Existence of barriers for surfaces with prescribed curvatures in M^2xR.

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Abstract

We present a deformation process of surfaces from a product space $\mathbb{M}_1 \times \mathbb{R}$ into another product space $\mathbb{M}_2 \times \mathbb{R}$ such that the relation of the principal curvatures of the deformed surfaces can be controlled in terms of the curvatures of $\mathbb{M}_1$ and $\mathbb{M}_2$. Thus, we obtain subsolutions for the existence or barriers for the non existence of surfaces with prescribed curvatures in a general product space $\mathbb{M} \times \mathbb{R}$.

1 Introduction.

The study of barriers for a class of surfaces $\mathcal{S}$ in an ambient space is used for giving results of existence or non existence of surfaces in $\mathcal{S}$ satisfying certain desired properties.

When possible, the rotational surfaces (or surfaces invariant under a 1-parametric group of isometries) in the class $\mathcal{S}$ are the most simple barriers if a maximum principle holds in $\mathcal{S}$. A classical example of this fact is the halfspace theorem for minimal surfaces in $\mathbb{R}^3$ [HM], where the catenoids are used as barriers in order to show that there is no properly immersed minimal surface in a halfspace of $\mathbb{R}^3$ different from a plane.

Our objective will be to describe a simple method for obtaining barriers in a product space $\mathbb{M} \times \mathbb{R}$, or more generally, in a warped space. For that, given two ambient spaces $\mathbb{M}_1 \times \mathbb{R}$ and $\mathbb{M}_2 \times \mathbb{R}$, we will start with a surface S in $\mathbb{M}_1 \times \mathbb{R}$ and obtain a new surface S^* in $\mathbb{M}_2 \times \mathbb{R}$ such

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that the principal curvatures of S and S^* can be related in terms of the curvatures of $\mathbb{M}_1$ and $\mathbb{M}_2$.

Thus, for instance, if we start with the well known rotational surfaces of constant mean curvature H in a homogeneous space $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ then we can obtain barriers for the existence or non existence of surfaces with constant mean curvature H in a general product space $\mathbb{M} \times \mathbb{R}$. Observe that the group of isometries of $\mathbb{M} \times \mathbb{R}$ could be very small.

The paper is organized as follows. Section 2 will be devoted to explain the deformation process for obtaining these barriers. In Section 3 we give some examples of how this construction can be used.

First, we extend in Theorem 1 a result by Espinar and Rosenberg [ER]. We prove that given a closed geodesic disk $D \subseteq \mathbb{M}$ then there is an explicit constant k_0, which only depends on the radius of D and the minimum of its curvature, such that there exists no graph over D in $\mathbb{M} \times \mathbb{R}$ with minimum of its mean curvature greater than or equal to k_0.

We also prove in Theorem 2 that under certain restrictions on the ambient space $\mathbb{M} \times \mathbb{R}$, for every properly embedded surface $\Sigma \subseteq \mathbb{M} \times \mathbb{R}$ with mean curvature $H \geq H_0 > 0$, its mean convex component can not contain a certain geodesic ball of radius r, where r only depends on H_0 and the infimum of the curvature of $\mathbb{M}$.

We show a halfspace type theorem for properly immersed surfaces with mean curvature satisfying $|H| \leq 1/2$ in $\mathbb{M} \times \mathbb{R}$, with curvature of $\mathbb{M}$ satisfying $K \leq -1$ (Theorem 3). This result is the extension to the corresponding one by Nelli and Sa Earp in $\mathbb{H}^2 \times \mathbb{R}$ [NS].

We also study in Theorem 4 some properties about the behavior at infinity of a minimal surface in $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a Hadamard surface with pinched curvature between two negative constants. This gives us a generalization of some results by Sa Earp and Toubiana in $\mathbb{H}^2 \times \mathbb{R}$ [ST].

Moreover, we show in Theorem 5 that, under some conditions, a compact surface with constant mean curvature and boundary in a slice of $\mathbb{M} \times \mathbb{R}$ must be a graph, when $\mathbb{M}$ is a Hadamard surface with pinched curvature between two negative constants. This extends the corresponding result in [NSST] for the homogeneous space $\mathbb{H}^2 \times \mathbb{R}$.

Finally we give in Theorem 6 a result of existence for graphs with positive constant extrinsic curvature in $\mathbb{M} \times \mathbb{R}$ which solves the Dirichlet problem for the associated Monge-Ampère equation with zero boundary conditions.

2 A comparison result.

Let $(\mathbb{M}, g)$ be a Riemannian surface, $\Omega \subset \mathbb{R}^2$ an open set and $\varphi : \Omega \longrightarrow \mathbb{M}$ a local parametrization such that the metric is locally written as

$$g = du^2 + G(u,v)\, dv^2, \qquad (u,v) \in \Omega.$$

This basically means that the coordinate curves $\alpha(u) = \varphi(u, v_0)$, for a constant v_0, are geodesics on $\mathbb{M}$ (parameterized by the arc length) which intersect orthogonally the coordinate curves $\beta(v) = \varphi(u_0, v)$, for a constant u_0.

