Complete surfaces with negative extrinsic curvature in $\mathbb{M}^2 \times \mathbb{R}$

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Abstract

We prove that there exist no complete vertical graphs in $\mathbb{M}^2 \times \mathbb{R}$ with extrinsic curvature bounded from above by a negative constant, when $\mathbb{M}^2$ is a Riemannian surface with a pole and non negative curvature.

1 Introduction.

In 1964, Efimov proved that no complete surface can be $\mathcal{C}^2$-immersed in the Euclidean 3-space $\mathbb{R}^3$ if its extrinsic curvature K_{ext} is bounded from above by a negative constant (see [3], [8]). This generalized Hilbert's classical result which stated that no complete smooth immersion exists with negative constant extrinsic curvature in $\mathbb{R}^3$.

Efimov's theorem has been tried to extend in several ways (see [4], [10], [9]). In this sense, not much is known about the behaviour of the extrinsic curvature of a complete surface in product spaces $\mathbb{M}^2 \times \mathbb{R}$ (see [1], [5]).

In this paper we prove that there exist no entire vertical graphs with extrinsic curvature bounded from above by a negative constant in $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a Riemannian surface with a pole and with non negative curvature $K_{\mathbb{M}}$. In our arguments we use Heinz's ideas, who showed this result when $\mathbb{R}^3$ is the ambient space (see [7]).

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As a consequence of a result in [6] we obtain that, if $\mathbb{M}^2$ is a Riemannian surface with $K_{\mathbb{M}} \geq 0$ and with a pole, then every entire vertical graph Σ in $\mathbb{M}^2 \times \mathbb{R}$ satisfies

$$\inf_{\Sigma} |K_{ext}| = 0,$$

where K_{ext} denotes the extrinsic curvature of Σ.

Moreover, we construct examples of the existence of complete surfaces with constant negative extrinsic curvature in certain product spaces $\mathbb{M}^2 \times \mathbb{R}$.

2 Graphs with negative extrinsic curvature.

Let $\mathbb{M}^2$ be a Riemmanian surface and (r, θ) local geodesic polar coordinates around a point $p_0 \in \mathbb{M}^2$ which are well defined for $r < R$, for a certain $R > 0$. The induced metric is given by

$$\langle \cdot, \cdot \rangle = dr^2 + G(r, \theta) d\theta^2.$$

We consider $\mathbb{M}^2 \times \mathbb{R}$ endowed with the product metric and a vertical graph Σ in $\mathbb{M}^2 \times \mathbb{R}$ over the geodesic disk $B(p_0, R)$ centered at p_0 and with radius R, given in local coordinates as

$$\psi(r, \theta) = \Big(exp_{p_0}\big(rcos(\theta), rsin(\theta)\big), z(r, \theta) \Big),$$

which we identify with $(r, \theta, z(r, \theta))$ in order to simplify notation.

We denote by $\{\partial_r, \partial_\theta\}$ and $\{\overline{\partial_r}, \overline{\partial_\theta}\}$ the partial derivatives with respect to r and θ on $\mathbb{M}^2$ and Σ respectively, where, for instance, $\partial_r \equiv \frac{\partial}{\partial r}$ and $\partial_\theta \equiv \frac{\partial}{\partial \theta}$. Then

$$\begin{aligned} \overline{\partial_r} &= \partial_r + z_r \partial_t \\ \overline{\partial_\theta} &= \partial_\theta + z_\theta \partial_t. \end{aligned}$$

