Complete Laguerre minimal surfaces in $\mathbf{R}^3$

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Abstract

We study from a global point of view the Laguerre minimal surfaces in $\mathbb{R}^3$. In particular, we generalize in a natural way the class of Laguerre minimal surfaces (introducing what we call *Generalized Laguerre minimal surfaces*) and obtain a conformal representation formula for this new class of surfaces. Besides, we study the completeness of the Laguerre metric and classify the flat Laguerre minimal surfaces. Finally, we solve the Björling problem for Generalized Laguerre minimal surfaces and, as an application, we classify the rotational ones.

Keywords: Complete Laguerre minimal surfaces, flat Laguerre minimal surfaces, conformal representation, Björling problem.

1 Introduction

An oriented surface $\psi : S \longrightarrow \mathbb{R}^3$ with non-zero Gauss curvature K and mean curvature H is called a *Laguerre minimal surface* (in short, *L-surface*) if

$$\Delta^{III}\frac{H}{K} = 0,$$

where Δ^{III} stands for the Laplacian with respect to the third fundamental form III of ψ. The study of these surfaces traces back to a series of papers by W. Blaschke [3, 4, 5, 6], where such surfaces appear as critical points of the functional

$$L(\psi) = \int \frac{H^2 - K}{K} dS,$$

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dS being the area element of the surface. This functional is invariant under the 10-dimensional group of Laguerre transformations, which are transformations of the space of oriented spheres that preserve oriented contact of spheres and take planes to planes (see, for instance, [13] for a detailed study).

This theory has recently received much attention (see, among others, [8, 12, 13, 15, 16, 17, 19, 21]), mostly from a local point of view.

In this paper we deal with the study of L-surfaces from a global point of view. Thus, in Section 2 we generalize the definition of L-surface to surfaces whose Gaussian curvature can vanish in a set of isolated points and are Laguerre minimal in the classic sense away from that set. This is a natural class of surfaces that we will call *Generalized Laguerre minimal immersions* (in short, *GL-immersion*). In particular, this class contains the one of minimal surfaces in $\mathbb{R}^3$.

In Section 3 we provide a conformal representation for GL-surfaces in terms of a real harmonic function, a meromorphic one and a holomorphic 1-form globally defined on the surface (Theorem 2). In Section 4, we use this representation in order to study the completeness of the Laguerre metric h_L. Thus, we prove that every L-immersion with complete Laguerre metric must have negative Euclidean Gaussian curvature K everywhere (Theorem 3). On the other hand, we show that every Euclidean complete immersion with negative K must have complete Laguerre metric h_L (Theorem 4); however the converse is not true. In addition, we prove that h_L can be extended on a GL-surface to the points where $K = 0$, and provide a complete classification of the flat GL-surfaces for the Laguerre metric (Theorem 5).

Finally, in Section 5 we give a Björling-like formula which solves the Björling problem for GL-surfaces, that is, we study the set of GL-surfaces containing a given curve with a given tangent plane along every point of such curve (Theorem 6). As an application of that formula, we classify the rotational GL-surfaces.

2 Preliminaries

Let $\psi : S \longrightarrow \mathbb{R}^3$ be an immersion from an orientable, connected surface S into $\mathbb{R}^3$, with Gaussian curvature K and mean curvature H. Let us assume that both curvatures are related by

$$H(p) = R(p) \cdot K(p) \quad \forall p \in S, \tag{1}$$

for a smooth function $R : S \longrightarrow \mathbb{R}$.

Under these conditions we can define the quadratic form

$$\widetilde{h} = I - 2\, R\, II, \tag{2}$$

where I and II are, respectively, the induced metric and the second fundamental form of the immersion.

This new quadratic form is conformal to the third fundamental form of the immersion if $K \neq 0$, since

$$III = -K\, I + 2\, H\, II = -K\, \widetilde{h}. \tag{3}$$

Besides, if $K = 0$ at a point $p \in S$ then, from (1), $H(p) = 0$ and $II = 0$ at p. In particular, $\widetilde{h} = I$ at p.

With all of this, $\widetilde{h}$ must be a definite quadratic form. In addition, since III is a Riemannian metric wherever $K \neq 0$ and $\widetilde{h} = I$ at the points with $K = 0$ then, from (3), $\widetilde{h}$ is a Riemannian metric if $K \leq 0$ and so $-\widetilde{h}$ is when $K > 0$.

Proposition 1 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an immersion. If $R : S \longrightarrow \mathbb{R}^3$ is a smooth function such that $H = R\,K$ then one of the following situations happens:*

- *$K > 0$ everywhere on S, or*
- *$K \leq 0$ on S, and in this case either K vanishes identically on S and so ψ lies on a plane, or $\{p \in S : K(p) = 0\}$ is a set of isolated points.*

Proof: Since the quadratic form $\widetilde{h}$ given by (2) is definite everywhere, then $\widetilde{h}$ must be globally positive definite or negative definite at every point. Hence, as $\widetilde{h}$ is positive definite at a point if and only if $K \leq 0$ at that point, we can deduce that $K > 0$ everywhere or $K \leq 0$ at every point.

On the other hand, from (3), $III = -K\,\widetilde{h}$, which is also true at the points with $K = 0$. Thus, since $III := \langle dN, dN \rangle$, where N is the Gauss map of the immersion, we obtain that N is a conformal map for S with the structure induced by $\widetilde{h}$. Hence, dN vanishes at isolated points or dN vanishes identically, or equivalently, $K = 0$ at isolated points or $\psi(S)$ lies on a plane.

□

Remark 1 *In the conditions of the above Proposition, we can define the Riemannian metric h given by $h = \widetilde{h}$ if $K \leq 0$ and $h = -\widetilde{h}$ if $K > 0$. In addition, we have proved that the Gauss map N of the immersion is conformal for the metric h since $III = |K|\,h$.*

Definition 1 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an immersion and $R : S \longrightarrow \mathbb{R}$ a smooth function such that $H = R\,K$. We say that ψ is a Generalized Laguerre minimal immersion (in short,* GL-immersion*) if the Laplacian of R with respect to the Riemannian metric h vanishes identically, that is,*

$$\Delta^h R = 0.$$

Observe that the family of immersions satisfying this condition basically agrees with the family of Laguerre minimal immersions (in short, *L-immersion*), by taking $R = H/K$ and bearing in mind that h and III are conformal.