Observe that there exist many coordinates of this form in any Riemannian surface. Geodesic polar coordinates $\varphi(u, v) = \exp_p(u \cos v, u \sin v)$ around a point $p \in \mathbb{M}$ are examples of these local coordinates.

Now, let us consider the generalized warped metric in $\varphi(\Omega) \times \mathbb{R} \subseteq \mathbb{M} \times \mathbb{R}$ given by

$$\langle , \rangle = f(u, t)\,(du^2 + G(u, v)\,dv^2) + dt^2, \qquad f(u, t) > 0.$$

Here, t is the coordinate in $\mathbb{R}$ and u can be understood as a distance function along the geodesics $\alpha(u)$. We remark that product metrics or, more generally, warped metrics $f(t)\,g + dt^2$ in $\mathbb{M} \times \mathbb{R}$ can be locally written in the previous way.

If we denote by ∇ the Levi-Civita connection in $(\mathbb{M} \times \mathbb{R}, \langle , \rangle)$ then a direct computation gives us

$$\begin{array}{lll}
\langle \nabla_{\partial_u} \partial_u, \partial_u \rangle = f_u/2, & \langle \nabla_{\partial_u} \partial_u, \partial_v \rangle = 0, & \langle \nabla_{\partial_u} \partial_u, \partial_t \rangle = -f_t/2, \\
\langle \nabla_{\partial_u} \partial_v, \partial_u \rangle = 0, & \langle \nabla_{\partial_u} \partial_v, \partial_v \rangle = (fG)_u/2, & \langle \nabla_{\partial_u} \partial_v, \partial_t \rangle = 0, \\
\langle \nabla_{\partial_u} \partial_t, \partial_u \rangle = f_t/2, & \langle \nabla_{\partial_u} \partial_t, \partial_v \rangle = 0, & \langle \nabla_{\partial_u} \partial_t, \partial_t \rangle = 0, \\
\langle \nabla_{\partial_v} \partial_v, \partial_u \rangle = -(fG)_u/2, & \langle \nabla_{\partial_v} \partial_v, \partial_v \rangle = fG_v/2, & \langle \nabla_{\partial_v} \partial_v, \partial_t \rangle = -f_t G/2, \\
\langle \nabla_{\partial_v} \partial_t, \partial_u \rangle = 0, & \langle \nabla_{\partial_v} \partial_t, \partial_v \rangle = f_t G/2, & \langle \nabla_{\partial_v} \partial_t, \partial_t \rangle = 0, \\
\langle \nabla_{\partial_t} \partial_t, \partial_u \rangle = 0, & \langle \nabla_{\partial_t} \partial_t, \partial_v \rangle = 0, & \langle \nabla_{\partial_t} \partial_t, \partial_t \rangle = 0,
\end{array} \tag{1}$$

where for instance ∂_u stands for $\frac{\partial}{\partial u}$.

Let us consider the graph $\psi(u, v) = (\varphi(u, v), h(u))$ in $\mathbb{M} \times \mathbb{R}$, with height function $h(u)$ only depending on the "distance function" u, and compute its principal curvatures.

We denote by $\overline{\partial}_u$ and $\overline{\partial}_v$ the partial derivatives with respect to u and v on the graph ψ. Thus,

$$\overline{\partial}_u = \partial_u + h'(u)\partial_t, \qquad \overline{\partial}_v = \partial_v,$$

and the induced metric of the graph is

$$I = (f + h'(u)^2)\,du^2 + f\,G\,dv^2.$$

Hence, the unit normal (pointing upwards) of the graph is given by

$$N = \frac{1}{\sqrt{f(f + h'(u)^2)}}(-h'(u)\partial_u + f\partial_t).$$

On the other hand,

$$\begin{aligned}
\nabla_{\overline{\partial}_u} \overline{\partial}_u &= \nabla_{\overline{\partial}_u}(\partial_u + h'(u)\partial_t) = \nabla_{\overline{\partial}_u} \partial_u + h''(u)\partial_t + h'(u)\nabla_{\overline{\partial}_u} \partial_t = \\
&= \nabla_{\partial_u} \partial_u + h'(u)\nabla_{\partial_t} \partial_u + h''(u)\partial_t + h'(u)(\nabla_{\partial_u} \partial_t + h'(u)\nabla_{\partial_t} \partial_t), \\
\nabla_{\overline{\partial}_u} \overline{\partial}_v &= \nabla_{\overline{\partial}_v} \overline{\partial}_u = \nabla_{\partial_v} \partial_u + h'(u)\nabla_{\partial_v} \partial_t, \\
\nabla_{\overline{\partial}_v} \overline{\partial}_v &= \nabla_{\partial_v} \partial_v.
\end{aligned} \tag{2}$$

Therefore, from (1) and (2), the second fundamental form can be computed as

$$\langle \nabla_{\overline{\partial}_u} \overline{\partial}_u, N \rangle = \frac{1}{\sqrt{f(f+h'(u)^2)}} \left(-\frac{1}{2} f_u h'(u) - h'(u)^2 f_t - \frac{1}{2} f f_t + f h''(u) \right),$$
$$\langle \nabla_{\overline{\partial}_u} \overline{\partial}_v, N \rangle = 0,$$
$$\langle \nabla_{\overline{\partial}_v} \overline{\partial}_v, N \rangle = \frac{1}{\sqrt{f(f+h'(u)^2)}} \left(\frac{1}{2}(f_u G + f G_u) h'(u) - \frac{1}{2} f f_t G \right).$$

