A D'Alembert formula for flat surfaces in the 3-sphere

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Keywords: flat Möbius strips, asymptotic curves, Björling problem, geometric Cauchy problem, wave equation
AMS Subject Classification: 53C42, 53A10

Abstract

We find all the flat surfaces in the unit 3-sphere $\mathbb{S}^3$ that pass through a given regular curve of $\mathbb{S}^3$ with a prescribed tangent plane distribution along this curve. The formula that solves this problem may be seen as a geometric analogue of the classical D'Alembert formula that solves the Cauchy problem for the homogeneous wave equation. We also provide several applications of this geometric D'Alembert formula, including a classification of the flat Möbius strips of $\mathbb{S}^3$.

1 Introduction

Given a Riemannian 3-manifold $\bar{M}^3$ together with a class of surfaces $\mathcal{A}$ that are immersed in $\bar{M}^3$, we can formulate the geometric Cauchy problem for the class $\mathcal{A}$ as follows:

> Let $\beta(s)$ denote a regular curve in $\bar{M}^3$, and let $\Pi(s)$ denote a distribution of oriented planes along β in the tangent bundle of $\bar{M}^3$, such that $\beta'(s) \in \Pi(s)$ for all s. Find all the surfaces belonging to the class $\mathcal{A}$ that pass through $\beta(s)$ and whose tangent plane distribution along this curve is precisely $\Pi(s)$.

It is clear that this problem is just a geometric version of the usual Cauchy problem for second order partial differential equations. It has its roots in the classical Björling problem [Bjo] for minimal surfaces in $\mathbb{R}^3$, and it has been considered in several geometric theories, such as minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^n$ [DHKW, GaMi2, Mir, Sch], maximal surfaces in the Lorentz-Minkowski space [ACM], surfaces with $H = 1$ in $\mathbb{H}^3$ [GaMi3], flat surfaces in $\mathbb{H}^3$ [GaMi4], or improper affine spheres in $\mathbb{R}^3$ [ACG].

The objective of the present paper is to solve the geometric Cauchy problem for the class of flat surfaces in the unit 3-sphere $\mathbb{S}^3$. It is important to emphasize that the situation here changes completely with respect to the previous works on the geometric Cauchy problem listed above. Indeed, all the above classes of surfaces have underlying elliptic PDEs with some associated holomorphic quantities. Consequently, the surfaces are real analytic, the geometric Cauchy problem for them is *ill-posed*, it is necessary to prescribe analytic initial data, and both existence and uniqueness results are available in that case for the problem. However, in the present setting of flat surfaces in $\mathbb{S}^3$, the underlying PDE is hyperbolic, what creates two basic difficulties. One is that the surfaces will not be analytic anymore, and thus by gluing procedures one can create many unhandy situations. The other one is that uniqueness of the solution to the Cauchy problem will break at *characteristic directions*. So, the study of the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ cannot follow the path suggested by other (elliptic) theories.

The theory of flat surfaces in $\mathbb{S}^3$ constitutes a quite singular situation. Its study traces back to Bianchi's works in the 19th century, and it has a very rich global theory, as evidenced by the existence of a large class of flat tori in $\mathbb{S}^3$. Indeed, these flat tori constitute the only examples of compact surfaces of constant curvature in space forms that are not totally umbilical round spheres. From a PDEs viewpoint, these surfaces are related to the homogeneous wave equation, while in general the constant curvature surfaces in $\mathbb{S}^3$ are related to elliptic or hyperbolic sin-Gordon or sinh-Gordon equations. So, flat surfaces in $\mathbb{S}^3$ admit a more explicit treatment than other constant curvature surfaces. Moreover, there are still important open problems regarding flat surfaces in $\mathbb{S}^3$, some of them unanswered for more than 30 years. All these facts together show that the geometry of flat surfaces in $\mathbb{S}^3$ is a worth studying topic. It is therefore somehow surprising that the number of contributions to the theory is not too large. In the authors' opinion, this is mainly due to two reasons: (1) the topic is not widely known, and (2) the techniques used to study these flat surfaces are specific to the theory, and must then be learned *ad hoc*. So, in order to contribute to a development of the theory in accordance to its interest, we have found it appropriate to include in Section 2 a brief survey on the geometry of flat surfaces in $\mathbb{S}^3$, including the known results, the fundamental techniques, and the most important open problems.