However, we allow the Gaussian curvature to vanish at some points. This is an important fact, because, for instance, minimal immersions are GL-immersions, but they are not L-immersions in general since the Gaussian curvature could vanish at some points. More generally, the Weingarten surfaces satisfying $H = c\,K$, $c \in \mathbb{R}$, are GL-surfaces.

Since the family of Laguerre minimal surfaces is invariant under the action of the Laguerre group, we can obtain GL-surfaces from a given one. Other examples can be obtained from dilations or normal translations:

Example 1 (Homothetic Surfaces) *Given a GL-surface $\psi : S \longrightarrow \mathbb{R}^3$, the homothetic surface $\widetilde{\psi} = c\,\psi$ for $c \in \mathbb{R}$, $c \neq 0$, is also a GL-surface with $\widetilde{R} = c\,R$. To check that, it is enough to observe that $\widetilde{K} = K/c^2$, $\widetilde{H} = H/c$ and the third fundamental forms of both immersions agree.* □

Example 2 (Parallel Surfaces) *Let $\psi : S \longrightarrow \mathbb{R}^3$ be a GL-surface with Gauss map N, Gaussian curvature K and mean curvature H. Then the parallel surface to ψ to a distance $a \in \mathbb{R}$, $\widetilde{\psi} = \psi + aN$, is an immersion wherever $1 - 2aH + a^2K \neq 0$, with Gaussian curvature and mean curvature given by (see, for instance, [20])*

$$\widetilde{K} = \frac{K}{1 - 2aH + a^2K},$$

$$\widetilde{H} = \frac{H - aK}{1 - 2aH + a^2K}.$$

Since the Gauss maps of both immersions agree then the third fundamental forms coincide. Moreover $\widetilde{R} = R - a$, and so we have that $\widetilde{\psi}$ is also a GL-immersion. □

We finish this Section by remarking that a compact GL-surface must be a round sphere. For that, it is enough to observe that any compact surface S in $\mathbb{R}^3$ must have a point with positive Gaussian curvature K. Thus, from Proposition 1, K is positive everywhere and the surface must be a topological sphere. On the other hand, since S is compact and R is harmonic on S, then R is a constant c_0. Therefore, the GL-surface is a Weingarten surface satisfying $H = c_0\,K$. In this case, it is classically known that S must be a round sphere (see, for instance, [7] or [11]).

3 A Conformal Representation

In this section we obtain an explicit conformal representation for GL-surfaces. A different representation for L-surfaces was given by Wang in [21]. In particular, Wang got a local conformal representation for an L-surface $u : \mathbb{D} \longrightarrow \mathbb{L}^4$, $u = (u_1, u_2, u_3, u_4)$, in the 4-dimensional Lorentz-Minkowski space $\mathbb{L}^4$. Then, from the 2-parameter family η of spheres in $\mathbb{R}^3$ centered at (u_1, u_2, u_3) with radius $-u_4$, he obtains two dual L-surfaces as the enveloping surfaces of η and proves that any L-surface can be locally obtained by this construction. We provide a global conformal representation which depends on three geometric data: a real harmonic function, a meromorphic one and a holomorphic 1-form.

We also refer the reader to [18], where a representation for L-isothermic surfaces which are L-minimal is obtained, and to [15, Proposition 8], where the authors provide a Weiertrass-like formula for nondegenerate L-minimal surfaces of spherical type.

Let $\psi : S \longrightarrow \mathbb{R}^3$ be an immersion with Gauss map $N = (N_1, N_2, N_3)$ and denote by $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ the composition of the stereographic projection from the north pole with N, that is,

$$g = \frac{N_1 + i\,N_2}{1 - N_3}.$$

As usual, we will also call g the Gauss map of the immersion ψ.

We observe that N can be obtained from g as

$$N = \left(\frac{g+\overline{g}}{1+|g|^2}, -i\frac{g-\overline{g}}{1+|g|^2}, \frac{-1+|g|^2}{1+|g|^2} \right). \tag{4}$$

From now on, if $\psi : S \longrightarrow \mathbb{R}^3$ is a GL-immersion then we will consider S as a Riemann surface with the conformal structure induced by h. Thus, from Remark 1, N is conformal for the metric h, or equivalently, $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ is a meromorphic function.

Moreover, R is a harmonic real function for the Riemann surface S, since $\Delta^h R = 0$. We denote by ∂R the (1,0)-part of dR for S, which is a holomorphic 1-form, that is, $\partial R = R_z\, dz$ for any local conformal parameter z for the Riemann surface S.

Theorem 2 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be a non flat GL-surface, with Gauss map g, such that $H = R\,K$. Then there exists a holomorphic 1-form ω such that the immersion ψ can be recovered as*

$$\psi = -R\,N + Re \int \left(\frac{\partial R + \omega}{g} + g\,(\partial R - \omega), i\,\frac{\partial R + \omega}{g} - i\,g\,(\partial R - \omega), 2\,\omega \right). \tag{5}$$

Conversely, let S be a simply connected Riemann surface, $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ a non constant meromorphic function, ω a holomorphic 1-form and $R : S \longrightarrow \mathbb{R}$ a real harmonic function, such that if g has a zero (resp. a pole) of order $n \in \mathbb{N}$ at $p \in S$, then $\partial R + \omega$ (resp $\partial R - \omega$) has a zero of order greater than or equal to n at p. Then (5) defines a GL-surface wherever

$$\left| \frac{\omega + \partial R}{g} + \bar{g}(\omega - \partial R) \right|^2 - R^2 \frac{4|dg|^2}{(1+|g|^2)^2} \neq 0 \tag{6}$$

with Gauss map g and $H = R\,K$.

