 0$ for all generators

⇒ non-degenerate minimum: *the* vacuum

– If $|0\rangle$ is such that $T_{a'}|0\rangle \neq 0$ for some (broken) generators a'

⇒ degenerate minimum: chose one (*true* vacuum) and $e^{-iT_{a'}\theta^{a'}}|0\rangle \neq |0\rangle$

⇒ excitations (particles) from $|0\rangle$ to $e^{-iT_{a'}\theta^{a'}}|0\rangle$ cost no energy: massless!

Spontaneous Symmetry Breaking | gauge symmetry

- Consider a U(1) gauge invariant Lagrangian for a complex scalar field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,, \quad D_\mu = \partial_\mu + \mathrm{i}eQA_\mu$$

inv. under $\phi(x) \mapsto \phi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\phi(x)\,, \quad A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$

If $\lambda > 0$, $\mu^2 < 0$, the $\mathcal{L}$ in terms of quantum fields η and χ with null VEVs:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)]\,, \quad \mu^2 = -\lambda v^2$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) \\ &\boxed{-\lambda v^2\eta^2} - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4 \\ &\boxed{+\,eQvA_\mu\partial^\mu\chi} + eQA_\mu(\eta\partial^\mu\chi - \chi\partial^\mu\eta) \\ &\boxed{+\,\frac{1}{2}(eQv)^2A_\mu A^\mu} + \frac{1}{2}(eQ)^2A_\mu A^\mu(\eta^2 + 2v\eta + \chi^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$

$m_\chi = 0$

(ii) $M_A = |eQv|$ (!)

(iii) Term $A_\mu\partial^\mu\chi$ (?)

(iv) Add $\mathcal{L}_{\mathrm{GF}}$

Spontaneous Symmetry Breaking | gauge symmetry

- Removing the cross term and the (new) gauge fixing Lagrangian:

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A \chi)^2$$

$$\Rightarrow \quad \mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overbrace{M_A \partial_\mu (A^\mu \chi)}^{\substack{M_A[\partial_\mu A^\mu \chi + A_\mu \partial^\mu \chi] \\ \text{total deriv.}}}$$
$$+ \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}\xi M_A^2 \chi^2 + \ldots$$

and the propagators of A_μ and χ are: 6

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\varepsilon}$$

$\Rightarrow$ χ has a gauge-dependent mass: actually it is not a physical field!

Spontaneous Symmetry Breaking | gauge symmetry

- A more transparent parameterization of the quantum field ϕ is

$$\phi(x) \equiv \mathrm{e}^{\mathrm{i}Q\zeta(x)/v} \frac{1}{\sqrt{2}}[v+\eta(x)] , \quad \langle 0|\, \eta \,|0\rangle = \langle 0|\, \zeta \,|0\rangle = 0$$

$$\phi(x) \mapsto \mathrm{e}^{-\mathrm{i}Q\zeta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}[v+\eta(x)] \quad \Rightarrow \quad \zeta \text{ gauged away!}$$

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) \\ & -\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \\ & + \frac{1}{2}(eQv)^2 A_\mu A^\mu + \frac{1}{2}(eQ)^2 A_\mu A^\mu (2v\eta + \eta^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\, v$

(ii) $M_A = |eQv|$

(iii) No need for $\mathcal{L}_{\mathrm{GF}}$

$\Rightarrow$ This is the unitary gauge ($\xi \to \infty$): just physical fields

$$\widetilde{D}_{\mu\nu}(k) \to \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_A^2}\right] \quad \text{and} \quad \widetilde{D}(k) \to 0$$

Spontaneous Symmetry Breaking | **gauge symmetry**

⇒ Brout-Englert-Higgs mechanism:

[Anderson '62]
[Higgs '64; Englert, Brout '64; Guralnik, Hagen, Kibble '64]

The gauge bosons associated with the spontaneously broken generators become massive, the corresponding would-be Goldstone bosons are unphysical and can be absorbed, the remaining massive scalars (Higgs bosons) are physical (the smoking gun!)

- The would-be Goldstone bosons are 'eaten up' by the gauge bosons ('get fat') and disappear (gauge away) in the unitary gauge ($\xi \to \infty$)
  ⇒ Degrees of freedom are preserved

Before SSB:	2 (massless gauge boson)	+ 1 (Goldstone boson)
After SSB:	3 (massive gauge boson)	+ 0 (absorbed would-be Goldstone)

- For loops calculations, 't Hooft-Feynman gauge ($\xi = 1$) is more convenient:
  ⇒ Gauge boson propagators are simpler, but
  ⇒ Goldstone bosons must be included in internal lines

Spontaneous Symmetry Breaking | gauge symmetry

- Comments:

  – After SSB the FP ghost fields (unphysical) acquire a gauge-dependent mass, due to interactions with the scalar field(s):

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 - \xi_a M_{W^a}^2 + \mathrm{i}\varepsilon}$$

  – Gauge theories with SSB are renormalizable ['t Hooft, Veltman '72]

  UV divergences appearing at loop level can be removed by renormalization of parameters and fields of the classical Lagrangian ⇒ predictive!

2. The Standard Model

Gauge group and field representations

[Glashow '61; Weinberg '67; Salam '68]
[D. Gross, F. Wilczek; D. Politzer '73]

- The Standard Model is a gauge theory based on the local symmetry group:

$$\underbrace{\mathrm{SU}(3)_c}_{\text{strong}} \otimes \underbrace{\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y}_{\text{electroweak}} \to \mathrm{SU}(3)_c \otimes \underbrace{\mathrm{U}(1)_Q}_{\text{em}}$$

with the electroweak symmetry spontaneously broken to the electromagnetic $\mathrm{U}(1)_Q$ symmetry by the Brout-Englert-Higgs mechanism

- The particle (field) content: (ingredients: 12 *flavors* + 12 gauge bosons + H)

		Fermions	I	II	III	Q
spin $\frac{1}{2}$	Quarks	f	uuu	ccc	ttt	$\frac{2}{3}$
		f'	ddd	sss	bbb	$-\frac{1}{3}$
	Leptons	f	ν_e	ν_μ	ν_τ	0
		f'	e	μ	τ	-1

$$Q_f = Q_{f'} + 1$$

	Bosons	
spin 1	8 gluons	strong interaction
	$W^\pm$, Z	weak interaction
	γ	em interaction
spin 0	Higgs	origin of mass

Gauge group and field representations

- The fields lay in the following representations (color, weak isospin, hypercharge):

Multiplets	$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$	I	II	III	$Q = T_3 + Y$
Quarks	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ $-\frac{1}{3} = -\frac{1}{2} + \frac{1}{6}$
	$(\mathbf{3}, \mathbf{1}, \frac{2}{3})$	u_R	c_R	t_R	$\frac{2}{3} = 0 + \frac{2}{3}$
	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$	d_R	s_R	b_R	$-\frac{1}{3} = 0 - \frac{1}{3}$
Leptons	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$	$0 = \frac{1}{2} - \frac{1}{2}$ $-1 = -\frac{1}{2} - \frac{1}{2}$
	$(\mathbf{1}, \mathbf{1}, -1)$	e_R	μ_R	τ_R	$-1 = 0 - 1$
	$(\mathbf{1}, \mathbf{1}, 0)$	ν_{e_R}	ν_{μ_R}	ν_{τ_R}	$0 = 0 + 0$
Higgs	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$	(3 families of quarks & leptons)			UNIVERSAL

$\Rightarrow$ Electroweak (QFD): $SU(2)_L \otimes U(1)_Y$ Strong (QCD): $SU(3)_c$

Electroweak interactions

The EWSM with one family (of quarks or leptons)

- Consider two massless fermion fields $f(x)$ and $f'(x)$ with electric charges $Q_f = Q_{f'} + 1$ in three irreps of SU(2)$_L$⊗U(1)$_Y$:

$$\mathcal{L}_F^0 = \mathrm{i}\overline{f}\partial\!\!\!/ f + \mathrm{i}\overline{f}'\partial\!\!\!/ f' \qquad f_{R,L} = \frac{1}{2}(1\pm\gamma_5)f\,, \quad f'_{R,L} = \frac{1}{2}(1\pm\gamma_5)f'$$

$$= \mathrm{i}\overline{\Psi}_1\partial\!\!\!/\Psi_1 + \mathrm{i}\overline{\psi}_2\partial\!\!\!/\psi_2 + \mathrm{i}\overline{\psi}_3\partial\!\!\!/\psi_3 \quad ; \quad \Psi_1 = \underbrace{\begin{pmatrix} f_L \\ f'_L \end{pmatrix}}_{(\mathbf{2},y_1)}, \quad \psi_2 = \underbrace{f_R}_{(\mathbf{1},y_2)}, \quad \psi_3 = \underbrace{f'_R}_{(\mathbf{1},y_3)}$$

- To get a Langrangian invariant under gauge transformations:

$$\Psi_1(x) \mapsto U_L(x)\mathrm{e}^{-\mathrm{i}y_1\beta(x)}\Psi_1(x), \quad U_L(x) = \mathrm{e}^{-\mathrm{i}T_i\alpha^i(x)}, \quad T_i = \frac{\sigma_i}{2} \quad \text{(weak isospin gen.)}$$

$$\psi_2(x) \mapsto \mathrm{e}^{-\mathrm{i}y_2\beta(x)}\psi_2(x)$$

$$\psi_3(x) \mapsto \mathrm{e}^{-\mathrm{i}y_3\beta(x)}\psi_3(x)$$

⇒ Introduce gauge fields $W^i_\mu(x)$ $(i=1,2,3)$ and $B_\mu(x)$ through covariant derivatives:

$$\left.\begin{aligned} D_\mu\Psi_1 &= (\partial_\mu - \mathrm{i}g\widetilde{W}_\mu + \mathrm{i}g'y_1B_\mu)\Psi_1\,, \quad \widetilde{W}_\mu \equiv \frac{\sigma_i}{2}W^i_\mu \\ D_\mu\psi_2 &= (\partial_\mu + \mathrm{i}g'y_2B_\mu)\psi_2 \\ D_\mu\psi_3 &= (\partial_\mu + \mathrm{i}g'y_3B_\mu)\psi_3 \end{aligned}\right\} \Rightarrow \boxed{\mathcal{L}_F} \quad (\not{P}, \not{C})$$

where two couplings g and g' have been introduced and

$$\widetilde{W}_\mu(x) \mapsto U_L(x)\widetilde{W}_\mu(x)U^\dagger_L(x) - \frac{\mathrm{i}}{g}(\partial_\mu U_L(x))U^\dagger_L(x)$$

$$B_\mu(x) \mapsto B_\mu(x) + \frac{1}{g'}\partial_\mu\beta(x)$$

⇒ Add Yang-Mills: gauge invariant kinetic terms for the gauge fields

$$\boxed{\mathcal{L}_{\rm YM}} = -\frac{1}{4}W^i_{\mu\nu}W^{i,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,, \quad W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon_{ijk}W^j_\mu W^k_\nu$$

(include self-interactions of the SU(2) gauge fields) and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

The EWSM with one family | mass terms forbidden

⇒ Note that mass terms are not invariant under $SU(2)_L \otimes U(1)_Y$, since LH and RH components do not transform the same:

$$m\overline{f}f = m(\overline{f_L}f_R + \overline{f_R}f_L)$$

⇒ Mass terms for the gauge bosons are not allowed either

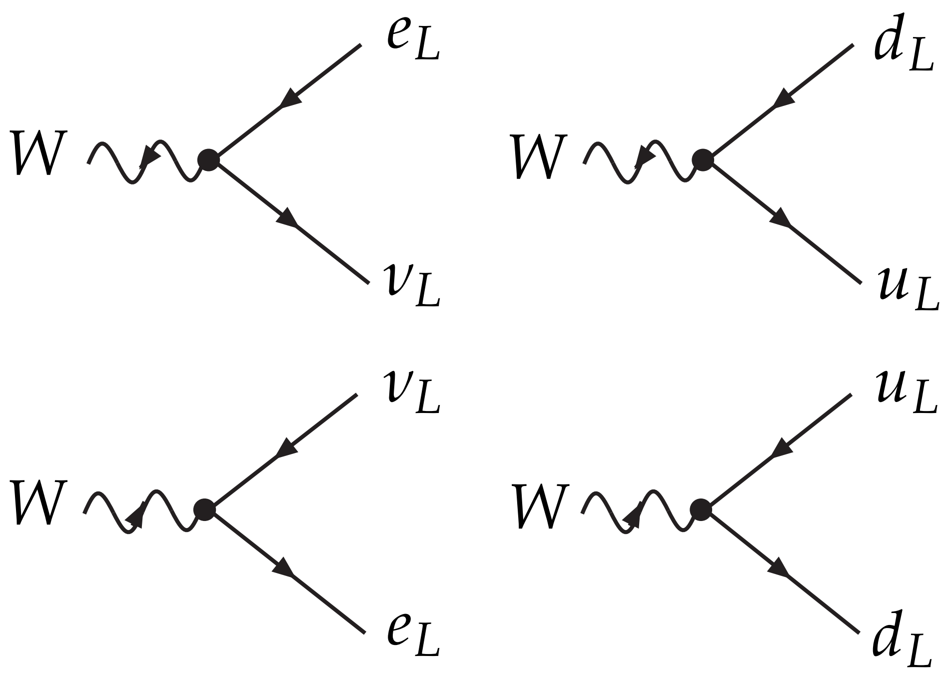
⇒ Next the different types of interactions are analyzed, and later the EWSB will be discussed

The EWSM with one family	charged current interactions

- $$\mathcal{L}_F \supset g\overline{\Psi}_1\gamma^\mu\widetilde{W}_\mu\Psi_1\,, \quad \widetilde{W}_\mu = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^\dagger \\ \sqrt{2}W_\mu & -W_\mu^3 \end{pmatrix}$$

$\Rightarrow$ charged current interactions of LH fermions with complex vector boson field W_μ:

$$\mathcal{L}_{\rm CC} = \frac{g}{2\sqrt{2}}\overline{f}\gamma^\mu(1-\gamma_5)f'W_\mu^\dagger + \text{h.c.}\,, \quad W_\mu \equiv \frac{1}{\sqrt{2}}(W_\mu^1 + \text{i}W_\mu^2)$$



- The diagonal part of

$$\mathcal{L}_F \supset g\overline{\Psi}_1\gamma^\mu \widetilde{W}_\mu \Psi_1 - g' B_\mu (y_1\overline{\Psi}_1\gamma^\mu\Psi_1 + y_2\overline{\psi}_2\gamma^\mu\psi_2 + y_3\overline{\psi}_3\gamma^\mu\psi_3)$$

$\Rightarrow$ neutral current interactions with neutral vector boson fields W^3_μ and B_μ

We would like to identify B_μ with the photon field A_μ but that requires:

$$y_1 = y_2 = y_3 \quad \text{and} \quad g' y_j = eQ_j \quad \Rightarrow \quad \text{impossible!}$$

$\Rightarrow$ Since they are both neutral, try a combination:

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \equiv \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad \begin{array}{c} s_W \equiv \sin\theta_W\,, \quad c_W \equiv \cos\theta_W \\ \theta_W = \text{weak mixing angle} \end{array}$$

$$\mathcal{L}_{\rm NC} = \sum_{j=1}^{3} \overline{\psi}_j\gamma^\mu \left\{ -\left[gT_3 s_W + g' y_j c_W\right] A_\mu + \left[gT_3 c_W - g' y_j s_W\right] Z_\mu \right\} \psi_j$$

with $T_3 = \dfrac{\sigma_3}{2}$ (0) the third weak isospin component of the doublet (singlet)

- To make A_μ the photon field:

$$(1)\ \boxed{e = g s_W = g' c_W} \qquad (2)\ \boxed{Q = T_3 + Y}$$

where the electric charge operator is: $Q_1 = \begin{pmatrix} Q_f & 0 \\ 0 & Q_{f'} \end{pmatrix}$, $\quad Q_2 = Q_f$, $\quad Q_3 = Q_{f'}$

⇒ (1) Electroweak unification: g of SU(2) and g' of U(1) related to $e = \dfrac{gg'}{\sqrt{g^2 + g'^2}}$

⇒ (2) The hyperchages are fixed in terms of electric charges and weak isospin:

$$y_1 = Q_f - \frac{1}{2} = Q_{f'} + \frac{1}{2}, \qquad y_2 = Q_f, \qquad y_3 = Q_{f'}$$

$$\mathcal{L}_{\text{QED}} = -e\, Q_f \overline{f} \gamma^\mu f\, A_\mu \quad + (f \to f')$$

⇒ RH neutrinos are sterile: $y_2 = Q_f = 0$

The EWSM with one family | neutral current interactions

- The Z_μ is the neutral weak boson field:

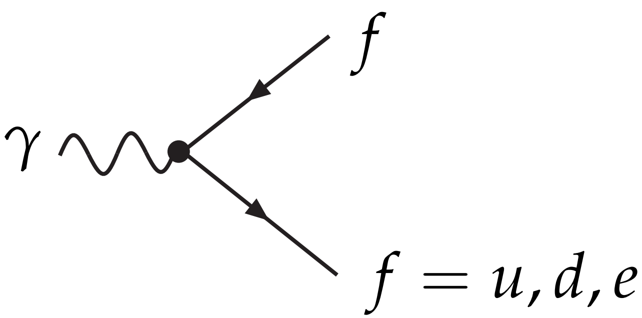
$$\mathcal{L}_{\rm NC}^{Z} = e\,\overline{f}\gamma^\mu(v_f - a_f\gamma_5)f\;Z_\mu \quad + (f \to f')$$

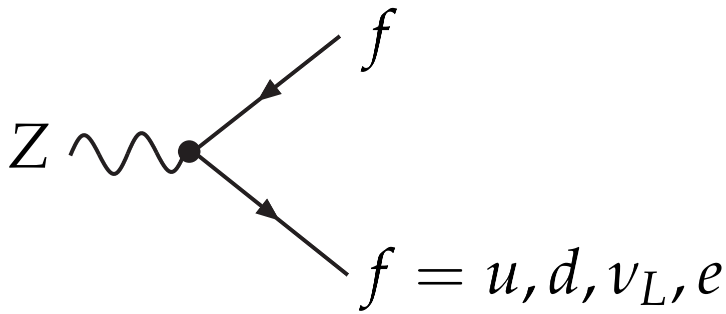
with

$$v_f = \frac{T_3^{f_L} - 2Q_f s_W^2}{2s_W c_W}\,, \quad a_f = \frac{T_3^{f_L}}{2s_W c_W}$$

- The complete neutral current Lagrangian reads:

$$\mathcal{L}_{\rm NC} = \mathcal{L}_{\rm QED} + \mathcal{L}_{\rm NC}^{Z}$$



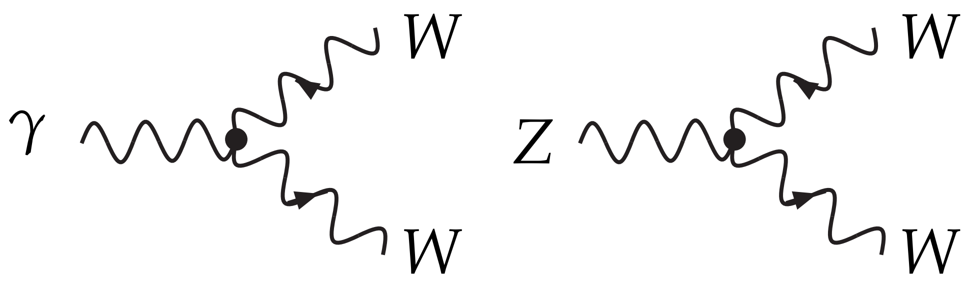


- Cubic:

$$\mathcal{L}_{\rm YM} \supset \mathcal{L}_3 = -\frac{\mathrm{i}ec_W}{s_W}\left\{W^{\mu\nu}W^\dagger_\mu Z_\nu - W^\dagger_{\mu\nu}W^\mu Z^\nu - W^\dagger_\mu W_\nu Z^{\mu\nu}\right\}$$
$$+\,\mathrm{i}e\left\{W^{\mu\nu}W^\dagger_\mu A_\nu - W^\dagger_{\mu\nu}W^\mu A^\nu - W^\dagger_\mu W_\nu F^{\mu\nu}\right\}$$

with

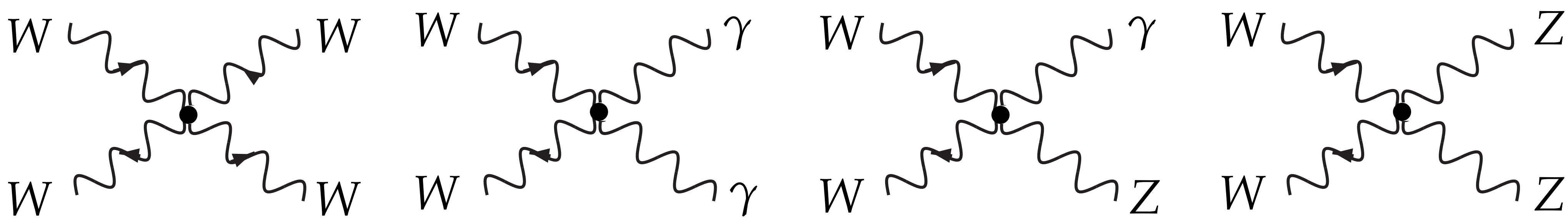
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$



The EWSM with one family | gauge boson self-interactions

- Quartic:

$$\mathcal{L}_{\rm YM} \supset \mathcal{L}_4 = -\frac{e^2}{2s_W^2}\left\{\left(W_\mu^\dagger W^\mu\right)^2 - W_\mu^\dagger W^{\mu\dagger} W_\nu W^\nu\right\}$$
$$-\frac{e^2 c_W^2}{s_W^2}\left\{W_\mu^\dagger W^\mu Z_\nu Z^\nu - W_\mu^\dagger Z^\mu W_\nu Z^\nu\right\}$$
$$+\frac{e^2 c_W}{s_W}\left\{2W_\mu^\dagger W^\mu Z_\nu A^\nu - W_\mu^\dagger Z^\mu W_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu Z^\nu\right\}$$
$$-e^2\left\{W_\mu^\dagger W^\mu A_\nu A^\nu - W_\mu^\dagger A^\mu W_\nu A^\nu\right\}$$



Note: even number of W and no vertex with just γ or Z

Electroweak symmetry breaking | setup

- Out of the 4 gauge bosons of $SU(2)_L \otimes U(1)_Y$ with generators $T_1,\ T_2,\ T_3,\ Y$ we need all to be broken except the combination $Q = T_3 + Y$ so that A_μ remains massless and the other three gauge bosons get massive after SSB

  $\Rightarrow$ Introduce a complex SU(2) Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \langle 0|\,\Phi\,|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

  with gauge invariant Lagrangian ($\mu^2 = -\lambda v^2$):

$$\boxed{\mathcal{L}_\Phi} = (D_\mu \Phi)^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 , \qquad D_\mu \Phi = (\partial_\mu - \mathrm{i} g \widetilde{W}_\mu + \mathrm{i} g' y_\Phi B_\mu) \Phi$$

$$\text{take } y_\Phi = \frac{1}{2} \quad \Rightarrow \quad (T_3 + Y) \begin{pmatrix} 0 \\ v \end{pmatrix} = Q \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

$$\{T_1, T_2, T_3 - Y\} \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0$$

Electroweak symmetry breaking | gauge boson masses

- Quantum fields in the unitary gauge:

$$\Phi(x) \equiv \exp\left\{\mathrm{i}\frac{\sigma_i}{2v}\theta^i(x)\right\}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$

$$\Phi(x) \mapsto \exp\left\{-\mathrm{i}\frac{\sigma_i}{2v}\theta^i(x)\right\}\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix} \quad \Rightarrow$$

1 physical Higgs field $H(x)$

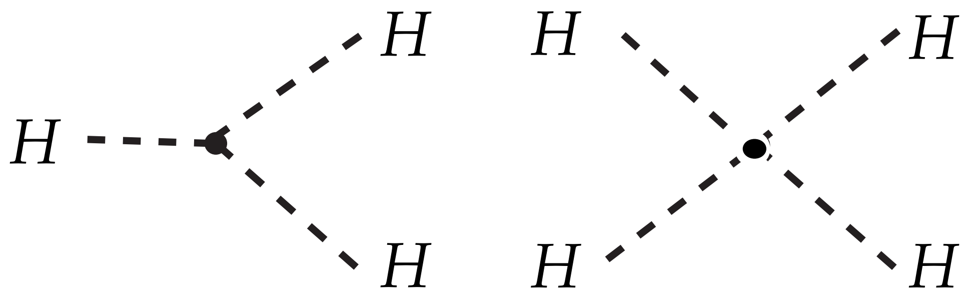
3 would-be Goldstones $\theta^i(x)$ gauged away

– The 3 dof apparently lost become the longitudinal polarizations of $\mathrm{W}^\pm$ and Z that get massive after SSB:

