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AN ONTO-SEMIOTIC ANALYSIS OF COMBINATORIAL PROBLEMS AND THE SOLVING PROCESSES BY UNIVERSITY STUDENTS

ABSTRACT. In this paper we describe an ontological and semiotic model for mathematical knowledge, using elementary combinatorics as an example. We then apply this model to analyze the solving process of some combinatorial problems by students with high mathematical training, and show its utility in providing a semiotic explanation for the difficulty of combinatorial reasoning. We finally analyze the implications of the theoretical model and type of analysis presented for mathematics education research and practice.

KEY WORDS: cognitive dualities, combinatorial reasoning, individual and institutional knowledge, mathematical objects, semiotic analysis, semiotic function

Nowadays there is growing interest in the use of semiotic ideas for mathematics education. Some examples are provided by the work presented at the Psychology of Mathematics Education conferences (Ernest, 1993; Vile and Lerman, 1996), and those carried out from the symbolic interactionism perspective (e.g., Cobb and Bauersfeld, 1995), which emphasize the ideas of meaning and negotiation of meanings. Works relating to representation (Duval, 1993), symbolization and communication (Pimm, 1995; Cobb et al., 2000), mathematics as semiosis (Anderson et al., 2003), and, in general, to the role of language (Ellerton and Clarkson, 1996) in the teaching and learning of mathematics, as well as research into understanding of mathematics (Sierpinska, 1994; Godino, 1996), cannot avoid concern about meaning either.

This interest is a natural consequence of the essential role played by the expressive tools in thinking processes. Thus Vygotsky (1934/1993) considers the meaning of words as the basic unit for thinking activity, and Cassirer (1971) suggests that signs are not mere eventual covers of thought, but they are essential and necessary to it.

All of this suggests that we need to analyze the role of signs and the idea of meaning itself, from the mathematics education perspective, and to articulate the semiotic and epistemological components of mathematical activity. It is urgent to reflect on the nature and type of objects involved in mathematical activity: “What we understand by meaning and understanding is far from being obvious, in spite of being two central terms in

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any discussion on the learning and teaching of mathematics at any level" (Pimm, 1995, p. 3).

Mathematical symbols (signifiers) refer or replace conceptual entities (meanings). In addition to the mathematical symbolic syntax, its semantics, and pragmatics – that is, the nature of mathematical concepts, propositions and their relationship to contexts and situation-problems – is crucial in instructional processes. It is, therefore, necessary to build theoretical models that articulate the semiotic, epistemological, psychological and sociocultural dimensions in mathematics education. This involves taking into account the following elements and assumptions:

- Diversity of objects involved in mathematical activity, on both the expression and content planes.
- Diversity of semiotic acts and processes (interpretation) among the different types of objects and in the production of signs.
- Diversity of psycho-social contexts and circumstances that determine and relativize the semiotic processes.

In this paper we summarize and extend the theoretical notions proposed by Godino (1996) and Godino and Batanero (1994, 1998), which represent an ontological and semiotical approach to mathematics cognition, particularly adapted to the study of teaching-learning processes and where we incorporate pragmatic and anthropological assumptions about mathematical activity.

We also apply this theoretical framework to analyze and explain the difficulties found by university students with high-level mathematical training when solving simple combinatorial problems. The empirical data and the problems analyzed are taken from Roa (2000). This author gave a questionnaire with 13 simple combinatorial problems to a sample of 118 students majoring in Mathematics (4th or 5th year of university study), who generally found it difficult to solve the problems (each student only solved an average number of 6 problems correctly).

The article is structured in the following sections:

- Theoretical model for mathematics cognition and its background
- Meanings of mathematical objects. Elementary combinatorics and its basic elements.
- Dual dimensions in solving combinatorial problems and mathematical activity.
- Comparative analysis of four students' personal meanings of elementary combinatorics. Implications for teaching.
- Other applications of semiotic analysis and implications for mathematics education.

1. Background

In this paper we describe an ontological and semiotic model, as a theoretical tool for jointly analysing mathematical thinking, the ostensives that support it and the situations and factors conditioning its development. Our approach is ontologic and semiotic, given the essential role we attribute to language and our categorization of the different types of objects that emerge in mathematical activity. We conceive of mathematical language in a wide sense, that includes a variety of expressions and consider as mathematical object any kind of real or imaginary entity to which we refer when performing, communicating or learning mathematics. The model comes within social-constructivist philosophy (Ernest, 1998), whereby mathematics is conceived of as a human activity and the objects emerging from this activity are seen as cultural entities. One central problem for any philosophy of mathematics is the existence and nature of mathematical objects. As Ernest said, we should recognize the objectivity of mathematics, without assuming an existence independent of people. In social constructivism mathematical objects are social constructs derived from mathematical discourse: "Mathematical signifiers and signifieds are mutually interacting and constituting, so the discourse of mathematics which seems to name objects outside of itself is in fact the agent of their creation, maintenance, and elaboration, through its use" (Ernest, 1998, p. 193).

We start from previous theoretical notions (Godino and Batanero, 1994, 1998) about institutional and personal meaning of mathematical objects, where, basing on pragmatic assumptions, we focused on institutional mathematical knowledge, without forgetting the individuals, which are the focus of educational effort.

In that work, we conceive the meaning of a mathematical object (e.g., real number, function, etc.), as "systems of practices carried out to solve certain types of problems." These operative and discursive practices can either be attributed to individuals – and then we speak of personal meaning of the object, or be shared in an institution – and then we consider the corresponding institutional meaning.

We also adopt Hjemslev's (1943) vision of meaning, as the content of a sign function, which is described by Eco (1979) as "semiotic function", and interpreted as that to which a subject refers at a given time and circumstance, and not just as a mental entity (basic assumption in Saussure's semiotic, 1916). We refer to "systems of practices", in some communicative acts, while at other times we refer to the constituent elements of such systems. Even ideas or abstractions can symbolize other ideas, in consonance with Peirce's semiotics (Eco, 1979).

In the following we summarize this ontological model, which is proposed as an analytical and explanatory tool for mathematical knowledge and includes six types of primary entities and five dual facets from which those primary entities can be considered.

Primary entities:

(1) Language (terms, expressions, notations, graphics);
(2) Situations (problems, extra or intra-mathematical applications, exercises, etc.);
(3) Subjects' actions when solving mathematical tasks (operations, algorithms, techniques, procedures);
(4) Concepts, given by their definitions or descriptions (number, point, straight line, mean, function, etc.);
(5) Properties or attributes, which usually are given as statements or propositions;
(6) Arguments used to validate and explain the propositions (deductive, inductive, etc.).

These six types of objects are primary constituents of more complex mathematical objects or organizations, such as conceptual systems or theories. We conceive them as functional entities, in such a way that the distinction between concept, property, action, argument is not absolute but relative to the language game: "The objects of mathematics are taken to exist only within systems of thought and culture. They are semiotic objects brought into being by conversation rooted in forms of life" (Ernest, 1998, p. 255).

Language is the focus of attention for didactics, without forgetting mathematical activity and the non-linguistic cultural objects emerging therefrom. The systems of operative and discursive practices identified in a previous article (Godino and Batanero, 1998) are still considered the essential theoretical objects of didactical analysis. Concepts are entities defined by mathematical discourse, whereas properties/propositions should be proven or justified (Brown, 1998). Linguistic entities play a representational and instrumental role. Even when part of mathematical activity is mental, it would be hard to perform mathematical work without the resource of writing and other material registers. Mathematical problems promote and contextualize mathematical activity and constitute its practical component, together with actions (algorithms, procedures). Discursive elements (concepts, properties, arguments) describe, generalize and justify the problem solutions. The possibility that each of these types of primary entities can be broken down into other subtypes is open, as suggested in the enumeration given when describing them: language (terms, expressions, notations, etc.), situations (problems, exercises, tasks, etc.), etc.