In particular, the coordinate curves are the lines of curvature of the graph. Moreover, the principal curvatures are given by

$$\begin{aligned} k_1 &= \frac{-f_u h'(u) - 2h'(u)^2 f_t - f f_t + 2 f h''(u)}{2(f+h'(u)^2)\sqrt{f(f+h'(u)^2)}}, \\ k_2 &= \frac{1}{2\sqrt{f(f+h'(u)^2)}} \left(\left(\frac{f_u}{f} + \frac{G_u}{G}\right) h'(u) - f_t \right). \end{aligned} \tag{3}$$

As a consequence of (3) we can state

Proposition 1. *Let* $(\mathbb{M}_1, g_1)$, $(\mathbb{M}_2, g_2)$ *be two Riemannian surfaces,* Ω *an open set in* $\mathbb{R}^2$ *and* $\varphi_i : \Omega \longrightarrow \mathbb{M}_i$, $i = 1, 2$, *two parameterizations such that*

$$g_i = du^2 + G^i(u,v) dv^2, \qquad (u,v) \in \Omega. \tag{4}$$

If we endow $\varphi_i(\Omega) \times \mathbb{R} \subseteq \mathbb{M}_i \times \mathbb{R}$ *with the warped metrics* $\langle , \rangle_i = f(u,t) g_i + dt^2$, $i = 1, 2$, *then the principal curvatures* k_1^i, k_2^i *of the graph* $\psi_i(u,v) = (\varphi_i(u,v), h(u))$, *for the metric* $\langle , \rangle_i$ *and the normal pointing upwards, satisfy* $k_1^1 = k_1^2$ *and*

$$k_2^1 \geq k_2^2 \quad \textit{if and only if} \quad \frac{G_u^1}{G^1} h'(u) \geq \frac{G_u^2}{G^2} h'(u). \tag{5}$$

We emphasize that the geometric quantity $\frac{G_u^i}{2G^i}$ appearing in (5) is the geodesic curvature of the coordinate curves $\beta_i(v) = \varphi_i(u_0, v)$ for the metric g_i given in (4). In order to obtain some natural conditions for (5) to be satisfied, we establish a comparison result which relates the geodesic curvature of the coordinate curves $\beta_i(v)$ and the Gaussian curvature K_i of $(\mathbb{M}_i, g_i)$.

Proposition 2. *In the conditions of Proposition 1, assume the domain* $\Omega = I \times J$ *is the product of two real intervals* $I, J \subseteq \mathbb{R}$. *Let us suppose that one of the following items holds:*

(i) There exists $(u_0, v_0) \in I \times J$ *such that*

$$\frac{G_u^1}{G^1}(u_0, v_0) \geq \frac{G_u^2}{G^2}(u_0, v_0), \qquad \textit{and} \qquad K_1(u, v_0) \leq K_2(u, v_0) \quad \textit{for all } u \geq u_0.$$

(ii) There exist points $p_i \in \mathbb{M}_i$ such that (u, v) are geodesic polar coordinates around p_i for the metric g_i, and an angle v_0 such that

$$K_1(u, v_0) \leq K_2(u, v_0) \quad \textit{for all } u \in I = (0, \rho_0).$$

Then

$$\frac{G_u^1}{G^1}(u, v_0) \geq \frac{G_u^2}{G^2}(u, v_0)$$

for $u \geq u_0$ in the first case, or for $u > 0$ in the second one.

Proof. From (4), the Gaussian curvature K_i of the metric g_i is given by

$$K_i = -\left(\frac{G_u^i}{2G^i}\right)_u - \left(\frac{G_u^i}{2G^i}\right)^2. \tag{6}$$

On the other hand, given four differentiable functions $f, g, k_f, k_g : [t_1, t_2) \longrightarrow \mathbb{R}$ satisfying

$$f'(t) + f(t)^2 = -k_f(t), \qquad g'(t) + g(t)^2 = -k_g(t), \tag{7}$$

with $f(t_1) \geq g(t_1)$, and $k_f(t) \leq k_g(t)$ for $t \in [t_1, t_2)$, then it is easy to show that $f(t) \geq g(t)$ for all $t \in [t_1, t_2)$.