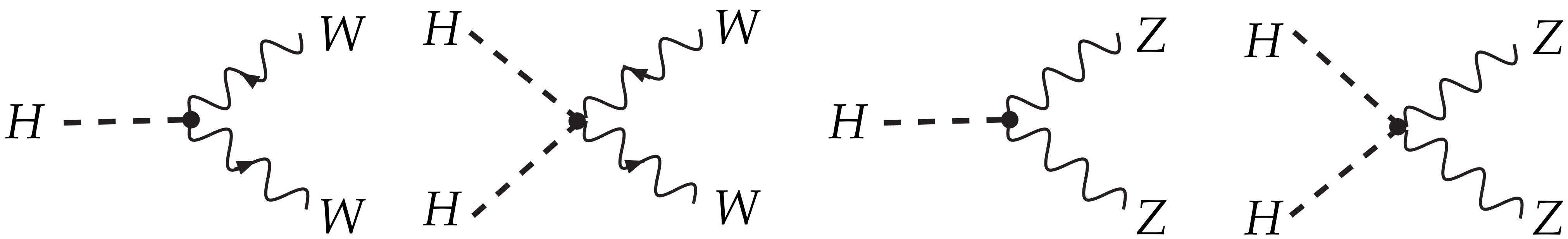
$$\mathcal{L}_\Phi \supset \mathcal{L}_M = \underbrace{\frac{g^2v^2}{4}}_{M_W^2} W_\mu^\dagger W^\mu + \underbrace{\frac{g^2v^2}{8c_W^2}}_{\frac{1}{2}M_Z^2} Z_\mu Z^\mu \quad \Rightarrow \quad \underbrace{M_W = M_Z c_W}_{\text{custodial symmetry}} = \frac{1}{2}gv$$

⇒ In the unitary gauge (just physical fields): $\mathcal{L}_\Phi = \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4$

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}M_H^2 H^2 - \frac{M_H^2}{2v}H^3 - \frac{M_H^2}{8v^2}H^4 \,, \quad M_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}\, v$$



$$\mathcal{L}_M + \mathcal{L}_{HV^2} = M_W^2 W_\mu^\dagger W^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\}$$



Electroweak symmetry breaking | Higgs sector

- Quantum fields in the R_ξ gauges:

$$\Phi(x) = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \equiv \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}[v + H(x) + i\chi(x)] \end{pmatrix}, \quad \phi^-(x) = [\phi^+(x)]^*$$

$$\mathcal{L}_\Phi = \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4$$

$$+ (\partial_\mu \phi^+)(\partial^\mu \phi^-) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi)$$

$$+ iM_W \left(W_\mu \partial^\mu \phi^+ - W_\mu^\dagger \partial^\mu \phi^-\right) + M_Z Z_\mu \partial^\mu \chi$$

$+$ trilinear interactions [SSS, SSV, SVV]

$+$ quadrilinear interactions [SSSS, SSVV]

Electroweak symmetry breaking | gauge fixing

- To remove the cross terms $W_\mu \partial^\mu \phi^+$, $W_\mu^\dagger \partial^\mu \phi^-$, $Z_\mu \partial^\mu \chi$ and define propagators add:

$$\boxed{\mathcal{L}_{\rm GF}} = -\frac{1}{2\xi_\gamma}(\partial_\mu A^\mu)^2 - \frac{1}{2\xi_Z}(\partial_\mu Z^\mu - \xi_Z M_Z \chi)^2 - \frac{1}{\xi_W}|\partial_\mu W^\mu + \mathrm{i}\xi_W M_W \phi^-|^2$$

$\Rightarrow$ Massive propagators for gauge and (unphysical) would-be Goldstone fields:

$$\widetilde{D}^\gamma_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_\gamma)\frac{k_\mu k_\nu}{k^2}\right]$$

$$\widetilde{D}^Z_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_Z^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_Z)\frac{k_\mu k_\nu}{k^2 - \xi_Z M_Z^2}\right] \quad ; \quad \widetilde{D}^\chi(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\varepsilon}$$

$$\widetilde{D}^W_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_W^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_W)\frac{k_\mu k_\nu}{k^2 - \xi_W M_W^2}\right] \quad ; \quad \widetilde{D}^\phi(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\varepsilon}$$

('t Hooft-Feynman gauge: $\xi_\gamma = \xi_Z = \xi_W = 1$)

- The SM is a non-Abelian theory $\Rightarrow$ add Faddeev-Popov ghosts $c_i(x)$ $(i=1,2,3)$

$$c_1 \equiv \frac{1}{\sqrt{2}}(u_+ + u_-)\,, \quad c_2 \equiv \frac{\mathrm{i}}{\sqrt{2}}(u_+ - u_-)\,, \quad c_3 \equiv c_W\, u_Z - s_W\, u_\gamma$$

$$\boxed{\mathcal{L}_{\rm FP}} = \underbrace{(\partial^\mu \bar{c}_i)(\partial_\mu c_i - g\epsilon_{ijk}c_j W^k_\mu)}_{\text{U kinetic + [UUV]}} + \underbrace{\text{interactions with } \Phi}_{\text{U masses + [SUU]}}$$

$\Rightarrow$ Massive propagators for (unphysical) FP ghost fields:

$$\widetilde{D}^{u_\gamma}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\,, \quad \widetilde{D}^{u_Z}(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\varepsilon}\,, \quad \widetilde{D}^{u_\pm}(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\varepsilon}$$

('t Hooft-Feynman gauge: $\xi_Z = \xi_W = 1$)

Electroweak symmetry breaking | Faddeev-Popov ghosts

$$\mathcal{L}_{\rm FP} = (\partial_\mu \overline{u}_\gamma)(\partial^\mu u_\gamma) + (\partial_\mu \overline{u}_Z)(\partial^\mu u_Z) + (\partial_\mu \overline{u}_+)(\partial^\mu u_+) + (\partial_\mu \overline{u}_-)(\partial^\mu u_-)$$

$$[\text{UUV}] \left\{ \begin{array}{l} + \mathrm{i}e[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]A_\mu - \dfrac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]Z_\mu \\ - \mathrm{i}e[(\partial^\mu \overline{u}_+)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_-]W^\dagger_\mu + \dfrac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_+)u_Z - (\partial^\mu \overline{u}_Z)u_-]W^\dagger_\mu \\ + \mathrm{i}e[(\partial^\mu \overline{u}_-)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_+]W_\mu - \dfrac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_-)u_Z - (\partial^\mu \overline{u}_Z)u_+]W_\mu \end{array} \right.$$

$$- \xi_Z M_Z^2\, \overline{u}_Z u_Z - \xi_W M_W^2\, \overline{u}_+ u_+ - \xi_W M_W^2\, \overline{u}_- u_-$$

$$[\text{SUU}] \left\{ \begin{array}{l} - e\xi_Z M_Z\, \overline{u}_Z \left[\dfrac{1}{2s_W c_W} H u_Z - \dfrac{1}{2s_W} (\phi^+ u_- + \phi^- u_+) \right] \\ - e\xi_W M_W\, \overline{u}_+ \left[\dfrac{1}{2s_W}(H + \mathrm{i}\chi)u_+ - \phi^+ \left(u_\gamma - \dfrac{c_W^2 - s_W^2}{2s_W c_W} u_Z \right) \right] \\ - e\xi_W M_W\, \overline{u}_- \left[\dfrac{1}{2s_W}(H - \mathrm{i}\chi)u_- - \phi^- \left(u_\gamma - \dfrac{c_W^2 - s_W^2}{2s_W c_W} u_Z \right) \right] \end{array} \right.$$

Electroweak symmetry breaking — fermion masses

- We need masses for quarks and leptons without breaking gauge symmetry

$\Rightarrow$ Introduce Yukawa interactions:

$$\mathcal{L}_\mathrm{Y} = -\lambda_d \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R - \lambda_u \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} u_R$$

$$- \lambda_\ell \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \ell_R - \lambda_\nu \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \nu_R \quad + \mathrm{h.c.}$$

where $\widetilde{\Phi} \equiv i\sigma_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ 7

$\Rightarrow$ After EW SSB, fermions acquire masses ($\overline{f}f = \overline{f_L}f_R + \overline{f_R}f_L$):

$$\mathcal{L}_\mathrm{Y} \supset -\frac{1}{\sqrt{2}}(v+H)\left\{\lambda_d\,\overline{d}d + \lambda_u\,\overline{u}u + \lambda_\ell\,\overline{\ell}\ell + \lambda_\nu\,\overline{\nu}\nu\right\} \quad \Rightarrow \quad m_f = \lambda_f \frac{v}{\sqrt{2}}$$

Additional generations — Yukawa matrices

- There are 3 generations of quarks and leptons in Nature. They are identical copies with the same properties under $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ differing only in their masses

  ⇒ Take a general case of n generations and let u_i^I, d_i^I, ν_i^I, ℓ_i^I be the members of family i ($i = 1, \ldots, n$). Superindex I (interaction basis) was omitted so far

  ⇒ General gauge invariant Yukawa Lagrangian:

$$\boxed{\mathcal{L}_\mathrm{Y}} = -\sum_{ij} \left\{ \begin{pmatrix} \overline{u}_{iL}^I & \overline{d}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(d)}\, d_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(u)}\, u_{jR}^I \right] + \begin{pmatrix} \overline{\nu}_{iL}^I & \overline{\ell}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(\ell)}\, \ell_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(\nu)}\, \nu_{jR}^I \right] \right\} + \mathrm{h.c.}$$

  where $\lambda_{ij}^{(d)}$, $\lambda_{ij}^{(u)}$, $\lambda_{ij}^{(\ell)}$, $\lambda_{ij}^{(\nu)}$ are arbitrary Yukawa matrices

Additional generations | mass matrices

- After EW SSB, in n-dimensional matrix form:

$$\mathcal{L}_{\rm Y} \supset -\left(1+\frac{H}{v}\right)\left\{\overline{\mathbf{d}}^I_L\,\mathbf{M}_d\,\mathbf{d}^I_R + \overline{\mathbf{u}}^I_L\,\mathbf{M}_u\,\mathbf{u}^I_R + \overline{\mathbf{l}}^I_L\,\mathbf{M}_\ell\,\mathbf{l}^I_R + \overline{\boldsymbol{\nu}}^I_L\,\mathbf{M}_\nu\,\boldsymbol{\nu}^I_R + \text{h.c.}\right\}$$

with mass matrices

$$(\mathbf{M}_d)_{ij} \equiv \lambda^{(d)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_u)_{ij} \equiv \lambda^{(u)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_\ell)_{ij} \equiv \lambda^{(\ell)}_{ij}\frac{v}{\sqrt{2}} \quad (\mathbf{M}_\nu)_{ij} \equiv \lambda^{(\nu)}_{ij}\frac{v}{\sqrt{2}}$$

⇒ Diagonalization determines mass eigenstates d_j, u_j, ℓ_j, ν_j in terms of interaction states d^I_j, u^I_j, ℓ^I_j, ν^I_j, respectively

⇒ Each $\mathbf{M}_f$ can be written as

$$\mathbf{M}_f = \mathbf{H}_f\,\mathcal{U}_f = \mathbf{V}_f^\dagger\,\mathcal{M}_f\,\mathbf{V}_f\,\mathcal{U}_f \quad \Longleftrightarrow \quad \mathbf{M}_f\mathbf{M}_f^\dagger = \mathbf{H}_f^2 = \mathbf{V}_f^\dagger\,\mathcal{M}_f^2\,\mathbf{V}_f$$

with $\mathbf{H}_f \equiv \sqrt{\mathbf{M}_f\mathbf{M}_f^\dagger}$ a Hermitian positive definite matrix and $\mathcal{U}_f$ unitary

– Every $\mathbf{H}_f$ can be diagonalized by a unitary matrix $\mathbf{V}_f$

– The resulting $\mathcal{M}_f$ is diagonal and positive definite

Additional generations | fermion masses and mixings

- In terms of diagonal mass matrices (mass eigenstate basis):

$$\mathcal{M}_d = \mathrm{diag}(m_d, m_s, m_b, \ldots)\,, \quad \mathcal{M}_u = \mathrm{diag}(m_u, m_c, m_t, \ldots)$$
$$\mathcal{M}_\ell = \mathrm{diag}(m_e, m_\mu, m_\tau, \ldots)\,, \quad \mathcal{M}_\nu = \mathrm{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}, \ldots)$$

$$\mathcal{L}_\mathrm{Y} \supset -\left(1 + \frac{H}{v}\right)\left\{ \overline{\mathbf{d}}\,\mathcal{M}_d\,\mathbf{d} + \overline{\mathbf{u}}\,\mathcal{M}_u\,\mathbf{u} + \overline{\mathbf{l}}\,\mathcal{M}_\ell\,\mathbf{l} + \overline{\boldsymbol{\nu}}\,\mathcal{M}_\nu\,\boldsymbol{\nu} \right\}$$

where fermion couplings to Higgs are proportional to masses and

$$\mathbf{d}_L \equiv \mathbf{V}_d\,\mathbf{d}^I_L \qquad \mathbf{u}_L \equiv \mathbf{V}_u\,\mathbf{u}^I_L \qquad \mathbf{l}_L \equiv \mathbf{V}_\ell\,\mathbf{l}^I_L \qquad \boldsymbol{\nu}_L \equiv \mathbf{V}_\nu\,\boldsymbol{\nu}^I_L$$
$$\mathbf{d}_R \equiv \mathbf{V}_d\mathcal{U}_d\,\mathbf{d}^I_R \qquad \mathbf{u}_R \equiv \mathbf{V}_u\mathcal{U}_u\,\mathbf{u}^I_R \qquad \mathbf{l}_R \equiv \mathbf{V}_\ell\mathcal{U}_\ell\,\mathbf{l}^I_R \qquad \boldsymbol{\nu}_R \equiv \mathbf{V}_\nu\mathcal{U}_\nu\,\boldsymbol{\nu}^I_R$$

$\Rightarrow$ Neutral Currents preserve chirality
$\overline{\mathbf{f}}^I_L\,\mathbf{f}^I_L = \overline{\mathbf{f}}_L\,\mathbf{f}_L$ and $\overline{\mathbf{f}}^I_R\,\mathbf{f}^I_R = \overline{\mathbf{f}}_R\,\mathbf{f}_R$
$\Bigg\} \Rightarrow \mathcal{L}_\mathrm{NC}$ does not change family

$\Rightarrow$ GIM mechanism [Glashow, Iliopoulos, Maiani '70]

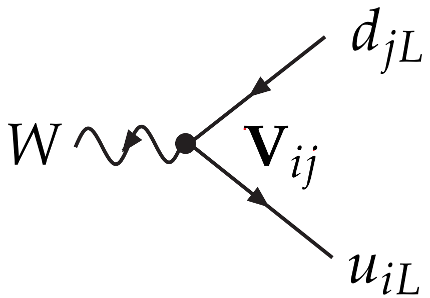
Additional generations — quark sector

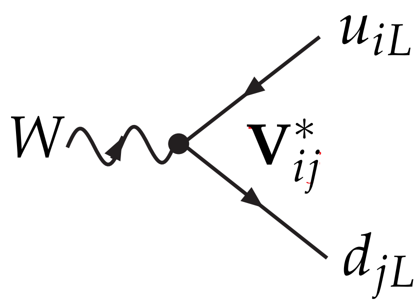
- However, in Charged Currents (also chirality preserving and only LH):

$$\overline{\mathbf{u}}^I_L\,\mathbf{d}^I_L = \overline{\mathbf{u}}_L\,\mathbf{V}_u\,\mathbf{V}_d^\dagger\,\mathbf{d}_L = \overline{\mathbf{u}}_L\mathbf{V}\mathbf{d}_L$$

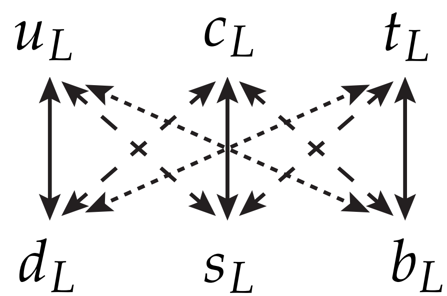
with $\mathbf{V}\equiv\mathbf{V}_u\,\mathbf{V}_d^\dagger$ the (unitary) CKM mixing matrix [Cabibbo '63; Kobayashi, Maskawa '73]

$$\Rightarrow\quad \mathcal{L}_{\rm CC} = \frac{g}{2\sqrt{2}}\sum_{ij}\overline{u}_i\gamma^\mu(1-\gamma_5)\,\mathbf{V}_{ij}\,d_j\,W^\dagger_\mu + \text{h.c.}$$





$\mathcal{L}_{\rm CC}$ changes family !!



$\Rightarrow$ If u_i or d_j had degenerate masses one could choose $\mathbf{V}_u=\mathbf{V}_d$ (field redefinition) and quark families would not mix. But they are *not degenerate*, so they mix!

$\Rightarrow$ $\mathbf{V}_u$ and $\mathbf{V}_d$ are not observable. Just masses and CKM mixings are observable

Additional generations — quark sector

- How many physical parameters in this sector?
  - Quark masses and CKM mixings determined by mass (or Yukawa) matrices
  - A general $n \times n$ unitary matrix, like the CKM, is given by

$$n^2 \text{ real parameters} = n(n-1)/2 \text{ moduli} + n(n+1)/2 \text{ phases}$$

Some phases are unphysical since they can be absorbed by field redefinitions:

$$u_i \to e^{i\phi_i}\, u_i\,, \quad d_j \to e^{i\theta_j}\, d_j \quad \Rightarrow \quad \mathbf{V}_{ij} \to \mathbf{V}_{ij}\, e^{i(\theta_j - \phi_i)}$$

Therefore $2n-1$ unphysical phases and the physical parameters are:

$$\boxed{(n-1)^2 = n(n-1)/2 \text{ moduli } + (n-1)(n-2)/2 \text{ phases}}$$

Additional generations — quark sector

⇒ Case of $n = 2$ generations: 1 parameter, the Cabibbo angle θ_C:

$$\mathbf{V} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}$$

⇒ Case of $n = 3$ generations: 3 angles + 1 phase. In the standard parameterization:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{ud} & \mathbf{V}_{us} & \mathbf{V}_{ub} \\ \mathbf{V}_{cd} & \mathbf{V}_{cs} & \mathbf{V}_{cb} \\ \mathbf{V}_{td} & \mathbf{V}_{ts} & \mathbf{V}_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}\mathrm{e}^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \Rightarrow$$

δ only source of CP violation in the SM !

with $c_{ij} \equiv \cos\theta_{ij} \geq 0, \quad s_{ij} \equiv \sin\theta_{ij} \geq 0 \quad (i < j = 1, 2, 3)$ and $0 \leq \delta \leq 2\pi$

Additional generations | lepton sector

- If neutrinos were massless we could redefine the (LH) fields $\Rightarrow$ no lepton mixing.

  However they *are* massive (though *very light* masses) $\Leftarrow$ neutrino oscillations!

  – $\boxed{\nu\text{SM}}$ (introduce ν_R and get masses from *tiny* Yukawa couplings like quarks)

  Alternatively ...

- Neutrinos are special:

  they *may* be their own antiparticle (Majorana) since they are neutral fermions

  $\Rightarrow$ New mechanisms for generation of masses and mixings

  - Mass terms are different to Dirac case
  - Neutrinos and antineutrinos *may* mix
  - Intergenerational mixings are richer (*more* CP *phases*)

  – *If* they are Majorana $\boxed{\text{physics Beyond }\nu\text{SM}}$ (*seesaw* mechanism?)

Additional generations | lepton sector

- What we know about neutrinos:
  - From Z lineshape: $n = 3$ generations of *active* ν_L $[\nu_i\ (i = 1, \ldots, n)]$ (but we do not know (*yet*) if neutrinos are Dirac or Majorana fermions)
  - From oscillations: active neutrinos are very light, non degenerate and mix PMNS matrix $\mathbf{U}$ [Pontecorvo '57; Maki, Nakagawa, Sakata '62; Pontecorvo '68]

$$|\nu_\alpha\rangle = \sum_i \mathbf{U}_{\alpha i} |\nu_i\rangle \quad \Longleftrightarrow \quad |\nu_i\rangle = \sum_\alpha \mathbf{U}^*_{\alpha i} |\nu_\alpha\rangle$$

mass eigenstates ν_i $(i = 1, 2, 3)$ / interaction states ν_α $(\alpha = e, \mu, \tau)$

$\Rightarrow$ ***If** neutrinos **were** Majorana* $\mathbf{U}$ *seems* unitary (for negligible light-heavy mixings) and analogous to $\mathbf{V}_u$, $\mathbf{V}_d$, $\mathbf{V}_\ell$ defined for quarks and charged leptons except for:

- ν fields have both chiralities: $\nu_i = \nu_{iL} + \eta_i \nu^c_{iL}$
- *If* ν's are Majorana, $\mathbf{U}$ *may* contain two additional physical (Majorana) phases that *cannot be absorbed* since then field phases are fixed by $\nu_i = \eta_i \nu^c_i$

Additional generations | lepton sector

⇒ Standard parameterization of the PMNS matrix:

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{e1} & \mathbf{U}_{e2} & \mathbf{U}_{e3} \\ \mathbf{U}_{\mu 1} & \mathbf{U}_{\mu 2} & \mathbf{U}_{\mu 3} \\ \mathbf{U}_{\tau 1} & \mathbf{U}_{\tau 2} & \mathbf{U}_{\tau 3} \end{pmatrix}$$

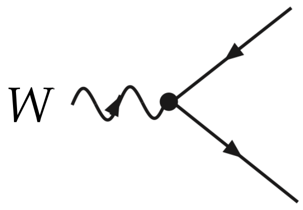
$$= \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, \mathrm{e}^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, \mathrm{e}^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, \mathrm{e}^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, \mathrm{e}^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, \mathrm{e}^{i\delta} & c_{23}\, c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{i\alpha_{21}/2} & 0 \\ 0 & 0 & \mathrm{e}^{i\alpha_{31}/2} \end{pmatrix}$$

(different values than in CKM) (Majorana phases)

[$\theta_{13} \equiv \theta_{\odot}$, $\theta_{23} \equiv \theta_{\text{atm}}$, θ_{13} and δ measured in oscillations]

Additional generations | lepton sector

- **U** introduces family mixings in $\mathcal{L}_{\rm CC}$ (like CKM), but in this case:



$$\nu_{\alpha L} = \sum_i \mathbf{U}_{\alpha i} \nu_{iL}$$

$e_{\alpha L}$

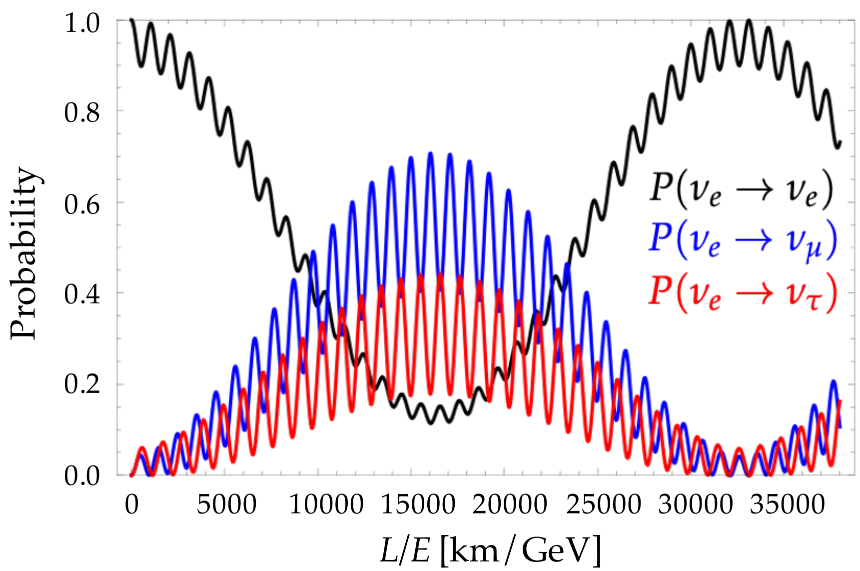
ν_α are *coherent* superpositions of mass eigenstates ν_i (produced/detected in association with ℓ_α)

ℓ_α (e, μ, τ) are mass eigenstates (do *not* oscillate) because $\Delta m_{ij}^2 \ll \Delta m_{\mu e}^2$ [0706.1216]

$$|\nu_\alpha; t\rangle = \sum_i \mathbf{U}_{\alpha i} \mathrm{e}^{-\mathrm{i}E_i t} |\nu_i\rangle , \quad E_i = E + \frac{m_i^2}{2E}$$

$$\langle \nu_\beta | \nu_\alpha; t\rangle = \sum_i \mathbf{U}_{\beta i}^* \mathbf{U}_{\alpha i} \mathrm{e}^{-\mathrm{i}E_i t}$$

$$P(\nu_\alpha \to \nu_\beta) = \sum_{ij} \mathbf{U}_{\beta i}^* \mathbf{U}_{\alpha i} \mathbf{U}_{\beta j} \mathbf{U}_{\alpha j}^* \exp\left(-\mathrm{i}\frac{\Delta m_{ij}^2}{2E} t\right)$$



Strong interactions

Strong interactions — Properties

- Quantum Chromodynamics (QCD) is *the* theory of strong interactions
- *Quarks* and *gluons* are the fundamental *dof* but they never show up as free states: they are bound in hadrons (confinement):

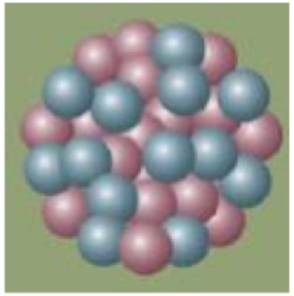
Baryons $(q_1q_2q_3$ or $\bar{q}_1\bar{q}_2\bar{q}_3)$				
name		content	Q [e]	m [GeV]
p	proton	uud	+1	0,938
$\bar{\text{p}}$	antiproton	$\bar{\text{u}}\bar{\text{u}}\bar{\text{d}}$	−1	
n	neutron	ddu	0	0,939
$\bar{\text{n}}$	antineutron	$\bar{\text{d}}\bar{\text{d}}\bar{\text{u}}$		
Λ	lambda	uds	0	1,116
$\bar{\Lambda}$	antilambda	$\bar{\text{u}}\bar{\text{d}}\bar{\text{s}}$		
. . . ∼ 120 . . .				