The ontological model is complemented with five cognitive facets from which the previous entities can be considered. Depending on the contextual circumstances and the language game, mathematical entities can be analyzed from the dual facets: *personal – institutional*, *ostensive – non-ostensive*, *example – type*, *elemental – systemic*, *expression – content*. This is certainly a complex model, but it is also a powerful descriptive and explanatory tool for analysing mathematical cognition and the semiosis processes.

Since our approach to mathematical cognition is semiotic, we should describe knowledge (or lack of knowledge) in semiotic terms. For us, a semiotic conflict is any discordance, disparity or mismatch between the meanings attributed to the same expression by two different subjects (people or institutions) in the interactive communication. Semiotic conflicts are identified not by the student himself, but by the lecturer or researcher who can compare the meanings with a reference point. We try to give an alternative explanation of students' misunderstanding and lack of mathematical competence in terms of potential semiotic conflicts. Our ontology is open to include new entities and facets, whenever they help to identify the meanings involved in mathematical practice and discourse and explain the origin of semiotic conflicts in didactical interaction.

In the following sections we first describe our onto-semiotic model using elementary combinatorics as an example and later apply the model to analyze the university students' processes in solving a sample of combinatoric problems. As a consequence we show the cognitive complexity of the elementary combinatorics and explain the difficulty of tasks by highlighting semiotic conflicts and incorrect application of elements of meaning that are not usually taken into account in the teaching of combinatorics.

2. Mathematical objects and elements of meaning: the case of elementary combinatorics

In the following we consider a mathematical object as anything that can be used, suggested or pointed to when doing, communicating or learning mathematics (Blumer, 1969). Below we analyze the different types of objects involved in mathematical activity for the specific case of elementary combinatorics.

2.1. *Combinatorial problems*

These are those problems and applications in mathematics or other areas that induce combinatorial activity, and from which combinatorial concepts have emerged. One example is given below:

TABLE I
Different possibilities in the selection model

	Ordered sample	Non-ordered sample
Replacement	$AR_{m,n}$	$CR_{m,n}$
No replacement	$A_{m,n}$	$C_{m,n}$

Problem 1: In a box there are four numbered marbles (with the digits 2, 4, 7, 9). We choose one of the marbles and note down its number. Then we put the marble back into the box. We repeat the process until we form a three-digit number. How many different three-digit numbers is it possible to obtain? For example, we might obtain the number 222.

Problem 1 is a typical example of a wider class of *selection* problems, where a set of m (usually distinct) objects is considered, from which a sample of n elements must be drawn. The keyword "choose", included in the problem statement, suggests to the student the idea of sampling marbles from a box. Other key verbs that usually refer to the idea of sampling are "select" "take", "draw", "gather", "pick", etc. We might substitute the marbles with people or objects.

In selecting a sample, sometimes it is permitted to repeat one or more elements in the sample, as in problem 1 and on other occasions this is not possible. According to this possibility and whether the order in which the sample taken is relevant or not, we obtain the four basic combinatorial operations shown in Table I: $AR_{m,n}$ (arrangements with repetition of m elements, taken n at a time), $A_{m,n}$ (arrangements of m elements, taken n at a time), $CR_{m,n}$ (combinations with repetition of m elements, taken n at a time) and $C_{m,n}$ (combinations of m elements, taken n at a time). We should also note that the permutation $A_{n,n} = P_n$ is a particular case of arrangement.

A second type of problem refers to the *distribution* of a set of n objects into m cells, such as in the following problem, in which each of the three identical cards must be introduced (placed) into one of four different envelopes.

Problem 2: Supposing we have three identical letters, we want to place them into four different colored envelopes: yellow, blue, red and green. It is only possible to introduce one letter into each different envelope. In how many ways can the three identical letters be placed into the four different envelopes? For example, we could introduce one letter into the yellow envelope, another into the blue envelope and the last one into the green envelope.

In this case, two different groups of objects intervene (letters and envelopes); therefore it is not easy to apply the rule "ordered/not ordered" to solve the problem. In the example the letters are identical and cannot be ordered, although we can order the group of envelopes.

Other key verbs that could be interpreted in the distribution model are "place", "introduce", "assign", "store", etc. The solution to this problem is $C_{4,3}$, but there are many different possibilities in this model, depending on the following features:

- Whether the objects to be distributed are identical (as in this problem) or not;
- Whether the containers are identical or not, as in the example;
- Whether the containers are ordered or not (in case of different containers);
- Whether we must order the objects placed into the containers; this makes no sense in problem 2 since the objects are identical;
- The conditions that you add to the distribution, such as the maximum number of objects in each cell, or the possibility of having empty cells and so on. In the problem proposed you may only introduce one letter into each envelope and there is one envelope left empty, but these conditions could be changed.

Assigning n objects to m cells is equivalent, from a mathematical point of view, to establishing an application from the set of n objects to the set of m cells. For injective applications we obtain the arrangements; in the case of a bijection we obtain the permutations. Nevertheless, there is no direct definition for the combinations using the idea of application. Moreover, if we consider a non-injective application, we could obtain a problem for which the solution is not a basic combinatorial operation, so there is not a different combinatorial operation for each different possible distribution. For example, if we consider the non-ordered distribution of n different objects into m identical cells, we obtain the second kind Stirling numbers $S_{n,m}$. Consequently, it is not possible to translate each distribution problem into a sampling problem. The reader may find a comprehensive study of Stirling numbers in Grimaldi (1989) and for the different possibilities in the distribution model in Dubois (1984).

Finally, we might also be interested in splitting a set of n objects into m subsets, that is, performing a *partition* of the set, as in problem 3.

Problem 3: A boy has four different colored cars (black, orange, white and green) and he decides to give out the cars to his friends Fernando, Luis and Teresa. In how many different ways can he distribute the cars? For example, he could give all the cars to Luis.

We could visualize the distribution of n objects into m cells as the partition of a set of n elements into m subsets (the cells). Consequently, there is a bijective correspondence between the models of partition and distribution considered by Dubois (1984), although for the pupils this might be not evident. Therefore, we cannot assume that the three types of problems described (selections, distributions and partitions) are equivalent in difficulty, though they may correspond to the same combinatorial operation.

Finally, in compound combinatorial problems, one or more simple problems are combined by the product or addition rules, as in the following case, where distribution and selection models are combined in a problem.

Problem 4: A boy has twelve playing cards: nine of them are numbered with the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9. The three remaining are the figures: jack, queen and king. In how many different ways can the boy arrange four of these cards in a row, with the condition that the three figures are always selected? Example: jack, queen, king, 1.

We note that all the problems to be analyzed in this paper are *counting* problems, where we are asked to count the number of combinatorial configurations with a given structure. We will not analyze existence, enumeration, classification and optimization combinatorial problems. These problems are described in detail elsewhere (Batanero et al., 1994).

2.2. *Combinatorial language*

When solving the above problems, or when describing them to another person, we use terms, expressions, notations, and graphics. For example, in the previous section we used the words ‘selection’, ‘partition’, ‘repetition’, ‘combinations’, ‘permutations’, ‘arrangements’, etc.

We represent abstract objects (combinations, combinations with repetition) and concrete situations (groups of two elements in a set with four elements) with symbolic notation, such as $C_{4,2}$, $CR_{m,n}$. We also use these symbols to represent both specific or variable numeric values, e.g., we can represent the number 6 by $C_{4,2}$. Later on, these notations will serve to operate with the represented quantities and variables and therefore, symbols play a representational and instrumental role.