Thus, the induced metric on Σ is given by

$$ds^2 = \left(1 + z_r^2\right) dr^2 + 2 z_r z_\theta dr d\theta + \left(G + z_\theta^2\right) d\theta^2$$

and its second fundamental form is given by

$$\begin{aligned} \left\langle \nabla_{\overline{\partial_r}} \overline{\partial_r}, N \right\rangle &= \frac{z_{rr}}{\sqrt{1 + z_r^2 + \frac{z_\theta^2}{G}}} \\ \left\langle \nabla_{\overline{\partial_\theta}} \overline{\partial_r}, N \right\rangle &= \frac{1}{\sqrt{1 + z_r^2 + \frac{z_\theta^2}{G}}} \left(-\frac{z_\theta G_r}{2G} + z_{r\theta} \right) \\ \left\langle \nabla_{\overline{\partial_\theta}} \overline{\partial_\theta}, N \right\rangle &= \frac{1}{\sqrt{1 + z_r^2 + \frac{z_\theta^2}{G}}} \left(\frac{z_r G_r}{2} - \frac{z_\theta G_\theta}{2} + z_{\theta\theta} \right), \end{aligned}$$

where ∇ is the Levi-Civita connection on $\mathbb{M}^2 \times \mathbb{R}$ and

$$N = \frac{-1}{\sqrt{1 + z_r^2 + \frac{z_\theta^2}{G}}} \left(z_r \partial_r + \frac{z_\theta}{G} \partial_\theta - \partial_t \right)$$

is the pointing upwards unit normal vector of the graph.

By using the above formulae, a straightforward computation gives

$$\begin{aligned} &\left(\sqrt{G}\right)_r d\left(\left(z_r^2 + \frac{z_\theta^2}{G}\right) d\theta\right) + d\left(z_r d\left(\frac{z_\theta}{\sqrt{G}}\right) - \frac{z_\theta}{\sqrt{G}} dz_r\right) \\ &= 2\sqrt{G}\left(1 + z_r^2 + \frac{z_\theta^2}{G}\right)^2 K_{ext}\left(dr \wedge d\theta\right), \end{aligned} \tag{1}$$

where K_{ext} denotes the extrinsic curvature of Σ.

Now we define the following auxiliar function which will be useful for our purposes

$$f(r) = \int_{B_r} \sqrt{G}\left(1 + \frac{z_\theta^2}{G}\right), \quad r > 0$$

where B_r denotes the ball centered at the origin of $\mathbb{R}^2$ and radius r, with $r < R$. Here, we are identifying $\mathbb{R}^2$ and the tangent plane of $\mathbb{M}^2$ at p_0, $T_{p_0}\mathbb{M}^2$, in the usual way.

We observe that along this section we will work with functions of type $\int_{B_r} h(\rho, \theta)$, where $h(\rho, \theta)$ is well defined in $B_r \setminus \{(0,0)\}$. However all these functions $h(\rho, \theta)$ can be continuously extended to the origin as it happens for the previous $f(r)$. This is due to the facts that $\lim_{p \to p_0} \sqrt{G}(p) = 0$, $\lim_{p \to p_0} \left(\sqrt{G}\right)_\rho (p) = 1$ (see [2]) and the functions $\left|\frac{z_\theta}{\sqrt{G}}\right|$ and $|z_\rho|$ are bounded in a neighbourhood of the origin.

Lemma 1. *Under these conditions, we have*

$$|B_r| \leq f(r) \leq \sqrt{|B_r|}\left(\int_{B_r} \sqrt{G}\left(1 + z_\rho^2 + \frac{z_\theta^2}{G}\right)^2\right)^{\frac{1}{2}}.$$

where $|B_r|$ denotes the area of the geodesic disk $B(p_0, r)$ in $\mathbb{M}^2$.

Proof. Since $\frac{z_\theta^2}{G} \geq 0$, the first inequality is clear. Moreover, by using the Cauchy-Schwarz inequality, one has

$$f(r) \leq \left(\int_{B_r} \sqrt{G}\right)^{\frac{1}{2}} \left(\int_{B_r} \sqrt{G}\left(1 + z_\rho^2 + \frac{z_\theta^2}{G}\right)^2\right)^{\frac{1}{2}} = \sqrt{|B_r|}\left(\int_{B_r} \sqrt{G}\left(1 + z_\rho^2 + \frac{z_\theta^2}{G}\right)^2\right)^{\frac{1}{2}}.$$