Mesons $(q_1\bar{q}_2)$				
name		content	Q [e]	m [GeV]
π^0	neutral pion	u$\bar{\text{u}}$, d$\bar{\text{d}}$	0	0,135
π^+	charged pion	u$\bar{\text{d}}$	+1	0,140
π^-		d$\bar{\text{u}}$	−1	
K^+	charged kaon	u$\bar{\text{s}}$	+1	0,494
K^-		s$\bar{\text{u}}$	−1	
K^0	neutral kaon	d$\bar{\text{s}}$	0	0,498
$\overline{K}^0$		s$\bar{\text{d}}$		
. . . ∼ 140 . . .				

and **exotics** (glueballs, tetraquarks, pentaquarks, . . .)

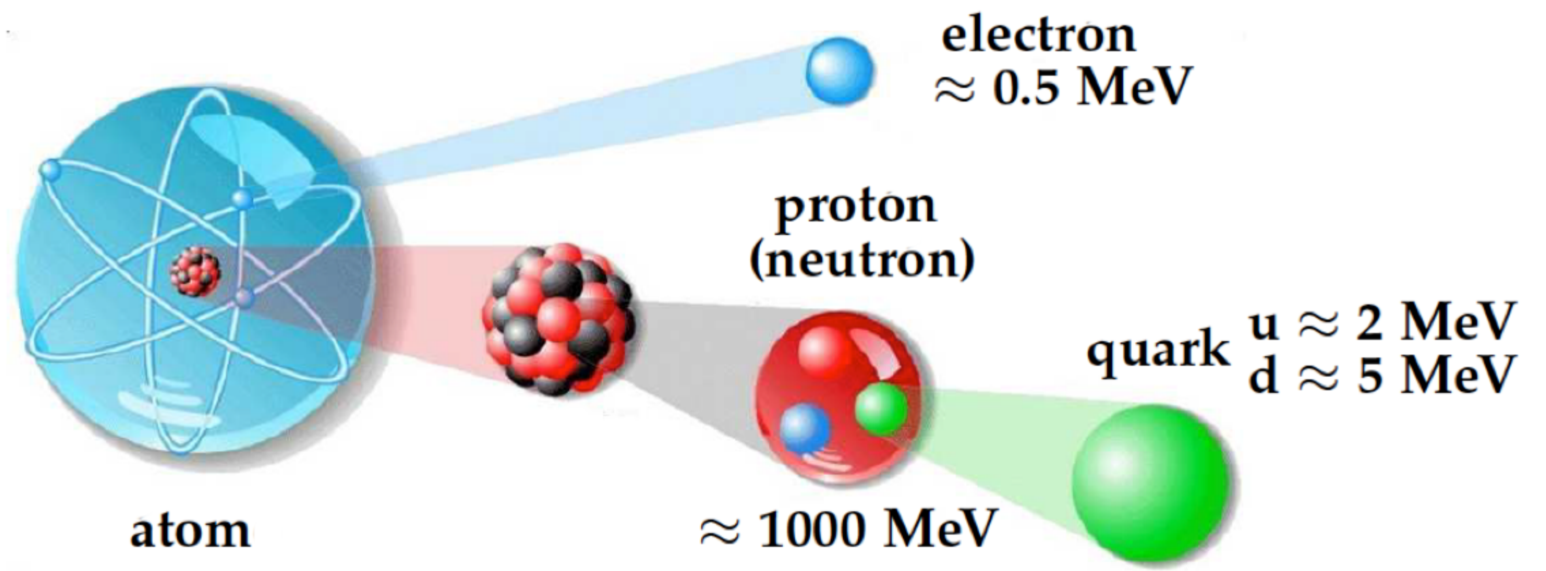
Strong interactions | Properties

- Strong interactions are responsible for:
  - Stability of nuclei (nucleon-nucleon interaction is a residual strong force)



strong attraction is greater than *electric* repulsion

  - $\sim$ 99% of nucleon mass is binding energy, i.e. most of the mass in everything!



QCD | Lagrangian | SU(3) gauge symmetry

$$\mathcal{L}_{\rm QCD} = \underbrace{\overline{\psi}_{fi}\left(\mathrm{i}\not{D}_{ij} - m\delta_{ij}\right)\psi_{fj}}_{\text{quarks}} - \underbrace{\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}}_{\text{gluons}} \qquad \text{(flavor diagonal)}$$

$$F^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g_s f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$$

- (Anti–)quarks ψ_f come in $N_c = 3$ colors (anticolors) and there are $n_f = 6$ flavors:

$$\psi_{fi} \qquad \begin{cases} f = \mathrm{u, d, s, c, b, t} & \text{(flavor index)} \\ i = 1, \ldots, N_c = 3 & \text{(color index)} \end{cases}$$

fundamental irrep **3** ($\bar{\mathbf{3}}$)

- Gluons $\mathcal{A}^a_\mu$ come in $N_c^2 - 1 = 8$ combinations of color and anticolor:

$$\mathcal{A}^a_\mu \qquad a = 1, \ldots, N_c^2 - 1 = 8 \quad \text{(color index)}$$

adjoint irrep **8**

QCD | Lagrangian | SU(3) gauge symmetry

$$\mathcal{L}_{\text{QCD}} = \underbrace{\overline{\psi}_{fi}\left(\mathrm{i}\not{D}_{ij} - m\delta_{ij}\right)\psi_{fj}}_{\text{quarks}} - \underbrace{\frac{1}{4}F^{a}_{\mu\nu}F^{a,\mu\nu}}_{\text{gluons}} \qquad \text{(flavor diagonal)}$$

$$F^{a}_{\mu\nu} = \partial_\mu \mathcal{A}^{a}_{\nu} - \partial_\nu \mathcal{A}^{a}_{\mu} + g_s f^{abc}\mathcal{A}^{b}_{\mu}\mathcal{A}^{c}_{\nu}$$

- Quark kinetic terms and quark-gluon interactions come from covariant derivative:

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - \mathrm{i}g_s t^{a}_{ij}\mathcal{A}^{a}_{\mu}\,, \quad t^{a}_{ij} = \frac{1}{2}\lambda^{a}_{ij} \quad \text{(8 Gell-Mann matrices } 3\times 3\text{)}$$

- Gluon kinetic terms and self-interactions fixed by SU(3) structure constants f^{abc}:

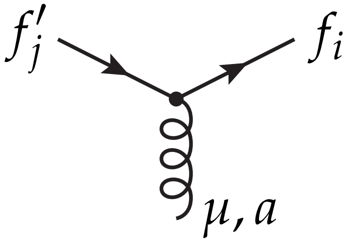
$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu \mathcal{A}^{a}_{\nu} - \partial_\nu \mathcal{A}^{a}_{\mu})(\partial^\mu \mathcal{A}^{a,\nu} - \partial^\nu \mathcal{A}^{a,\mu})$$

$$\mathcal{L}_{\text{cubic}} = -\frac{1}{2}g_s f_{abc}\,(\partial_\mu \mathcal{A}^{a}_{\nu} - \partial_\nu \mathcal{A}^{a}_{\mu})\mathcal{A}^{b,\mu}\mathcal{A}^{c,\nu}$$

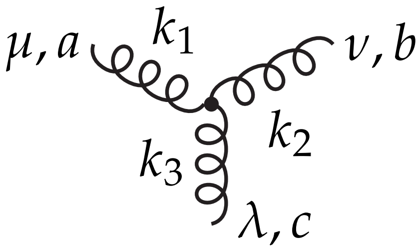
$$\mathcal{L}_{\text{quartic}} = -\frac{1}{4}g_s^2 f_{abe}f_{cde}\,\mathcal{A}^{a}_{\mu}\mathcal{A}^{b}_{\nu}\mathcal{A}^{c,\mu}\mathcal{A}^{d,\nu}$$

- Quark and gluon external legs and propagators are as usual

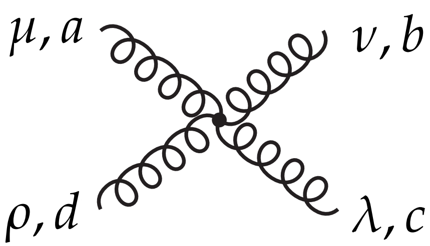
- Vertices:



$$= \mathrm{i} g_s t^a_{ij} \gamma^\mu \delta_{ff'}$$



$$= g_s f_{abc} \left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu \right]$$



$$= -\mathrm{i} g_s^2 \left[\begin{array}{r} f_{abe} f_{cde} \left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) \\ + f_{ace} f_{dbe} \left(g_{\mu\rho} g_{\nu\lambda} - g_{\mu\nu} g_{\lambda\rho} \right) \\ + f_{ade} f_{bce} \left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} \right) \end{array} \right]$$

(interactions with Faddeev-Popov ghosts omitted here)

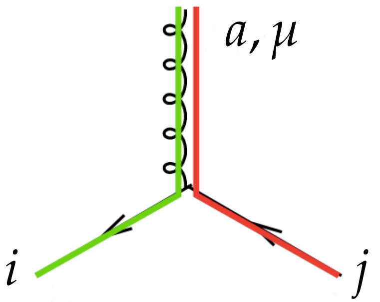
QCD | About color charges

- Quarks carry color charge: $\psi = \psi(x) \otimes \begin{pmatrix} R \\ G \\ B \end{pmatrix}$

- Antiquarks carry anticolor charge: $\overline{\psi} = \overline{\psi}(x) \otimes \begin{pmatrix} \overline{R} & \overline{G} & \overline{B} \end{pmatrix}$

- Gluons carry color and anticolor. A gluon emission *repaints* the quark:

e.g.



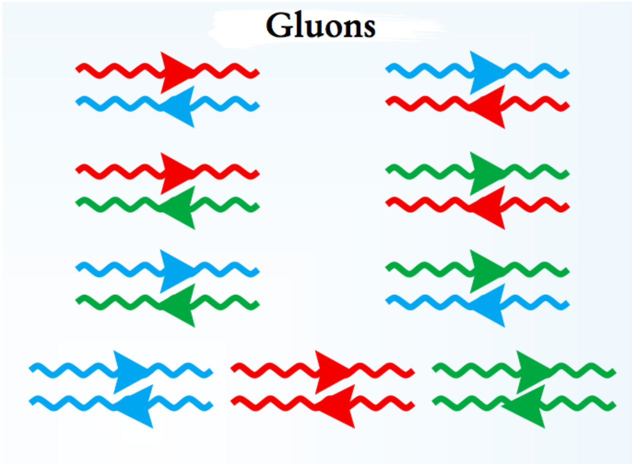
$$\overline{\psi}_i t^1_{ij} \psi_j \sim \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix},$$

QCD — About color charges

- 8 gluons:



$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

(1 in 9 combinations is color-neutral)

$$\left\{ \begin{array}{l} R\overline{R} - G\overline{G} \\ R\overline{R} + G\overline{G} - 2B\overline{B} \end{array} \right.$$

If the color-singlet massless gluon state $R\overline{R} + G\overline{G} + B\overline{B}$ existed, it would give rise to a strong force of infinite range!

- Likewise, only color-singlet states can exist as free particles:

$$q\bar{q}' \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus 8 \quad \textbf{: mesons} \quad \frac{1}{\sqrt{3}}\delta^{ij}\left|q_i\bar{q}'_j\right\rangle$$

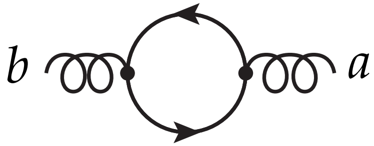
$$qq'q'' \quad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \quad \textbf{: baryons} \quad \frac{1}{\sqrt{6}}\epsilon^{ijk}\left|q_i q'_j q''_k\right\rangle$$

$$i, j, k \in \{R, G, B\}$$

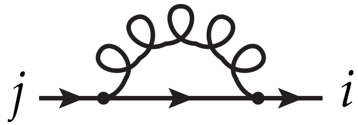
but qq' color singlets do not exist, since $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$

QCD — About color charges

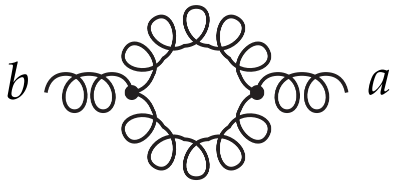
- Color algebra (useful identities): $t^a = \frac{1}{2}\lambda^a, \quad N_C = 3$



$\propto \mathrm{Tr}(t^a t^b) = T_R \delta_{ab}, \quad T_R = \dfrac{1}{2}$



$\propto t^a_{ik} t^a_{kj} = C_F \delta_{ij}, \quad C_F = \dfrac{N_c^2 - 1}{2N_c} = \dfrac{4}{3}$



$\propto f_{acd} f_{bcd} = C_A \delta_{ab}, \quad C_A = N_c = 3$

probability of a gluon to emit $q\bar{q}$ < quark to emit a gluon < gluon to emit gluons

(gluons interact more strongly than quarks)

QED vs QCD | running coupling

- All coupling constants *run*:

  $\alpha \equiv \dfrac{g^2}{4\pi} = \alpha(Q^2)$, where Q is the momentum scale of the process

$$Q^2 \frac{\partial \alpha}{\partial Q^2} = \beta(\alpha) \,, \quad \beta(\alpha) \equiv -\alpha^2(\beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 + \ldots)$$

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 + \beta_0 \alpha(Q_0^2) \ln \dfrac{Q^2}{Q_0^2}} \qquad \text{(Leading Order)}$$

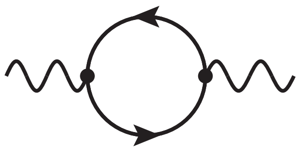
- Physically, this is related to the (anti-)screening of the fundamental charges by quantum fluctuations, depending on the sign of β_0:

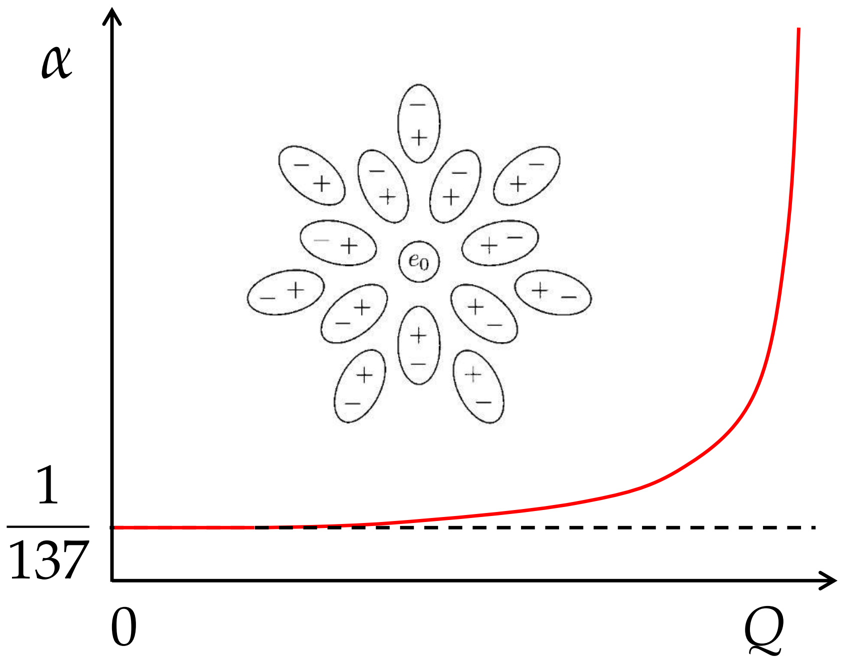
  - In QED: $\alpha_{\text{em}} = \dfrac{e^2}{4\pi}\,, \beta_0(\alpha_{\text{em}}) = -\dfrac{1}{3\pi} \quad (< 0)$

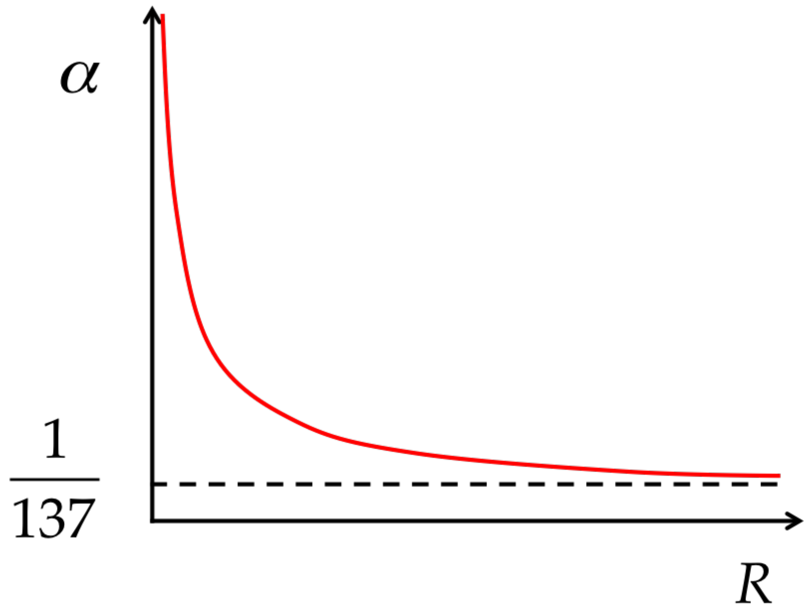
  - In QCD: $\alpha_s = \dfrac{g_s^2}{4\pi}\,, \beta_0(\alpha_s) = \dfrac{11 C_A - 4 T_R N_f}{12\pi} = \dfrac{33 - 2N_f}{12\pi} \quad (> 0 \text{ for } N_f \leq 16)$

QED running coupling

- In QED, the fluctuating vacuum behaves like a dielectric medium, screening the bare electric charge e_0 at increasing distances $R \sim 1/Q$:



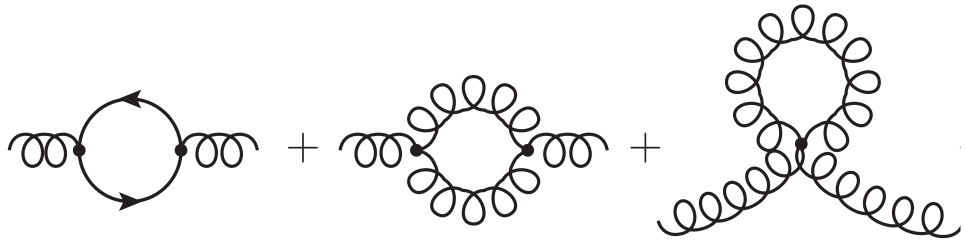




e.g. $\alpha(M_Z^2) \approx 1/128$

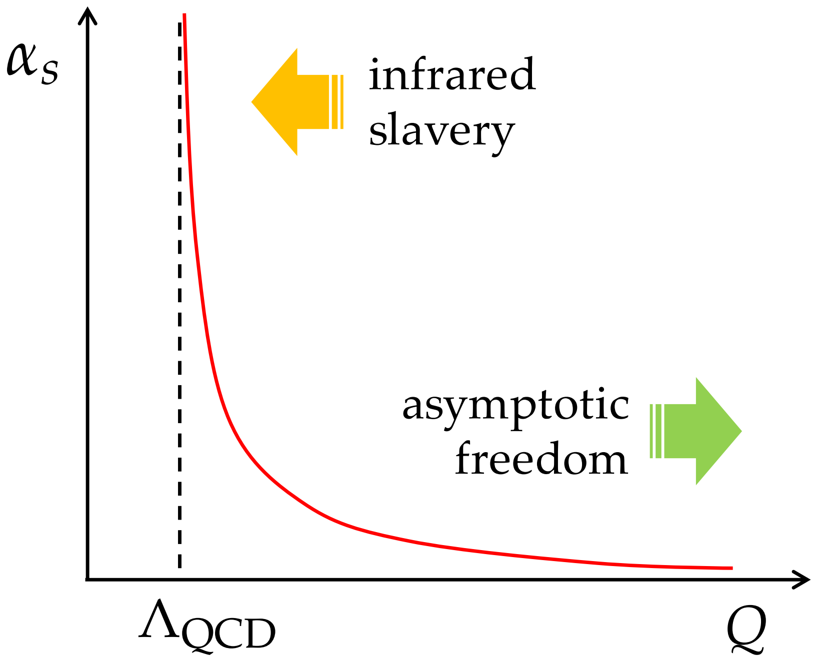
QCD running coupling

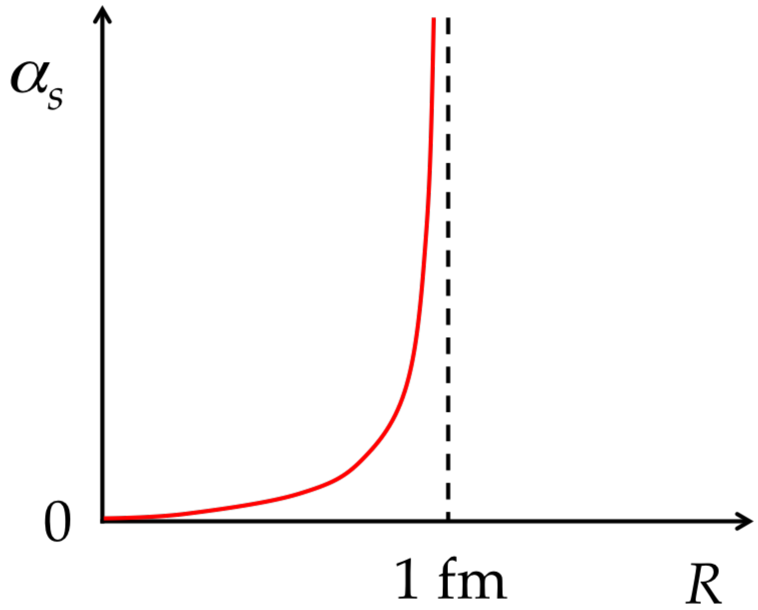
- Contributions to the QCD beta function $\beta(\alpha_s)$ (from QCD vacuum polarization):



(1) screening

(2), (3) anti-screening (non-abelian!)





$\Lambda_{QCD} \approx 200$ MeV

QCD | running coupling

⇒ There is a scale $\Lambda_{\rm QCD}$ where $\alpha_s \to \infty$ (dimensional transmutation) given at LO by

$$\Lambda^2_{\rm QCD} = Q^2 \exp\left\{-\frac{1}{\beta_0 \alpha_s(Q^2)}\right\} \quad \Leftrightarrow \quad \alpha_s(Q^2) = \frac{1}{\beta_0 \ln \dfrac{Q^2}{\Lambda^2_{\rm QCD}}} \qquad (Q^2 > \Lambda^2_{\rm QCD})$$

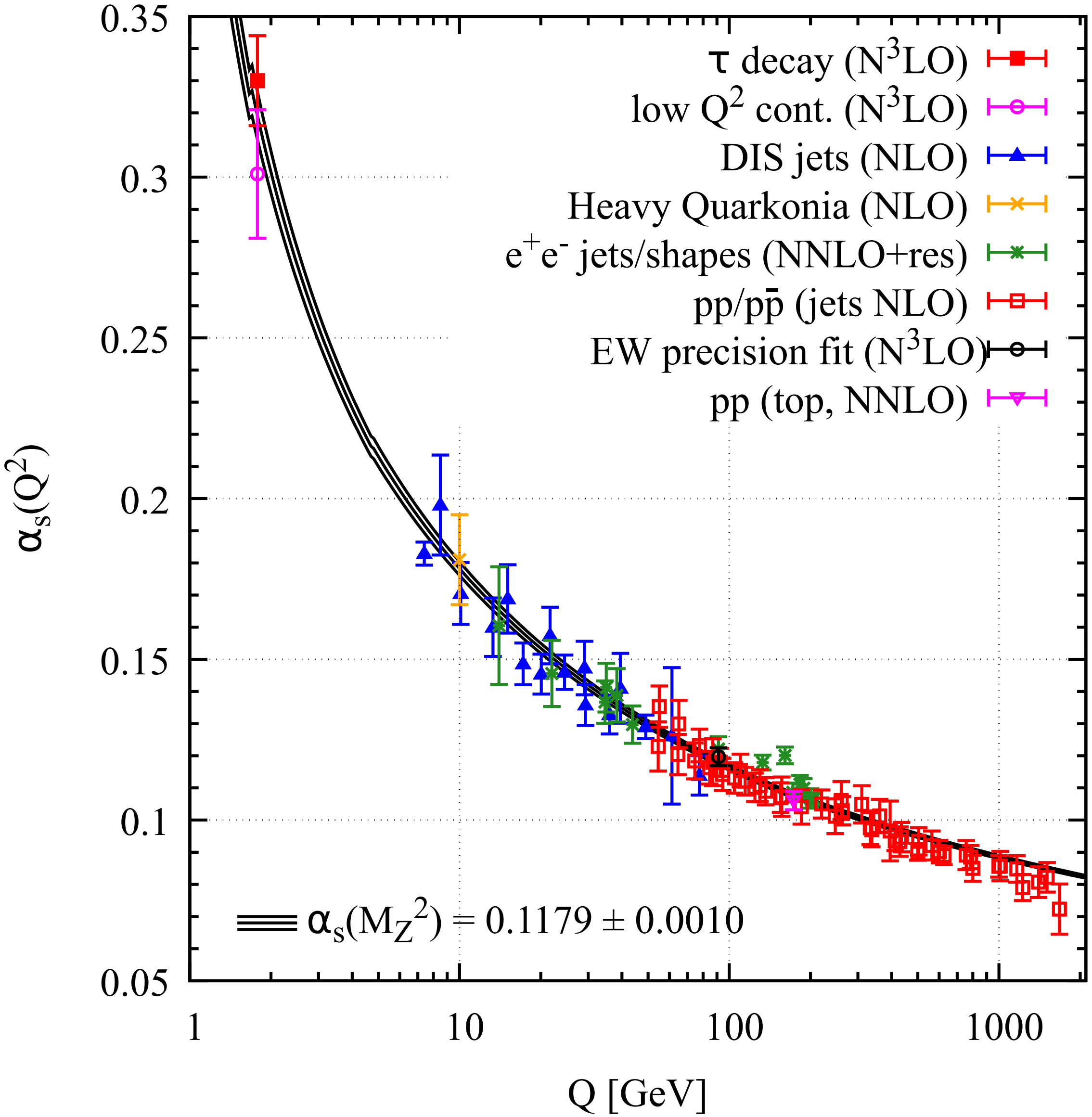
$\Lambda_{\rm QCD} \approx 200$ MeV, that is $R \sim 1/Q \approx 1$ fm (the size of a proton!)