The scheme of the paper is the following one. Section 2 will revise the theory of flat surfaces in $\mathbb{S}^3$. In Section 3 we will solve the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ when the initial data are non-characteristic. Its solution will be given by a certain construction process, that we have christened as the *D'Alembert formula* for flat surfaces in $\mathbb{S}^3$, since it may be regarded as a geometric analogue of the solution to the Cauchy problem for the 2-dimensional wave equation. In Section 4 we will extend our discussion by allowing the initial data to point at characteristic directions. In that way we will lose in general both existence and uniqueness, but we will be able to consider more general applications. Probably the most important of these applications will be the classification of the flat surfaces in $\mathbb{S}^3$ with the topology of a Möbius strip. This will be accomplished in Section 5.

2 Flat surfaces in the 3-sphere

This section is devoted to expose the basics of the theory of flat surfaces in the unit 3-sphere $\mathbb{S}^3$. We will begin by describing the fundamental equations of flat surfaces in terms of asymptotic parameters. Then we will describe $\mathbb{S}^3$ as well as the usual Hopf fibration in terms of quaternions. By means of this model for $\mathbb{S}^3$, we will explain the classical Bianchi method via which flat surfaces in $\mathbb{S}^3$ are constructed by multiplying two intersecting asymptotic curves. We will also describe a refinement of this method due to Kitagawa [Kit1], which has been the fundamental tool for studying flat surfaces in $\mathbb{S}^3$ from a global viewpoint. Afterwards, we will expose the most significative global results regarding complete flat surfaces and flat tori in $\mathbb{S}^3$. Finally, we will discuss some of the most important open problems of the theory. The basic references for most of what follows are [GaMi1, Kit1, Spi, Wei2].

2.1 Asymptotic parameters

Generally, the fundamental equations of a flat surface in $\mathbb{S}^3$ are better understood by means of parameters whose coordinate curves are asymptotic curves on the surface. First, let us observe that, as the surface is flat, its intrinsic Gauss curvature vanishes identically. Consequently, by the Gauss' equation, the extrinsic (or Gauss-Kronecker) curvature of the surface is $K_{\text{ext}} = -1$. In this situation, as K_{ext} is negative, it is classically known (see [Spi]) that there exist *Tschebyscheff coordinates* around every point. This simply means that we can choose local coordinates (u, v) such that: (a) the u-curves and the v-curves are asymptotic curves of the surface, and (b) these curves are parametrized by arclength. In this way, the first, second and third fundamental forms of the surface are given by

$$\begin{aligned} I &= du^2 + 2\cos\omega\, dudv + dv^2, \\ II &= 2\sin\omega\, dudv, \\ III &= du^2 - 2\cos\omega\, dudv + dv^2. \end{aligned} \tag{2.1}$$

for a certain smooth function ω, called the *angle function*. This function ω has two basic properties: Firstly, as I is regular, we must have $0 < \omega < \pi$. Secondly, the Gauss equation of the surface translates into $\omega_{uv} = 0$. In other words, the angle ω verifies the homogeneous wave equation, and thus it can be locally decomposed as $\omega(u, v) = \omega_1(u) + \omega_2(v)$, where ω_1 and ω_2 are smooth real functions. Let us point out here that as the flat surfaces in $\mathbb{S}^3$ are described by the homogeneous wave equation, which is hyperbolic, it turns out that flat surfaces in $\mathbb{S}^3$ will not be real analytic in general. This will prove to be essential for our purposes later on.