□

Lemma 2. *Let us denote by $K_{\mathbb{M}}$ the Gauss curvature of $\mathbb{M}^2$. Then in the previous conditions we have,*

$$\begin{aligned}\frac{d}{dr}\int_0^{2\pi}\left(\frac{z_\theta^2}{\sqrt{G}}\right)d\theta &= \int_0^{2\pi}\left(\sqrt{G}\right)_r z_r^2 d\theta + \int_{B_r}\left(z_\rho^2+\frac{z_\theta^2}{G}\right)K_{\mathbb{M}}\sqrt{G}\\ &- 2\int_{B_r}\sqrt{G}\left(1+z_\rho^2+\frac{z_\theta^2}{G}\right)^2 K_{ext}.\end{aligned}$$

Proof. From (1) we can write

$$\begin{aligned}&2\int_{B_r}\sqrt{G}\left(1+z_\rho^2+\frac{z_\theta^2}{G}\right)^2 K_{ext}\\ &=\int_{B_r}\left[\left(\sqrt{G}\right)_\rho d\left(\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)+d\left(z_\rho d\left(\frac{z_\theta}{\sqrt{G}}\right)-\frac{z_\theta}{\sqrt{G}}dz_\rho\right)\right]\\ &=\int_{B_r}\left(\left(\sqrt{G}\right)_\rho-1\right)d\left(\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)\\ &+\int_{B_r}\left[d\left(\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)+d\left(z_\rho d\left(\frac{z_\theta}{\sqrt{G}}\right)-\frac{z_\theta}{\sqrt{G}}dz_\rho\right)\right].\end{aligned} \tag{2}$$

Using the Green theorem and integrating by parts one gets

$$\begin{aligned}&\int_{B_r}\left(\left(\sqrt{G}\right)_\rho-1\right)d\left(\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)\\ &=\int_{B_r}d\left(\left(\left(\sqrt{G}\right)_\rho-1\right)\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)-\int_{B_r}\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\left(\left(\left(\sqrt{G}\right)_\rho-1\right)d\theta\right)\\ &=\int_0^{2\pi}\left(\left(\sqrt{G}\right)_r-1\right)\left(z_r^2+\frac{z_\theta^2}{G}\right)d\theta-\int_{B_r}\left(z_\rho^2+\frac{z_\theta^2}{G}\right)\left(\left(\sqrt{G}\right)_{\rho\rho}\right)\\ &=\int_0^{2\pi}\left(\sqrt{G}\right)_r z_r^2 d\theta-\int_0^{2\pi}z_r^2 d\theta+\int_0^{2\pi}\left(\left(\sqrt{G}\right)_r-1\right)\frac{z_\theta^2}{G}d\theta\\ &+\int_{B_r}\left(z_r^2+\frac{z_\theta^2}{G}\right)K_{\mathbb{M}}\sqrt{G},\end{aligned} \tag{3}$$

where we have used that $K_{\mathbb{M}}=\frac{-\left(\sqrt{G}\right)_{rr}}{\sqrt{G}}$.

On the other hand, using again the Green theorem one obtains

$$\begin{aligned}&\int_{B_r}\left[d\left(\left(z_\rho^2+\frac{z_\theta^2}{G}\right)d\theta\right)+d\left(z_\rho d\left(\frac{z_\theta}{\sqrt{G}}\right)-\frac{z_\theta}{\sqrt{G}}dz_\rho\right)\right]\\ &=\int_0^{2\pi}z_r^2+\frac{z_\theta^2}{G}d\theta+\int_0^{2\pi}z_r d\left(\frac{z_\theta}{\sqrt{G}}\right)-\frac{z_\theta}{\sqrt{G}}dz_r d\theta\\ &=\int_0^{2\pi}z_r^2+\frac{z_\theta^2}{G}+\frac{z_r z_{\theta\theta}}{\sqrt{G}}-\frac{z_r z_\theta G_\theta}{2G\sqrt{G}}-\frac{z_\theta z_{r\theta}}{\sqrt{G}}d\theta.\end{aligned} \tag{4}$$