- Asymptotic freedom:
  At short distances ($Q \gg \Lambda_{\rm QCD}$) quarks and gluons are almost free, they interact *weakly*: perturbative regime

- Infrared slavery:
  At long distances ($Q \sim \Lambda_{\rm QCD}$) the coupling diverges (Landau pole), quarks and gluons interact very *strongly* (**confinement into hadrons**): non-perturbative regime

⇒ Strong interactions are short-range, despite of gluon being massless

QCD running coupling

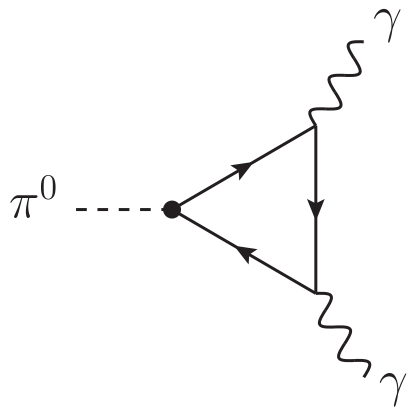


Anomalies?

About anomalies

- Anomaly: a symmetry of the classical Lagrangian broken by quantum corrections.
- Anomalies appear when *both* axial ($\psi\gamma^\mu\gamma_5\psi$) and vector ($\psi\gamma^\mu\psi$) currents involved.

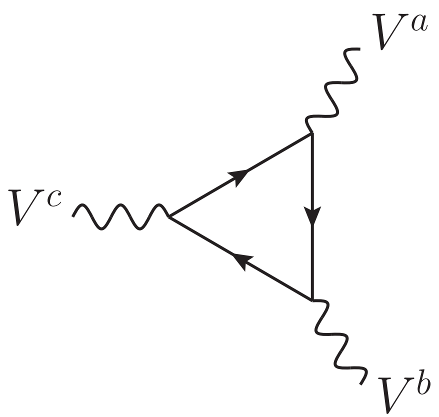
– Anomalies of global symmetries are welcome. For example:
  $\pi^0 \to \gamma\gamma$ thanks to coupling of an axial current $j_A^\mu = (\overline{u}\gamma^\mu\gamma_5 u - \overline{d}\gamma^\mu\gamma_5 d)$ to two electromagnetic (vector) currents, breaking the conservation of the axial current ($\partial^\mu j_A^\mu \neq 0$) at 1 loop, even in the limit of massless quarks.



– However, gauge anomalies are a disaster:
  they break Ward-Takahashi identities spoiling renormalizability.

Gauge anomalies

- The gauge anomalies are generated by **triangle diagrams** connecting three gauge bosons V^a, V^b, V^c, each coupled to fermions by $(\overline{\Psi}_L\gamma^\mu T_L^a\Psi_L + \overline{\Psi}_R\gamma^\mu T_R^a\Psi_R)V_\mu^a$ with T_L^a (T_R^a) the associated generators:



+ crossed

$$\mathcal{A}^{abc} = \mathrm{Tr}\left(\{T_L^a, T_L^b\}T_L^c\right) - \mathrm{Tr}\left(\{T_R^a, T_R^b\}T_R^c\right)$$

[traces include summation over *all* fermions]

Gauge symmetry is preserved at quantum level if *every* $\mathcal{A}^{abc} = 0$.

- In $SU(3)_C \times SU(2)_L \times U(1)_Y$ we have $T^a \in \{\frac{1}{2}\lambda^i, \frac{1}{2}\sigma^i, Y\}$ with

$$\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}\,\mathbb{1}$$

$$\mathrm{Tr}(\lambda^i) = \mathrm{Tr}(\sigma^i) = 0$$

Gauge anomalies cancel!

- Since $SU(3)_C$ is non-chiral (not anomalous), the only non trivial combinations are

$$SU(3)^2U(1): \quad \mathrm{Tr}(\{\lambda^i,\lambda^j\}Y\}) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{quarks}} (Y_L - Y_R) = 0 \quad \checkmark$$

$$SU(2)^2U(1): \quad \mathrm{Tr}(\{\sigma^i,\sigma^j\}Y) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{leptons}} Y_L + N_C \sum_{\text{quarks}} Y_L = 0 \quad \checkmark$$

$$U(1)^3: \quad \mathrm{Tr}(Y^3) \quad \Rightarrow \quad \mathcal{A}^{abc} \propto \sum_{\text{leptons}} (Y_L^3 - Y_R^3) + N_C \sum_{\text{quarks}} (Y_L^3 - Y_R^3) = 0 \quad \checkmark$$

where

	ν_e	e	u	d
Y_L	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
Y_R	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$

and anomalies cancel if $N_C = 3$.

- In particular the second constraint is equivalent to

$$Q_\nu + Q_e + N_C(Q_u + Q_d) = -1 + \frac{1}{3}N_C = 0 \quad \Rightarrow \quad N_C = 3 \quad (!!)$$

$\Rightarrow$ The electroweak SM needs leptons + quarks in every generation !!

$\Rightarrow$ The electroweak SM needs the QCD part !!

Electroweak Pheno

The full Lagrangian — fields and interactions

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_{\rm YM} + \mathcal{L}_\Phi + \mathcal{L}_{\rm Y} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$$

$$\mathcal{L}_F \supset \mathcal{L}_{\rm CC} + \mathcal{L}_{\rm NC}$$

$$\mathcal{L}_{\rm YM} \supset \mathcal{L}_{VVV} + \mathcal{L}_{VVVV}$$

$$\mathcal{L}_\Phi \supset \text{gauge boson masses}$$

$$\mathcal{L}_Y \supset \text{fermion masses and mixings}$$

- Fields: [F] fermions; [S] scalars (Higgs and unphysical Goldstones); [V] vector bosons; [U] unphysical ghosts

- Interactions: [FFV] [FFS] [SSV] [SVV] [SSVV]
  [VVV] [VVVV] [SSS] [SSSS]
  [SUU] [UUVV]

Full SM Lagrangian — types of interactions

- Lorentz structure of generic interactions (normalized to e):

$$\mathcal{L}_{\rm FFV} = e\,\overline{\psi}_i\gamma^\mu(g_V - g_A\gamma_5)\psi_j\,V_\mu = e\,\overline{\psi}_i\gamma^\mu(g_L P_L + g_R P_R)\psi_j\,V_\mu$$

$$\mathcal{L}_{\rm FFS} = e\,\overline{\psi}_i(g_S - g_P\gamma_5)\psi_j\,\phi = e\,\overline{\psi}_i(c_L P_L + c_R P_R)\psi_j\,\phi$$

$$\mathcal{L}_{\rm VVV} = -\mathrm{i}e\,c_{VVV}\left(W^{\mu\nu}W^\dagger_\mu V_\nu - W^\dagger_{\mu\nu}W^\mu V^\nu - W^\dagger_\mu W_\nu V^{\mu\nu}\right)$$

$$\mathcal{L}_{\rm VVVV} = e^2\,c_{VVVV}\left(2W^\dagger_\mu W^\mu V_\nu V'^\nu - W^\dagger_\mu V^\mu W_\nu V'^\nu - W^\dagger_\mu V'^\mu W_\nu V^\nu\right)$$

$$\mathcal{L}_{\rm SSV} = -\mathrm{i}e\,c_{SSV}\,\phi\overleftrightarrow{\partial_\mu}\phi'\,V^\mu$$

$$\mathcal{L}_{\rm SVV} = e\,c_{SVV}\,\phi\,V^\mu V'_\mu$$

$$\mathcal{L}_{\rm SSVV} = e^2\,c_{SSVV}\,\phi\phi' V^\mu V'_\mu$$

$$\mathcal{L}_{\rm SSS} = e\,c_{SSS}\,\phi\phi'\phi''$$

$$\mathcal{L}_{\rm SSSS} = e^2\,c_{SSSS}\,\phi\phi'\phi''\phi''',$$

where $\phi\overleftrightarrow{\partial_\mu}\phi' \equiv \phi_i\partial_\mu\phi' - (\partial_\mu\phi_i)\phi'$ and $V_\mu \in \{A_\mu, Z_\mu, W_\mu, W^\dagger_\mu\}$.

Full SM Lagrangian | Feynman Rules

- Feynman rules for generic vertices normalized to e (all momenta incoming):

$(\mathrm{i}\mathcal{L})$

$$
\begin{aligned}
[\mathrm{FFV}_\mu] &= \mathrm{i}e\gamma^\mu(g_L P_L + g_R P_R) \\
[\mathrm{FFS}] &= \mathrm{i}e(c_L P_L + c_R P_R) \\
[\mathrm{V}_\mu(k_1)\mathrm{V}_\nu(k_2)\mathrm{V}_\rho(k_3)] &= \mathrm{i}e\,c_{VVV}\left[g_{\mu\nu}(k_2-k_1)_\rho + g_{\nu\rho}(k_3-k_2)_\mu + g_{\mu\rho}(k_1-k_3)_\nu\right] \\
[\mathrm{V}_\mu\mathrm{V}_\nu\mathrm{V}_\rho\mathrm{V}_\sigma] &= \mathrm{i}e^2\,c_{VVVV}\left[2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right] \\
[\mathrm{S}(p)\mathrm{S}(p')\mathrm{V}_\mu] &= \mathrm{i}e\,c_{SSV}\,(p_\mu - p'_\mu) \\
[\mathrm{SV}_\mu\mathrm{V}_\nu] &= \mathrm{i}e\,c_{SVV}g_{\mu\nu} \\
[\mathrm{SSV}_\mu\mathrm{V}_\nu] &= \mathrm{i}e^2\,c_{SSVV}\,g_{\mu\nu} \\
[\mathrm{SSS}] &= \mathrm{i}e\,c_{SSS} \\
[\mathrm{SSSS}] &= \mathrm{i}e^2\,c_{SSSS}
\end{aligned}
$$

Note: $g_{L,R} = g_V \pm g_A$ $\qquad$ $\partial_\mu \to -\mathrm{i}p_\mu$ $\qquad$ Attention to symmetry factors!

$c_{L,R} = g_S \pm g_P$ $\qquad$ *e.g.* $2 \times HZZ$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

FFV	$\bar{f}_i f_j \gamma$	$\bar{f}_i f_j Z$	$\bar{u}_i d_j W^+$	$\bar{d}_j u_i W^-$	$\bar{\nu}_i \ell_j W^+$	$\bar{\ell}_j \nu_i W^-$
g_L	$-Q_f \delta_{ij}$	$g_+^f \delta_{ij}$	$\frac{1}{\sqrt{2} s_W} \mathbf{V}_{ij}$	$\frac{1}{\sqrt{2} s_W} \mathbf{V}_{ij}^*$	$\frac{1}{\sqrt{2} s_W} \delta_{ij}$	$\frac{1}{\sqrt{2} s_W} \delta_{ij}$
g_R	$-Q_f \delta_{ij}$	$g_-^f \delta_{ij}$	0	0	0	0

$$g_\pm^f \equiv v_f \pm a_f \qquad v_f = \frac{T_3^{f_L} - 2Q_f s_W^2}{2 s_W c_W} \qquad a_f = \frac{T_3^{f_L}}{2 s_W c_W}$$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

FFS	$\bar{f}_i f_j H$	$\bar{f}_i f_j \chi$	$\bar{u}_i d_j \phi^+$	$\bar{d}_j u_i \phi^-$
c_L	$-\frac{1}{2s_W}\frac{m_{f_i}}{M_W}\delta_{ij}$	$-\frac{\mathrm{i}}{2s_W}2T_3^{f_L}\frac{m_{f_i}}{M_W}\delta_{ij}$	$+\frac{1}{\sqrt{2}s_W}\frac{m_{u_i}}{M_W}\mathbf{V}_{ij}$	$-\frac{1}{\sqrt{2}s_W}\frac{m_{d_j}}{M_W}\mathbf{V}_{ij}^*$
c_R	$-\frac{1}{2s_W}\frac{m_{f_i}}{M_W}\delta_{ij}$	$+\frac{\mathrm{i}}{2s_W}2T_3^{f_L}\frac{m_{f_i}}{M_W}\delta_{ij}$	$-\frac{1}{\sqrt{2}s_W}\frac{m_{d_j}}{M_W}\mathbf{V}_{ij}$	$+\frac{1}{\sqrt{2}s_W}\frac{m_{u_j}}{M_W}\mathbf{V}_{ij}^*$

$(f = u, d, \ell)$

FFS	$\bar{\nu}_i \ell_j \phi^+$	$\bar{\ell}_j \nu_i \phi^-$
c_L	$+\frac{1}{\sqrt{2}s_W}\frac{m_{\nu_i}}{M_W}\delta_{ij}$	$-\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_j}}{M_W}\delta_{ij}$
c_R	$-\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_j}}{M_W}\delta_{ij}$	$+\frac{1}{\sqrt{2}s_W}\frac{m_{\nu_i}}{M_W}\delta_{ij}$

Full SM Lagrangian **Feynman rules** ('t Hooft-Feynman gauge)

SVV	HZZ	HW^+W^-	$\phi^\pm W^\mp\gamma$	$\phi^\pm W^\mp Z$
c_{SVV}	$M_W/(s_W c_W^2)$	M_W/s_W	$-M_W$	$-M_W s_W/c_W$

SSV	χHZ	$\phi^\pm\phi^\mp\gamma$	$\phi^\pm\phi^\mp Z$	$\phi^\mp H W^\pm$	$\phi^\mp \chi W^\pm$
c_{SSV}	$-\dfrac{\mathrm{i}}{2s_W c_W}$	∓ 1	$\pm\dfrac{c_W^2 - s_W^2}{2s_W c_W}$	$\mp\dfrac{1}{2s_W}$	$-\dfrac{\mathrm{i}}{2s_W}$

VVV	$W^+W^-\gamma$	W^+W^-Z
c_{VVV}	-1	c_W/s_W

Full SM Lagrangian | Feynman rules ('t Hooft-Feynman gauge)

VVVV	$W^+W^+W^-W^-$	W^+W^-ZZ	$W^+W^-\gamma Z$	$W^+W^-\gamma\gamma$
c_{VVVV}	$\frac{1}{s_W^2}$	$-\frac{c_W^2}{s_W^2}$	$\frac{c_W}{s_W}$	-1

SSVV	HHW^-W^+	$HHZZ$
c_{SSVV}	$\frac{1}{2s_W^2}$	$\frac{1}{2s_W^2c_W^2}$

SSS	HHH
c_{SSS}	$-\frac{3M_H^2}{2M_Ws_W}$

SSSS	$HHHH$
c_{SSSS}	$-\frac{3M_H^2}{4M_W^2s_W^2}$

- Would-be Goldstone bosons in [SSVV], [SSS] and [SSSS] omitted
- Faddeev-Popov ghosts in [SUU] and [UUVV] omitted
- All Feynman rules from FeynArts (same conventions; $\chi,\phi^\pm \to G^0,G^\pm$):

http://www.ugr.es/local/jillana/SM/FeynmanRulesSM.pdf

Input parameters

- Parameters:

$17\ {+9}$	$=$	1	1	1	1	$9\ {+3}$	4	6
	formal:	g	g'	v	λ	λ_f	$\mathbf{V}_{\text{CKM}}$	$\mathbf{U}_{\text{PMNS}}$
	practical:	α	M_W	M_Z	M_H	m_f		

where $g = \dfrac{e}{s_W}$ $g' = \dfrac{e}{c_W}$ and

$$\underbrace{\alpha = \frac{e^2}{4\pi} \qquad M_W = \frac{1}{2}gv \qquad M_Z = \frac{M_W}{c_W}}_{g,\ g',\ v} \qquad M_H = \sqrt{2\lambda}\,v \qquad m_f = \frac{v}{\sqrt{2}}\lambda_f$$

$\Rightarrow$ Many (more) experiments

$\Rightarrow$ After Higgs discovery, for the first time *all* parameters measured!

Input parameters

- Experimental values [Particle Data Group '20]

  - Fine structure constant:

    $\alpha^{-1} = 137.035\,999\,150\,(33)$ Harvard cyclotron (g_e) [1712.06060]

    $\alpha^{-1} = 137.035\,999\,046\,(27)$ atom interferometry (Cesium) [1812.04130]

    $\alpha^{-1} = 137.035\,999\,206\,(11)$ atom interferometry (Rubidium) [Nature 588, 61(2020)]

  - The SM predicts $M_W < M_Z$ in agreement with measurements:

    $M_Z = (91.1876 \pm 0.0021)$ GeV LEP1/SLD

    $M_W = (80.379 \pm 0.012)$ GeV LEP2/Tevatron/LHC

  - Top quark mass:

    $m_t = (172.76 \pm 0.30)$ GeV Tevatron/LHC

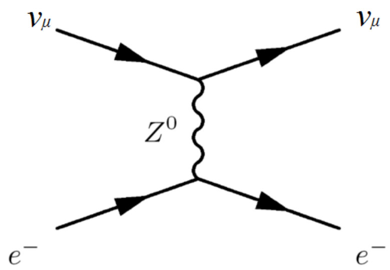
  - Higgs boson mass:

    $M_H = (125.25 \pm 0.17)$ GeV LHC

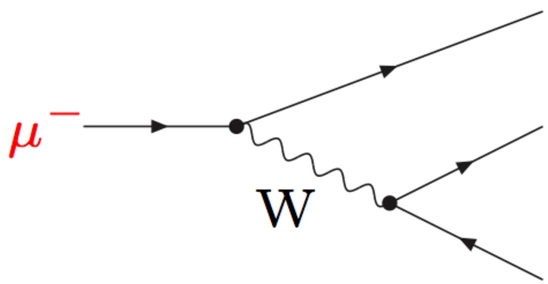
  - ...

Observables and experiments

Weak NC discovery (1973)



- Low energy observables ($Q^2 \ll M_Z^2$)
  - ν-nucleon (NuTeV) and νe (CERN) scattering
    asymmetries CC/NC and $\nu/\bar{\nu}$ $\Rightarrow$ $\boxed{s_W^2}$
  - Parity and Atomic Parity violation (SLAC, CERN, Jefferson Lab, Mainz)
    LR asymmetries $\mathrm{e}_{R,L}N \to \mathrm{e}X$ and Z effects on atomic transitions $\Rightarrow$ $\boxed{s_W^2}$
  - muon decay: $\mu \to e\,\overline{\nu}_e \nu_\mu$ (PSI)
    lifetime



$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2)$$

$$f(x) \equiv 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \qquad \Rightarrow \boxed{G_F}$$

$$= 0.99981295$$

$$\mathrm{i}\mathcal{M} = \left(\frac{\mathrm{i}e}{\sqrt{2}s_W}\right)^2 \bar{e}\gamma^\rho \nu_L \frac{-\mathrm{i}g_{\rho\delta}}{q^2 - M_W^2} \overline{\nu_L}\gamma^\delta \mu \equiv -\mathrm{i} \overbrace{\frac{4G_F}{\sqrt{2}} (\bar{e}\gamma^\rho \nu_L)(\overline{\nu_L}\gamma_\rho \mu)}^{\text{Fermi theory } (-q^2 \ll M_W^2)} \; ; \quad \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2 s_W^2 M_W^2}$$

Observables and experiments

- Low energy observables

$\Rightarrow$ Fermi constant provides the Higgs VEV (electroweak scale):

$$v = \left(\sqrt{2}G_F\right)^{-1/2} \approx 246\,\text{GeV}$$

and constrains the product $M_W^2 s_W^2$, which implies

$$M_Z^2 > M_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F s_W^2} > \frac{\pi\alpha}{\sqrt{2}G_F} \approx (37.4\ \text{GeV})^2$$

$\Rightarrow$ Consistency checks: e.g. from muon lifetime:

$$G_F = 1.166\,378\,7(6) \times 10^{-5}\ \text{GeV}^{-2}$$

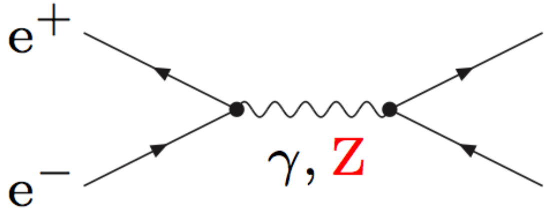
If one compares with (tree level result)

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 M_W^2} = \frac{\pi\alpha}{2(1 - M_W^2/M_Z^2)M_W^2} \approx 1.125 \times 10^{-5}$$

a discrepancy that disappears when *quantum corrections* are included

Observables and experiments

- $e^+e^- \to \bar{f}f$ (PEP, PETRA, TRISTAN, ..., LEP1, SLD)



$$\frac{d\sigma}{d\Omega} = N_c^f \frac{\alpha^2}{4s}\beta_f \Big\{ \left[1+\cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2-1)\, G_2(s) + 2\beta_f \cos\theta\, G_3(s) \Big\}$$

$$G_1(s) = Q_e^2Q_f^2 + 2Q_eQ_f v_e v_f \mathrm{Re}\chi_Z(s) + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$

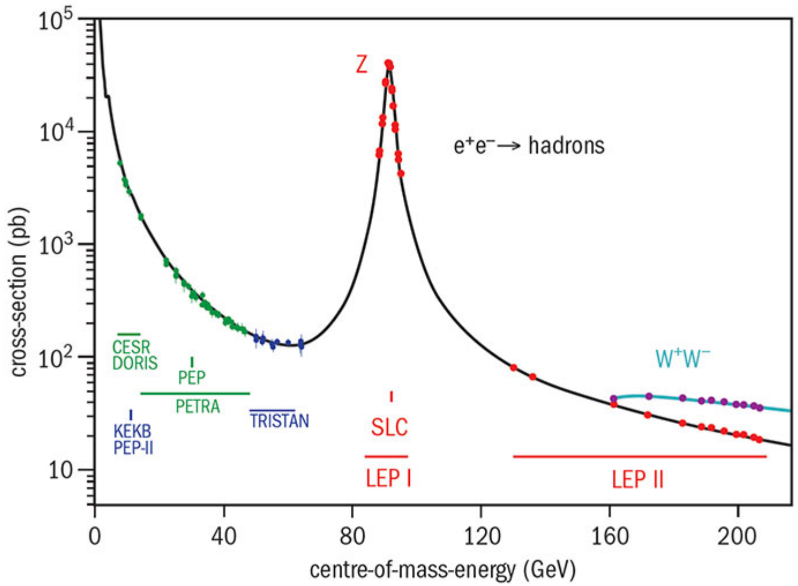
$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$

$$G_3(s) = 2Q_eQ_f a_e a_f \mathrm{Re}\chi_Z(s) + 4v_e v_f a_e a_f |\chi_Z(s)|^2 \quad \Rightarrow \quad \boxed{A_{FB}(s)}$$

$$\chi_Z(s) \equiv \frac{s}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}, \; N_c^f = 1 \text{ (3) for } f = \text{lepton (quark)}$$

$$\sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]$$

$$\beta_f = \sqrt{1-4m_f^2/s}$$

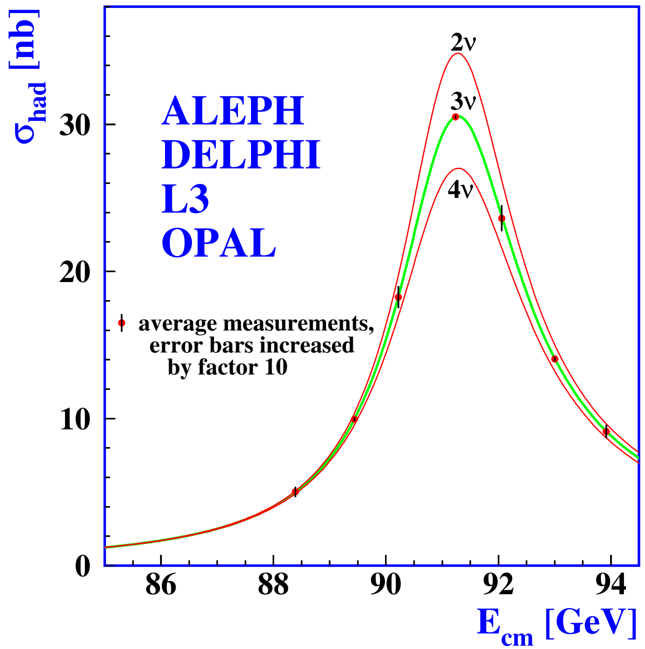


Observables and experiments

- Z pole observables (LEP1/SLD)