Proof: Since ψ is a non flat immersion then, from Proposition 1, the points where the Gaussian curvature vanishes are isolated. Thus, if we denote by $S_0 = \{p \in S : \; K(p) \neq 0\}$ then $\psi : S_0 \longrightarrow \mathbb{R}^3$ is an L-immersion and

$$\Delta^{III}(\psi + R\,N) = 0$$

for all $p \in S_0$ (see, for instance [13]).

Since h and III are conformal Riemannian metrics on S_0, then $\Delta^h(\psi + R\,N) = 0$ on S_0 and so it is on S by continuity. Thus, we can consider the three holomorphic 1-forms

$$\Phi_k = \partial\,(\psi_k + RN_k)\,, \qquad k = 1, 2, 3.$$

On the other hand, if z is a local conformal parameter for h then the third fundamental form and the second fundamental form of the immersion can be written as

$$\begin{aligned} III &= 2\,\langle N_z, N_{\bar{z}}\rangle\,|dz|^2 \\ II &= \langle \psi_z, -N_z\rangle\,dz^2 + 2\langle \psi_z, -N_{\bar{z}}\rangle\,|dz|^2 + \langle \psi_{\bar{z}}, -N_{\bar{z}}\rangle\,d\bar{z}^2, \end{aligned}$$

where again we have used that h and III are conformal, that is, $\langle N_z, N_z\rangle \equiv 0$.

Hence, a straightforward computation gives (see also [14, Eq. 8])

$$\frac{\langle \psi_z, -N_{\bar z}\rangle}{\langle N_z, N_{\bar z}\rangle} = \frac{H}{K} = R$$

on S_0 and so, by continuity, $\langle \psi_z + R\, N_z, N_{\bar z}\rangle = 0$ on S.

This let us obtain

$$\langle (\phi_1, \phi_2, \phi_3)\,, N_{\bar z}\rangle = \langle (\psi + RN)_z\,, N_{\bar z}\rangle = \langle \psi_z + R\, N_z, N_{\bar z}\rangle = 0, \tag{7}$$

where $\Phi_k = \phi_k\, dz$, $k = 1, 2, 3$.

From (4),

$$N_{\bar z} = \frac{\overline{g_z}}{(1+|g|^2)^2}\left(1-g^2, i(1+g^2), 2g\right)$$

and from (7) one gets

$$(1-g^2)\phi_1 + i(1+g^2)\phi_2 + 2g\phi_3 = 0. \tag{8}$$

On the other hand, we obtain $R_z = \langle (\psi + R\, N)_z\,, N\rangle = \langle (\phi_1, \phi_2, \phi_3)\,, N\rangle$, that is, using (4)

$$(g+\overline{g})\,\phi_1 - i(g-\overline{g})\,\phi_2 + (|g|^2-1)\,\phi_3 = R_z(1+|g|^2). \tag{9}$$

By using (8) and (9), the functions ϕ_1 and ϕ_2 can be obtained as

$$\begin{aligned} 2\,\phi_1 &= \frac{R_z+\phi_3}{g} + g\,(R_z-\phi_3), \\ 2\,\phi_2 &= i\,\frac{R_z+\phi_3}{g} - i\,g\,(R_z-\phi_3). \end{aligned}$$

Therefore, by taking $\omega = \Phi_3$, we obtain from

$$\psi_z + (R\, N)_z = (\phi_1, \phi_2, \phi_3)$$

the representation formula (5).

Conversely, let S be a Riemann surface and consider $\psi : S \longrightarrow \mathbb{R}^3$ given by (5). From the conditions between the zeroes and poles of g, $\partial R + \omega$ and $\partial R - \omega$, we obtain that the three 1-forms which must be integrated in (5) are well-defined on S and so the map ψ is.

Given a local conformal parameter z for S, then

$$\langle \psi_z, \psi_{\bar z}\rangle^2 - \langle \psi_z, \psi_z\rangle\langle \psi_{\bar z}, \psi_{\bar z}\rangle =$$

$$\left(\frac{(1+|g|^2)^4|\phi_3|^2 - 4|g|^2|g_z|^2R^2 + (1-|g|^4)^2|R_z|^2 + (1+|g|^2)^3(1-|g|^2)(\phi_3 R_{\bar z} + \overline{\phi_3} R_z)}{2|g|^2(1+|g|^2)^2}\right)^2,$$

where $\omega = \phi_3 dz$. Thus, ψ is an immersion if and only if (6) is satisfied.

A direct computation gives $\langle \psi_z, N\rangle = 0$, that is, N, or equivalently g, is the Gauss map of the immersion.

Moreover, the second and third fundamental forms are given by

$$\begin{aligned} II &= -\left(\omega + \frac{1-|g|^2}{1+|g|^2}\,\partial R\right)\frac{dg}{g} + \frac{4\,R\,|dg|^2}{(1+|g|^2)^2} - \left(\overline{\omega} + \frac{1-|g|^2}{1+|g|^2}\,\overline{\partial} R\right)\frac{d\overline{g}}{\overline{g}} \\ III &= \frac{4\,|dg|^2}{(1+|g|^2)^2} \end{aligned} \tag{10}$$

Hence, at any point where $K \neq 0$ (or equivalently $g_z \neq 0$) we have

$$\frac{H}{K} = \frac{\langle \psi_z, -N_{\bar z}\rangle}{\langle N_z, N_{\bar z}\rangle} = R,$$

and so $H = R\,K$ on S by continuity.

In addition, since the third fundamental form and the Riemannian metric h are conformal, and $\langle N_z, N_z\rangle \equiv 0$, the conformal structure given by h is the one of S. Therefore, $\Delta^h R = 0$, that is, ψ is a GL-immersion.

□

This conformal representation coincides with the classical Weierstrass representation for minimal surfaces when $R = 0$, and is also valid for the linear Weingarten surfaces satisfying $H = c_0\,K$, $c_0 \in \mathbb{R}$.

Moreover, if ψ lies on a plane, the previous representation also works by taking g as a constant.