As a consequence, the result is proven in case (i) from (6) and (7) by replacing

$$f(t) = \frac{G_u^1}{2G^1}(t, v_0), \quad g(t) = \frac{G_u^2}{2G^2}(t, v_0), \quad k_f(t) = K_1(t, v_0), \quad k_g(t) = K_2(t, v_0).$$

Now, let us assume (u, v) are geodesic polar coordinates. In this case it is well known that (see for instance [dC])

$$\lim_{u \to 0} G^i(u, v_0) = 0 \qquad \text{and} \qquad \lim_{u \to 0} (\sqrt{G^i})_u(u, v_0) = 1. \tag{8}$$

In addition, if $G_u^i \neq 0$ then (6) can be rewritten as

$$\left(\frac{2G^i}{G_u^i}\right)_u = 1 + K_i \left(\frac{2G^i}{G_u^i}\right)^2. \tag{9}$$

On the other hand, given four real functions $f, g, k_f, k_g : [0, t_0) \longrightarrow \mathbb{R}$, satisfying

$$f'(t) = 1 + k_f(t) f(t)^2, \qquad g'(t) = 1 + k_g(t) g(t)^2, \tag{10}$$

with $k_f(t) \leq k_g(t)$, $t \in [0, t_0)$, and $f(0) = g(0) = 0$, then it is easy to show that $f(t) \leq g(t)$ for $t \in [0, t_0)$.

Observe that $K_i(0, v)$ is well defined as the Gaussian curvature at the point $p_i \in \mathbb{M}_i$ where the geodesic polar coordinates are centered, and from the initial conditions about G^i we have

$$\lim_{u \to 0} \frac{2G^i}{G^i_u} = \lim_{u \to 0} \frac{\sqrt{G^i}}{(\sqrt{G^i})_u} = 0.$$

Thus, from (8), there exists $\varepsilon_0 > 0$ such that $(G^i)_u(u, v_0) > 0$ for $0 < u < \varepsilon_0$, and the case (ii) can be proven from (9) and (10) by replacing

$$f(t) = \frac{2G^1}{G^1_u}(t, v_0), \quad g(t) = \frac{2G^2}{G^2_u}(t, v_0), \quad k_f(t) = K_1(t, v_0), \quad k_g(t) = K_2(t, v_0)$$

for $0 < u < \varepsilon_0$.

Finally, from case (i), the result is true for any $u \in I = (0, \rho_0)$. $\square$

Combining these two propositions we have a strong comparison result in order to deform a surface from an ambient space to another in such a way that its principal curvatures are controlled in the process. This fact will be shown in the next section.

3 Existence of barriers in MxR.

In this section we give some existence and non existence results for surfaces in product spaces and generalize some known results in homogeneous product spaces to general product spaces.

From now on we will denote by $\mathbb{M}(c)$ the complete simply connected 2-dimensional space form of constant curvature c, that is, a hyperbolic plane if $c < 0$, the Euclidean plane if $c = 0$ or a sphere if $c > 0$.

Let us start with a topological sphere S of constant mean curvature H_0 in $\mathbb{M}(c) \times \mathbb{R}$ (see, for instance, [AR, AEG2]). Observe that S is unique up to isometries of the ambient space and only exists for $H_0 > \sqrt{-c}/2$ if $c < 0$. Moreover, S is rotational and symmetric with respect to a horizontal slice. In particular, S is a bigraph over a geodesic disk of $\mathbb{M}(c)$ of radius $r_c(H_0) > 0$.

Thus, let $p \in \mathbb{M}(c)$ and (ρ, θ) be geodesic polar coordinates around p. Since S is a rotational surface the lower part of S can be considered as a graph over the geodesic disk centered at p and radius $r_c(H_0)$, with height function $h(\rho)$ which only depends on ρ. Moreover, $h(\rho)$ is strictly increasing, see [AR]. Hence, this part of the constant mean surface S can be described as

$$\psi_1(\rho, \theta) = (\rho, \theta, h(\rho)) \in \mathbb{M}(c) \times \mathbb{R}.$$

Note that, for convenience, we have deleted the parameterization φ in the previous expression.

Now, given a general product space $\mathbb{M} \times \mathbb{R}$ and geodesic polar coordinates (ρ, θ) around a point $q \in \mathbb{M}$, which are well defined for $0 < \rho \leq r_c(H_0)$, we can consider the new immersion

$$\psi_2(\rho, \theta) = (\rho, \theta, h(\rho)) \in \mathbb{M} \times \mathbb{R}.$$

Applying the same process for the upper part of S, we obtain a sphere S^* in $\mathbb{M} \times \mathbb{R}$ which is a bigraph over the geodesic disk of radius $r_c(H_0)$ centered at q.

We remark that S^* is symmetric with respect to a horizontal slice as S is, and any vertical translation of S^* is congruent to S^*. However, S^* depends strongly on the point $q \in \mathbb{M}$, i. e. if we start with another point $\widetilde{q} \in \mathbb{M}$ and obtain a new surface $\widetilde{S^*}$ following the same process then S^* and $\widetilde{S^*}$ are not isometric in general.

Now, assume the Gaussian curvature K of $\mathbb{M}$ is bigger than or equal to c. Thus, using Proposition 1 and Proposition 2 we have that S^* has mean curvature $H \leq H_0$ for its inner normal.

With all of this, we obtain

Theorem 1. *Let D_r be a closed geodesic disk of radius $r > 0$ in a Riemannian surface $\mathbb{M}$, and $c := \min_{D_r}(K)$ be the minimum of the Gaussian curvature of $\mathbb{M}$ in the disk D_r. Consider $H_0 > 0$ such that $r_c(H_0) = r$, then there is no vertical graph over D_r with minimum of its mean curvature $\min(H) \geq H_0$.*

Proof. Assume $\Sigma \subseteq \mathbb{M} \times \mathbb{R}$ is a graph over D_r with $\min(H) \geq H_0$ for a unit normal N. Without loss of generality we can assume the unit normal N points upwards.