Other useful representations are tabular arrangements (Pascal’s combinatorial triangle is a typical example), tree and Venn diagrams, figurative and iconic elements. In Figures 1 and 2 we reproduce the solutions given by a student (Pedro, case 1) to problems 1 and 2 in Roa’s research, which were incorrect, since the use of figurative elements was unproductive. In problem 1 (Figure 1), Pedro lacked the systematic enumeration ability to complete the tree diagram. In problem 2 (Figure 2), the representation used did not take into account the fact that the letters were indistinguishable.

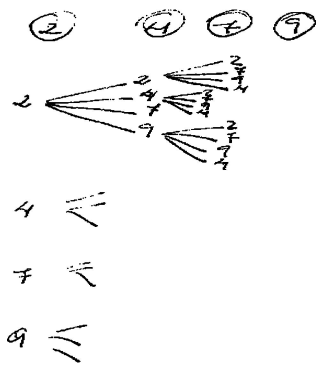

Figure 1. Pedro's solution to problem 1.

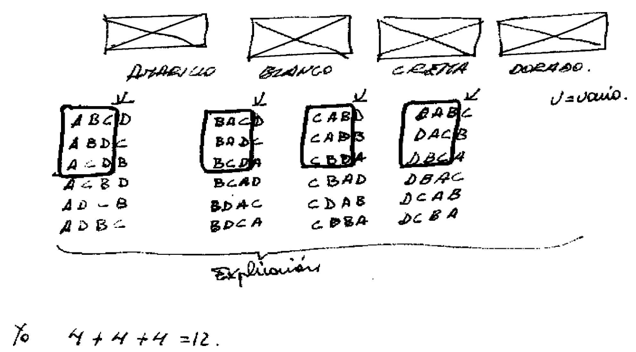

Figure 2. Pedro's solution to problem 2.

In the above examples, linguistic elements are given in written or graphic form within a text, although we also use oral and gestural registers in mathematical work; for example, a deaf student might use sign language.

2.3. *Subjects' actions when confronted with mathematical tasks*

When faced with a mathematical problem, subjects need to carry out diverse actions (algorithms, techniques, computations, etc.) to obtain the solution. Let us consider problem 3, which was the most difficult problem in Roa's research (only 9.4% of students provided the correct solution).

In the problem statement, it is easy to identify the partition scheme (distributing the group of four different cars into one or several subsets, to give them to one, two or three of the children, which are also distinguishable). There are no additional conditions about the distribution of cars. We therefore have to divide a group of 4 distinguishable elements (the cars) into 3 distinguishable sets (the children), one of which at least is not empty, and the union of them should produce the original set. The order of the elements within each subset of the partition is not considered.

In Spanish schools and textbooks combinatorial operations are defined by means of the selection scheme, using Table I. Students in our research were familiar with these definitions. Therefore, we expect them to carry out the following actions to solve problem 3:

- *Translating* the problem statement (*partition model*) into an equivalent problem in the *selection scheme*, where it is easier for students to apply the combinatorial formulae. In doing this translation, students need to exchange the parameters m and n.
- *Identifying the sampling conditions:* this is a sampling with a replacement situation, since the same child can receive more than a car; the selection order is relevant, as it influences the result: the first child chosen receives the black car; the second the orange car and so on.
- *Recognizing* the conditions in which it is possible to apply the concept of "arrangements with repetition of 3 objects taken 4 at a time."
- *Remembering and operating with* the formula $R_{3,4} = 3^4$.
- Carrying out the arithmetic operations $3^4 = 3 \times 3 \times 3 \times 3 = 81$.

This is one possible way to solve the problem. Those students who have studied advanced combinatorics might directly apply the partition scheme and find a direct formula. Students without any previous instruction or students who forget the formula might apply a direct combinatorial or arithmetic reasoning, or carry out a systematic enumeration of the different possibilities.

2.4. *Combinatorial concepts*

In the previous description of mathematical activity, it is easy to note that students should not just carry out actions on the symbols or material objects they operate with, but that they also need to evoke the definitions of different mathematical objects, such as combinatorial configuration, partition, selection, arrangement, group, subset, parameters, permutations, combinations, etc. These concepts emerge as a result of people carrying out some types of actions (e.g., ordering a group of objects, selection samples, doing partitions) and reflecting about these actions.

2.5. *Properties or attributes*

Concepts are specified by different *properties* and attributes that refer to the conditions for carrying out the actions, the specific characteristics of the situations and the relationships between objects. For example, the selection of samples may be with or without replacement, the samples may be ordered or not; combinations are obtained as the quotient between arrangements and permutations.

2.6. *Arguments*

Finally, all these actions and objects are linked by arguments or reasoning that are used to check the problem solutions and to explain these solutions to others. These arguments are not limited to deductive proofs. For example, a student might carry out a partial or total enumeration of all the ways in which it is possible to distribute the cars in problem 4 to validate the solution obtained by a combinatorial formula.

The six types of objects described intervene in mathematical activity as the basic primary constituents for other more complex mathematical objects, such as conceptual systems or theories.

Mathematical problems promote and contextualize mathematical activity, and together with actions, constitute the praxemic or phenomenological component of mathematics (*praxis*) as proposed by Chevallard (1997). The three remaining components (concept-definitions, properties, arguments) are produced by reflective practice, and constitute the theoretical or discursive component (*logos*).

This grouping of mathematical entities in praxis and logos does not mean their mutual independence. Language is an intrinsic element of both praxis and logos; the logos is justified by praxis, and the latter is developed and governed by logos.

3. Facets of Mathematical Knowledge

In addition to the different types of mathematical entities described in the previous section, we consider different dual facets, when analyzing didactical processes.

3.1. *Personal – institutional*

In each category above we may refer to a personal or institutional object. Problems, language, actions, concepts, properties and arguments might be idiosyncratic to a particular person or be shared inside an institution. For example, we might be interested in the personal point of view when analyzing the students' answers to problems 1 and 4. When preparing the tests to evaluate the students' knowledge, when analyzing textbooks or curricular documents, we focus on the institutional point of view, which is the reference to understand and evaluate the teaching and learning process.

400 040 004

301 130 013 310 031 103

220 022 202

121 211 112

Figure 3. Pedro's solution to problem 3.

From the institutional point of view, the cars to be distributed in problem 3 are distinguishable, and this fact conditions the problem solution, and therefore, the combinatorial operation that solves the problem. This interpretation was not obvious, however, for the students in Roa (2000) who found problem 3 to be very difficult. To explain the difficulty of this and other problems, Roa carried out a semiotic analysis of the students' written answers, which was complemented with interviews. Below, we analyze Pedro's solution to problem 3. Pedro's personal meaning for this problem does not coincide with the foreseen institutional meaning.

Pedro found great difficulty in combinatorial problems (he only solved correctly 5 out of the 13 problems in the questionnaire, in spite of having studied combinatorics both at high school and at university – as part of a course on Probability and Statistics). In the interview, he was unsure about the definitions of combinatorial operations, and confused their formulas, being unable to apply them to solve the problems. In general, he tried to solve the problems by total or partial enumeration of the configurations. We reproduce and analyze his solution to problem 3 in Figure 3.

Pedro introduces a notation to indicate the number of cars that each child can receive, but he does not take into account which car each child receives. He was unable to identify the configurations to be counted. Although he does not specify this explicitly, the groups of three digits used to represent the number of cars that each child receives suggest that he is implicitly using the partition scheme suggested in the problem statement.