Finally, we observe that the Gauss curvature of the GL-immersion can be easily computed from (10) as

$$K = \frac{\frac{4|dg|^2}{(1+|g|^2)^2}}{R^2\,\frac{4|dg|^2}{(1+|g|^2)^2} - \left|\frac{\omega+\partial R}{g} + \bar g(\omega - \partial R)\right|^2}. \tag{11}$$

4 Completeness of the Laguerre metric

Let ψ be an L-immersion in $\mathbb{R}^3$. It is well-known that the quadratic form

$$h_L = \frac{H^2-K}{K^2}\;III$$

is invariant under the Laguerre group and, in fact, it is a Riemannian metric if ψ has no umbilical point, that is, if $H^2 - K \neq 0$. This is called the *Laguerre metric* of the immersion.

From (10) and (11) the Laguerre metric can be computed as

$$h_L = \frac{H^2-K}{K^2}\;III = \left(R^2 - \frac{1}{K}\right)\;III = \left|\frac{\omega+\partial R}{g} + \bar g(\omega - \partial R)\right|^2. \tag{12}$$

As a consequence we obtain a global property on complete L-surfaces.

Theorem 3 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an L-immersion with complete Laguerre metric. Then the Euclidean Gaussian curvature of the immersion is negative everywhere.*

Proof: Let us assume that the immersion has a point of positive Euclidean Gaussian curvature. Then, from Proposition 1, $K > 0$ everywhere. So, from (12), the conformal metric $R^2\, III$ satisfies $h_L \leq R^2\, III$ and then $R^2\, III$ is a complete metric on S (with R different from 0 everywhere).

Since R is harmonic and III is a metric of constant curvature 1, we obtain that the curvature of the conformal metric $R^2\, III$ must be positive at every point. Thus, using the classical Huber Theorem, we obtain that S is conformally equivalent to either a compact surface or a compact surface minus a finite number of points. In particular, S is a parabolic Riemann surface and R is a harmonic function on S with $R \neq 0$. Therefore, R must be a constant c_0.

In such a case, $R^2\, III = c_0^2\, III$ is a complete metric on S of positive constant curvature and so, using Gauss-Bonnet, S is a topological sphere.

On the other hand, from (10), the (2,0)-part of II with respect to the Riemann surface S is the holomorphic quadratic form $-\omega\, dg/g$. But a quadratic holomorphic form on a sphere must vanish identically. Therefore, from (10), II and III are conformal metrics, or equivalently, ψ is a totally umbilical immersion. But this is impossible since h_L is not degenerate, which proves the result.

$\square$

Remark 2 *A GL-immersion with Euclidean Gaussian curvature $K > 0$ at any point and associated complete metric h must be a totally umbilical round sphere. For proving that, it is enough to observe that the conformal metric $R^2\ III$ is complete since*

$$h = 2\, R\, II - I = \frac{1}{K}\ III \leq R^2\ III,$$

and the proof follows as in the previous theorem.

In particular we have an alternative proof of the fact that a compact GL-surface must be a totally umbilical round sphere.

We also study the relation between the Euclidean completeness of an immersion and the completeness of the Laguerre metric.

Theorem 4 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an Euclidean complete immersion with negative curvature. Then the Laguerre metric h_L of the immersion is complete.*

Proof: Let p be a point of S. Since $K(p) < 0$, there exist doubly orthogonal coordinates (u, v) in a neighbourhood of p such that the first fundamental form and the second fundamental form of the immersion are given by

$$I = E\, du^2 + G\, dv^2, \qquad II = k_1\, E\, du^2 + k_2\, G\, dv^2$$

where E, G are positive functions and k_1, k_2 are its principal curvatures.

Thus, the third fundamental form is written as $III = k_1^2\,E\,du^2 + k_2^2\,G\,dv^2$ and the Laguerre metric as

$$\begin{aligned} h_L &= \frac{H^2-K}{K^2}\,III = \frac{1}{4}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)^2\left(k_1^2\,E\,du^2 + k_2^2\,G\,dv^2\right) = \\ &= \frac{1}{4}\left(\left(1-\frac{k_1}{k_2}\right)^2 E\,du^2 + \left(1-\frac{k_2}{k_1}\right)^2 G\,dv^2\right) \geq \frac{1}{4}\,I, \end{aligned} \tag{13}$$

where we have used that $k_1k_2 < 0$.

This inequality proves that the completeness of I implies that the Laguerre metric is also complete.

□

Let $\psi : S \longrightarrow \mathbb{R}^3$ be a non flat GL-immersion with Euclidean Gaussian curvature $K \leq 0$. It is worth pointing out that the Laguerre metric is a well defined metric on the set of isolated points with $K = 0$. For that, let us first observe that, from (12), h_L is a well-defined quadratic form on every GL-immersion since $(\omega + \partial R)/g$ and $g(\omega - \partial R)$ are holomorphic 1-forms (Theorem 2). Besides, if $p \in S$ with $K(p) = 0$ then there exists a punctured neighborhood of p where $K < 0$. Thus, from (13), $I \leq 4h_L$ in this neighborhood, and so $I(p) \leq 4h_L(p)$. Therefore, h_L is also a Riemannian metric at p.

On the other hand, it is important to bear in mind that the converse of Theorem 4 is not true, that is, the Laguerre completeness does not imply the Euclidean completeness. Let us show this in the following example.

Example 3 *Let us consider the L-immersion given by Theorem 2 for $S = \mathbb{C} \equiv \mathbb{R}^2$ with*

$$R = \frac{z^2+\bar{z}^2}{2}, \qquad \omega = -z\,dz, \qquad g(z) = -\frac{1}{z}, \qquad z \in \mathbb{C}.$$

That is,

$$\psi(u,v) = \left(\frac{-2u(1+2v^2)}{1+u^2+v^2}, \frac{-2v(1+2u^2)}{1+u^2+v^2}, \frac{-2u^2+2v^2}{1+u^2+v^2}\right)$$

where $z = u + i\,v$ (see Figure 1).

A direct computation gives us

$$I = \frac{4(1+2v^2)^2}{(1+u^2+v^2)^2}\,du^2 + \frac{4(1+2u^2)^2}{(1+u^2+v^2)^2}\,dv^2$$

and a complete Laguerre metric $h_L = 4(du^2 + dv^2)$.