Let $q \in \mathbb{M}$ be the center of the geodesic disk D_r and consider the sphere S^* centered at q obtained previously, which has mean curvature less than or equal to H_0 for its inner normal.

Move the sphere S^* up until Σ is below S^*, and go down till S^* intersects Σ for the first time. Then, the classical maximum principle for mean curvature asserts that both surfaces must agree locally. In particular, Σ and S^* have constant mean curvature H_0 and Σ agrees with the lower hemisphere of S^*. However, this is a contradiction because S^* is not a strict graph over the boundary of D_r since its unit normal is horizontal at those points. $\square$

This result can be seen as an extension of the one by Espinar and Rosenberg [ER] which asserts that if $\mathbb{M}$ is a complete Riemannian surface with $\inf(K) = -1$, then there is no complete entire vertical graph in $\mathbb{M} \times \mathbb{R}$ with $\inf(H) > 1/2$. The proof in their case is different and they study the spectrum of the Laplacian of $\mathbb{M}$ in order to estimate the Cheeger constant of $\mathbb{M}$.

Theorem 1 can be generalized in different ways. On one hand, the hypothesis on the mean curvature of the surface is not a basic condition since it is only used in order to apply a maximum principle to this functional. Thus, a similar result to Theorem 1 can be analogously proven for other functionals with a maximum principle. For instance, positive extrinsic curvature or Gaussian curvature with positive extrinsic curvature (see [AEG1, EGR] for the revolution spheres in $\mathbb{M}(c) \times \mathbb{R}$ to be deformed in these cases). On the other hand, the same ideas can be used for warped products. For example, the hyperbolic 3-space can be seen as a warped product and a similar result can be shown for mean curvature in the so called pseudo-hyperbolic spaces (see [T]).

Next, we will give more applications to our comparison result for the mean curvature functional, bearing in mind that analogous results are possible for other functionals.

Let us denote by $\mathrm{diam_c(H_0)}$ the diameter of the unique sphere in $\mathbb{M}(c)\times\mathbb{R}$ of constant mean curvature $H_0 > 0$. Then we obtain,

Theorem 2. *Let $\mathbb{M}$ be a complete, simply connected surface with infimum of its Gaussian curvature $c \in \mathbb{R}$ and injectivity radius $i \in (0,\infty]$. Consider a properly embedded surface Σ in $\mathbb{M}\times\mathbb{R}$ with mean curvature $H \geq H_0 > 0$ ($H_0 > \sqrt{-c}/2$ if $c < 0$). If $r_c(H_0) < i$ then the mean convex component can not contain a closed geodesic ball in $\mathbb{M}\times\mathbb{R}$ of radius $\mathrm{diam_c(H_0)}/2$.*

Proof. Let S be a sphere of constant mean curvature H_0 in $\mathbb{M}(c)\times\mathbb{R}$. We will call center of S to the unique point $(p_0, t_0) \in \mathbb{M}(c)\times\mathbb{R}$ such that S is a revolution surface with respect to the axis given by the vertical geodesic $\gamma(t) = (p_0, t)$ and symmetric with respect to the slice of height $t = t_0$. Thus, if we consider the corresponding sphere S^* in $\mathbb{M}\times\mathbb{R}$, using geodesic polar coordinates at a point $q_0 \in \mathbb{M}$, we will say that S^* is centered at (q_0, t_0).

Assume Σ is a properly embedded surface in $\mathbb{M}\times\mathbb{R}$ with mean curvature $H \geq H_0 > 0$ and B is a closed geodesic ball of radius $\mathrm{diam_c(H_0)}/2$ contained in the mean convex component of Σ. Let $\alpha : [0,1] \longrightarrow \mathbb{M}\times\mathbb{R}$ be a continuous curve such that $\alpha(0)$ is the center of B and $\alpha(1) \in \Sigma$. So, since the injectivity radius $i > r_c(H_0)$, for each $s \in [0,1]$ we can consider the sphere $S^*(s)$ centered at $\alpha(s)$ which satisfies

$$H \leq H_0 \qquad \text{for all } z \in S^*(s) \quad \text{for all } s \in [0,1].$$

Though the spheres $S^*(s)$ are not isometric in general, they constitute a continuous deformation when s varies. So, it must exist a first $s_0 \in (0,1)$ such that $S^*(s_0)$ intersects Σ at a point z_0. Hence, as $S^*(s_0)$ is contained in the mean convex component of Σ, the maximum principle can be used in order to show that Σ and $S^*(s_0)$ must agree. But this is a contradiction, as we wanted to prove. $\square$

It is well-known that the exponential map is a global diffeomorphism in a Hadamard manifold $\mathbb{M}$ for any point $p \in \mathbb{M}$. In addition, given a complete geodesic γ in $\mathbb{M}$, we will consider the vertical plane $P = \gamma\times\mathbb{R}$ in the Hadamard manifold $\mathbb{M}\times\mathbb{R}$, which is isometric to $\mathbb{R}^2$. It is clear that the geodesics of $\mathbb{M}\times\mathbb{R}$ which are orthogonal to P give a foliation of $\mathbb{M}\times\mathbb{R}$. Thus, we will say that a surface is a horizontal graph over the vertical plane if its projection on P is one to one.