Pedro correctly produces all the ordered decompositions of number 4 in three addends. However, he considered that the objects to be distributed were indistinguishable and this is an incorrect interpretation, from the institutional point of view, as is shown in the following extract of his interview:

Researcher: Did you understand the problem statement?
P: Yes, there are four cars and it is necessary to distribute them to three children. You can give as many of these cars as you prefer to each child.
Researcher: What do you need to find out?
P: The number of different ways in which I can distribute the cars to the children.

Researcher: Are there some superfluous data?
P: The colors, for example.
I: The colors? Could we omit the fact that the cars have different colors?
P: Yes, this does not change the solution.

3.2. *Elementary/systemic*

In the study of combinatorics, the concepts of arrangements, combinations and permutations are compound entities, with a given structure. Therefore, we can consider these concepts as systems. For example, we study the relationships between combinations and arrangements without replacement, the properties of combinatorial numbers, the generation of these numbers from the Pascal's triangle, and their relationship to the coefficients in the binomial power.

In other cases, we just consider the concept as a unitary entity, for example, in the expressions "the combinations of 4 elements taken 2 at a time." This elementary – systemic (or unitary-compound) distinction is also applicable to the other types of elements.

3.3. *Ostensive/non-ostensive*

Any object has an ostensive facet, which is perceptible and another non-ostensive facet. Usually the ostensive facet of concepts, properties, problems, arguments and actions is given by language, which is just one way to express the non-ostensive objects, and also a tool for its constitution and development. In principle, linguistic objects are directly perceptible and are ostensive (writing, sound and gesture). Nonetheless, from a personal point of view, linguistic objects can be imagined: for example, we can think about the word "permutation" or the notation "$n!$" and such mental objects constitute the non-ostensive facet of linguistic elements. This paradox is solved by assuming that linguistic objects (the various language registers) are primary functional entities, and then can be considered to be ostensive or non-ostensive from both the personal and institutional points of view.

In the previous example, we assume that Pedro used the partition scheme, because this is suggested in the layout of the symbols he used in the enumeration. For example, we assume that Pedro is imagining all the different partitions of the four cars among the three children, with the condition that all the cars are given to the same child (Fernando, Luis or Teresa), when he writes 400 040 004 in the first line. We also assume that 400 indicates that Fernando is the one receiving all the cars. Identical interpretation may be given to the remaining enumeration presented by Pedro. While the student uses the numerals 0 and 4 – perceptible notation – to refer to the corresponding numbers (non-perceptible concepts), in this example he also uses other concepts (partition) and phenomenological objects (children, cars) that are not directly perceptible in his answer.

It is typical of mathematical activity to operate with both ostensive and non-ostensive objects and, in particular, language is the ostensive facet of mathematical objects. Praxemic and discursive entities are intrinsically different from language, although they need language for their constitution and operation. On the other hand, even when linguistic entities are mainly ostensive, a person might just imagine or think about a linguistic object without making it perceptible to others.

3.4. *Example – type*

The example/type distinction is classical in language theory. We use this distinction here to propose a linguistic interpretation of the concrete/abstract duality, which is frequent in mathematical work, in which it is applied not just to conceptual objects but to any of the six different types of primary entities (as well as to the secondary types). This notion might be used to describe the mathematical tendency to generalize and to explain some conflicts in the teaching and learning of mathematics, as confusion between example and type. In analysing mathematical activity or a particular mathematical study process, we should specify in each circumstance whether we refer to a concrete object (which intervenes by itself) or to the said object as a representative of a wider class, that is, an example of a given type.

In the study of mathematics we are always interested in generalizing the problems, the solutions found and the discourse we use to describe and organize them. We are not interested in solving isolated problems but rather we solve types of problems and develop general techniques. Such solutions are organized and justified in progressively more global structures.

Let us consider another student's solution (Adolfo, case 2) to problem 4 that we have split into analysis units. This student did not remember the formulas or the definitions of combinatorial operations (in units U1, U6 he refers to permutations as combinations, that is, this students' language does not match the institutional language). This student however is able to provide a correct solution (he was able to correctly solve 12 out of the 13 problems, although he never used the combinatorial formulas).

Adolfo uses actions such as enumeration (U2), arguments, such as generalization (U5), mathematical properties, such as the product rule (U4, U7) and a variety of expressive tools, including numbers, words and symbols. We also observe the coexistence of ostensive and non-ostensive entities (such as some of the mathematical properties he uses, and the pack of cards that are supposed to be arranged).

The student has obtained a concrete solution to problem 4. Moreover, once he understood the process for reaching the solution, he was able to generalize it to other situations, as we can see in the following transcript:

Researcher: How would you solve the problem if there were many more cards, for example thirty different numbers and the four figures, and you had to arrange them in groups of seven, instead of four?

A: Just as I did it here. The only difference is that there are thirty different numbers and there would be seven cards in the groups. This is not a problem, since I would write the combinations of the four figures, and would put three more numbers between them in the different positions. The solution would be similar and, if I remembered the formulas, I would give you a quick solution.

As shown in this example, in mathematical activity we sometimes refer to concrete objects (something that is interesting in itself, like Adolfo's solution to problem 4). On other occasions (like in the interview), we consider this object to be a representative of a wider class of objects. This distinction between concrete and abstract, that is, between an example (something determined in itself) and the type (a class or group of objects) is essentially relative to the language game.

3.5. *Expression – content (signifier – meaning)*

Mathematical activity is essentially relational. The different objects described are not isolated, but rather they are related in the mathematical language and activity by means of semiotic functions.

According to Hjemslev (1943/1971), there is a semiotic function when a person or institution establishes a correspondence between an antecedent (expression, significant) and a consequent (content or meaning), according to some correspondence criteria. These criteria may be habits or agreements about the terms (functive) that should be put in correspondence in certain circumstances.

We add the aforementioned mathematical ontology to the notion of semiotic function and then postulate that each of the entities considered can play the role of expression or content. Besides, the dependence relationships between expression and content can be representational (an object is replaced by another), operative (an object is using other objects as tools), or cooperative (two or more objects compose a new system from which a new object emerges). Therefore, the semiotics proposed radically generalizes the notion of representations, as currently used in mathematics education and cognitive research. The objects put into correspondence may be personal (mental) or institutional (this includes Saussure's semiotics);

U1 We need all the possible combinations with jack, queen and king and the order is relevant

U2 jack, queen, king; queen, jack, king; king, jack, queen

jack, king, queen; queen, king, jack; king, queen, jack

U3 There are 6 possibilities

U4 If I put number 1 in the 1st position, for each of the 6 combinations above, I get 6 different arrangements

U5 I can do the same putting number 1 in second, third, fourth positions. There are in total 24 possible arrangements with 1, jack, queen and king.

U6 Since we have got 9 different numbers, in the same way I can produce 24 different possible arrangements for each of these numbers

U7 Therefore, there are a total of 9.24 = 216 different arrangements.

Figure 4. Adolfo's solution to problem 4.

they may be linguistic entities, which may participate as exemplary ones or types; they may be particular representations or registers (this includes Duval's theory).

For example, in the transcription of Adolfo's solution to problem 4 (Figure 4), we can identify examples of semiotic functions, where different entities perform the expression and content roles.

U1: the word '*combinations*' (linguistic element) refers to the different possible arrangements of the three given cards (arrangement concept). The words *jack*, *queen*, *king* refer to physical objects, in this case, the cards to be arranged in the problem.

U2: The student has produced an enumeration of all the permutations of the words *jack*, *queen* and *king*. Each one of these permutations of the three words makes reference to a permutation of the three real cards; that is to say one problem (producing all the permutations of three words) is put into correspondence with another (permutations of three physical objects). Equally the group of all the permutations of three words is put into correspondence with the group of all the permutations of the three objects. The student's action (writing down the enumeration) makes reference to another action (carrying out the permutations physically).