It is easy to see that the curve $\alpha(t) = \psi(t,0)$, $t \geq 0$, has Euclidean length

$$L(\alpha) = \int_0^\infty \frac{2}{1+t^2}\,dt = \pi,$$

which proves that the induced metric I is not complete. □

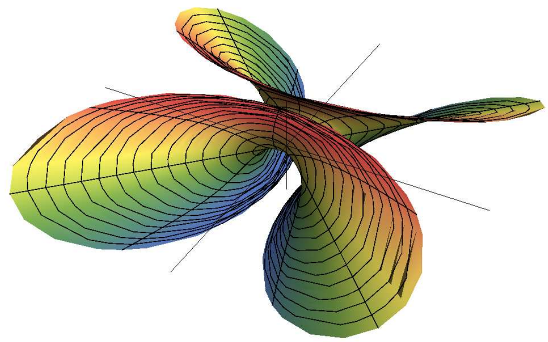

Figure 1: Complete flat L-surface.

This example gives us a complete flat immersion for the Laguerre metric with vanishing Laguerre form (see [12, Theorem 5.1]). In [19], Song and Wang prove that an L-surface with constant Laguerre curvature must be flat and this surface is Laguerre equivalent to a surface with vanishing mean curvature in a degenerate hyperplane $\mathbb{R}^3_0$ of the 4-dimensional Lorentz-Minkowski space.

We use the conformal representation given by Theorem 2 in order to classify the GL-surfaces in $\mathbb{R}^3$ with flat Laguerre metric.

Theorem 5 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be a GL-surface with flat Laguerre metric. Then, up to an isometry of $\mathbb{R}^3$, the immersion can be locally parameterized as*

$$\psi(u,v) = \left(2u - \frac{4\,R\,R_u}{4+R_u^2+R_v^2}, -2v + \frac{4\,R\,R_v}{4+R_u^2+R_v^2}, \frac{8\,R}{4+R_u^2+R_v^2}\right) \qquad (14)$$

for suitable local conformal parameters $(u,v) \in \Omega \subseteq \mathbb{R}^2$, where the harmonic function $R(u,v)$ must satisfy $|R\,R_{zz}| \neq 1 + |R_z|^2$, $z = u + iv$.

Moreover, if h_L is complete then ψ is, up to an isometry of $\mathbb{R}^3$, globally parameterized as in (14) with $(u,v) \in \mathbb{R}^2$ and $|R\,R_{zz}| < 1 + |R_z|^2$.

Proof: Let us take $p \in S$ with Euclidean Gaussian curvature $K(p) \neq 0$. Up to a rotation in $\mathbb{R}^3$ we can assume $N(p) = (0,0,-1)$, or equivalently, $g(p) = 0$. Now, using (12) and that h_L is flat, we have that the metric agrees locally with the modulus of a holomorphic 1-form, that is,

$$\left|\frac{\omega + \partial R}{g} + \bar{g}(\omega - \partial R)\right|^2 = |a(z)dz|^2 \qquad (15)$$

in a neighborhood of p, where $a(z)$ is a holomorphic function for the local conformal parameter z.

Since $K(p) \neq 0$ we have that $dg \neq 0$ at p, and using that g is meromorphic with $g(p) = 0$ then we can take $\zeta = g$ as a local parameter in a neighborhood of p.

On the other hand, since h_L cannot vanish at p and $g(p) = 0$, the holomorphic 1-form $(\omega + \partial R)/g$ cannot vanish in a neighborhood of p. Thus, from (15), we obtain

$$|1 + \bar{\zeta}\, b(\zeta)|^2 = |c(\zeta)|^2 \tag{16}$$

in a neighborhood of $\zeta = 0$, where $b(\zeta)$ and $c(\zeta)$ are holomorphic functions with

$$b(\zeta) = g\, \frac{\omega - \partial R}{\omega + \partial R}. \tag{17}$$

If we evaluate (16) at $\zeta = 0$ we obtain $|c(0)| = 1$. Moreover, if we derive (16), with respect to ζ, $n \geq 2$ times, we have

$$\bar{\zeta}\, \frac{d^n b(\zeta)}{d\zeta^n} + \overline{\zeta\, b(\zeta)}\, \frac{d^n(\zeta\, b(\zeta))}{d\zeta^n} = \overline{c(\zeta)}\, \frac{d^n c(\zeta)}{d\zeta^n}.$$

Thus, for $\zeta = 0$, we obtain $\frac{d^n c(\zeta)}{d\zeta^n}(0) = 0$ for any $n \geq 2$, because $|c(0)| = 1$. Hence, $c(\zeta) = c_0 + c_1\, \zeta$, and so $b(\zeta)$ must be a constant b_0 from (16).

Then, from (17), we obtain that

$$g\, \frac{\omega - \partial R}{\omega + \partial R} = b_0$$

in a neighborhood of p. But this must be globally true on S by analyticity.

Therefore, the holomorphic 1-form ω can be recovered from g and R as

$$\omega = \frac{g + b_0}{g - b_0}\, \partial R,$$

and

$$h_L = 4 \left| \frac{(1 + b_0\, \bar{g})\partial R}{g - b_0} \right|^2 = 4 \left| \frac{(1 + \overline{b_0}\, g)\partial R}{g - b_0} \right|^2.$$

Since h_L is a flat metric, given $q \in S$, there exists a conformal parameter $w = u + iv$ in a neighborhood of q such that $h_L = 4|dw|^2 = 4(du^2 + dv^2)$. Moreover, if h_L is complete then the parameter w is globally defined in $\mathbb{C} \equiv \mathbb{R}^2$.