We observe that

$$\int_0^{2\pi} \frac{z_r z_{\theta\theta}}{\sqrt{G}} - \frac{z_r z_\theta G_\theta}{2G\sqrt{G}} + \frac{z_\theta z_{r\theta}}{\sqrt{G}} d\theta = \int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\frac{z_r z_\theta}{\sqrt{G}}\right) d\theta = 0. \tag{5}$$

Moreover, it is easy to check

$$2\int_0^{2\pi} \frac{z_\theta z_{r\theta}}{\sqrt{G}} d\theta = \frac{d}{dr}\int_0^{2\pi} \frac{z_\theta^2}{\sqrt{G}} d\theta + \int_0^{2\pi} \left(\sqrt{G}\right)_r \frac{z_\theta^2}{G} d\theta. \tag{6}$$

Using formulae (4)-(6) we have

$$\begin{aligned} &\int_{B_r}\left[d\left(\left(z_\rho^2 + \frac{z_\theta^2}{G}\right) d\theta\right) + d\left(z_\rho d\left(\frac{z_\theta}{\sqrt{G}}\right) - \frac{z_\theta}{\sqrt{G}} dz_\rho\right)\right] \\ &= \int_0^{2\pi} z_r^2 d\theta + \int_0^{2\pi}\left(1 - \left(\sqrt{G}\right)_r\right)\frac{z_\theta^2}{G} d\theta - \frac{d}{dr}\int_0^{2\pi}\frac{z_\theta^2}{\sqrt{G}} d\theta. \end{aligned} \tag{7}$$

Finally, by adding (3) and (7), we obtain (2). □

Recall that a point p_0 in a manifold $\mathbb{M}^2$ is said to be a pole if the exponential map is a diffeomorphism from the tangent plane at p_0 onto $\mathbb{M}^2$. In such a case, $\mathbb{M}^2$ is complete.

Remark 2.1. *Let $\mathbb{M}^2$ be a Riemannian surface as above with a pole at p_0. Let us observe that, in these conditions, $\left(\sqrt{G}\right)_r$ is a non-negative function if the Gauss curvature $K_{\mathbb{M}}$ of $\mathbb{M}^2$ is non negative. Since*

$$\left(\sqrt{G}\right)_{rr} = -K_{\mathbb{M}}\sqrt{G} \le 0,$$

one has that $\sqrt{G}$ is concave as a function of r. Fixed an arbitrary θ_0, if there existed a point $r_0 > 0$ such that $\left(\sqrt{G}\right)_r (r_0, \theta_0) < 0$ then, from the concavity of $\sqrt{G}(\cdot, \theta_0)$ we would have that $\sqrt{G}(r, \theta_0) = 0$ for a certain $r > r_0$.

Now, we can establish the main Theorem of this section.

Theorem 1. *Let $\mathbb{M}^2$ be a Riemannian surface with a pole. If the Gaussian curvature of $\mathbb{M}^2$ is non negative then there is no entire vertical graph Σ in $\mathbb{M}^2 \times \mathbb{R}$ with extrinsic curvature bounded from above by a negative constant.*

Proof. Differentiating f twice and using Lemma 2 one has

$$\begin{aligned} f''(r) &= \frac{d}{dr}\int_0^{2\pi}\sqrt{G}\left(1 + \frac{z_\theta^2}{G}\right) d\theta = \int_0^{2\pi}\left(\sqrt{G}\right)_r d\theta + \frac{d}{dr}\int_0^{2\pi}\frac{z_\theta^2}{\sqrt{G}} d\theta \\ &= \int_0^{2\pi}\left(\sqrt{G}\right)_r \left(1 + z_r^2\right) d\theta + \int_{B_r}\left(z_\rho^2 + \frac{z_\theta^2}{G}\right) K_{\mathbb{M}}\sqrt{G} \\ &- 2\int_{B_r}\sqrt{G}\left(1 + z_\rho^2 + \frac{z_\theta^2}{G}\right)^2 K_{ext}. \end{aligned}$$