Throughout the interview, the student describes the solving process he would perform in the event of increasing the number of letters. This solution refers to the real solution that the student would carry out in the new problem.

In these examples, we may see that any of the diverse entities described in the previous sections can play the expression or content role (signifier and meaning) in semiotic functions, e.g., when we speak of "proving the relationship between combinatorial numbers and the binomial power", the verbal expression refers to an argument.

On the other hand, the relationships between expressions and contents can be representational, instrumental and cooperative. In some cases, one object replaces another, such as in the words "jack" "queen" "king" in U2, and the relationship is representational. On other occasions, an object uses others as instruments, such as in U7, where the student carries out operations with the symbols in the expression $9.24 = 216$, instead of operating directly with the objects. Instead of really repeating a given number of objects a certain number of times, this action is replaced by the algorithm for multiplication, which is automatically carried out.

In unit U2, each group of the three words "jack, queen, king" is used in a componential or cooperative way. Here, two or more objects (words) compose a system from which new objects (permutations of three words) emerge.

3.6. *Synthesis of the model*

As we have shown in the previous paragraphs, our model (summarized in Figure 5) generalizes the notion of *representation*, which is receiving great attention in mathematics education cognitive research.

In our model we interpret knowledge and understanding of any mathematical object O by a subject X (a person or an institution) in terms of the semiotic functions that X is able to establish as regards O. Each semiotic function constitutes a *knowledge* (we speak of a *meaning*).

Beyond procedural (techniques) and conceptual knowledge (concepts and propositions), we also consider phenomenological (problems, tasks), linguistic – notational (language, representations) and argumentative (arguments) knowledge. We introduce a variety of types of knowledge, in correspondence with the diversity of semiotic functions.

On the other hand, we distinguish personal from institutional knowledge. "Personal knowledge" emerges from an individual's thinking and actions when faced with a class of problems, while "institutional knowledge" is the result of agreement and regulation within a group of individuals. The study of the complex dialectical relationships established between personal and institutional knowledge is essential to mathematics education.

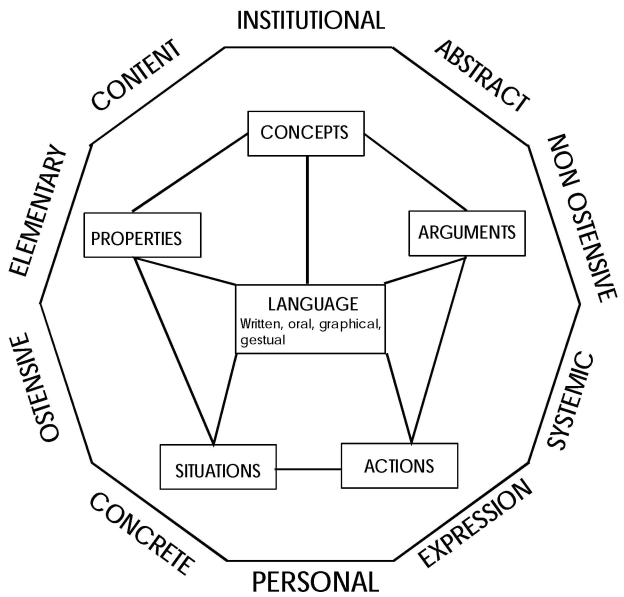

Figure 5. Components and facets of mathematical knowledge.

In our previous works (Godino and Batanero, 1994, 1998), we defined a meaningful practice for a person (or an institution), as that which is carried out to solve the problem, to communicate, validate or generalize its solution. We considered the meaning of an object as the system of meaningful practices related to that object. That notion is widened and specified in the present work, in which we also postulate a typology for mathematical objects and facets.

4. An application to analyze the process for solving a combinatorial problem

As a result of the previous sections, assessing understanding should be considered as the study of the correspondence between personal and institutional meanings. In the following, we introduce an analytical technique based on the ideas of semiotic function and the types of objects and facets described. We also apply this technique to analyze some students' solutions to problem 3 in Roa (2000).

4.1. *Adolfo's case*

First, we analyze Adolfo's correct solution (the student analyzed previously as case 2) to this problem that is transcribed in Figure 6. Adolfo interpreted

U1 Case 1

a) Fernando receives all 4 cars.

b) Luis receives all 4 cars.

c) Teresa receives all 4 cars.

U2 *Case 2: Distribute all 4 cars to 2 children. These are the possibilities:*

A) a) Fernando and Luis b) Fernando and Luis c) Fernando and Luis

(3 cars) (1 c.) (2 cars) (2 c.) (1 car) (3 c.)

ABV R AB VR

ABR V AV BR there are 4 ways

ARV B AR VB (same as a))

BRV A BV AR

BR AV

BR AB

there are 4 ways there are 6 ways

U3 *Therefore, there are 14 different ways to distribute them between Fernando and Luis.*

U4 *B) In the same way, there are 14 different ways to distribute them between Fernando and Teresa.*

U5 *C) In the same way, there are 14 different ways to distribute them between Luis and*

U6 *Teresa.*

Case 3: All the children receive at least one car.

A) a)Fernando 2, Luis 1, Teresa 1 b)Fernando 2, Luis 1, Teresa 1

AB V R BV A R

AB R V BV R A

AV B R BR A V

AV R B BR V A

AR B V BR A B

AR V B RV B A

U7 *Therefore, there are 12 different possibilities in case A.*

U8 *In the same way, case B) Luis 2, Fernando 1, Teresa 1, have 12 different possibilities*

U9 *In the same way, case C) Teresa 2, Fernando 1, Luis 1, have 12 different possibilities*

When adding all the options, I got:

Case 1 + case 2 + case 3 = 3 + (14+14+14) + (12+12+12) = 81 different ways

Figure 6. Adolfo's solution to problem 3.

the problem as the partition of a set (the cars to be distributed) into subsets (groups of cars each child receives), without additional conditions. That is to say, he uses the scheme suggested in the problem statement, without trying to make a translation to the selection scheme. He identifies all the problem data correctly.

In his solution Adolfo uses the diverse types of objects described. Adolfo needs mathematical language to represent the problem data (the numeral 4 to represent the number of objects to be distributed, letters to represent the objects), the actions he carries out (the words *gives* and *to be distributed*, the sum symbol), and even the results of such actions (tabular arrangement to represent the possible distributions). He also needs language to operate with the objects; for example, to produce each partition in U2.

Adolfo carries out actions to solve the problem: he applies algorithms (such as systematic enumeration) and strategies (such as analyzing all the different partitions of number 4). In each step he set some problem variables. For example, in U2 he sets the number of cars that each one of the two children receives, which may be 3 and 1, 2 and 2 or 1 and 3. For each one of these cases he solves a related subproblem by means of recursion (which is used to produce the permutation of the four objects to be distributed). In each subproblem the technique of systematic enumeration is applied, supported by a symbolic notation to represent the cars and a tabular display to prove systematicity.

Throughout the solving process, the student poses new problems related to the initial problem, although less complex, such as finding the different forms of distributing the cars to the same child. In steps U1, U2 and U6 Adolfo uses the strategy of breaking down the problem into simpler subproblems: distributing all 4 cars to the same child (case 1), distributing them between two children (case 2) and among all three (case 3).

Adolfo applies properties: in step U9 Adolfo recognizes and applies the sum rule. Finally he is able to produce arguments to validate his solution. In step U6 he generalizes the number of ways to distribute the cars so that one child receives 1 car and the other 3, which is independent of the particular children. Generalization is also used in steps U4 and U5, as well as in steps U7 and U8.