  $M_Z,\ \Gamma_Z,\ \sigma_{\rm had},\ A_{FB},\ A_{LR},\ R_b,\ R_c,\ R_\ell \quad \Rightarrow \boxed{M_Z,\ s_W^2}$

  from $e^+e^- \to \bar{f}f$ at the Z pole ($\gamma - Z$ interference vanishes). Neglecting m_f:



$$\sigma^0_{\rm had} = 12\pi \frac{\Gamma(e^+e^-)\Gamma(\rm had)}{M_Z^2\Gamma_Z^2}$$

9

$$R_b = \frac{\Gamma(b\bar{b})}{\Gamma(\rm had)} \quad R_c = \frac{\Gamma(c\bar{c})}{\Gamma(\rm had)} \quad R_\ell = \frac{\Gamma(\rm had)}{\Gamma(\ell^+\ell^-)}$$

$$\left[\Gamma(Z \to f\bar{f}) \equiv \Gamma(f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)\right]$$

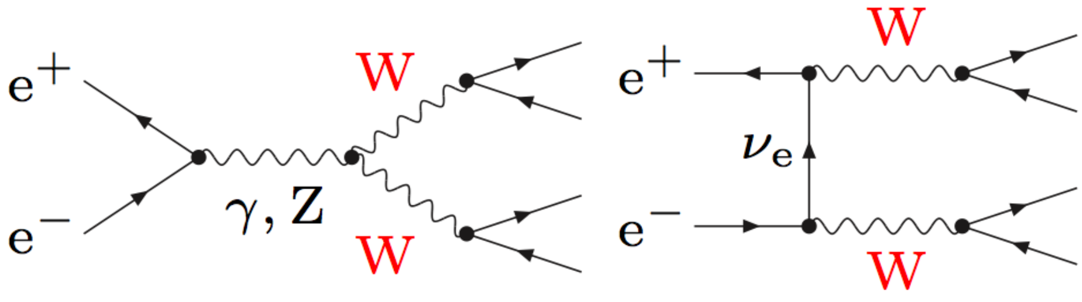
$$\Gamma_Z \simeq 2.5 \text{ GeV} \Rightarrow N_\nu \simeq 3 \checkmark$$

Forward-Backward and (if polarized e^-) Left-Right asymmetries due to Z:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4} A_f \frac{A_e + P_e}{1 + P_e A_e} \qquad A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = A_e P_e \quad \text{with } A_f \equiv \frac{2 v_f a_f}{v_f^2 + a_f^2}$$

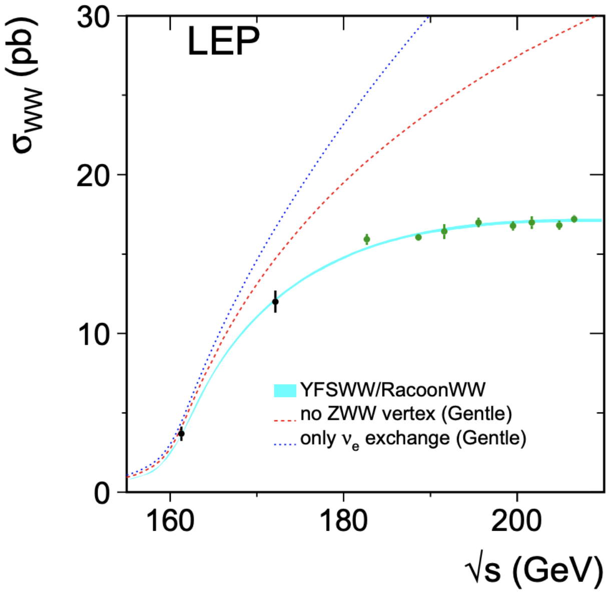
Observables and experiments

- W-pair production (LEP2)
  $e^+e^- \to WW \to 4\,f\ (+\gamma)$



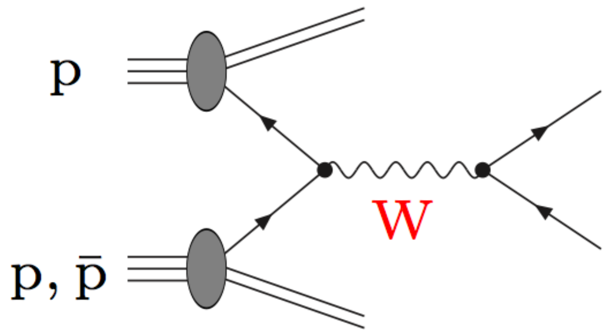
$\Rightarrow \boxed{M_W}$

$\Downarrow$



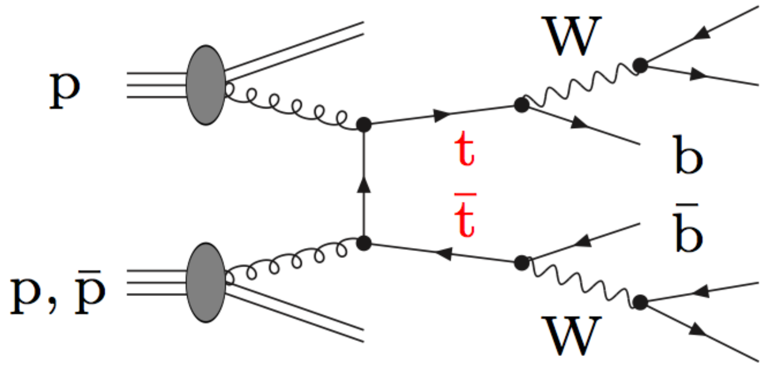
[1302.3415]

- W production (Tevatron/LHC)
  $pp/p\bar{p} \to W \to \ell\nu_\ell\ (+\gamma)$



$\Rightarrow \boxed{M_W}$

- Top-quark production (Tevatron/LHC)
  $pp/p\bar{p} \to t\bar{t} \to 6\,f$



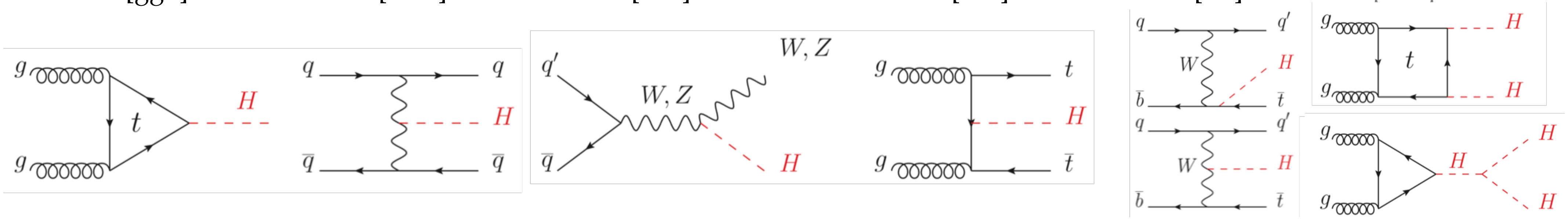
$\Rightarrow \boxed{m_t}$

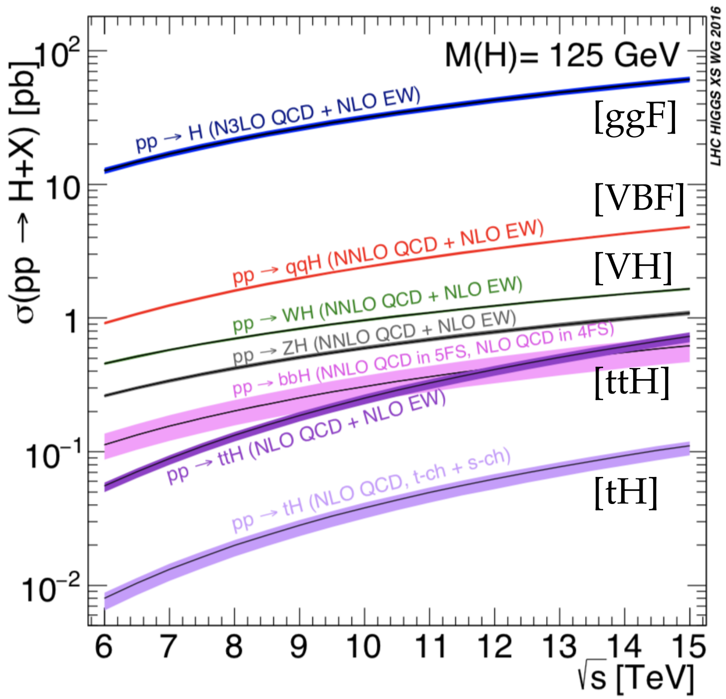
Observables and experiments

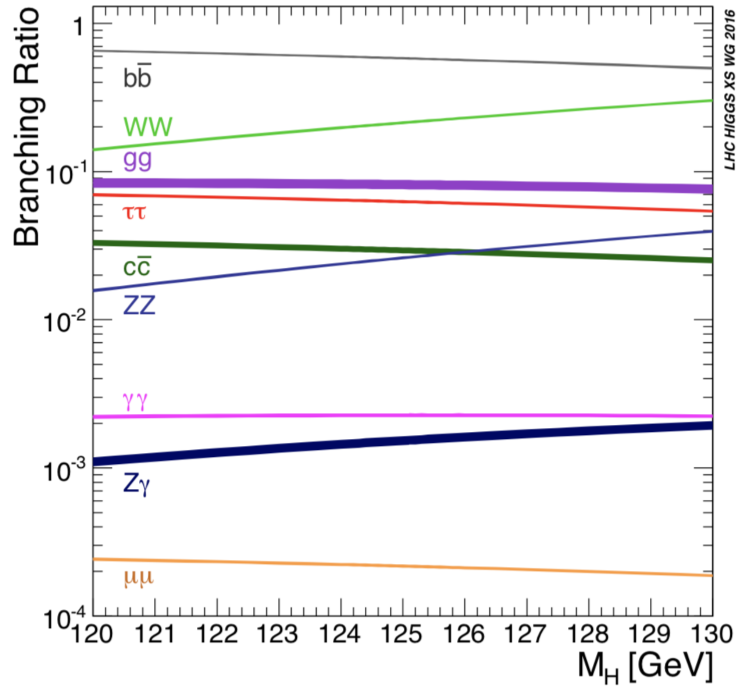
- Higgs (LHC) [PDG '20]

  Single and Double H production and decay to different channels ⇒ $\boxed{M_H}$

[ggF] [VBF] [VH] [ttH] [tH] [HH]







Observables and experiments

- Higgs (LHC) [PDG '20]

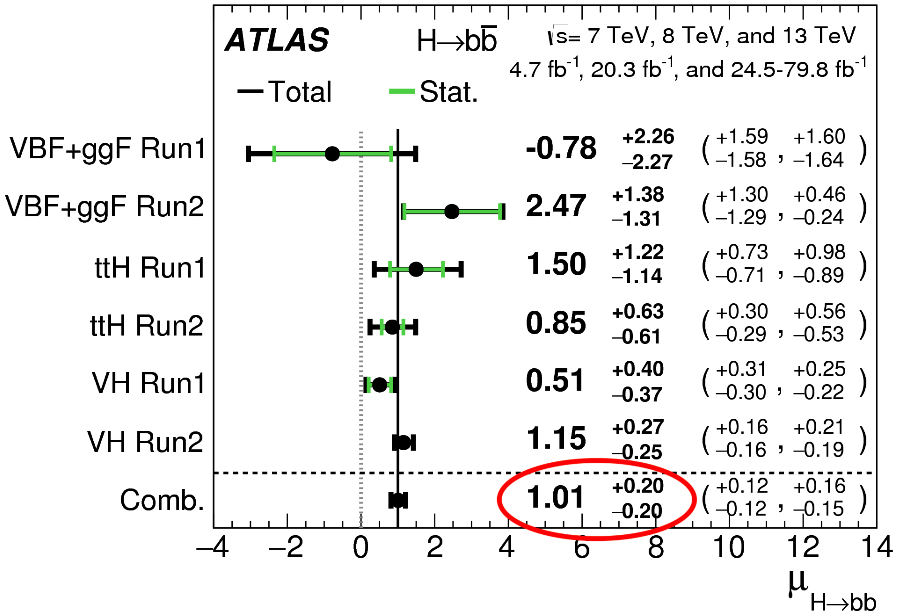
Signal strength $\mu = \frac{(\sigma \cdot \mathrm{BR})_{\mathrm{obs}}}{(\sigma \cdot \mathrm{BR})_{\mathrm{SM}}}$	Run 1	Run 2
ATLAS	1.17 ± 0.27	1.02 ± 0.14
CMS	$1.18^{+0.26}_{-0.23}$	$1.18^{+0.17}_{-0.14}$

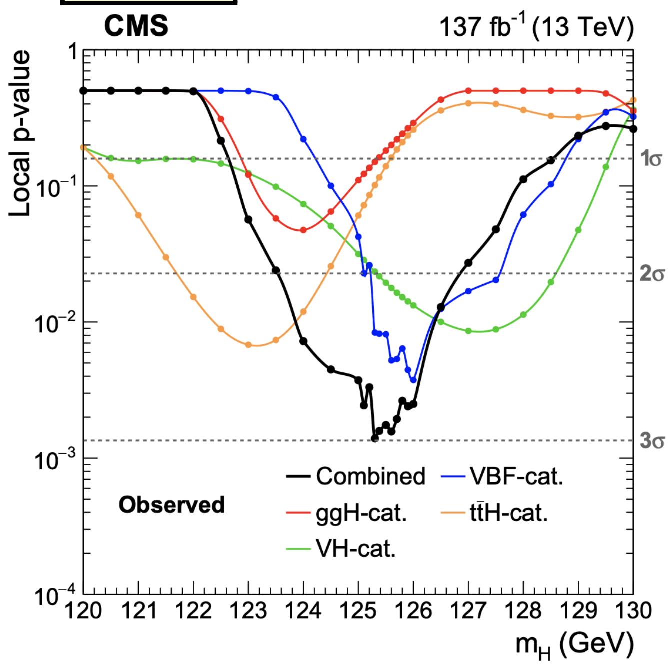
Per channel:

$\gamma\gamma, ZZ, W^+W^-, \tau^+\tau^- > 5\sigma$

$b\bar{b} > 5\sigma$ [Jul '18!]

$\mu^+\mu^- \sim 3\sigma$ [Jul '20!!]

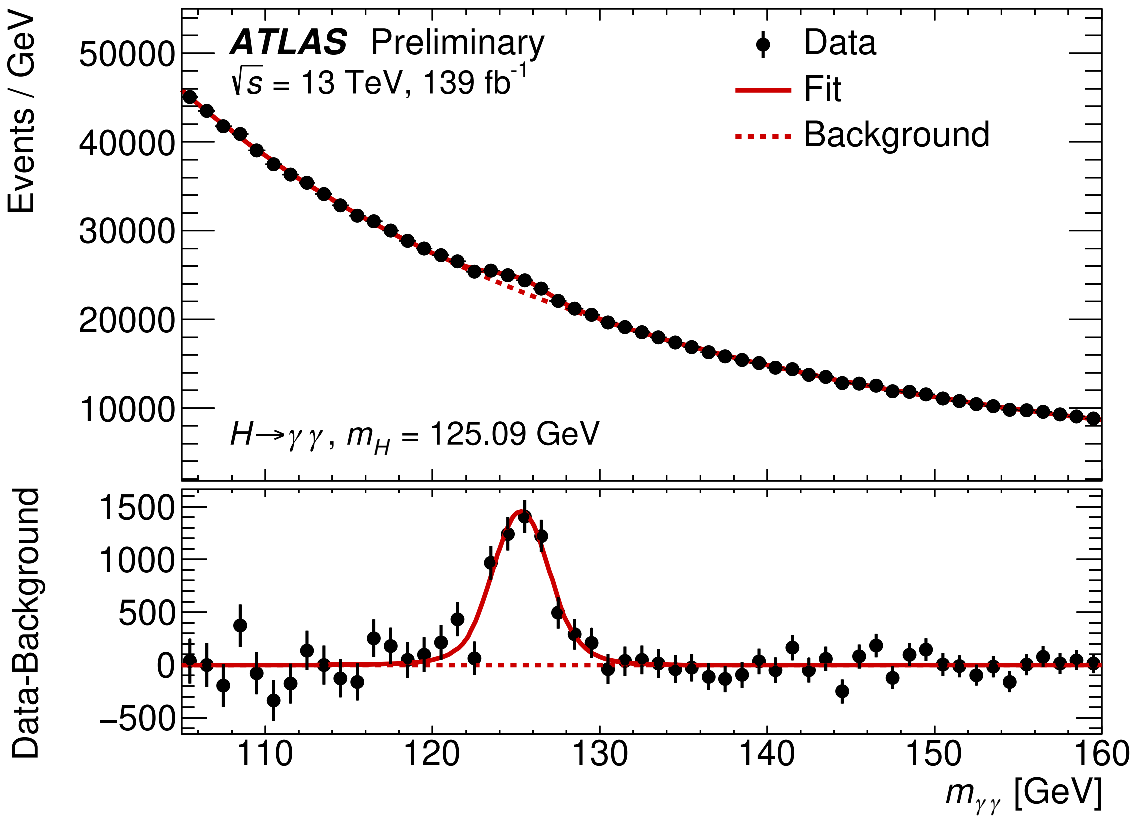


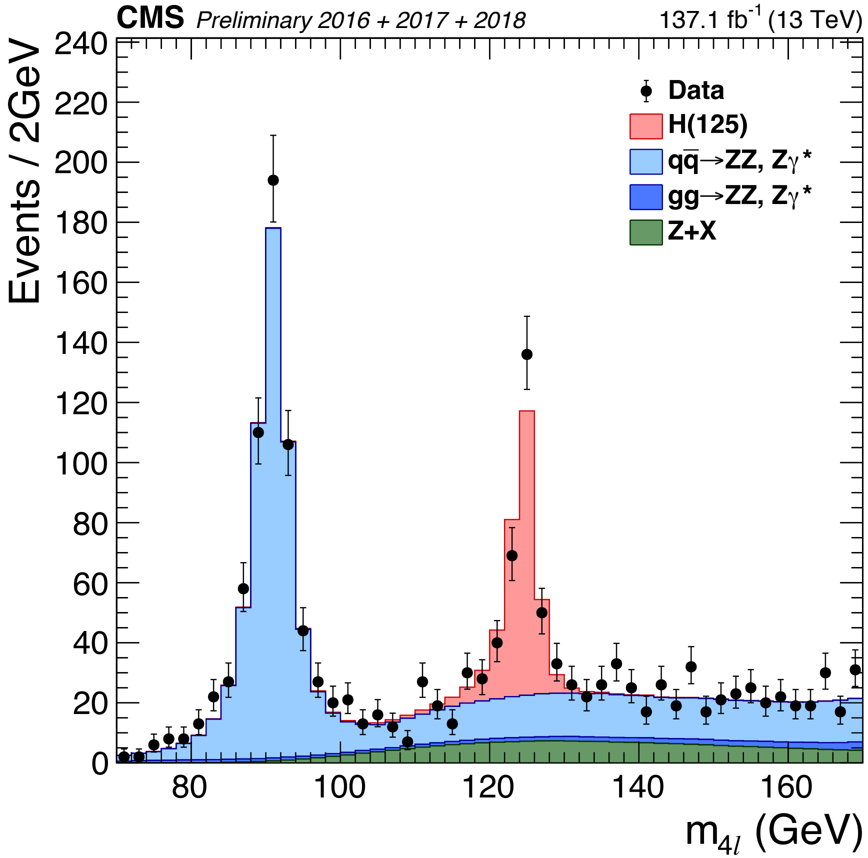


Observables and experiments

- Higgs mass (LHC) [PDG '20]

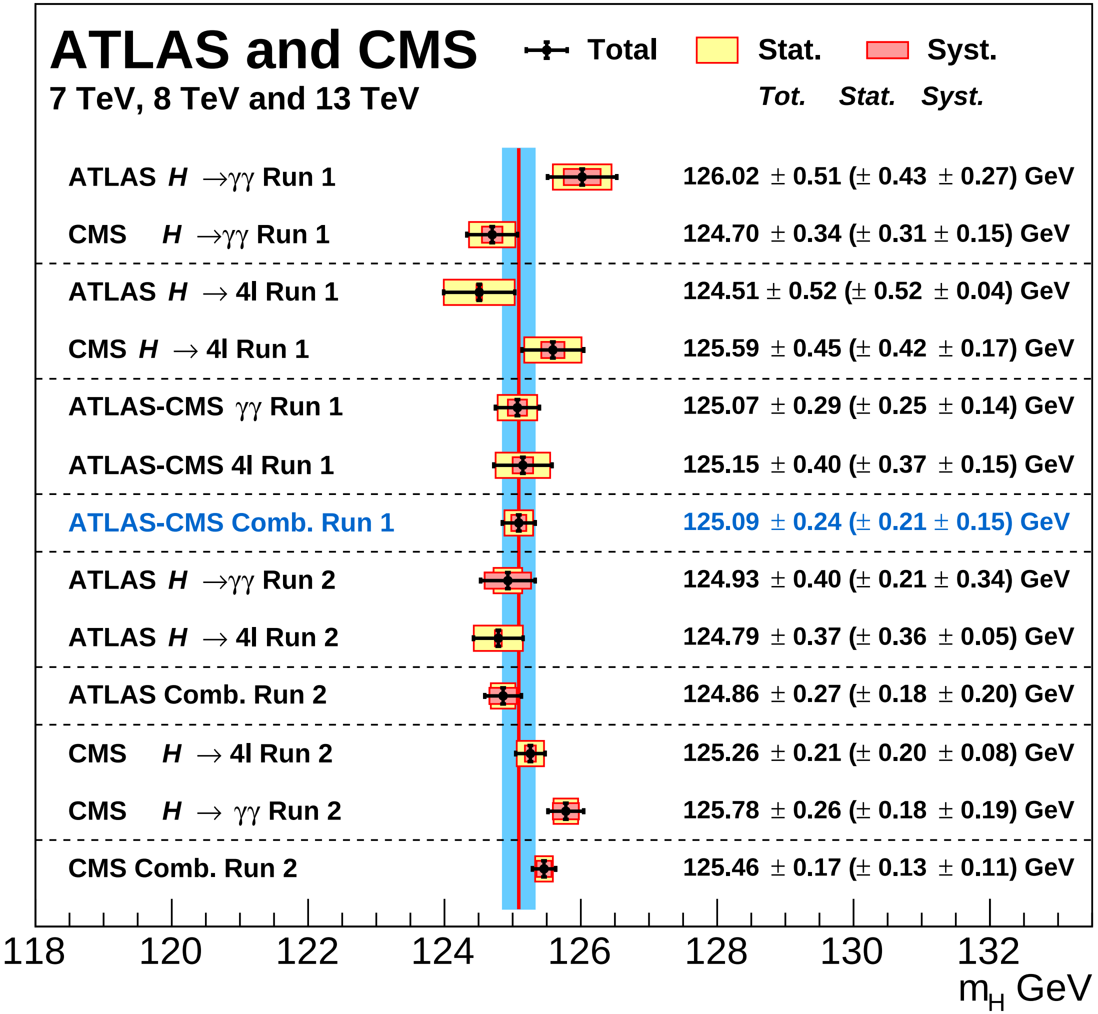
$H \to \gamma\gamma$ $H \to ZZ^* \to 4\ell$





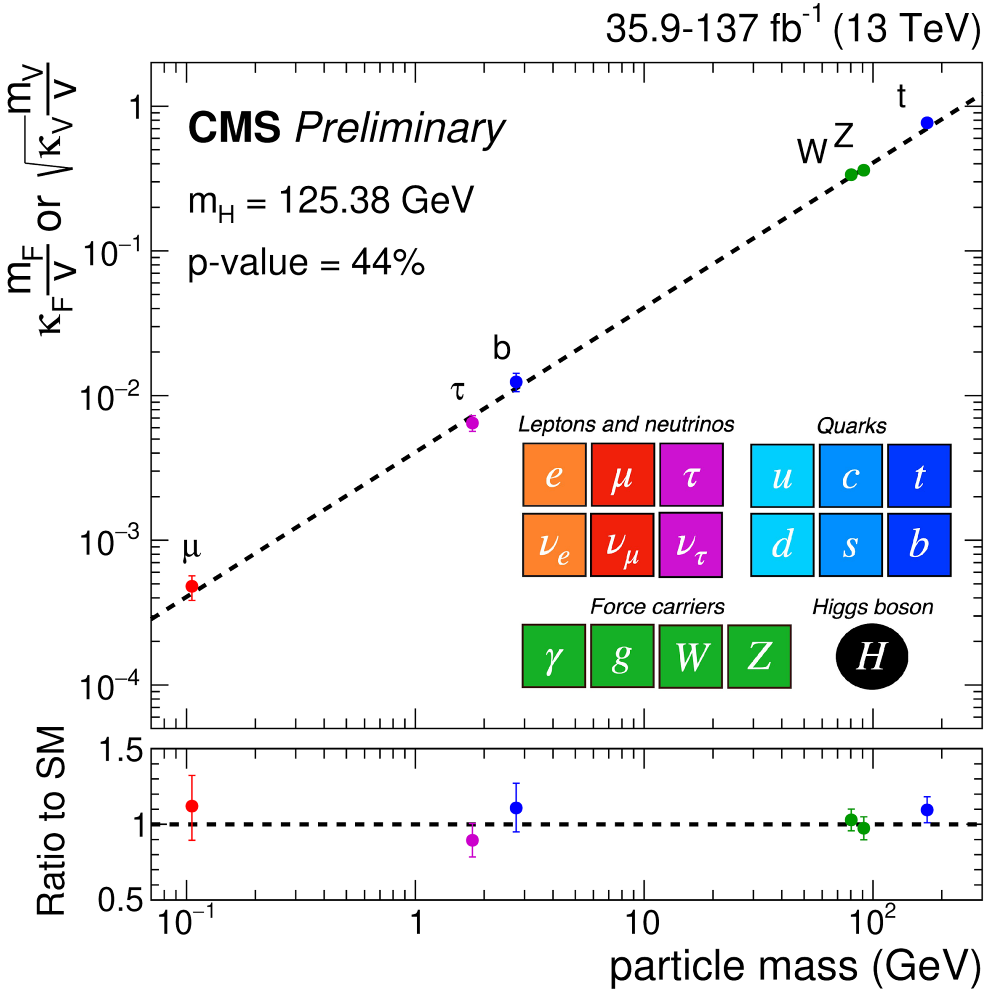
Observables and experiments

- Higgs mass (LHC) [PDG '20]



Observables and experiments

- Higgs couplings (LHC) [2009.04363]



H self-couplings not yet observed

$\Leftarrow \kappa_F$ or $\kappa_V = \dfrac{\text{obs}}{\text{SM}}$

proben over more that 3 orders of mangitude!