Since $K_{\mathbb{M}} \geq 0$, it is clear from Remark 2.1 that the following inequality holds

$$\int_0^{2\pi} \left(\sqrt{G}\right)_r \left(1+z_r^2\right) d\theta + \int_{B_r} \left(z_\rho^2 + \frac{z_\theta^2}{G}\right) K_{\mathbb{M}} \sqrt{G} \geq 0.$$

Thus, using that $K_{ext} \leq -\alpha$ for a certain positive constant α

$$f''(r) \geq -2\int_{B_r} \sqrt{G}\left(1+z_\rho^2+\frac{z_\theta^2}{G}\right)^2 K_{ext} \geq 2\alpha \int_{B_r} \sqrt{G}\left(1+z_\rho^2+\frac{z_\theta^2}{G}\right)^2 \geq \frac{2\alpha}{|B_r|} f(r)^2$$

by Lemma 1.

From the last inequality

$$\frac{d}{d\rho}\left(f'(\rho)^2\right) = 2f'(\rho)f''(\rho) \geq 2f'(\rho)\frac{2\alpha}{|B_\rho|}f(\rho)^2 \geq f'(\rho)\frac{4\alpha}{|B_r|}f(\rho)^2 = \frac{4\alpha}{3|B_r|}\frac{d}{d\rho}\left(f(\rho)^3\right).$$

By integrating both sides of the above inequality one gets

$$\int_\epsilon^r \frac{d}{d\rho}\left(f'(\rho)^2\right) \geq \int_\epsilon^r \frac{4\alpha}{3|B_r|}\frac{d}{d\rho}\left(f(\rho)^3\right),$$

where $0 < \epsilon < r$. Thus,

$$f'(r)^2 \geq \frac{4\alpha}{3|B_r|}\left(f(r)^3 - f(\epsilon)^3\right) + f'(\epsilon)^2 \geq \frac{4\alpha}{3|B_r|}\left(f(r)^3 - f(\epsilon)^3\right).$$

It is clear from the definition of f that $f(\epsilon) \to 0$ when $\epsilon \to 0$, hence

$$f'(r)^2 \geq \frac{4\alpha}{3|B_r|} f(r)^3$$

and so

$$-\frac{d}{dr}\left(f(r)^{-\frac{1}{2}}\right) = \frac{1}{2}f(r)^{-\frac{3}{2}}f'(r) \geq \frac{1}{2}f(r)^{-\frac{3}{2}}\sqrt{\frac{4\alpha}{3|B_r|}}f(r)^{\frac{3}{2}} = \sqrt{\frac{\alpha}{3|B_r|}}.$$

Let R_1 and R_2 be real numbers such that $0 < R_1 < R_2$. By integrating we have on one hand

$$\int_{R_1}^{R_2} -\frac{d}{dr}\left(f(r)^{-\frac{1}{2}}\right) dr = -f(R_2)^{-\frac{1}{2}} + f(R_1)^{-\frac{1}{2}}.$$

On the other hand, since $K_{\mathbb{M}} \geq 0$, from the volume comparison theorem we have $|B_r| \leq \pi r^2$, and one has from the above inequality

$$\int_{R_1}^{R_2} -\frac{d}{dr}\left(f(r)^{-\frac{1}{2}}\right) dr \geq \int_{R_1}^{R_2} \sqrt{\frac{\alpha}{3|B_r|}} dr = \sqrt{\frac{\alpha}{3}} \int_{R_1}^{R_2} \frac{1}{\sqrt{|B_r|}} dr$$
$$\geq \sqrt{\frac{\alpha}{3}} \int_{R_1}^{R_2} \frac{1}{r\sqrt{\pi}} dr = \sqrt{\frac{\alpha}{3\pi}} \log\left(\frac{R_2}{R_1}\right).$$