Hence, as a straightforward consequence we obtain the following result.

Corollary 1. *Let $H_0 > 0$ and $\mathbb{M}$ be a Hadamard surface with $c = \inf(K) > -\infty$. Then there exists no entire horizontal graph in $\mathbb{M}\times\mathbb{R}$ with mean curvature $H \geq H_0 > \sqrt{-c}/2$.*

Now, we construct different families of barriers in $\mathbb{M}\times\mathbb{R}$ in order to obtain some theorems which generalize the corresponding theorems in homogeneous product spaces. Once the barriers are obtained the proof of the results will follow as its analogous result in a homogeneous space. Thus, we will only present the surfaces which are the desired barriers and omit the rest of details of the proof.

First, we generalize a certain analogous to the half-space theorem for minimal surfaces of $\mathbb{R}^3$ given by Nelli and Sa Earp for surfaces with the critical value of the mean curvature $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Here, $\mathbb{H}^2$ denotes the hyperbolic plane of constant curvature -1.

Let us denote by S_0 the entire rotational graph with constant mean curvature $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Nelli and Sa Earp proved that a complete surface with constant mean curvature $H = 1/2$, different from a rotational simply connected one, can not be properly immersed in the mean convex side of S_0 [NS].

Let $\mathbb{M}$ be a Hadamard surface with curvature $K \leq -1$. Given a point $p \in \mathbb{M}$, we denote by $S_0^*(p) \subseteq \mathbb{M} \times \mathbb{R}$ the corresponding simply connected graph obtained from the rotational entire graph $S_0 \subseteq \mathbb{H}^2 \times \mathbb{R}$, using polar coordinates at p. Since vertical translations are isometries of the ambient space $\mathbb{M} \times \mathbb{R}$, we will also denote by $S_0^*(p)$ the vertical translations of the previous entire graph.

From Proposition 1 and Proposition 2 the surface $S_0^*(p)$ has mean curvature $H \geq 1/2$ for the unit normal pointing upwards.

Theorem 3. *Let $\mathbb{M}$ be a Hadamard surface with curvature $K \leq -1$. Consider a complete surface Σ in $\mathbb{M} \times \mathbb{R}$ with mean curvature satisfying $|H| \leq 1/2$. Then, given $p \in \mathbb{M}$, Σ can not be properly immersed in the mean convex side of $S_0^*(p)$, unless $H \equiv 1/2$, $\mathbb{M} = \mathbb{H}^2$ and $\Sigma = S_0$.*

Proof. Following [NS], consider the family $\mathcal{H}_\alpha$, $\alpha > 1$, of non embedded rotational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H = 1/2$ (see also [NSST]).

In this case the height function of the profile curve Γ_α of the surface $\mathcal{H}_\alpha$ is not monotonous. We only need to consider the non bounded connected component Γ_α^1 of Γ_α for which the height function is strictly increasing as a function of the distance to the revolution axis.

Let $p \in \mathbb{M}$ and Σ be a surface which is properly immersed in the mean convex side of $S_0^*(p)$, with mean curvature satisfying $|H| \leq 1/2$.

Let us denote by $\mathcal{G}_\alpha$ the piece of $\mathcal{H}_\alpha$ obtained by rotating the curve Γ_α^1. Thus, given polar coordinates at the point $p \in \mathbb{M}$, the graphs $\mathcal{G}_\alpha$ in $\mathbb{H}^2 \times \mathbb{R}$ can be deformed to graphs $\mathcal{G}_\alpha^*$ in $\mathbb{M} \times \mathbb{R}$. From the conditions of the height function of Γ_α^1 and using Proposition 1 and Proposition 2, we obtain that the mean curvature of these surfaces is bigger than or equal to $1/2$ everywhere.

Thus, we can use the family of graphs $\mathcal{G}_\alpha^*$ in order to show that Σ is a vertical translation of $S_0^*(p)$ or there must be an $\alpha_0 > 1$ such that Σ and $\mathcal{G}_{\alpha_0}^*$ intersect at a point and Σ is above $\mathcal{G}_{\alpha_0}^*$. The latter possibility can not happen from the maximum principle for the mean curvature of both surfaces.