We noted the complexity of Adolfo's solution, as compared to applying the variations with repetition formula directly. We also highlight Adolfo's ability to divide the problem into subproblems, to enumerate and in using recursion.

In addition to words and symbols (ostensive entities), we identify in Adolfo's solution a variety of non-ostensive objects that are evoked by semiotic functions, both elementary (the word *Fernando* makes reference to an imaginary child) and systemic (in the expression "case A has 12 possibilities," he refers to a group of partitions). In this example, the objects are used in a concrete way, it is a specific partition, a concrete number of cars and children to be distributed. However, the student is able to consider this concrete example as a prototype for an abstract type of problem:

Researcher: How would you solve this problem, if instead of four cars and three children, we had thirty cars and seven children?
A: I would apply the formula.
Researcher: What would you do if you did not remember the formula?
A: I would repeat the process. I would try to divide it into smaller problems as I did here. It is the same type of problem, then I would try to find all the cases. However, it would be more complicated because the number of possibilities is greater.

We have just analyzed Pedro's and Adolfo's solutions from a global point of view, since we only tried to show the diversity of objects they used in their solutions. However, a more detailed analysis would show how these objects are related by semiotic functions, and would reveal semiotic conflicts, where the teacher and the student give different meanings attributed to the same expression.

One such semiotic conflict appears between the meaning we gave to problem 3 and the meaning assigned by Pedro (case 1) to that problem in his solution. These types of conflicts may appear in the linguistic interaction between people or institutions, and they frequently explain the difficulties and limitations of teaching and learning mathematics. Below, we will apply this analytical technique to the solutions given to problem 4 by two other students.

4.2. *Luisa's case*

In Figure 7 we transcribe Luisa's solution (case 3). This girl had studied Combinatorics at high school, as well as in the first course at University, as a part of a course on Statistics. She remembered the combinatorial formulas, although it was hard for her to find the solution to the problems, since the combinatorial scheme that was used in the definitions of the combinatorial operations in her studies was that of selection. She had not studied the distribution scheme or the applications that were related to combinatorial operations in her studies.

Luisa correctly solved 12 out of the 13 problems, by directly applying the definition of the combinatorial operations she learnt and by translating the problem statement when needed to the selection scheme. In fact, she only failed to solve problem 4 correctly, for which she was unable to make an appropriate translation of the problem that allowed her to identify the

U1	B, O, W, G
U2	*We should take into account that the cars are distinguishable.*
U3	*Suppose only one child receives all 4 cars: 4 possibilities*
	Suppose one child receives 3 cars: 4. C_4^3
	Suppose we give one child 2 cars and the other 2 cars to another 2: C_4^2 *. 4 . 3.*
	One child receives 2 cars and two other children receive one car each: C_4^2 *. 4 . 3.* C_3^2
U4	*Each child receives one car:* P_4
U5	*We add up all the possibilities.*
	F L T 4 0 0 3 1 0 3 0 1 2 1 1 2 2 0 2 0 2 1 3 0 1 2 1 1 1 2 1 0 3 0 4 0 0 0 4 0 3 1 0 2 2 0 1 3

Figure 7. Luisa's solution to problem 3.

combinatorial operation. In this case we will analyze each step in her solution to show the points at which she needed interpretative processes and semiotic conflicts took place that presumably originated an erroneous final solution.

U1: Luisa begins to interpret the data in the problem statement and she introduces a symbolic notation (language) to designate the four cars to be distributed (problem situation). There is a correspondence between each letter and each verbal expression of the objects to be distributed in the problem, as well as between them and the real cars. For example, the letter B (language) is used in the representation of the expression 'black car' (language) and this expression is used in the representation of a real physical object.

U2: Luisa continues to interpret the statement and she recognizes that the objects to be distributed are distinguishable (a property). This property is not explicit in the problem statement, and the student should recognize it by considering the objects' color, which is important in this problem, because it induces an order (property) in the group of objects (definition of ordered and non-ordered set). This affects the problem solution.

U3: To start solving the problem, Luisa breaks it down into new subproblems that should be correctly identified (new problem situations). Each of them corresponds to a possible partition of number 4 into addends. Therefore, she was able to formulate and solve a new auxiliary problem, which consists of the enumeration of all the possible partitions of the number 4, which she did by using systematic enumeration (action). We observe that Luisa did not translate the problem statement, but rather she used the partition scheme (concept) suggested in the problem statement directly.

She tried to solve each problem using the formulas she already knew, although she addressed some problems by interpreting the problem data. Below, we analyze her reasoning.

"*Suppose only one child receives all 4 cars: 4 possibilities*." We observe she is confusing the number of children (3), with the number of cars to be distributed (4) since the number of possibilities of giving all the cars to the same boy is only 3. A first semiotic conflict has taken place when interpreting the problem data, which will lead to an erroneous solution. We also observe the complexity of the partition problem, where two different sets intervene: the set of cars and the set of children. Each of these sets may be formed by distinguishable or indistinguishable objects, whether ordered or not. All these data, as well as the number of objects in each group, are only implicit in the problem statement and require interpretive processes from the student.

We will not take into account this first error and will continue our analysis considering that in fact there are four children.

"*Suppose one child receives 3 cars: 4.* C_4^3." Luisa suggests it is necessary to select (action) the boy to which one car will be given, giving the rest to another of the children (4 cases according to the interpretation given). She uses linguistic elements 'one', 'car', '4', '3' that refer to a boy, the cars and also to concepts (the numbers 1, 3, 4). We also observe that Luisa suggests the different forms of selecting 3 cars among the four (a new action, and a symbolic notation C_4^3), which in this case has a systemic meaning, because she refers both to a differentiated group of possibilities, the action of distributing the cars, and the results of the distribution.

This notation also refers to the concept of combination and its definitions: the idea of combination as the selection of a sample of 3 objects among 4 given objects, without taking the order into account. The student was able to interpret this definition and to apply it to the problematic situation she has posed herself (giving three cars to one of the children and one to another of them). She assigned an appropriate value (action) to the parameters (concept) and also identified and correctly applied (action) the rule of product (property). She also expressed her solution with the help of symbols (language) to represent the combinations and their parameters, as well as the product of combinations by an integer.

"*Suppose we give one child 2 cars and the other 2 cars to another 2:* C_4^2*.4.3*." Luisa's reasoning and her interpretive processes are similar to those in the previous step, and therefore we only summarize them. Luisa again applied the product rule (property) and the idea of combination (definition) C_4^2 assigning appropriate values to the parameters (action and definition) (selecting 2 out of 4 cars); she also applies recursion (action), so there are 4 cases to choose the first boy and there are only 3 to choose the second one. Finally, she correctly applies the property $C_m^n = C_m^{m-n}$. She used linguistic elements to express her solution again.

"*A child receives 2 cars and two other children receive one car each: C_4^2. 4. 3. C_3^2.*" She continues with similar reasoning, although now a semiotic conflict arises. In selecting the two cars that she would give to the first child, she correctly uses the idea of combination C_4^2; the number of possibilities to choose the child (4 cases) is also correct. However, in choosing the two other children to whom we would give the remaining cars, there is confusion in the combinatorial operation C_3^2, which should be the arrangements and not the combinations, since the children are distinguishable. Here, there is an incorrect identification of the relevance of order (concept), which is also necessary to take into account in the group of children, and not just in the group of cars, as the student assumed.

This semiotic conflict has probably been induced by the teaching received by Luisa, since, when learning the definition for combinatorial operations she was taught to use the selection scheme to differentiate between combinations and arrangements. In the selection scheme there is only a reference to one group of objects and therefore only one possible order is relevant, while in the partition scheme two groups of objects, each of which may be ordered or not, are considered. Again the student correctly uses the notation of the combinations that is a linguistic object.