So, by Lemma 1

$$|B_{R_1}|^{-\frac{1}{2}} \geq f(R_1)^{-\frac{1}{2}} \geq f(R_1)^{-\frac{1}{2}} - f(R_2)^{-\frac{1}{2}} \geq \sqrt{\frac{\alpha}{3\pi}} \log\left(\frac{R_2}{R_1}\right).$$

Hence

$$\sqrt{\frac{3\pi}{\alpha}}\, |B_{R_1}|^{-\frac{1}{2}} \geq \log\left(\frac{R_2}{R_1}\right),$$

that is,

$$R_2 \leq R_1 \exp\left(\sqrt{\frac{3\pi}{\alpha}} \frac{1}{\sqrt{|B_{R_1}|}}\right)$$

for every $0 < R_1 < R_2$, with R_2 an arbitrary real value. □

We observe that if $\mathbb{M}^2$ is compact then, for any entire vertical graph Σ in $\mathbb{M}^2 \times \mathbb{R}$ there exist points in Σ with non negative extrinsic curvature. This happens at the points where the height function achieves a local maximum or a local minimum.

Finally, as a consequence of Theorem 1 and [6, Theorem 1] we have

Corollary 1. *Let $\mathbb{M}^2$ be a Riemannian surface with a pole and non negative Gauss curvature. If Σ is an entire vertical graph in $\mathbb{M}^2 \times \mathbb{R}$ with extrinsic curvature K_{ext} then*

$$\inf_{\Sigma} |K_{ext}| = 0.$$

3 Complete surfaces with constant negative extrinsic curvature.

In this section we will give an example of existence of complete surfaces with strictly negative extrinsic curvature in product manifolds.

Consider $\mathbb{M}^2$ as the surface $\mathbb{R}^2$ with the induced metric

$$\langle \cdot, \cdot \rangle = d\rho^2 + G(\rho) d\theta^2$$

which is well defined in $\mathbb{R}^2 \setminus \{(0,0)\}$, where (ρ, θ) are the usual polar coordinates of $\mathbb{R}^2$. Here, the function $G(\rho)$ will be defined later.

Let Σ be a rotation surface in $\mathbb{M}^2 \times \mathbb{R}$ parametrized by

$$\Psi(r, \theta) = \Big(k(r)\cos(\theta), k(r)\sin(\theta), z(r)\Big) \equiv \Big(k(r), \theta, z(r)\Big)$$

where $k(r) > 0$ and $z(r)$ are smooth functions, and r denotes the arc length parameter of the curve $\beta(r) = \Psi(r, 0)$. Thus

$$k'(r)^2 + z'(r)^2 = 1. \quad (8)$$

If we denote by $\{\overline{\partial_r}, \overline{\partial_\theta}\}$ and $\{\partial_\rho, \partial_\theta\}$ the respective partial derivatives of Σ and $\mathbb{M}^2$ we have

$$\begin{aligned}\overline{\partial_r}(r,\theta) &= k'(r)\partial_\rho\big(k(r),\theta\big) + z'(r)\partial_t\big(k(r),\theta\big)\\ \overline{\partial_\theta}(r,\theta) &= \partial_\theta\big(k(r),\theta\big).\end{aligned}$$

So from (8), the induced metric in our surface is given by

$$ds^2 = dr^2 + G\big(k(r)\big)d\theta^2,$$

and its second fundamental form is

$$II(r,\theta) = \Big(-k''(r)z'(r) + k'(r)z''(r)\Big)dr^2 + \left(\frac{z'(r)G_\rho\big(k(r)\big)}{2}\right)d\theta^2.$$

Hence, the principal curvatures of Σ can be written as

$$\begin{aligned}\lambda_1 &= -k''(r)z'(r) + k'(r)z''(r)\\ \lambda_2 &= z'(r)\frac{G_\rho\big(k(r)\big)}{2G\big(k(r)\big)},\end{aligned}$$

and the extrinsic curvature of the graph is

$$K_{ext} = -z'(r)\frac{G_\rho\big(k(r)\big)}{2G\big(k(r)\big)}\big(k''(r)z'(r) - k'(r)z''(r)\big). \tag{9}$$