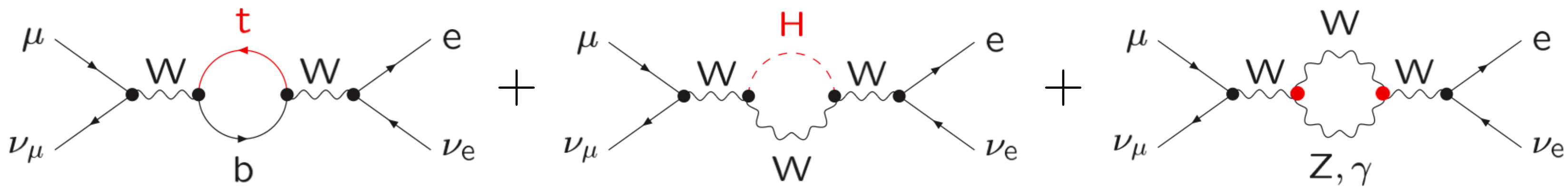
Precise determination of parameters

- Experimental precision requires accurate predictions ⇒ quantum corrections (complication: loop calculations involve renormalization)

– Correction to G_F from muon lifetime:

$$\frac{G_F}{\sqrt{2}} \to \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2(1-M_W^2/M_Z^2)M_W^2}[1+\Delta r(m_t, M_H)]$$

when loop corrections are included:



Since muon lifetime is measured more precisely than M_W, it is traded for G_F:

$$\Rightarrow M_W^2(\alpha, G_F, M_Z, m_t, M_H) = \frac{M_Z^2}{2}\left(1+\sqrt{1-\frac{4\pi\alpha}{\sqrt{2}G_F M_Z^2}[1+\Delta r(m_t, M_H)]}\right)$$

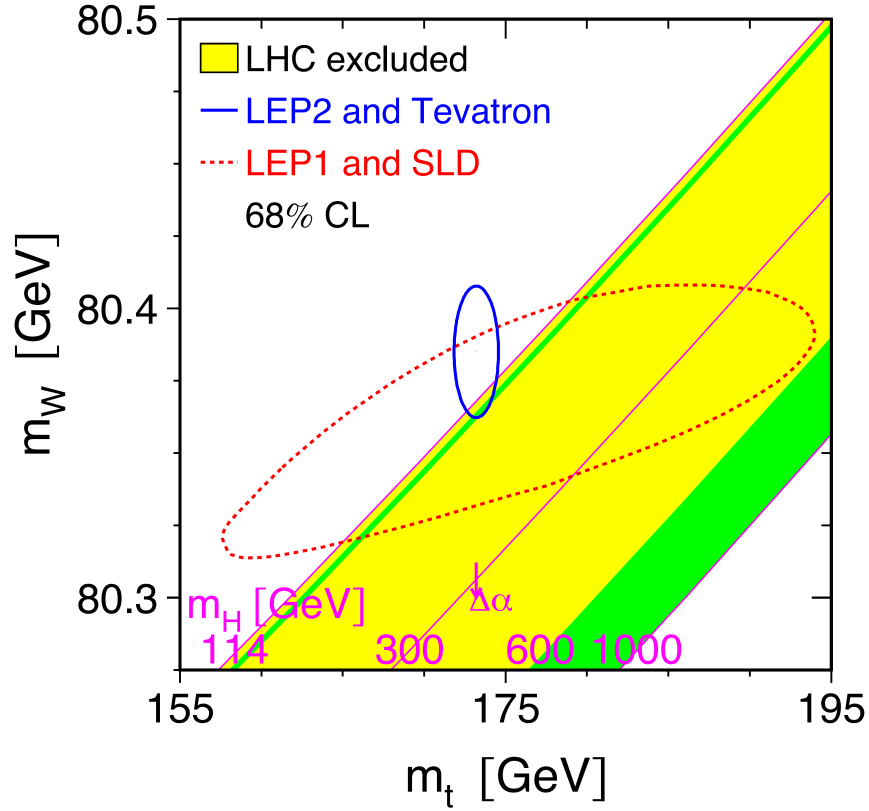
(correlation between M_W, m_t and M_H, given α, G_F and M_Z)

Precise determination of parameters

Indirect constraints from LEP1/SLD — Direct measurements from LEP2/Tevatron

$M_H(M_W, m_t)$ — Allowed regions for M_H allowed by direct searches — LHC excluded

[LEPEWWG 2013]



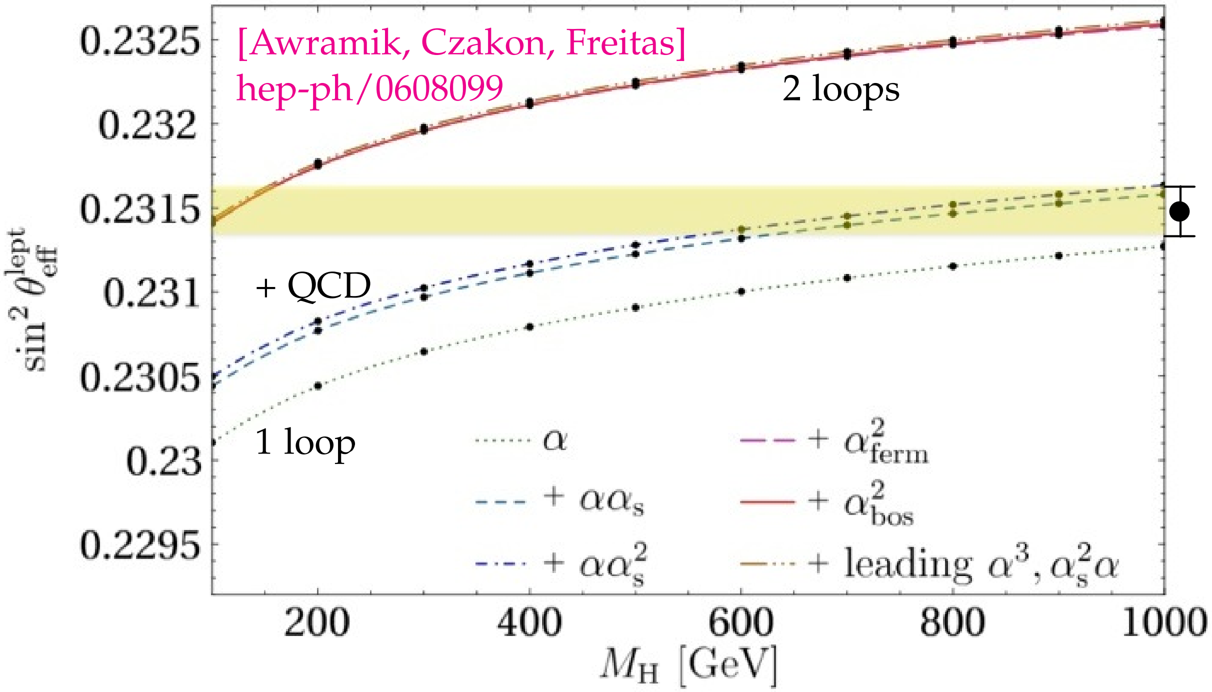
Precise determination of parameters

– Corrections to vector and axial couplings from Z pole observables:

$$v_f \to g_V^f = v_f + \Delta g_V^f \qquad a_f \to g_A^f = a_f + \Delta g_A^f$$

$$\Rightarrow \sin^2\theta_{\rm eff}^f \equiv \frac{1}{4|Q_f|}\left|1 - \mathrm{Re}(g_V^f/g_A^f)\right| \equiv \overbrace{(1 - M_W^2/M_Z^2)}^{s_W^2}\ \kappa_Z^f$$

(Two) loop calculations are crucial and point to a light Higgs:



$s_W^2 = 0.22290 \pm 0.00029$ (tree)

$\sin^2\theta_{\rm eff}^{\rm lept} = 0.23148 \pm 0.000017$ (exp)

Precise determination of parameters

- In addition, experiments and observables testing the flavor structure of the SM: flavor conserving: dipole moments, … flavor changing: $b \to s\gamma$, …

  $\Rightarrow$ very sensitive to new physics through loop corrections

  Extremely precise measurements are:

  – electron magnetic moment:

$$\left.\begin{array}{rl} \text{exp:} & g_e/2 = 1.001\,159\,652\,182\,032\,(720) \\ \text{theo:} & \text{QED (5 loops!)} \end{array}\right\} \Rightarrow \alpha^{-1} = 137.035\,999\,150\,(33)$$

  – muon anomalous magnetic moment: $a_\mu = (g_\mu - 2)/2$

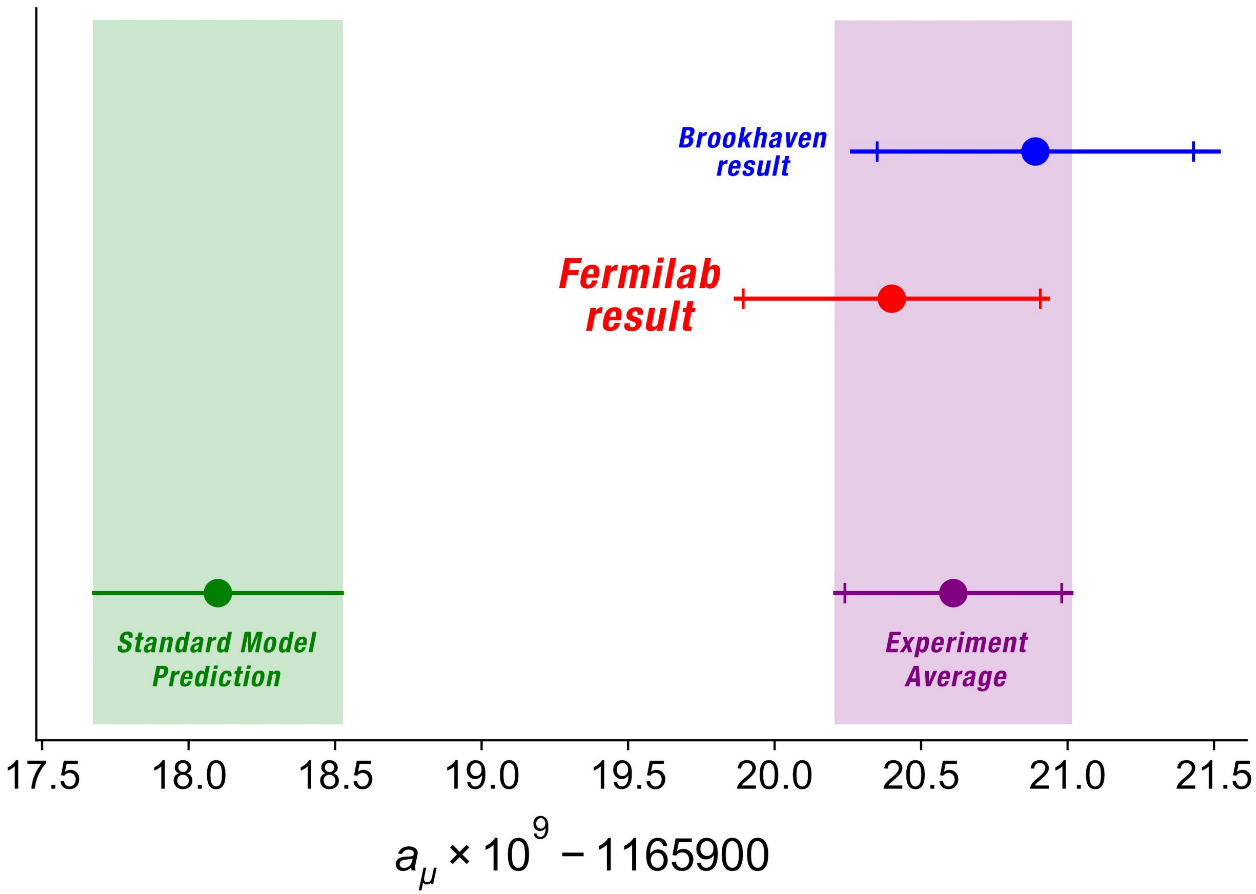
$a_\mu^{\text{exp}} = 116\,592\,089\,(63) \times 10^{-11}$	[Brookhaven '06]	
$a_\mu^{\text{QED}} = 116\,584\,719 \times 10^{-11}$	[QED: 5 loops]	
$a_\mu^{\text{EW}} = 154\,(1) \times 10^{-11}$	[W, Z, H: 2 loops]	
$a_\mu^{\text{had}} = 6\,937\,(43) \times 10^{-11}$	[$e^+e^- \to$ had]	
$\boldsymbol{a_\mu^{\text{SM}} = 116\,591\,810\,(43) \times 10^{-11}}$	**[Theory Initiative '20]**	

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 279\,(76) \times 10^{-11}$$

3.7σ !

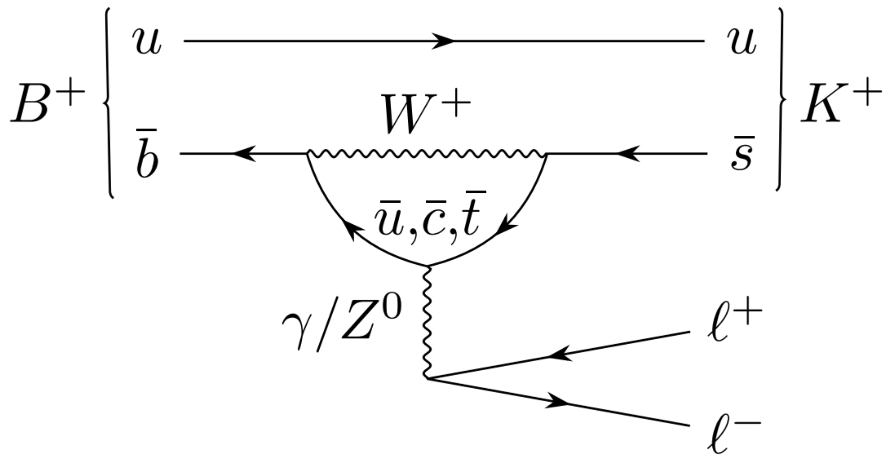
- New Muon $g-2$ Experiment at Fermilab. First results! [2104.03281]

$$a_\mu^{\rm exp} = 116\,592\,061\,(41) \times 10^{-11}$$

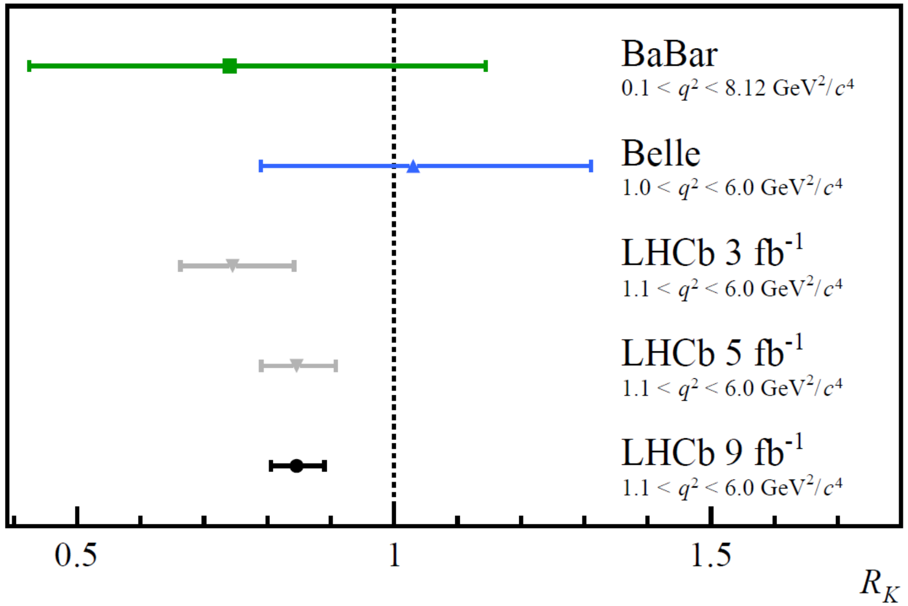


$\Rightarrow 4.2\sigma$!!

- Test of lepton universality in b decays at LHCb [2103.11769]



$$R_K = \frac{\mathcal{B}(B^+ \to K^+\mu^+\mu^-)}{\mathcal{B}(B^+ \to K^+e^+e^-)} = 0.846^{+0.044}_{-0.041} \quad \Rightarrow$$



Global fits

- Fit parameters from a list of observables: find the $\chi^2_{\rm min}$ varying some of them [$n_{\rm dof}$ = # of observables minus # of parameters]

http://gfitter.desy.de [1803.01853]

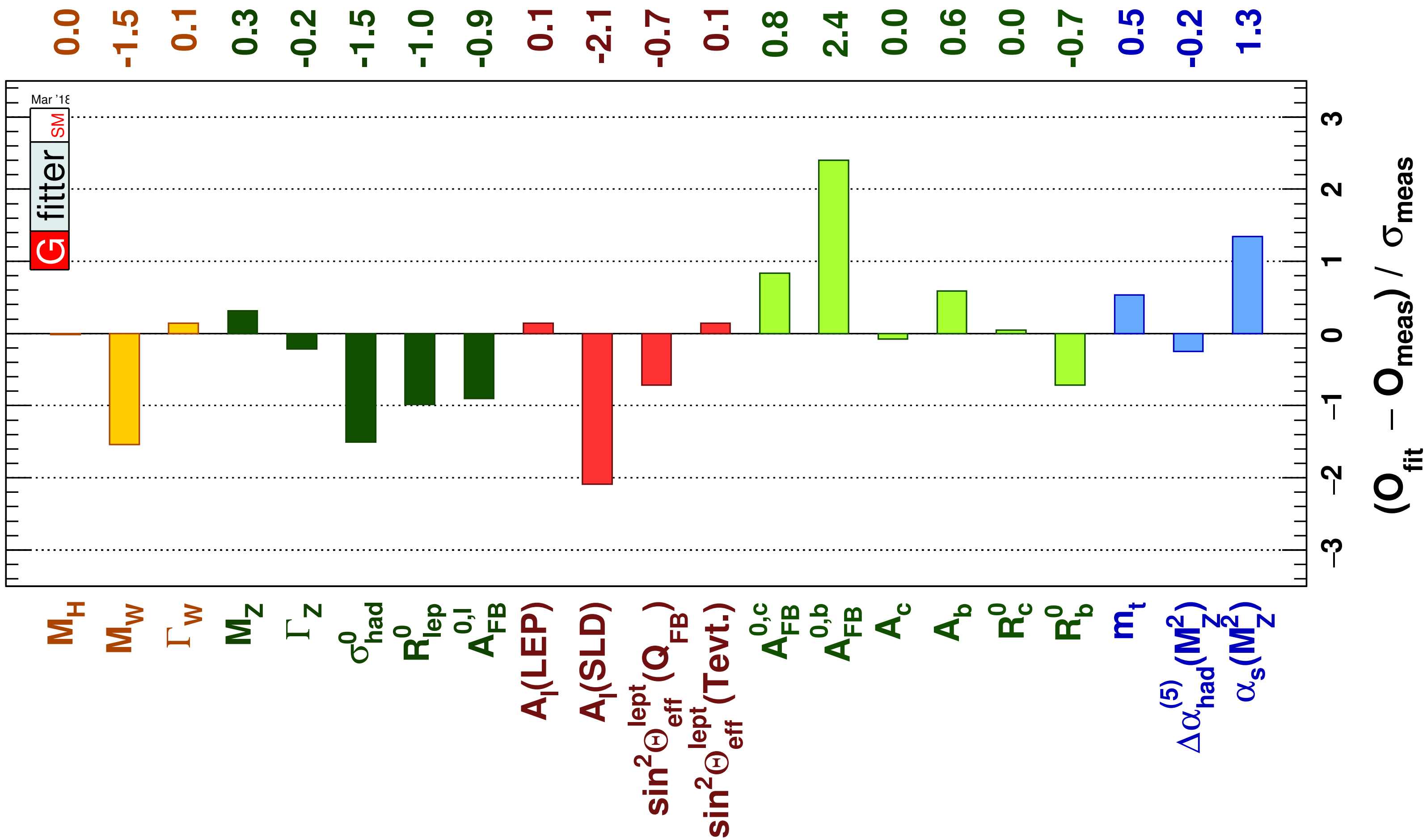
$n_{\rm dof}$	$\chi^2_{\rm min}$	p-value
15	18.6	0.23

(goodness of fit)

Parameter	Input value	Free in fit
M_H [GeV]	125.1 ± 0.2	Yes
M_W [GeV]	80.379 ± 0.013	–
Γ_W [GeV]	2.085 ± 0.042	–
M_Z [GeV]	91.1875 ± 0.0021	Yes
Γ_Z [GeV]	2.4952 ± 0.0023	–
$\sigma^0_{\rm had}$ [nb]	41.540 ± 0.037	–
R^0_ℓ	20.767 ± 0.025	–
$A^{0,\ell}_{\rm FB}$	0.0171 ± 0.0010	–
A_ℓ $^{(\star)}$	0.1499 ± 0.0018	–
$\sin^2\theta^\ell_{\rm eff}(Q_{\rm FB})$	0.2324 ± 0.0012	–
$\sin^2\theta^\ell_{\rm eff}$(Tevt.)	0.23148 ± 0.00033	–
A_c	0.670 ± 0.027	–
A_b	0.923 ± 0.020	–
$A^{0,c}_{\rm FB}$	0.0707 ± 0.0035	–
$A^{0,b}_{\rm FB}$	0.0992 ± 0.0016	–
R^0_c	0.1721 ± 0.0030	–
R^0_b	0.21629 ± 0.00066	–
$\overline{m}_c$ [GeV]	$1.27^{+0.07}_{-0.11}$	Yes
$\overline{m}_b$ [GeV]	$4.20^{+0.17}_{-0.07}$	Yes
m_t [GeV]$^{(\triangledown)}$	172.47 ± 0.68	Yes
$\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ $^{(\dagger\triangle)}$	2760 ± 9	Yes
$\alpha_s(M_Z^2)$	–	Yes

Global fits (Comparisons)

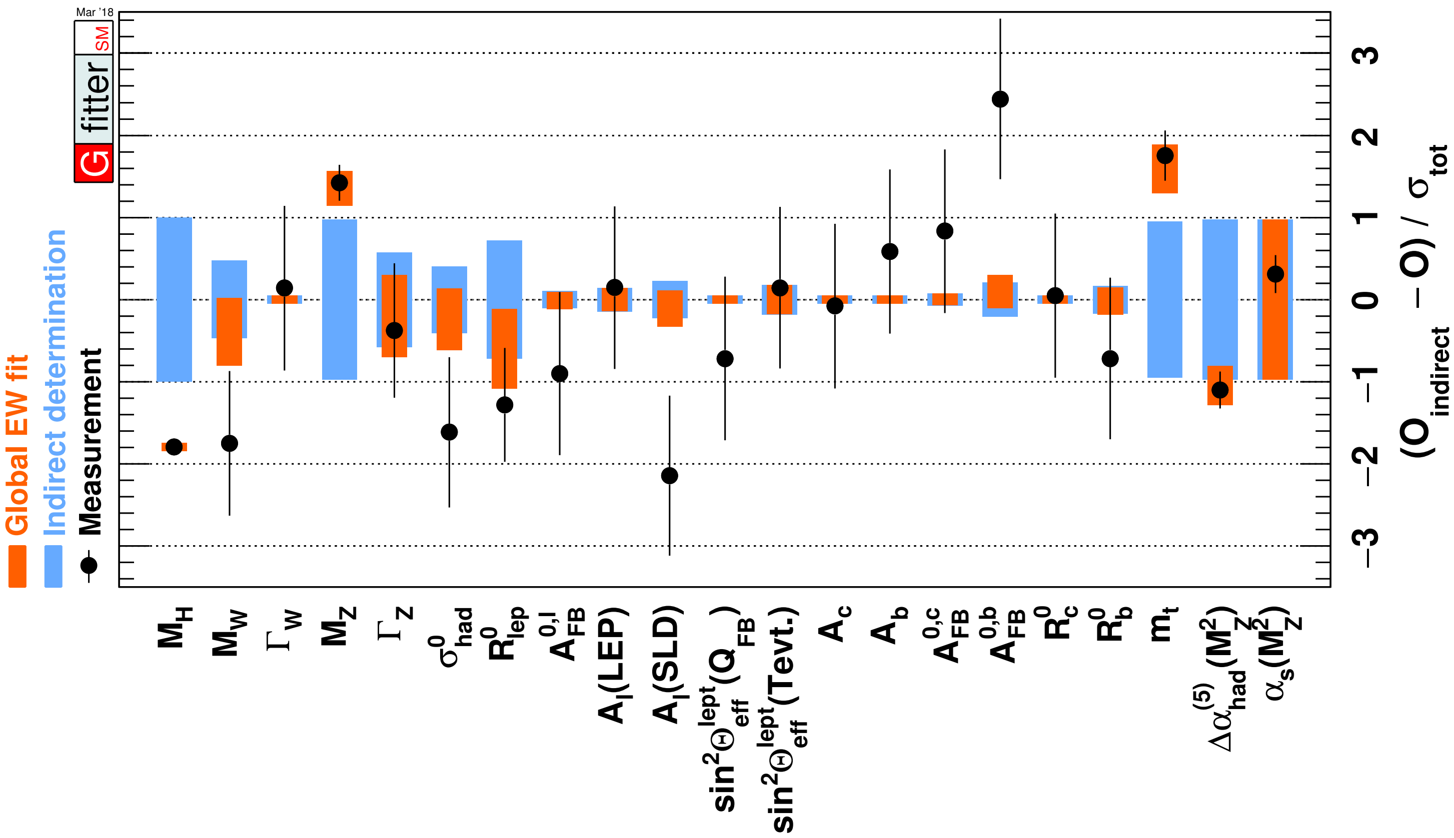
- Compare direct measurements of the observables with fit values:



$\Rightarrow$ some tensions (none above 3σ): A_ℓ(SLD), A^b_{FB}(LEP), R_b, ...