"*Each child receives one car: P_4.*" The student uses the idea of permutation (concept) and her notation, which are correctly applied in the case she interprets that there are 4 children. She uses linguistic elements to refer to concepts, actions and phenomenological elements.

U4: "*We add up all the possibilities*." The student identifies the sum rule (property) and makes a correct application of the same. To do this, she needs to recognize that the previous partitions (concept) are exhaustive and incompatible (properties).

U5: The student tries to check her solution (therefore we can consider this step as an argument). For this purpose, she enumerates (action) the different partitions (the total number of partitions was not finally computed in the previous steps). She uses a new notation and a tabular arrangement to refer to the children and possible partitions of the objects. The student enumerates the different decompositions of the number 4 into addends (concepts); each of them represents a possible partition of the cars to be distributed (result of an action).

The enumeration carried out is not systematic, although it is complete. One semiotic conflict is that this enumeration is not consistent with the interpretation that there are four children, that is, with the previously contributed solution. Another semiotic conflict is that she does not use the fact that the cars are distinguishable that was recognized in unit U2. She just takes into account the number of cars that each child receives.

To sum up, diverse semiotic conflicts have contributed to an erroneous solution of the problem, even when Luisa has shown her good combinatorial ability, when posing new subproblems, identifying concepts and properties (sum and product rule, set, subset, combinations, permutations, selection, partition, order, parameters), using language and symbolic notation for combinations and permutations, linking all of these by arguments which were generally correct. The student's enumeration is not systematic and there were semiotic conflicts in interpreting the number of elements in the two groups, as well as in the need for taking the order into account in the group of children. In spite of not having found the correct solution, this student's combinatorial knowledge and ability is high, which coincides with the fact that she correctly solved the 12 remaining problems.

4.3. *Juan's case*

In Figure 8, finally we present the solution given by Juan (case 4) to problem 3. In the interview this student said he remembered the definitions of combinatorial operations and he tried to solve all the problems using them, although he was only able to solve four problems correctly. When asking him the definitions he had difficulty remembering them and he only gave one correct definition for permutations. Below we analyze his solution.

U1: Juan introduces a double symbolic notation to designate the different elements in the problem, that is, the cars, whose color is represented by letters, and the siblings, who are represented by the abbreviations of their names (language making reference to phenomenological elements in the problem). This notation suggests that Juan interprets the problem statement correctly.

U1	B, O, W, G Fer., Luis, Ter.
U2	*The order and the nature of the elements is relevant (the order is needed to keep in mind which car each child would be given. That is to say, it is different to give one child the green car or the blue car) and, therefore:*
U3	$V_{4,3} = 4!/(4\text{-}3)! = 4!/1! = 24$ *possible ways.*

Figure 8. Juan's solution to problem 3.

U2: Juan tries to solve the problem by directly applying the selection scheme where he studied the combinatorial operations definition, which is consistent with his general technique of trying to directly identify the combinatorial operations to solve the problems. To do this, he first translates the problem statement (action) into a selection scheme (concept). He was the only one of the four students analyzed who performed this translation.

Juan translates the condition of the cars being different (property) to taking the order (concept) into account in selecting the cars (action). This is a correct and nontrivial translation, since the order idea does not appear explicitly in the problem statement. He also identifies correctly the fact that the elements are distinguishable (property).

U3: Juan tries to directly identify the combinatorial operation (concept) that solves the problem, by applying its definition. The identification of the combinatorial operation is incorrect (there is a semiotic conflict with respect to the use of an incorrect definition). The student correctly discriminates the relevance of order (concept) in the definitions of variations and combinations (concepts).

But there is a semiotic conflict in not identifying the possibility of replacement in this problem, which makes arrangements with replacement instead of simple arrangements be the appropriate operation in this case. We should acknowledge that it is not easy to recognize this condition in the problem statement, because the student should translate the condition that the same child can receive more than one car (the cars are never repeated) to a new equivalent condition, that each child may be selected more than once in the distribution (and thus, the children may be "repeated"). Again the fact of dealing with two different groups (children and cars) in each of which it is possible to repeat or not the elements, makes the definition the student knows useless (selection model), that refers to a single group, thus producing semiotic conflicts, when applying this definition in a partition context.

There is also another semiotic conflict in the incorrect identification of the parameters (concept), since the parameters should be exchanged (property) when translating from the partition scheme into that of selection. Juan did not exchange them, that is, he is imagining sampling from the group of cars, instead of sampling from the group of children. He does not identify the population and sample that intervene in the problem (concepts). The problem solution involves selecting 4 out of 3 children with replacement, $VR_{3,4}$, instead of selecting 3 objects out of 4, without replacement $V_{4,3}$ as Juan has erroneously assumed.

Juan tries to develop the formula of variations to do the computations (actions). Here, a potential conflict is that the denominator is not needed in the development of the formula. Its use suggests that the student is, in fact, insecure, as regards the formulas, but he finally obtains a correct number of variations of four elements taking three at a time.

4.4. *Synthesis of knowledge used by the students in solving the problem*

The analysis carried out allows us to identify the knowledge used correctly or incorrectly by the students in solving problem 3, which is summarized in Table II, to show the complexity of solving this apparently simple problem. It also suggests the varied combinatorial ability of these four students that reflects the variety of personal meaning of elementary combinatorics for the same. In solving the problem, semiotic conflicts leading to errors arise, due to the disparity between the selection scheme, where these students learned the definitions for combinatorial operations and the various situations (such as a partition problem) where these definitions should be applied.

Only one student (Juan) directly tried to apply the definition of combinatorial operations he studied in the selection scheme, where he was taught to remember a mnemotechnic rule (taking into account – not taking into account order) to distinguish between arrangements and combinations; (there is or not replacement) to distinguish between the two types of arrangements and the two types of combinations. These rules are unproductive for problems presented in a different scheme (partition or distribution), such as problem 3, if the student is unable to translate the problem to the selection scheme.

In partition and distribution schemes the essential fact for distinguishing between arrangements and combinations is the fact that the objects to be distributed are identical or different, and also repetition is translated by the fact that each group in the partition or arrangement is allowed to have more than one element. This is not intuitive for three of the students analyzed (Luisa, Juan, Pedro).

Since in problem 3, each child (subset of the partition) can receive more than one car (objects to be distributed) the arrangement should be with replacement. In partition and distribution situations, the rules to identify the values of the parameters are also unproductive, since the parameters are exchanged, as regards the selection scheme. This is the reason why Juan was unable to give correct values for the parameters.

Given the impossibility of identifying the combinatorial operations directly, the students have recourse to dividing the problem, and formulate new, related and simpler problems, as well as the problem of partition of the number 3 into addends (Adolfo, Luisa and Pedro). Only one was able to correctly solve all these subproblems, by correctly identifying the conditions in the original problem and in each subproblem. Failures in taking into account that the persons receiving the objects (Pedro) or the objects

TABLE II
Knowledge used by the students in solving problem 3

	Adolfo	Luisa	Pedro	Juan
Problems:				
Posing similar problems with lower size	Correct	Incorrect		
Posing the problem of partition of number 4 into addends	Correct	Correct	Correct	
Linguistic elements:				
Algebraic or numerical symbolizing of elements to count	Correct	Correct	Partial	Correct
Tabular arrangements	Correct	Incorrect	Correct	
Combinatorial notations to refer to combinatorial operations		Correct		Correct
Combinatorial formulas				Correct
Correspondence between the problem data and the parameters of combinatorial formulae		Correct		Incorrect
Expression of addition, product and quotient rules	Correct	Correct		
Expression of arithmetical operations	Correct			
Actions:				
Translation of combinatorial schemes				Incorrect
Developing combinatorial formulae				Correct
Enumeration	Systematic	Non systematic	Systematic	
Setting variable values	Correct			
Recursive solution of combinatorial problems	Correct	Correct		
Carrying out arithmetical operations	Correct			Correct
Definitions and properties:				
Partition scheme	Correct	Correct	Correct	
Conditions given (influence of order, replacement)	Correct	Incorrect	Incorrect	Incorrect
Sampling scheme		Correct		Incorrect

(*Continued on next page*.)