By differentiating (8) we have

$$k'(r)k''(r) + z'(r)z''(r) = 0,$$

and so (9) becomes the following ODE for k

$$k''(r) = -\frac{2G\big(k(r)\big)}{G_\rho\big(k(r)\big)}K_{ext}. \tag{10}$$

Now we take, for instance, the function $G(\rho) = \rho^4 e^{\rho^4}$ and impose that the extrinsic curvature of Σ is constant -1.

Under these conditions, (10) becomes

$$k''(r) = \frac{k(r)}{2\,(1 + k(r)^4)} \tag{11}$$

Now, let $k(r)$ be a solution of the previous ODE with initial conditions $k(0) = k_0 > 0$, $k'(0) = 0$, where k_0 is a positive value which will be determined later.

It is well known that the solution $k(r)$ to the previous problem is uniquely determined in a neighbourhood of the origin. Let us see in fact that $k(r)$ is well defined for every $r \in \mathbb{R}$.

First, we observe that, from the uniqueness to the previous problem, $k(r) = k(-r)$ since $k(-r)$ is also a solution to (11) satisfying the same initial conditions. Thus, in order to study the behaviour of $k(r)$, we will only work for $r \geq 0$.

From (11) we have

$$\frac{d}{dr}\left(k'(r)\right)^2 = \frac{d}{dr}\left(\frac{1}{2}\arctan\left(k(r)^2\right)\right)$$

and so

$$k'(r)^2 = c_0 + \frac{1}{2}\arctan\left(k(r)^2\right). \tag{12}$$

Hence there exists a constant c_1 such that $|k'(r)| \leq c_1$, that is $k(r)$ is globally defined as we required.

Since $k(0) = k_0 > 0$ and $k'(0) = 0$, if we take $k_0 > 0$ small enough we would have $c_0 \in \left(\frac{\pi}{4} - 1, 0\right)$. Thus, from (12) one gets $|k'(r)| < 1$ for every $r \in \mathbb{R}$. Therefore, from (8), $z(r)$ is a well defined smooth function in $\mathbb{R}$.

Let us see that $k(r) \geq k(0) = k_0 > 0$ for every $r \in \mathbb{R}$. In such a case, Σ is a regular surface due to the generatrix curve $\beta(r) = \Psi(r, 0)$ does not touch the rotation axis. For that purpose, let us first observe that $k(r) > 0$, for every $r > 0$.

Let us assume that there exists a first point $\epsilon > 0$ such that $k(\epsilon) = 0$. Since $k(r)$ is a positive function in the interval $[0, \epsilon)$, from (11) we obtain that $k(r)$ is a convex function in that interval. Moreover, $k'(0) = 0$, thus $k(\epsilon) \geq k(0) = k_0 > 0$, and it is contradiction with $k(\epsilon) = 0$.

On the other hand, since from (11) the function $k(r)$ is a global convex function in $\mathbb{R}$ and $k'(0) = 0$, then $k(r) \geq k(0) = k_0$ for every $r \in \mathbb{R}$ and

$$\lim_{r \to \pm\infty} k(r) = +\infty.$$

In particular, this shows that Σ is properly immersed in $\mathbb{M}^2 \times \mathbb{R}$.

One can easily check that our metric $\langle \cdot, \cdot \rangle$ is not well defined at the origin. Since $k(r) \geq k_0 > 0$ for every $r \in \mathbb{R}$, we can perturb the metric $\langle \cdot, \cdot \rangle$ for instance for $0 \leq \rho < \frac{k_0}{2}$, and have a complete metric in $\mathbb{M}^2$ well defined at the origin and such that Σ preserves the same induced metric. As Σ is properly immersed in $\mathbb{M}^2 \times \mathbb{R}$, then Σ is a complete surface with negative constant extrinsic curvature as we claimed.

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