Hence, Σ is a vertical translation of $S_0^*(p)$. So, Σ and $S_0^*(p)$ are congruent and they must have constant mean curvature $1/2$. Finally, from (5) and (6), it is easy to see that since the mean curvature of the entire graphs $S_0^*(p)$ and S_0 agree then the curvature of $\mathbb{M}$ is $K = -1$ for any $q \in \mathbb{M}$, and so $\mathbb{M}$ is, up to isometries, the hyperbolic plane. $\square$

As usual, we say that an end E of a surface in $\mathbb{M} \times \mathbb{R}$ is cylindrically bounded if there exists a compact set $D \subseteq \mathbb{M}$ such that E is contained in the vertical "cylinder" $D \times \mathbb{R}$. As a straightforward consequence of Theorem 3 we have,

Corollary 2. *Let $\mathbb{M}$ be a Hadamard surface with curvature $K \leq -1$. Assume Σ is a properly immersed surface in $\mathbb{M} \times \mathbb{R}$ with mean curvature $|H| \leq 1/2$ and cylindrically bounded ends. Then Σ must have more than one end.*

As in the Euclidean space $\mathbb{R}^3$, there also exist catenoids in $\mathbb{H}^2 \times \mathbb{R}$. These surfaces have been used by Sa Earp and toubiana [ST] in order to obtain an interesting result about the behavior at infinity of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. We can generalize that result as follows.

Theorem 4. *Let $\mathbb{M}$ be a Hadamard surface with pinched curvature $-1 \leq K \leq b < 0$. Let $\gamma \subseteq \partial_\infty \mathbb{M} \times \mathbb{R}$ be an arc. Assume there exist a vertical straight line $L \subseteq \partial_\infty \mathbb{M} \times \mathbb{R}$ and a subarc $\gamma' \subseteq \gamma$ such that*

(1) $\gamma' \cap L \neq \emptyset$ and $\partial\gamma' \cap L = \emptyset$,

(2) γ' stays on one side of L,

(3) $\gamma' \subseteq \partial_\infty \mathbb{M} \times (t_0, t_0 + \pi)$, for some real number t_0.

Then there is no properly immersed minimal surface (maybe with finite boundary), $\Sigma \subseteq \mathbb{M} \times \mathbb{R}$, with asymptotic boundary γ and such that $\Sigma \cup \gamma$ is a continuous surface with boundary.

Proof. We first observe that since $K \leq b < 0$, given two points $p_1, p_2 \in \partial_\infty \mathbb{M}$, there exists a unique geodesic $\Gamma \subseteq \mathbb{M}$ with p_1 and p_2 as its points at infinity. Thus, the idea of the proof follows as in [ST] where we need to consider now as a family of barriers the surfaces C^* in $\mathbb{M} \times \mathbb{R}$ obtained from the family of catenoids in $\mathbb{H}^2 \times \mathbb{R}$. Since $K \geq -1$, from Proposition 1 and Proposition 2 the new surfaces C^* in $\mathbb{M} \times \mathbb{R}$ have mean curvature $H \leq 0$ for the unit normal pointing to the interior of the neck, and they can be used as in [ST]. □

As a consequence of this result one has,

Corollary 3. *Let $\mathbb{M}$ be a Hadamard surface with $-1 \leq K \leq b < 0$. Consider a Jordan curve $\gamma \subseteq \partial_\infty \mathbb{M} \times \mathbb{R}$ homologous to zero (in $\partial_\infty \mathbb{M} \times \mathbb{R}$), which is contained in a open slab between two horizontal circles of $\partial_\infty \mathbb{M} \times \mathbb{R}$ with width equal to π. Then, there is no properly immersed minimal surface with asymptotic boundary γ.*

It is also possible to show that under certain conditions a constant mean curvature surface with boundary on a slice must be a graph. This result was proven by Nelli, Sa Earp, Santos and Toubiana [NSST] in $\mathbb{H}^2 \times \mathbb{R}$. For that, they use the rotational entire graphs of constant mean curvature $H \in [0, 1/2]$. By deforming these graphs we can also obtain,

Theorem 5. *Let $\mathbb{M}$ be a Hadamard surface with curvature satisfying $-c^2 \leq K \leq -1$, $c \geq 1$. Consider a compact surface Σ immersed in $\mathbb{M} \times \mathbb{R}$ with boundary a $\mathcal{C}^{2,\alpha}$ Jordan curve Γ with geodesic curvature greater than c, contained in a horizontal slice. Assume that Σ has constant mean curvature $H_0 \in [0, 1/2]$. Then Σ is a vertical graph.*

Proof. First of all, we observe that the result is obvious for $H_0 = 0$ from the maximum principle for Σ and the minimal surfaces given by the horizontal slices.

Thus, let Σ be a compact surface immersed in $\mathbb{M} \times \mathbb{R}$ with boundary $\Gamma \subseteq \mathbb{M} \times \{0\}$ and constant mean curvature $H_0 \in (0, 1/2]$. Let D be the bounded closed domain determined by $\Gamma \subseteq \mathbb{M} \equiv \mathbb{M} \times \{0\}$. Let us see that Σ is contained in the cylinder $D \times \mathbb{R}$.

Let $p_0 \in \mathbb{M}$ be a point in Γ. Consider the unique horocycle in $\mathbb{M}$ which is tangent to Γ at p_0 with the same unit normal. Since the curvature of $\mathbb{M}$ satisfies $K \in [-c^2, -1]$, we have that the geodesic curvature of this horocycle is between 1 and c. Using that Γ is a Jordan curve with geodesic curvature greater than c, it is easy to see that Γ must be contained in the convex closed domain $\overline{\Omega}$ determined by the previous horocycle.