So, up to a change of parameter $\widetilde{w} = e^{i\theta}\, w$, $\theta \in \mathbb{R}$, if necessary, we have

$$\frac{(1 + \overline{b_0}\, g)\partial R}{g - b_0} = dw.$$

Hence, we can calculate g and ω in terms of R as

$$g = \frac{b_0 + R_w}{1 - \overline{b_0}\, R_w}, \qquad \omega = \frac{2b_0 + (1 - |b_0|^2)R_w}{1 + |b_0|^2}. \tag{18}$$

Thus, using Theorem 2, the immersion is, up to a translation, given by

$$\begin{aligned}
\psi_1(u,v) &= \frac{(1-b_1^2+b_2^2)(2u-\frac{4RR_u}{4+R_u^2+R_v^2})-2b_1b_2(-2v+\frac{4RR_v}{4+R_u^2+R_v^2})-2b_1\frac{8R}{4+R_u^2+R_v^2}}{1+b_1^2+b_2^2},\\
\psi_2(u,v) &= \frac{-2b_1b_2(2u-\frac{4RR_u}{4+R_u^2+R_v^2})+(1+b_1^2-b_2^2)(-2v+\frac{4RR_v}{4+R_u^2+R_v^2})-2b_2\frac{8R}{4+R_u^2+R_v^2}}{1+b_1^2+b_2^2},\\
\psi_3(u,v) &= \frac{2b_1(2u-\frac{4RR_u}{4+R_u^2+R_v^2})+2b_2(-2v+\frac{4RR_v}{4+R_u^2+R_v^2})+(1-b_1^2-b_2^2)\frac{8R}{4+R_u^2+R_v^2}}{1+b_1^2+b_2^2},
\end{aligned}$$

where $b_0 = b_1 + ib_2$.

Therefore, the immersion is given by (14) when we compose with the isometry of $\mathbb{R}^3$, $\Phi(x) = A\,x$, for the orthogonal matrix

$$A = \frac{1}{1+b_1^2+b_2^2}\begin{pmatrix} 1-b_1^2+b_2^2 & -2b_1b_2 & 2b_1 \\ -2b_1b_2 & 1+b_1^2-b_2^2 & 2b_2 \\ -2b_1 & -2b_2 & 1-b_1^2-b_2^2 \end{pmatrix}.$$

Let us observe that the formula (14) is obtained from Theorem 2 when we take $b_0 = 0$ in (18). Hence, from (6), the immersion is not degenerated at any point if, and only if, $|R\,R_{zz}| \neq 1 + |R_z|^2$. In addition, if h_L is complete then, from Theorem 3 and using that h_L is well-defined at the isolated points with $K = 0$, its Euclidean Gaussian curvature must be non positive. So we finally get $|R\,R_{zz}| < 1 + |R_z|^2$ (see (11)).

□

5 The Björling Problem for GL-Surfaces

Let $\psi : S \longrightarrow \mathbb{R}^3$ be a non flat GL-surface. Consider a real interval I, a regular analytic curve $\alpha(s) : I \longrightarrow S$ and take $\beta(s) = (\psi\circ\alpha)(s)$, $V(s) = (N\circ\alpha)(s)$, $r(s) = (R\circ\psi\circ\alpha)(s)$.

By the inverse function theorem there exists a local conformal parameter z for the metric h, defined in a complex domain Ω containing I, such that its real part is s, i.e. $z = s + it$. Thus,

$$\psi(s,0) = \beta(s) \quad \forall s \in I, \tag{19}$$
$$N(s,0) = V(s) \quad \forall s \in I, \tag{20}$$
$$R(s,0) = r(s) \quad \forall s \in I. \tag{21}$$

Our aim is to recover the immersion ψ in a neighborhood of the curve $\alpha(s)$ from (19), (20) and (21). However, we will observe that this information does not provide uniqueness, and so we will need to add another condition. In particular we will see that adding the initial data

$$R_t(s,0) := d(s) \quad \forall s \in I$$

to the previous ones, we can recover ψ univocally. That is, we want to recover completely the immersion ψ in terms of the value of ψ, N, R and R_t along a curve $\alpha(s)$. Among others, we refer the reader to [1, 2, 9, 10] where different Björling type problems are studied.

In order to do that, let us consider the holomorphic map

$$\begin{aligned}\Phi(z) &= (\psi(z)+R(z)N(z))_z \\ &= \frac{1}{2}\left\{\frac{\partial\psi}{\partial s}(z)+R(z)\frac{\partial N}{\partial s}(z)+\frac{\partial R}{\partial s}(z)N(z)\right\} \\ &\quad -\frac{i}{2}\left\{\frac{\partial\psi}{\partial t}(z)+R(z)\frac{\partial N}{\partial t}(z)+\frac{\partial R}{\partial t}(z)N(z)\right\}.\end{aligned}$$

This map can be computed along the curve α as

$$\begin{aligned}\Phi(s) &= \frac{1}{2}\{\beta'(s)+r(s)V'(s)+r'(s)V(s)\} \\ &\quad -\frac{i}{2}\left\{\frac{\partial\psi}{\partial t}(s)+r(s)\frac{\partial N}{\partial t}(s)+d(s)V(s)\right\},\end{aligned}$$

where $\psi_t(s), N_t(s)$ must be calculated.

Let us write the second and third fundamental forms as

$$II = eds^2+2fdsdt+gdt^2, \quad III=\lambda(ds^2+dt^2),$$

where we have used that z is a conformal parameter for h and so for III.

Since $\langle N_s, N_s\rangle = \langle N_t, N_t\rangle$ and $\langle N_s, N_t\rangle = 0$ we have $-N_t = N\times N_s$ and so

$$-N_t(s) = V(s)\times V'(s).$$

Up to at the isolated points where the Euclidean Gauss curvature vanishes, N_s and N_t are a basis of the tangent plane. Hence, from the expressions of II and III we have

$$\psi_t = \frac{1}{\lambda}\left(-f\,N_s - g\,N_t\right),$$

wherever $K\neq 0$.