TABLE II
(Continued)

	Adolfo	Luisa	Pedro	Juan
Combinatorial operations (variations, permutations, combinations, arrangements)		Correct		Incorrect
Addition, product and quotient rules	Correct	Correct		
Combinatorial properties		Correct		
Arguments:				
Justifying the model conditions				Incorrect
Enumeration to check the solution			Incorrect	
Generalization	Correct			

to be distributed (Luisa and Pedro) are distinguishable lead to incorrect solutions for problem 3.

When solving the original problem or the subproblems, the students carry out different types of actions. The first one is translating the partition scheme in the problem statement to the selection scheme, that only Pedro incorrectly carries out. The other students recur to enumeration that it is not always systematic as in Luisa's solution.

Adolfo and Luisa are able successfully to set values in the problem variables to state simpler problems. Adolfo correctly solves the series of problems generated using recursion. Luisa tries to solve the intermediate steps with combinatorial formulas, although she does not develop them or compute their value. While Juan develops and computes the value of the combinatorial operation, he made an incorrect identification of the same, and therefore his solution is incorrect.

Depending on the solving procedure, the students use different definitions and properties. Only Adolfo and Juan use concepts related to the selection scheme (Juan incorrectly), while three students use concepts related with the partition scheme. Since only Adolfo correctly identifies the conditions of the scheme given into problem statement, this point will be highly relevant to the solution obtained. Adolfo and Luisa correctly link partial solutions by means of the sum rule and Luisa also uses a property of the combinatorial numbers.

The students need ostensive representations to visualize the concepts and data they are working with, such as algebraic or numeric symbolization of the elements to be combined, according to their nature, use of tabular arrangements, notations for the combinatorial formulas and their parameters, expression of sum, product and quotient rules and arithmetic operations. Here semiotic conflicts also appear.

Finally the students use arguments, such as correct generalization (Adolfo), enumeration (Luisa) and justification of the scheme conditions (Juan); incorrectly in these last two cases.

It is clear from the analysis of these students' protocols that problem solving activity involves a diversity of objects that we have made explicit in our analysis; these objects vary from one student to another, as summarized in Table II. Even when, due to space limitations, we present the analysis of a single problem here, this same process was repeated with the 12 remaining problems in Roa's research. That served to show the plurality of knowledge used by the students in solving combinatorial problems and the diversity of students' personal (idiosyncratic and systemic) meaning for elementary combinatorics.

The notion of *semiotic function* we have introduced, allows us to remember the essentially relational nature of mathematical activity and teaching/learning processes. In the semiotic functions, the correspondence between the expression and the content is fixed by explicit or implicit codes, rules, habits or agreements, about what and how knowledge elements should be put into correspondence, according to different circumstances. An agreement that in our example was made explicit in the teaching is to associate the expression $V_{4,3}$ to the arrangements of 4 elements taken 3 at a time. In the problem analyzed there are also implicit agreements, such as the fact that the partition is exhaustive; this agreement is not explicit in the problem statement, however, it was recognized and applied by all four students.

In the above section, we applied our theoretical model of mathematics cognition to describe the mathematics activity of a sample of university students when solving elementary combinatoric problems. As a result we provided original and relevant information to better understand the students' combinatorics thinking. Our theoretical tools served to identify the variety of mathematical objects involved in combinatorial problem solving, the cognitive dualities from which they can be considered, and the semiotic functions that can be established among them. The students' errors and difficulties were explained by semiotic conflicts, i.e. as disparities between the subject's interpretation and the meaning in the mathematics institution. This interpretation suggests some ways to improve the teaching and learning process and also help to overcome certain "transparency illusions" in the practice of teaching combinatorics and in assessing the students' combinatoric reasoning ability.

5. IMPLICATIONS FOR FURTHER RESEARCH

The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research. We then give an affirmative answer to Ernest's question (1997) about whether semiotics can offer the base for a unified theory of mathematics education (and mathematics), whenever we adopt appropriate semiotics and supplement it with an ontology that take into account the multiple objects involved in mathematical activity.

The theoretical model described in this paper incorporates elements from the pragmatic (operational) and realist (referential) theories of meaning. The meaning of terms and expressions is found in their use in institutional contexts and language games although these do not hinder the possibility of considering prototypical uses that would be denoted with new terms and expressions, and considered as new emergent objects. As suggested by Ullmann (1962, p. 76), researchers should first gather an appropriate sample of contexts and approach them later with an open mind, thus allowing the meaning or meanings to emerge from these contexts. Once this phase has been concluded, we can safely enter into the "referential" phase and try to formulate the meaning or meanings identified in this way. Our meaning begins by being pragmatic, relative to the context, but there are typical uses that allow us to guide mathematical teaching and learning processes. These types are objectified by language and constitute the referents of the institutional lexicon.

The model for mathematics cognition described in this paper results from extended work, starting from 1994 (Godino and Batanero, 1994, 1998, 1999; Godino, 2002). This work was useful as a theoretical framework in various doctoral theses and publications, intended to characterize elementary and systemic meanings involved in teaching and learning mathematics. Likewise, it provided explanations for the difficulties and limitations in mathematical learning based on the nature and complexity of different mathematical objects. At a more theoretical level it allowed confrontation of tools proposed by other theoretical models for mathematical knowledge. Some examples are given below:

(a) In our model we can interpret the notion of *scheme* as the interiorized (non-ostensive) facet of the personal practices system, and *concept-in-act*s, *theorem-in-acts* and *conceptions* (Vergnaud, 1990) as partial components of this system (corresponding to primary entities, concept-rules and properties). All these notions play an important role in cognitive analysis, although the theoretical notion "*system of personal practices*" is an instrument with more descriptive and explanatory possibilities because it includes an organized system of operative, situational, discursive and linguistic components.
(b) The *sense* notion in the Theory of Didactic Situations (Brousseau, 1997) is restricted to the correspondence between a mathematical object and

the class of situations from which it emerges, and "receives its sense" (our "situational meaning"). This correspondence is certainly crucial in our theoretical model since it gives the "reason of being" of such an object, its justification or phenomenological origin. However, we also take into account correspondences or semiotic functions between that object and the other operative and discursive components of the system of practices from which we consider the object emerges. For us, the meaning of a mathematical object is the content of any semiotic function and, therefore, can be an ostensive or non-ostensive, concrete or abstract, personal or institutional object; it can refer to a system of practices, or to a component (situation-problem, a notation, a concept, etc.), depending of the communicative act.

(c) The notions of *representation* (Goldin, 1998) and *semiotic register* (Duval, 1993) refer in our model to particular types of semiotic representational function between ostensive and not ostensive mental objects. The notion of semiotic function generalizes this correspondence to any type of objects and, it also considers other types of dependences among objects (instrumental and componential).

The theoretical notions described in this article are centered in the cognitive dimension (individual and institutional) of mathematics education processes. However, we are aware that it is necessary to extend this theoretical model to include the instructional, affective (beliefs, attitudes and emotions), axiological (values and ends of mathematical education), politics, and curricular dimensions that globally condition the teaching and learning of mathematics. These dimensions should be objects of attention in a unified approach to mathematics education.

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