Global fits (Comparisons)

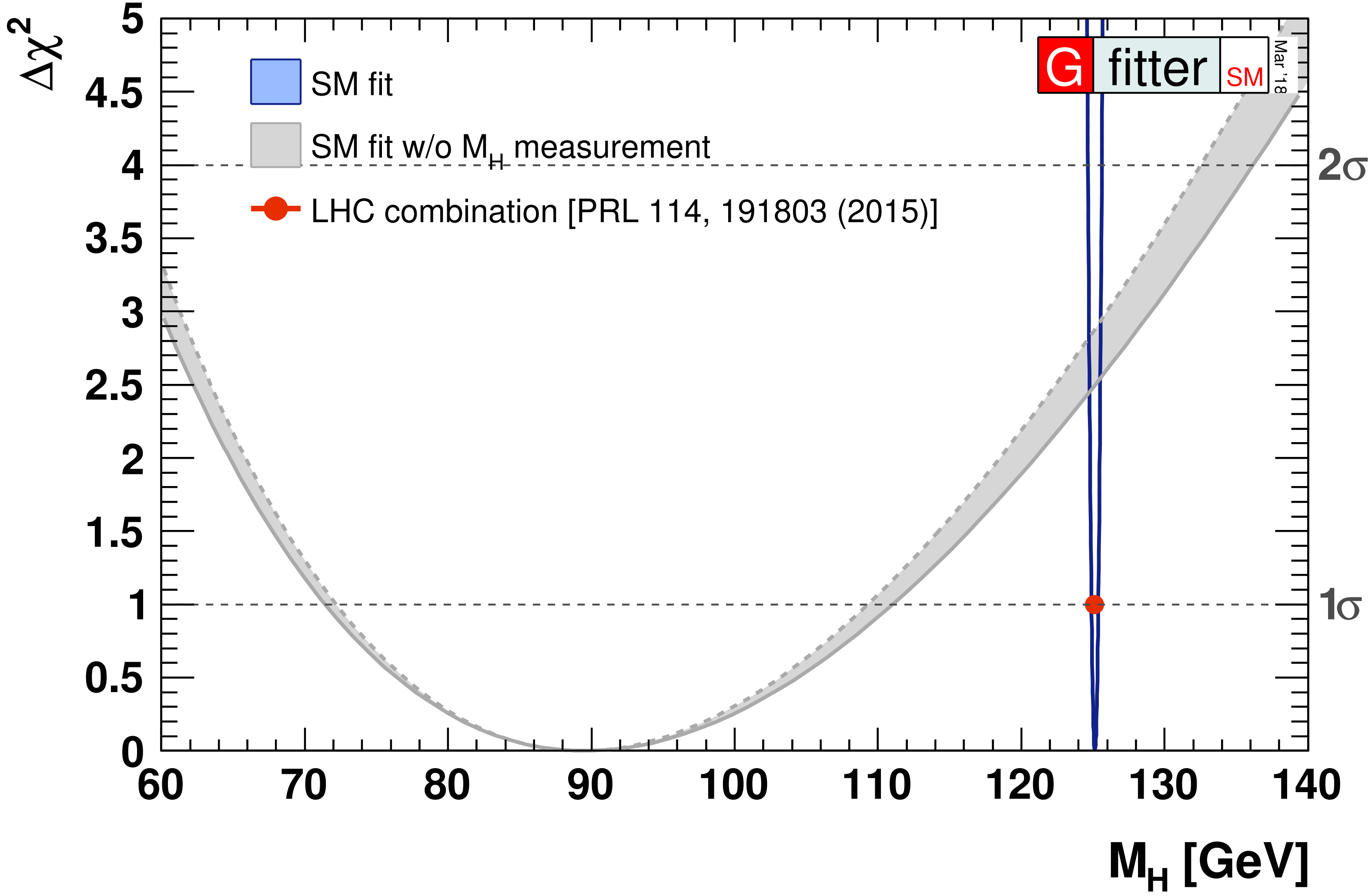
- Compare indirect determinations with fit values (error bars are direct measmts.):



[indirect determination means fit without using constraint from given direct measurement]

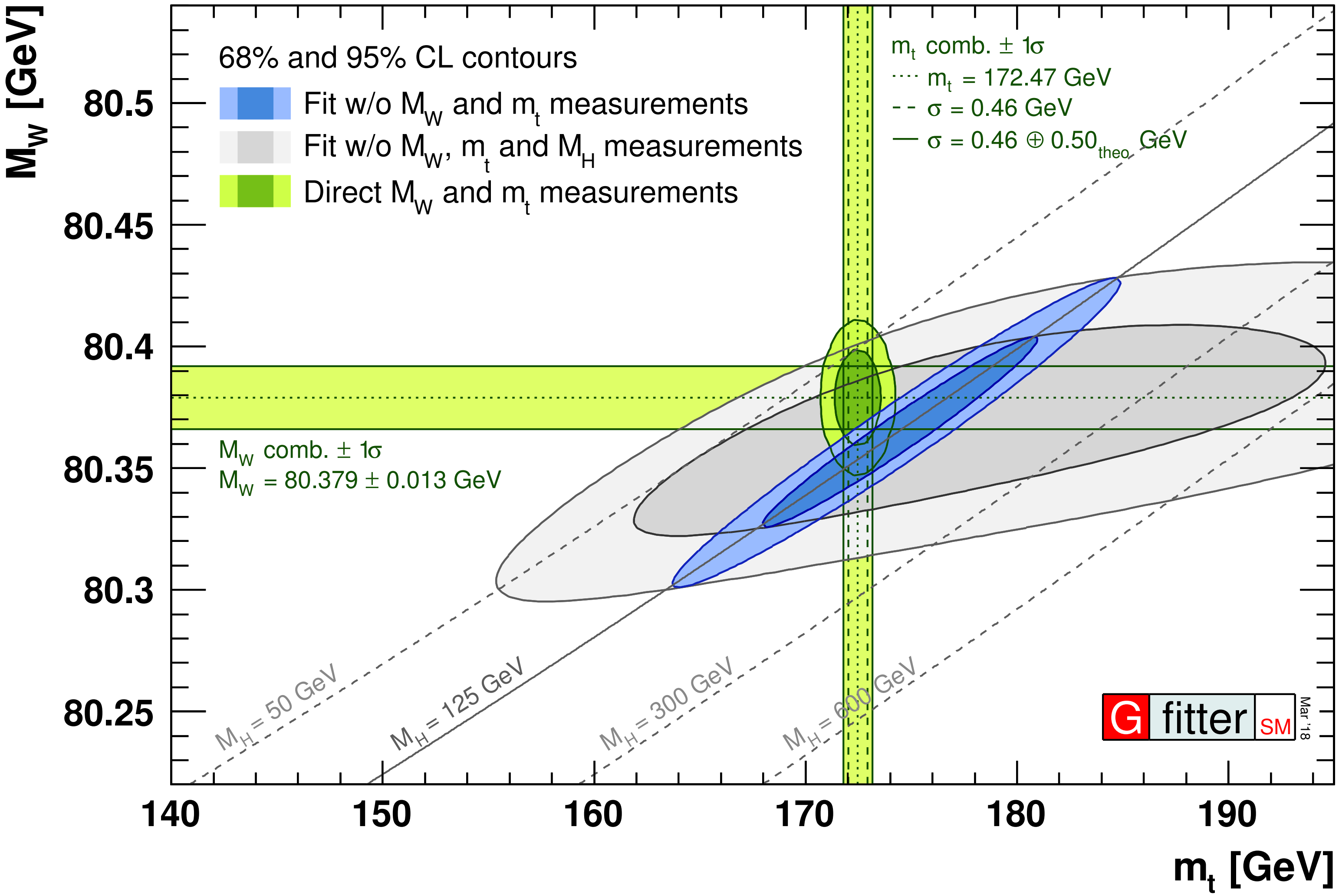
Global fits (Conclusions)

⇒ Fits prefer a somewhat lighter Higgs:



Global fits (Conclusions)

⇒ In general, impressive consistency of the SM, e.g.:



Final remarks

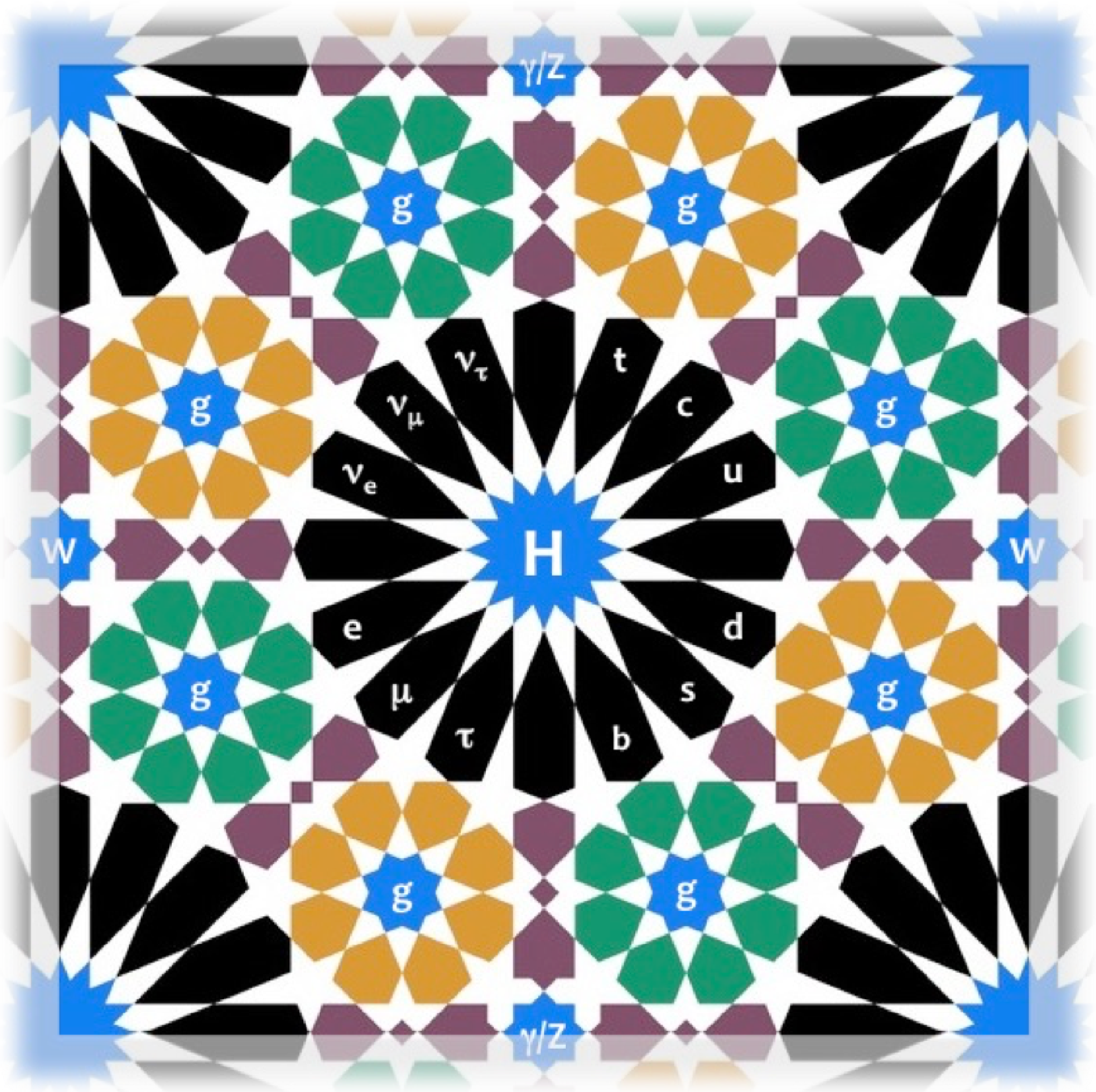
Summary

- The SM is a gauge theory with spontaneous symmetry breaking (renormalizable)
- Confirmed by many low and high energy experiments with remarkable accuracy, at the level of quantum corrections, with (almost) no significant deviations
- In spite of its tremendous success, it leaves fundamental questions unanswered:

  why 3 generations? why the observed pattern of fermion masses and mixings?

- And there are several hints for physics beyond:
  - phenomenological:
    - $(g_\mu - 2)$
    - neutrino masses
    - flavor anomalies
    - baryon asymmetry
    - dark matter
    - dark energy
  - conceptual:
    - gravity is not included
    - hierarchy problem
    - cosmological constant

    The SM is an Effective Theory valid up to electroweak scale?

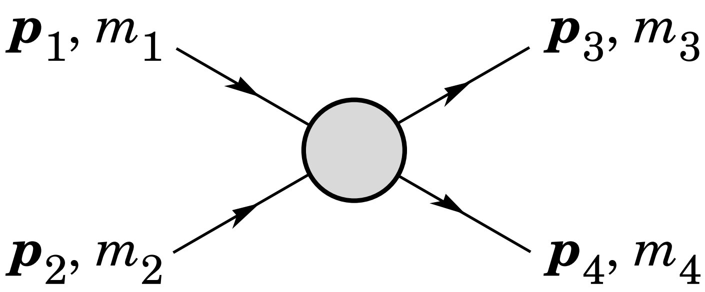


Thank You!

APPENDIX

Kinematics

$2 \to 2$ Kinematics



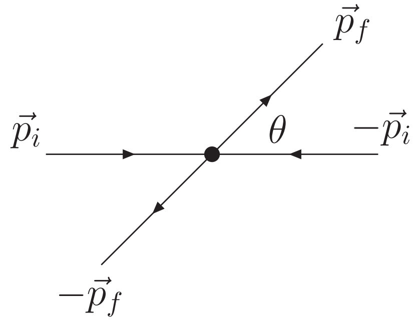
$$p_1 + p_2 = p_3 + p_4$$

Mandelstam variables

(Lorentz invariant)

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 = m_1^2 + m_2^2 + 2(p_1 \cdot p_2) \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_1^2 + m_3^2 - 2(p_1 \cdot p_3) \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_1^2 + m_4^2 - 2(p_1 \cdot p_4) \\
s + t + u &= \sum_{i=1}^{4} m_i^2 .
\end{aligned}$$

$2 \to 2$ Kinematics | Center of mass frame (CM)



Consider particular case: $m_1 = m_2 \equiv m_i, \quad m_2 = m_4 \equiv m_f$

$$
\begin{aligned}
p_1 &= (E, 0, 0, |\boldsymbol{p_i}|) \\
p_2 &= (E, 0, 0, -|\boldsymbol{p_i}|) \\
p_3 &= (E, |\boldsymbol{p_f}| \sin\theta, 0, |\boldsymbol{p_f}| \cos\theta) \\
p_4 &= (E, -|\boldsymbol{p_f}| \sin\theta, 0, -|\boldsymbol{p_f}| \cos\theta)
\end{aligned}
$$

$$
\begin{aligned}
s &= 4E^2 = E^2_{\text{CM}} \\
t &= -\frac{s}{2}(1 - \beta_i\beta_f \cos\theta) + m_i^2 + m_f^2 \\
u &= -\frac{s}{2}(1 + \beta_i\beta_f \cos\theta) + m_i^2 + m_f^2
\end{aligned}
$$

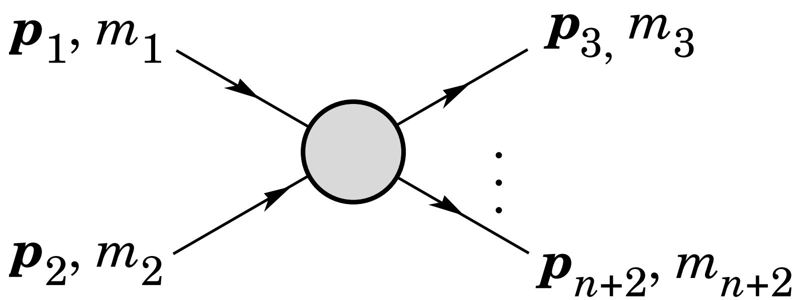
$$(s \geq \text{máx}\{4m_i^2, 4m_f^2\};\; t, u \leq -|m_i^2 - m_f^2|)$$

where $E^2 - |\boldsymbol{p}_{i,f}|^2 = m_{i,f}^2, \quad \beta_{i,f} = |\boldsymbol{p}_{i,f}|/E = \sqrt{1 - 4m_{i,f}^2/s}$.

Scalar products:

$$
\begin{aligned}
m_i^2 + (p_1 \cdot p_2) = m_f^2 + (p_3 \cdot p_4) &= 2E^2 = \frac{s}{2} \\
(p_1 \cdot p_3) = (p_2 \cdot p_4) &= E^2(1 - \beta_i\beta_f \cos\theta) = \frac{m_i^2 + m_f^2 - t}{2} \\
(p_1 \cdot p_4) = (p_2 \cdot p_3) &= E^2(1 + \beta_i\beta_f \cos\theta) = \frac{m_i^2 + m_f^2 - u}{2}
\end{aligned}
$$

Cross-section



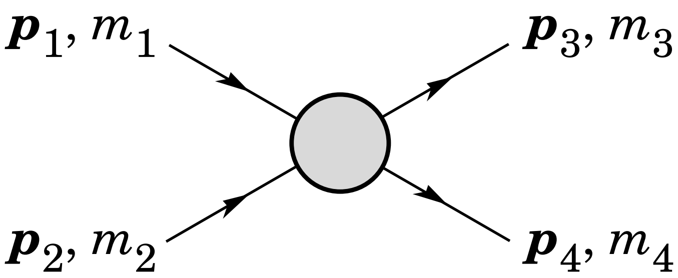
$$\mathrm{d}\sigma(i \to f) = \frac{1}{4\,\{(p_1 p_2)^2 - m_1^2 m_2^2\}^{1/2}} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_i - p_f) \prod_{j=3}^{n+2} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}$$

▷ Sum over initial polarizations and/or average over final polarizations if the initial state is unpolarized and/or the final state polarization is not measured

▷ Divide the total cross-section by a symmetry factor $S = \prod_i n_i!$ if there are n_i identical particles of species i in the final state

Cross-section — case $2 \to 2$ in CM frame

$$\boldsymbol{q} = \boldsymbol{p}_1 = -\boldsymbol{p}_2$$



$$\boldsymbol{p} = \boldsymbol{p}_3 = -\boldsymbol{p}_4$$

$$\Rightarrow \int \mathrm{d}\Phi_2 \equiv (2\pi)^4 \int \delta^4(p_1 + p_2 - p_3 - p_4) \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4} = \int \frac{|\boldsymbol{p}|\mathrm{d}\Omega}{16\pi^2 E_{\mathrm{CM}}}$$

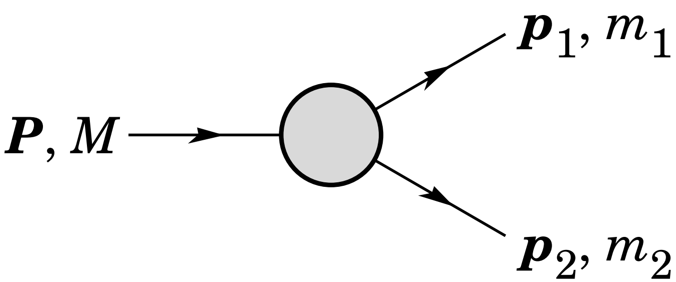
and if $m_1 = m_2 \quad \Rightarrow \quad 4\{(p_1 p_2)^2 - m_1^2 m_2^2\}^{1/2} = 4E_{\mathrm{CM}}|\boldsymbol{q}|$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(1,2 \to 3,4) = \frac{1}{64\pi^2 E_{\mathrm{CM}}^2} \frac{|\boldsymbol{p}|}{|\boldsymbol{q}|} |\mathcal{M}|^2}$$

Decay width

$$\mathrm{d}\Gamma(i \to f) = \frac{1}{2M} |\mathcal{M}|^2 (2\pi)^4 \delta^4(P - p_f) \prod_{j=1}^{n} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}$$

case $1 \to 2$



$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}(i \to 1, 2) = \frac{1}{32\pi^2} \frac{|\boldsymbol{p}|}{M^2} |\mathcal{M}|^2$$

▷ Note that masses M, m_1 and m_2 fix final energies and momenta:

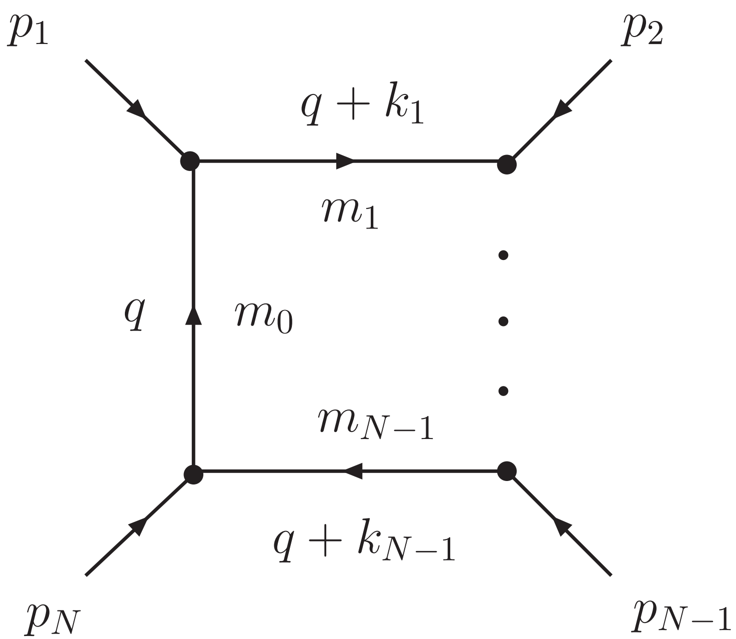
$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} \qquad E_2 = \frac{M^2 - m_1^2 + m_2^2}{2M}$$

$$|\boldsymbol{p}| = |\boldsymbol{p}_1| = |\boldsymbol{p}_2| = \frac{\left\{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]\right\}^{1/2}}{2M}$$

Loop calculations

Structure of one-loop amplitudes

- Consider the following generic one-loop diagram with N external legs:



$$k_1 = p_1, \quad k_2 = p_1 + p_2, \quad \ldots \quad k_{N-1} = \sum_{i=1}^{N-1} p_i$$

- It contains general integrals of the kind:

$$\frac{\mathrm{i}}{16\pi^2} T^N_{\mu_1 \ldots \mu_P} \equiv \mu^{4-D} \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{q_{\mu_1} \cdots q_{\mu_P}}{[q^2 - m_0^2][(q+k_1)^2 - m_1^2] \cdots [(q+k_{N-1})^2 - m_{N-1}^2]}$$

Structure of one-loop amplitudes

- D dimensional integration in dimensional regularization
- Integrals are symmetric under permutations of Lorentz indices
- Scale μ introduced to keep the proper mass dimensions
- P is the number of q's in the numerator and determines the tensor structure of the integral (scalar if $P = 0$, vector if $P = 1$, etc.). Note that $P \leq N$
- Notation: A for T^1, B for T^2, etc. For example, the scalar integrals A_0, B_0, etc.
- The tensor integrals can be decomposed as a linear combination of the Lorentz covariant tensors that can be built with $g_{\mu\nu}$ and a set of linearly independent momenta [Passarino, Veltman '79]
- The choice of basis is not unique

  Here we use the basis formed by $g_{\mu\nu}$ and the momenta k_i, where the the tensor coefficients are totally symmetric in their indices [Denner '93]

  This is the basis used by the computer package LoopTools [www.feynarts.de/looptools]

Structure of one-loop amplitudes

- We focus here on:

$$
\begin{aligned}
B_\mu &= k_{1\mu} B_1 \\
B_{\mu\nu} &= g_{\mu\nu} B_{00} + k_{1\mu} k_{1\nu} B_{11} \\
C_\mu &= k_{1\mu} C_1 + k_{2\mu} C_2 \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + \sum_{i,j=1}^{2} k_{i\mu} k_{j\nu} C_{ij} \\
C_{\mu\nu\rho} &= \ldots
\end{aligned}
$$

- We will see that the scalar integrals A_0 and B_0 and the tensor integral coefficients B_1, B_{00}, B_{11} and C_{00} are divergent in $D = 4$ dimensions (ultraviolet divergence, equivalent to take cutoff $\Lambda \to \infty$ in q)