Since $\lambda(s) = \langle V'(s), V'(s)\rangle$, $f(s) = \langle\beta'(s), V(s)\times V'(s)\rangle$ and $g(s)$ can be calculated from the equality

$$R=\frac{H}{K}=\frac{e+g}{2\lambda}$$

as $g(s) = 2r(s)\langle V'(s), V'(s)\rangle + \langle V'(s), \beta'(s)\rangle$, we obtain

$$\psi_t(s) = -\frac{\langle\beta'(s), V(s)\times V'(s)\rangle}{\langle V'(s), V'(s)\rangle}V'(s) - \left(2r(s)+\frac{\langle V'(s),\beta'(s)\rangle}{\langle V'(s), V'(s)\rangle}\right)V(s)\times V'(s).$$

With all of this,

$$\Phi(s) = \frac{1}{2}\beta'(s)+a_1(s)\,V(s)+a_2(s)\,V'(s)+a_3(s)\,V(s)\times V'(s) \tag{22}$$

where

$$a_1(s) = \frac{r'(s) - i\,d(s)}{2}, \quad a_2(s) = \frac{1}{2}\left(r(s) + i\,\frac{\langle \beta'(s), V(s) \times V'(s)\rangle}{\langle V'(s), V'(s)\rangle}\right),$$
$$a_3(s) = -\frac{i}{2}\left(r(s) + \frac{\langle V'(s), \beta'(s)\rangle}{\langle V'(s), V'(s)\rangle}\right).$$

Therefore, the holomorphic function $\Phi(z) = (\psi(z) + R(z)N(z))_z$ can be recovered as the complex analytic extension of $\Phi(s)$ in a complex domain $D \subseteq \Omega$ with $I \subseteq D$.

Analogously, the holomorphic function $g(z)$ can be obtained as the complex analytic extension of

$$g(s) = \frac{V_1(s) + iV_2(s)}{1 - V_3(s)}. \tag{23}$$

So, $N(z)$ can be calculated from $g(z)$ using (4).

In the same way, since R_z is a holomorphic function and

$$R_z(s) = \frac{1}{2}(R_s(s) - iR_t(s)) = \frac{1}{2}(r'(s) - id(s)),$$

we get

$$R(z) = Re(r(z)) + Im\left(\int_{z_0}^{z} d(w)\,\mathrm{d}w\right), \tag{24}$$

with $z_0 \in I$.

This allows us to recover the immersion in terms of the initial data:

Theorem 6 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be a non flat GL-surface, I a real interval, $\alpha(s) : I \longrightarrow S$ a regular analytic curve, $\beta(s) = (\psi \circ \alpha)(s)$, $V(s) = (N \circ \alpha)(s)$ and $r(s) = (R \circ \psi \circ \alpha)(s)$. Let z be a local conformal parameter for the metric h, defined in a complex domain Ω containing I, such that its real part is s, i.e. $z = s + it$, and take $d(s) = (R \circ \psi)_t \circ \alpha(s)$. Then we can recover ψ in a certain domain $D \subseteq \Omega$, $I \subseteq D$, as*

$$\psi(z) = \beta(z_0) + r(z_0)\,V(z_0) - R(z)\,N(z) + 2\,Re\int_{z_0}^{z} \Phi(w)\,\mathrm{d}w, \tag{25}$$

where $\Phi(z)$ is the complex analytic extension of (22), $N(z)$ is obtained from the complex analytic extension $g(z)$ of (23), $R(z)$ is given by (24) and $z_0 \in I$.

Conversely, let $\beta(s), V(s) : I \longrightarrow \mathbb{R}^3$ be two non constant analytic curves with $\langle \beta'(s), V(s)\rangle = 0$ and let $r(s), d(s) : I \longrightarrow \mathbb{R}$ be two analytic functions. Then the map $\psi(z)$ given by (25) is, at its regular points, a GL-surface with Gauss map N and conformal parameter z, satisfying $H = R\,K$. Moreover, for $s \in I$ one gets $\psi(s) = \beta(s)$, $N(s) = V(s)$, $R(s) = r(s)$ and $R_t(s) = d(s)$, where $z = s + i\,t$.

Proof: For the converse part, let us start by observing that

$$\langle (\psi + R\,N)_z, N_{\bar z}\rangle = \frac{\overline{g_z}}{(1 + |g|^2)^2}\langle \Phi(z), (1 - g^2, i(1 + g^2), 2g)\rangle.$$

As $\langle \Phi(z), (1-g^2, i(1+g^2), 2g)\rangle$ is a holomorphic function and a direct computation gives $\langle (\psi + R\,N)_z, N_{\bar z}\rangle = 0$ for $z = s \in I$ then

$$\langle (\psi + R\,N)_z, N_{\bar z}\rangle \equiv 0. \tag{26}$$

From this equality, and using that $R(z)$ is harmonic and $|N(z)| = 1$, we have

$$\begin{aligned} \tfrac{\partial}{\partial \bar z}\langle \psi_z, N\rangle &= \langle \psi_{z\bar z}, N\rangle + \langle \psi_z, N_{\bar z}\rangle = \langle -RN_{z\bar z}, N\rangle + \langle -(RN)_z, N_{\bar z}\rangle \\ &= -R(\langle N_{z\bar z}, N\rangle + \langle N_z, N_{\bar z}\rangle) = 0. \end{aligned}$$

In addition, since $\langle \psi_z, N\rangle = 0$ on I, we have that $\langle \psi_z, N\rangle \equiv 0$ and N is the unit normal of ψ.

On the other hand, since $g(z)$ is holomorphic, or equivalently, $N(z)$ is a conformal map, one has that z is a conformal parameter for III at the points where $\psi(z)$ is an immersion with $K(z) \neq 0$.

From (26), $\langle \psi_z + RN_z, N_{\bar z}\rangle \equiv 0$, and so at the regular points with $K(z) \neq 0$ one gets

$$\frac{H}{K} = \frac{\langle \psi_z, -N_{\bar z}\rangle}{\langle N_z, N_{\bar z}\rangle} = R.$$

But since $K(z) = 0$ only happens at the points where $g'(z) = 0$, we have that the regular points with vanishing K are isolated. Therefore, $H = R\,K$ at the regular points of ψ, z is a conformal parameter for h (because h and III are conformal) and $\Delta^h R = 0$ as we wanted to show.

Finally, it is a straightforward computation to prove that the initial conditions for $z = s \in I$ are satisfied.