Let $\gamma(s)$ be the oriented geodesic parameterized by the arc length which is normal to Γ with $\gamma(0) = p_0$ and $\gamma'(0)$ pointing to $\overline{\Omega}$. For each $s > 0$ consider the closed geodesic disk D_s centered at $\gamma(s)$ and radius s. Since $\overline{\Omega}$ is the limit of the disks D_s when s goes to infinity and Γ is a compact set, then there must exist $s_0 > 0$ such that D_{s_0} contains Γ.

Let S be the rotational entire graph with constant mean curvature H_0 in $\mathbb{H}^2 \times \mathbb{R}$. Consider its associated entire graph $S^* \subseteq \mathbb{M} \times \mathbb{R}$ when we use geodesic polar coordinates at $\gamma(s_0) \in \mathbb{M}$. Let us also denote by $\overline{S}^*$ the reflection of S^* with respect to a horizontal slice. The surfaces S^* and $\overline{S}^*$ are congruent, and from Proposition 1 and Proposition 2 we obtain that they have mean curvature $H \geq H_0$ for its unit normal pointing to the mean convex component.

Since S^* and $\overline{S}^*$ have been obtained using polar coordinates at $\gamma(s_0)$ we can assume that, up to vertical translations, $S^* \cap \mathbb{M} \times \{0\} = \partial D_{s_0} = \overline{S}^* \cap \mathbb{M} \times \{0\}$. In particular, Γ is contained in the mean convex components of S^* and $\overline{S}^*$, because $\Gamma \subseteq D_{s_0}$.

As Σ is compact we can move vertically S^* in such a way that Σ is completely contained in the mean convex component of S^*. Now, from the maximum principle, if we move back S^* then the surfaces Σ and S^* do not intersect until S^* is in its initial position (that is, when $S^* \cap \mathbb{M} \times \{0\} = \partial D_{s_0}$).

The same argument can be used for $\overline{S}^*$. Therefore, Σ is contained in the cylinder $D_{s_0} \times \mathbb{R}$ and there is no point of Σ in the vertical line $\{p_0\} \times \mathbb{R}$ except the point $(p_0, 0) \in \Gamma$.

Repeating the same argument for any point $(p_0, 0)$ in Γ we get that Σ is contained in the vertical cylinder $D \times \mathbb{R}$.

Now, let Σ_0 be the graph of constant mean curvature H_0 with boundary Γ (see [S]). Then, from the balancing formula for graphs with constant mean curvature $H_0 > 0$ [HLR, Proposition 3], it follows (see [NSST]) that Σ must agree with Σ_0 or the reflection of Σ_0 with respect to the slice $\mathbb{M} \times \{0\}$. □

The obtained comparison result can be also used for obtaining barriers which are subsolutions of certain partial differential equations. These subsolutions will be sufficient for showing existence of solutions to that equation. In fact, this work was motivated by the construction in [GR] of some specific barriers for minimal surfaces in $\mathbb{M} \times \mathbb{R}$. Similar barriers have been recently used in [FR], which can be seen as a case of our general description in Section 2.

Now, we give an example of existence of graphs with constant extrinsic curvature, that is, we obtain solutions to a Dirichlet problem for the Monge-Ampère equation associated to the graphs of positive constant extrinsic curvature. Let $r_c(k_0)$ be the radius of the geodesic disk of the space form $\mathbb{M}^2(c)$ where the rotational sphere of constant extrinsic curvature $k_0 > 0$ of $\mathbb{M}^2(c) \times \mathbb{R}$ is defined (see [EGR]).

Theorem 6. *Let D_r be a closed geodesic disk of radius $r > 0$ in $\mathbb{M}$, and c the maximum of the curvature of $\mathbb{M}$ in D_r. Take $k_0 > 0$ such that $r_c(k_0) = r$, then there exists a graph h of constant extrinsic curvature $k > 0$ in $\mathbb{M} \times \mathbb{R}$ and*

$$h_{|\partial D_r} = 0$$

for any $k < k_0$.

Proof. Let S_k be the sphere in $\mathbb{M}(c) \times \mathbb{R}$ with constant extrinsic curvature $k > 0$ (see [EGR]). Observe that, up to translations, S_{k_1} is contained in the open bounded component determined by S_{k_2} when $k_1 > k_2$. Hence, since $r = r_c(k_0)$ is the radius of the closed disk where the sphere S_{k_0} is defined then, up to vertical translations, for $k < k_0$ there exists a spherical cup C_k of S_k such that C_k is the strict graph of a function h_k in a closed geodesic disk of radius r, with $h_k = 0$ at the boundary.

If we consider the associated graph C_k^* in $\mathbb{M} \times \mathbb{R}$ using geodesic polar coordinates at the center of D_r then from Proposition 1 and Proposition 2 the surface C_k^* has extrinsic curvature greater than or equal to k. Thus, we have a subsolution for our Dirichlet problem.

Therefore, from [G] (see also [S]), there exists a graph of constant extrinsic curvature k and zero boundary data. $\square$

Lastly, we must remark that the process of obtaining barriers can be also carried out in higher dimensions in $\mathbb{M}^n \times \mathbb{R}$, though in this case the process is more intricate [GL].

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