- It is possible to express every tensor coefficient in terms of scalar integrals (scalar reduction) [Denner '93]

Explicit calculation

- Basic ingredients:

– Euler Gamma function:

$$\Gamma(x+1) \;=\; x\Gamma(x)$$

Taylor expansion around poles at $x = 0, -1, -2, \ldots$:

$$\begin{aligned} x=0: \quad \Gamma(x) &= \frac{1}{x} - \gamma + \mathcal{O}(x) \\ x=-1: \quad \Gamma(x) &= -\frac{1}{(x+1)} + \gamma - 1 + \cdots + \mathcal{O}(x+1) \end{aligned}$$

where $\gamma \approx 0.5772\ldots$ is Euler-Mascheroni constant

– Feynman parameters:

$$\frac{1}{a_1 a_2 \cdots a_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \delta\left(\sum_{i=1}^{n} x_i - 1\right) \frac{(n-1)!}{[x_1 a_1 + x_2 a_2 + \cdots x_n a_n]^n}$$

Explicit calculation

– The following integrals (with $\varepsilon \to 0^+$) will be needed:

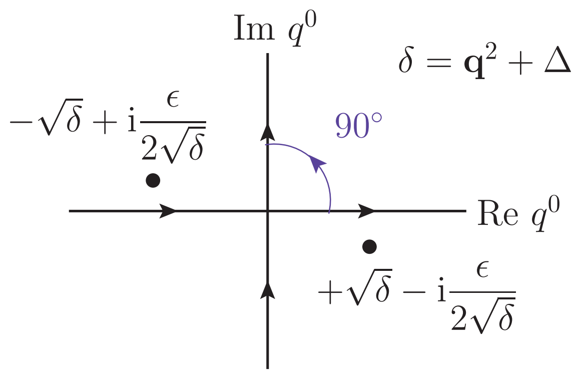
$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - \Delta + i\varepsilon)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-D/2}$$

$$\Rightarrow \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - \Delta + i\varepsilon)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n - D/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-D/2-1}$$

▷ Let's solve the first integral in Euclidean space: $q^0 = iq_E^0$, $\boldsymbol{q} = \boldsymbol{q}_E$, $q^2 = -q_E^2$,

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - \Delta + i\varepsilon)^n} = i(-1)^n \int \frac{d^D q_E}{(2\pi)^D} \frac{1}{(q_E^2 + \Delta)^n}$$

(equivalent to a Wick rotation of 90°). The second integral follows from this one



Explicit calculation

In D-dimensional spherical coordinates:

$$\int \frac{\mathrm{d}^D q_E}{(2\pi)^D} \frac{1}{(q_E^2+\Delta)^n} = \int \mathrm{d}\Omega_D \int_0^\infty \mathrm{d}q_E q_E^{D-1} \frac{1}{(q_E^2+\Delta)^n} \equiv \mathcal{I}_A \times \mathcal{I}_B$$

where $$\mathcal{I}_A = \int \mathrm{d}\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

$$\text{since } (\sqrt{\pi})^D = \left(\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2}\right)^D = \int \mathrm{d}^D x\, \mathrm{e}^{-\sum_{i=1}^D x_i^2} = \int \mathrm{d}\Omega_D \int_0^\infty \mathrm{d}x\, x^{D-1} \mathrm{e}^{-x^2}$$

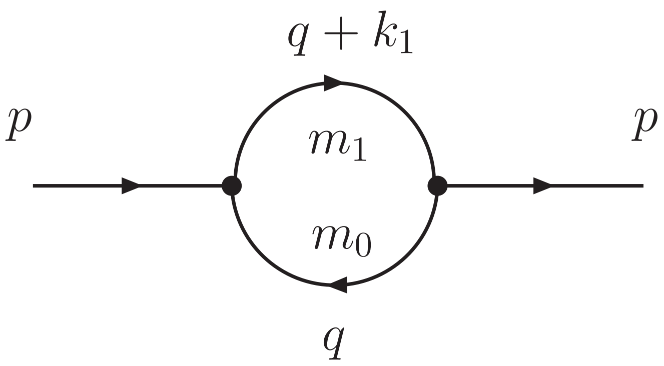
$$= \left(\int \mathrm{d}\Omega_D\right) \frac{1}{2} \int_0^\infty \mathrm{d}t\, t^{D/2-1} \mathrm{e}^{-t} = \left(\int \mathrm{d}\Omega_D\right) \frac{1}{2} \Gamma(D/2)$$

and, changing variables: $t = q_E^2$, $z = \Delta/(t+\Delta)$, we have

$$\mathcal{I}_B = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-D/2} \int_0^1 \partial z\, z^{n-D/2-1}(1-z)^{D/2-1} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-D/2} \frac{\Gamma(n-D/2)\Gamma(D/2)}{\Gamma(n)}$$

where Euler Beta function was used: $B(\alpha,\beta) = \int_0^1 \mathrm{d}z\, z^{\alpha-1}(1-z)^{\beta-1} = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Explicit calculation — Two-point functions



$$\frac{\mathrm{i}}{16\pi^2}\{B_0,\ B^\mu,\ B^{\mu\nu}\}(\text{args}) = \mu^{4-D}\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\frac{\{1,\ q^\mu,\ q^\mu q^\nu\}}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]}$$

▷ $k_1 = p$

▷ The integrals depend on the masses m_0, m_1 and the invariant p^2:

$$(\text{args}) = (p^2; m_0^2, m_1^2)$$

Explicit calculation | Two-point functions

- Using Feynman parameters,

$$\frac{1}{a_1 a_2} = \int_0^1 \mathrm{d}x \frac{1}{[a_1 x + a_2(1-x)]^2}$$

$$\Rightarrow \frac{\mathrm{i}}{16\pi^2}\{B_0,\ B^\mu,\ B^{\mu\nu}\} = \mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1,\ -A^\mu,\ q^\mu q^\nu + A^\mu A^\nu\}}{(q^2-\Delta_2)^2}$$

with

$$\Delta_2 = x^2 p^2 + x(m_1^2 - m_0^2 - p^2) + m_0^2$$

$$a_1 = (q+p)^2 - m_1^2$$
$$a_2 = q^2 - m_0^2$$

and a loop momentum shift to obtain a perfect square in the denominator:

$$q^\mu \to q^\mu - A^\mu, \quad A^\mu = x p^\mu$$

Explicit calculation | Two-point functions

- Then, the scalar function is:

$$\frac{\mathrm{i}}{16\pi^2} B_0 = \mu^{4-D} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{(q^2-\Delta_2)^2}$$

$$\Rightarrow \quad B_0 = \Delta_\epsilon - \int_0^1 \mathrm{d}x \, \ln \frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]$$

where $\Delta_\epsilon \equiv \frac{2}{\epsilon} - \gamma + \ln 4\pi$ and the Euler Gamma function was expanded around $x=0$ for $D=4-\epsilon$, using $x^\epsilon = \exp\{\epsilon \ln x\} = 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2)$:

$$\mu^{4-D} \frac{\mathrm{i}\Gamma(2-D/2)}{(4\pi)^{D/2}} \left(\frac{1}{\Delta_2}\right)^{2-D/2} = \frac{\mathrm{i}}{16\pi^2} \left(\Delta_\epsilon - \ln \frac{\Delta_2}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

- Comparing with the definitions of the tensor coefficientes we have:

$$\frac{\mathrm{i}}{16\pi^2} B^\mu = -\mu^{4-D} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{A^\mu}{(q^2-\Delta_2)^2}$$

$$\Rightarrow \quad B_1 = -\frac{1}{2}\Delta_\epsilon + \int_0^1 \mathrm{d}x \, x \ln \frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]$$

Explicit calculation | Two-point functions

and

$$\frac{\mathrm{i}}{16\pi^2}B^{\mu\nu} = \mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{(q^2/D)g^{\mu\nu}+A^\mu A^\nu}{(q^2-\Delta_2)^2}$$

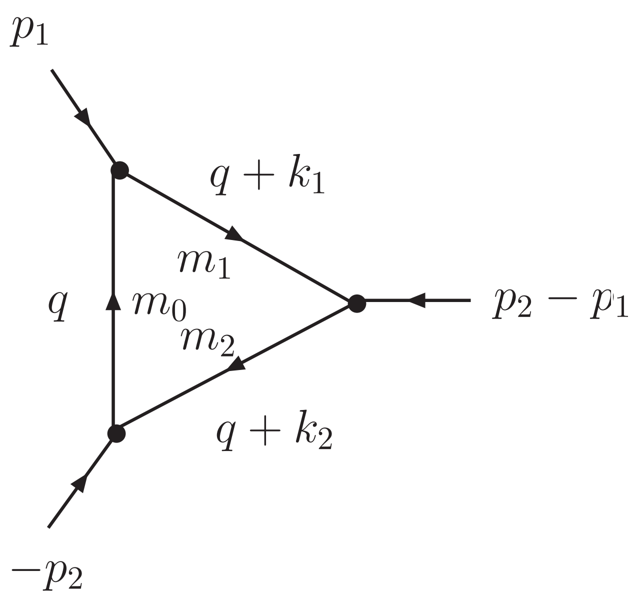
$$\Rightarrow \quad B_{00} = -\frac{1}{12}(p^2-3m_0^2-3m_1^2)\Delta_\epsilon + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]$$

$$B_{11} = \frac{1}{3}\Delta_\epsilon - \int_0^1 \mathrm{d}x\; x^2 \ln\frac{\Delta_2}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D=4-\epsilon]$$

where $q^\mu q^\nu$ have been replaced by $(q^2/D)g^{\mu\nu}$ in the integrand and the Euler Gamma function was expanded around $x=-1$ for $D=4-\epsilon$:

$$-\mu^{4-D}\frac{\mathrm{i}\Gamma(1-D/2)}{(4\pi)^{D/2}2\Gamma(2)}\left(\frac{1}{\Delta_2}\right)^{1-D/2} = \frac{\mathrm{i}}{16\pi^2}\frac{1}{2}\left(\Delta_\epsilon - \ln\frac{\Delta_2}{\mu^2}+1\right)\Delta_2 + \mathcal{O}(\epsilon)$$

Explicit calculation | Three-point functions



$$\frac{\mathrm{i}}{16\pi^2}\{C_0,\ C^\mu,\ C^{\mu\nu}\}\,(\text{args}) = \mu^{4-D}\int\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\{1,\ q^\mu,\ q^\mu q^\nu\}}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]}$$

▷ It is convenient to choose the external momenta so that:

$$k_1 = p_1, \quad k_2 = p_2.$$

▷ The integrals depend on the masses m_0, m_1, m_2 and the invariants:

$$(\text{args}) = (p_1^2, Q^2, p_2^2; m_0^2, m_1^2, m_2^2), \quad Q^2 \equiv (p_2 - p_1)^2.$$

Explicit calculation | Three-point functions

- Using Feynman parameters,

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{[a_1 x + a_2 y + a_3(1-x-y)]^3}$$

$$\Rightarrow \frac{\mathrm{i}}{16\pi^2}\{C_0,\ C^\mu,\ C^{\mu\nu}\} = 2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1,\ -A^\mu,\ q^\mu q^\nu + A^\mu A^\nu\}}{(q^2-\Delta_3)^3}$$

with

$$\Delta_3 = x^2 p_1^2 + y^2 p_2^2 + xy(p_1^2 + p_2^2 - Q^2) + x(m_1^2 - m_0^2 - p_1^2) + y(m_2^2 - m_0^2 - p_2^2) + m_0^2$$

$$\begin{aligned} a_1 &= (q+p_1)^2 - m_1^2 \\ a_2 &= (q+p_2)^2 - m_2^2 \\ a_3 &= q^2 - m_0^2 \end{aligned}$$

and a loop momentum shift to obtain a perfect square in the denominator:

$$q^\mu \to q^\mu - A^\mu, \quad A^\mu = x p_1^\mu + y p_2^\mu$$

Explicit calculation | Three-point functions

- Then the scalar function is:

$$\frac{i}{16\pi^2} C_0 = 2\mu^{4-D} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2-\Delta_3)^3}$$

$$\Rightarrow \quad C_0 = -\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{\Delta_3} \qquad [D=4]$$

- Comparing with the definitions of the tensor coefficientes we have:

$$\frac{i}{16\pi^2} C^\mu = -2\mu^{4-D} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D q}{(2\pi)^D} \frac{A^\mu}{(q^2-\Delta_3)^3}$$

$$\Rightarrow \quad C_1 = \int_0^1 dx \int_0^{1-x} dy \, \frac{x}{\Delta_3} \qquad [D=4]$$

$$C_2 = \int_0^1 dx \int_0^{1-x} dy \, \frac{y}{\Delta_3} \qquad [D=4]$$

Explicit calculation | Three-point functions

$$
\begin{aligned}
\frac{\mathrm{i}}{16\pi^2} C^{\mu\nu} &= 2\mu^{4-D} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(q^2/D) g^{\mu\nu} + A^\mu A^\nu}{(q^2 - \Delta_3)^3} \\
\Rightarrow \quad C_{11} &= -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{x^2}{\Delta_3} \qquad [D=4] \\
C_{22} &= -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y^2}{\Delta_3} \qquad [D=4] \\
C_{12} &= -\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{xy}{\Delta_3} \qquad [D=4] \\
C_{00} &= \frac{1}{4}\Delta_\epsilon - \frac{1}{2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \ln \frac{\Delta_3}{\mu^2} + \mathcal{O}(\epsilon) \qquad [D = 4-\epsilon]
\end{aligned}
$$

where $\Delta_\epsilon \equiv \dfrac{2}{\epsilon} - \gamma + \ln 4\pi$ and $q^\mu q^\nu$ was replaced by $(q^2/D) g^{\mu\nu}$ in the integrand

In C_{00} the Euler Gamma function was expanded around $x = 0$ for $D = 4 - \epsilon$:

$$
\mu^{4-D} \frac{\mathrm{i}\Gamma(2 - D/2)}{(4\pi)^{D/2}\Gamma(3)} \left(\frac{1}{\Delta_3} \right)^{2-D/2} = \frac{\mathrm{i}}{16\pi^2} \frac{1}{2} \left(\Delta_\epsilon - \ln \frac{\Delta_3}{\mu^2} \right) + \mathcal{O}(\epsilon)
$$

Note about Diracology in D dimensions

- Attention should be paid to the traces of Dirac matrices when working in D dimensions (dimensional regularization) since

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}_{4\times 4}, \quad g^{\mu\nu}g_{\mu\nu} = \mathrm{Tr}\{g^{\mu\nu}\} = D$$

Thus, the following identities involving contractions of Lorentz indices can be proven:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= D \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(D-2)\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-D)\gamma^\nu\gamma^\rho\gamma^\sigma
\end{aligned}$$

Neutrinos are special

Dirac vs Majorana fermions

- A Dirac fermion field is a spinor with 4 independent components: 2 LH+2 RH (left/right-handed particles and antiparticles)

$$\psi_L = P_L\psi\ , \quad \psi_R = P_R\psi\ , \quad \psi^c_L \equiv (\psi_L)^c = P_R\psi^c\ , \quad \psi^c_R \equiv (\psi_R)^c = P_L\psi^c$$

where $\psi^c \equiv C\overline{\psi}^{\mathsf{T}} = -\mathrm{i}\gamma^2\psi^*$ (charge conjugate), $C = -\mathrm{i}\gamma^2\gamma^0$, $P_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$

- A Majorana fermion field has just 2 independent components since $\psi^c \equiv \eta^*\psi$:

$$\psi_L = \eta\psi^c_R\ , \quad \psi_R = \eta\psi^c_L$$

where $\eta = -\mathrm{i}\eta_{\mathrm{CP}}$ (CP parity) with $|\eta|^2 = 1$. Only possible if neutral

[*Useful relations*: $\quad C^\dagger = C^T = C^{-1} = -C, \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T, \quad \overline{\psi^c} = \psi^T C$]

General mass terms

- Lorentz invariant terms:

$$\overline{\psi_R}\psi_L = \overline{\psi_L^c}\psi_R^c \quad \stackrel{\text{hc}}{\longleftrightarrow} \quad \overline{\psi_L}\psi_R = \overline{\psi_R^c}\psi_L^c \quad (\Delta F = 0)$$

$$\left.\begin{array}{ccc} \overline{\psi_L^c}\psi_L & = & \overline{\psi_L}\psi_L^c \\ \overline{\psi_R}\psi_R^c & = & \overline{\psi_R^c}\psi_R \end{array}\right\} \quad (|\Delta F| = 2)$$

$$\Rightarrow \quad -\mathcal{L}_m = \underbrace{m_D\,\overline{\psi_R}\psi_L}_{\text{Dirac term}} + \underbrace{\frac{1}{2}m_L\,\overline{\psi_L^c}\psi_L + \frac{1}{2}m_R\,\overline{\psi_R}\psi_R^c}_{\text{Majorana terms}} + \text{h.c.}$$

  – A Dirac fermion can only have a Dirac mass term (fermion number preserving)
  – Majorana fermions may have Majorana mass terms

⇒ In the SM:
* m_D from Yukawa coupling after EW SSB $\qquad (m_D = \lambda_\nu\, v/\sqrt{2})$
* m_L forbidden by gauge symmetry
* m_R compatible with gauge symmetry! (ν_R are sterile)

General mass terms (a more transparent parameterization)

- Rewrite previous mass terms introducing an array of two Majorana fermions:

$$\chi^0_L = \begin{pmatrix} \psi_L \\ \psi^c_R \end{pmatrix}, \quad \chi^0 = \chi^{0c} = \chi^0_L + \chi^{0c}_L \equiv \begin{pmatrix} \chi^0_1 \\ \chi^0_2 \end{pmatrix}, \qquad \begin{array}{l} \chi^0_1 = \chi^{0c}_1 = \chi^0_{1L} + \chi^{0c}_{1L} \equiv \psi_L + \psi^c_L \\ \chi^0_2 = \chi^{0c}_2 = \chi^0_{2L} + \chi^{0c}_{2L} \equiv \psi^c_R + \psi_R \end{array}$$

$$\Rightarrow \quad -\mathcal{L}_m = \frac{1}{2}\overline{\chi^{0c}_L}\,\mathbf{M}\,\chi^0_L + \text{h.c.} \quad \text{with} \quad \mathbf{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

M is a square symmetric matrix ⇒ diagonalizable by a unitary matrix $\widetilde{\mathcal{U}}$:

$$\widetilde{\mathcal{U}}^{\mathsf{T}}\mathbf{M}\,\widetilde{\mathcal{U}} = \mathcal{M} = \text{diag}(m'_1, m'_2)\,, \quad \chi^0_L = \widetilde{\mathcal{U}}\chi_L \quad (\chi^{0c}_L = \widetilde{\mathcal{U}}^*\chi^c_L)$$

To get positive eigenvalues $m_i = \eta_i m'_i$ (physical masses) replace $\chi_{iL} = \sqrt{\eta_i}\,\xi_{iL}$

$$\chi^0_L = \mathcal{U}\xi_L, \quad \mathcal{U} = \widetilde{\mathcal{U}}\text{diag}(\sqrt{\eta_1}, \sqrt{\eta_2})\,, \qquad \begin{array}{l} \xi_1 = \xi_{1L} + \xi^c_{1L} \\ \xi_2 = \xi_{2L} + \xi^c_{2L} \end{array} \quad \text{(physical fields)}$$

General mass terms

♣ Case of only Dirac term $\boxed{\chi_L^0 = (\nu_L, \nu_R^c)}$

$(m_L = m_R = 0)$

$$\mathbf{M} = \begin{pmatrix} 0 & m_D \\ m_D & 0 \end{pmatrix} \quad \Rightarrow \quad \widetilde{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \quad m_1' = -m_D , \quad m_2' = m_D$$

Eigenstates $\Rightarrow$ Physical states

$$\chi_{1L} = \frac{1}{\sqrt{2}}(\nu_L - \nu_R^c) \qquad \xi_{1L} = -\mathrm{i}\chi_{1L} \quad [\eta_1 = -1]$$

$$\chi_{2L} = \frac{1}{\sqrt{2}}(\nu_L + \nu_R^c) \qquad \xi_{2L} = \chi_{2L} \quad [\eta_2 = +1]$$

with masses $m_1 = m_2 = m_D$

$$\Rightarrow \quad -\mathcal{L}_m = m_D(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R) = \frac{1}{2}m_D(\overline{\xi_{1L}^c}\xi_{1L} + \overline{\xi_{2L}^c}\xi_{2L}) + \text{h.c.}$$

One Dirac fermion = two Majorana of equal mass and opposite CP parities

General mass terms

♣ Case of seesaw (type I) $\boxed{\chi_L^0 = (\nu_L, N_R^c)}$

$(m_D \ll m_R)$ [Yanagida '79; Gell-Mann, Ramond, Slansky '79; Mohapatra, Senjanovic '80]

$$\mathbf{M} = \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \quad \Rightarrow \quad \widetilde{\mathcal{U}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$m_1 \equiv m_\nu \simeq \frac{m_D^2}{m_R} \ll m_2 \equiv m_N \simeq m_R$$

$$\theta \simeq \frac{m_D}{m_R} \simeq \sqrt{\frac{m_\nu}{m_N}} \ll 1$$

$$\chi_{1L} \approx \nu_L - \frac{m_D}{m_R} N_R^c \approx \nu_L \quad \Rightarrow \quad \xi_{1L} \approx -i\nu_L$$

$$\chi_{2L} \approx \frac{m_D}{m_R} \nu_L + N_R^c \approx N_R^c \quad \Rightarrow \quad \xi_{2L} \approx N_R^c$$

$$\Rightarrow \quad -\mathcal{L}_m = \underbrace{\frac{1}{2} m_\nu \, \overline{\nu_L^c} \nu_L}_{\text{gauge invariant??}} + \frac{1}{2} m_N \, \overline{N_R^c} N_R + \text{h.c.}$$

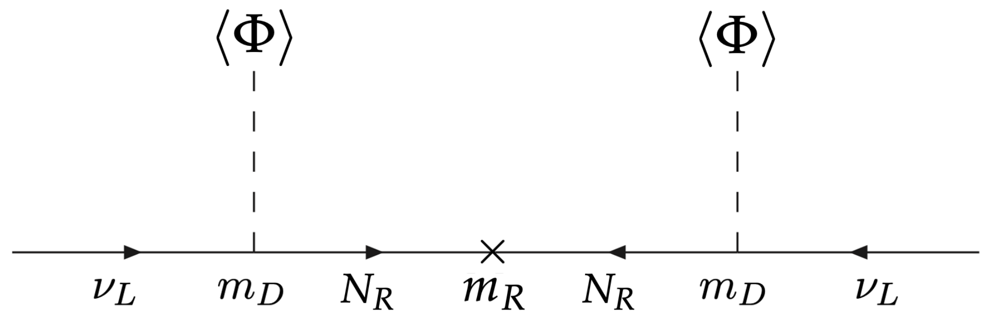
General mass terms

♣ Case of seesaw (type I) $\chi_L^0 = (\nu_L, N_R^c)$

$\frac{1}{2} m_\nu \, \overline{\nu_L^c} \nu_L$ comes after EW SSB from a dim-5 effective interaction, that is gauge-invariant but lepton-number violating (Weinberg operator):

$$\mathcal{L}_{\text{Weinberg}} = -\frac{1}{2}\frac{\lambda_\nu^2}{m_R}(\overline{L}\widetilde{\Phi})(\widetilde{\Phi}^T L^c) + \text{h.c.}$$

with $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$



Perhaps the observed neutrino ν_L is the LH component of a light Majorana ν (then $\overline{\nu}$ = RH) and light because of a very heavy Majorana neutrino N

e.g. $\quad m_D = \lambda_\nu \frac{v}{\sqrt{2}} \sim 100 \text{ GeV}\,, \quad m_R \sim m_N \sim 10^{14} \text{ GeV} \quad \Rightarrow \quad m_\nu \sim 0.1 \text{ eV}$ ✓

General mass terms

♣ Case of seesaw (type I): $\boxed{\chi_L^0 = (\nu_{\alpha L}, N_{Rj}^c)}$ **several generations**

$$\alpha = e, \mu, \tau \text{ (active)} \qquad j = 1, \ldots, n_R \geq 2 \text{ (sterile)}$$

$$\mathbf{M} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \qquad \text{with blocks} \quad \begin{cases} 0 : 3 \times 3 & M_D : 3 \times n_R \\ M_D^T : n_R \times 3 & M_R : n_R \times n_R \end{cases}$$

For $M_D \ll M_R$, and taking M_R diagonal to simplify:

$$\mathcal{U}^T \mathbf{M} \mathcal{U} \approx \begin{pmatrix} \mathbf{U}^T M_D M_R^{-1} M_D^T \mathbf{U} & 0 \\ 0 & M_R \end{pmatrix} \equiv \begin{pmatrix} M_\nu^{\text{diag}} & 0 \\ 0 & M_N^{\text{diag}} \end{pmatrix}$$

The 3×3 block $\mathbf{U}$ is *approximately unitary* because it is contained in $\mathcal{U}$:

$$\mathcal{U} \approx \begin{pmatrix} \mathbf{U} & \mathcal{O}(m_D/m_R) \\ \mathcal{O}(m_D/m_R) & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \nu_\alpha = \nu_{\alpha L} + \nu_{\alpha L}^c \quad \text{with} \quad \nu_{\alpha L} = \mathbf{U}_{\alpha i} \nu_{iL}$$

$$(\chi_L^0 = \mathcal{U} \xi_L)$$