□

As an example of the use of the above theorem, we will study the rotational GL-surfaces. It is important to observe that the following process can be realized not only for revolution GL-surfaces but for any GL-surface invariant under an 1-parameter group of the 10-dimensional Laguerre group.

Let S be a rotational surface with respect to the axis $OZ = \{(0,0,x_3) : x_3 \in \mathbb{R}\}$. Up to a Laguerre transformation given by a vertical translation and a dilation, we can assume that $(1,0,0) \in S$.

Since S is a rotational surface, the curve given up to a reparameterization by

$$\beta(s) = (\cos s, \sin s, 0), \qquad s \in \mathbb{R},$$

is contained in S, and the unit normal along $\beta(s)$ must be given by the rotation of $V(0) = (\cos\rho, 0, \sin\rho)$ for a certain constant $\rho \in [0, 2\pi)$, that is,

$$V(s) = (\cos\rho\cos s, \cos\rho\sin s, \sin\rho), \qquad s \in \mathbb{R}.$$

Observe that if $\cos\rho = 0$ then $V(s) = (0,0,\pm 1)$ and so the Gauss map $g(z)$ is a constant, that is, S lies on a plane. Otherwise, $g(s) = e^{i\,s}\cos\rho/(1-\sin\rho)$ and

$$g(z) = \frac{e^{i\,z}\cos\rho}{(1-\sin\rho)}.$$

Hence, the unit normal is given by

$$N(s+i\,t)=\left(\frac{2ae^t\cos s}{a^2+e^{2t}},\frac{2ae^t\sin s}{a^2+e^{2t}},\frac{a^2-e^{2t}}{a^2+e^{2t}}\right),$$

where $a=\cos\rho/(1-\sin\rho)$.

Let us note that the image of the unit normal along a parallel of S must be a parallel of the unit sphere. So, from the expression of N we have that the parallels of S are obtained when t is constant. On the other hand, R must be constant along any parallel of the revolution surface S. Hence R only depends on t and so

$$R(s+i\,t)=r+d\,t$$

for certain constants $r,d\in\mathbb{R}$, since R must be harmonic with respect to the conformal parameter $z=s+i\,t$.

Now, the map $\Phi(s)$ given by (22) can be computed as

$$\Phi(s)=\left(\frac{-A\sin s-i\,B\,\cos s}{2},\frac{A\cos s-i\,B\,\sin s}{2},-\frac{i}{2}\,C\right)$$

where $A=1+r\cos\rho$, $B=-\sin\rho+\cos\rho(d-r\,\sin\rho)$ and $C=\cos\rho(1+r\cos\rho)+d\sin\rho$.

Therefore, S can be recovered as

$$\begin{aligned}
\psi_1&=\left(A\cosh t+B\sinh t-(r+d\,t)\frac{2\,a\,e^t}{a^2+e^{2t}}\right)\cos s\\
\psi_2&=\left(A\cosh t+B\sinh t-(r+d\,t)\frac{2\,a\,e^t}{a^2+e^{2t}}\right)\sin s\\
\psi_3&=C\,t+D-(r+d\,t)\frac{a^2-e^{2t}}{a^2+e^{2t}}
\end{aligned}$$

with $D=r\sin\rho$.

In order to simplify the expression of ψ we consider the reparameterization $\psi^*(s,t)=\psi(s,t+t_0)$ where we choose t_0 such that $e^{t_0}=a$ if $a>0$. Then, using the relations of the constants a,d,A,B,C

$$A=\frac{C+d}{2\,a}+\frac{a(C-d)}{2},\qquad B=\frac{C+d}{2\,a}-\frac{a(C-d)}{2},$$

we obtain, up to a vertical translation,

$$\begin{aligned}
\psi_1^*&=\left(C\cosh t+d\left(\sinh t-\frac{t}{\cosh t}\right)-r^*\frac{1}{\cosh t}\right)\cos s\\
\psi_2^*&=\left(C\cosh t+d\left(\sinh t-\frac{t}{\cosh t}\right)-r^*\frac{1}{\cosh t}\right)\sin s\\
\psi_3^*&=C\,t+(d\,t+r^*)\tanh t,
\end{aligned}$$

where $r^*=r+d\,t_0$. If $a<0$ then we choose t_0 such that $e^{t_0}=-a$. In this case, we obtain, up to a reparameterization, one of the previous examples. Therefore, we have proved

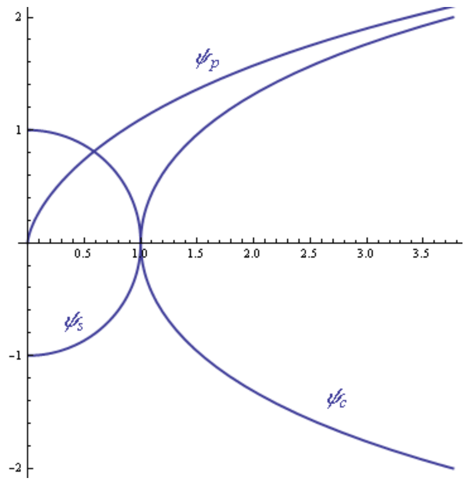

Figure 2: Generatrices of ψ_c, ψ_s and ψ_p

Proposition 7 *Any non flat revolution GL-surface with respect to the OZ axis is, up to a vertical translation, given by*

$$a_1\psi_c + a_2\psi_s + a_3\psi_p$$

for three real constants a_1, a_2, a_3*, where* ψ_c *is a catenoid,* ψ_s *is a sphere and* ψ_p *is a revolution surface with a pick (see Figure 2), parameterized as,*

$$\begin{aligned}
\psi_c(s,t) &= (\cosh t \cos s, \cosh t \sin s, t), \\
\psi_s(s,t) &= \left(\frac{\cos s}{\cosh t}, \frac{\sin s}{\cosh t}, -\tanh t\right), \\
\psi_p(s,t) &= \left((\sinh t - \frac{t}{\cosh t}) \cos s, (\sinh t - \frac{t}{\cosh t}) \sin s, t \tanh t\right).
\end{aligned}$$

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