As these Tschebyscheff coordinates (T-coordinates from now on) are essential for the local study of flat surfaces, it is important to understand when are they *globally available* on a surface, in order to develop a global theory. The most classical step in this direction is to observe that *any simply-connected complete flat surface in* $\mathbb{S}^3$ *has globally defined T-coordinates*. A proof of this fact can be found in [Spi], for instance. If we drop the trivial topology assumption, this is no longer true. Alternatively, the situation in which completeness is dropped is discussed in [GaMi1]. It turns out that simply connected flat surfaces in $\mathbb{S}^3$ do not possess in general globally defined T-coordinates, but instead, they admit a globally defined *Tschebyscheff immersion*. In other words, we can take two maps u, v from the surface into $\mathbb{R}$ that verify all the properties of T-coordinates, except for the fact that the map (u, v) into $\mathbb{R}^2$ may not be injective. The existence of this T-immersion is enough in many cases to deal globally with non-complete flat surfaces in $\mathbb{S}^3$. At last, it is also proved in [GaMi1] that *T-coordinates are globally available on any (not necessarily complete) simply connected real-analytic flat surface in* $\mathbb{S}^3$.

2.2 The quaternionic model for $\mathbb{S}^3$

The best way to describe explicitly flat surfaces in $\mathbb{S}^3$ is to regard the 3-sphere as the set of unit quaternions. Let us explain this model for $\mathbb{S}^3$ briefly.

We begin by identifying $\mathbb{R}^4$ with the quaternions in the standard way, that is, (x_1, x_2, x_3, x_4) is viewed as the quaternion $x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$. In that way the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is regarded as the space of unit quaternions, i.e. quaternions x with unit norm, $||x|| = 1$. We also point out that $\mathbb{S}^2 \equiv \mathbb{S}^3 \cap \{x_1 = 0\}$ can be seen as the space of purely imaginary unit quaternions.

The advantage of this model is that, by using the usual product of quaternions, the space $\mathbb{S}^3$ receives in a natural way a Lie group structure. Indeed, if $x, y \in \mathbb{S}^3$, then $x\,y \in \mathbb{S}^3$. Moreover, the left and right translations $x \mapsto x\,a$ and $x \mapsto a\,x$ turn out to be isometries for the standard Riemannian metric of $\mathbb{S}^3$. In other words, the metric of $\mathbb{S}^3$ is bi-invariant with respect to this Lie group structure. Thus we have for any $x, y, a \in \mathbb{S}^3$ that $\langle x, y\rangle = \langle a\,x, a\,y\rangle = \langle x\,a, y\,a\rangle$.

Apart from multiplication, there is another operation with quaternions that will be useful to us: the *conjugation* $x \mapsto \bar{x}$. It turns out that conjugation is geometrically an orientation reversing isometry in $\mathbb{S}^3$. Moreover we have $\bar{x} = x^{-1}$ whenever $x \in \mathbb{S}^3$, and in addition $\bar{x} = -x$ if $x \in \mathbb{S}^2 \subset \mathbb{S}^3$. Let us remark that $\overline{x\,y} = \bar{y}\,\bar{x}$

We end up the description of the geometry of $\mathbb{S}^3$ via quaternions with the introduction of the usual *Hopf fibration*. Let us define $\mathrm{Ad}(x)y := x\,y\,\bar{x}$, where $x, y \in \mathbb{S}^3$. Then the

Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $h(x) = \mathrm{Ad}(x)\mathbf{i} = x\,\mathbf{i}\,\bar{x}$. It follows immediately that the fibers $h^{-1}(p)$ are great circles of $\mathbb{S}^3$. The Hopf fibration will be crucial for describing flat surfaces in $\mathbb{S}^3$. Let us also remark that one can define *skew Hopf fibrations* by means of $h_\xi(x) := \mathrm{Ad}(x)\xi : \mathbb{S}^3 \to \mathbb{S}^2$, where $\xi \in \mathbb{S}^2$ is fixed but arbitrary. The fibers will still be great circles of $\mathbb{S}^3$.

2.3 First examples of flat surfaces in $\mathbb{S}^3$

The simplest way to obtain flat surfaces in $\mathbb{S}^3$ is by means of the Hopf fibration. This is due to the following remark by H.B. Lawson (see [Spi, Pin]): *if c is a regular curve in $\mathbb{S}^2$, then $h^{-1}(c)$ is a flat surface in $\mathbb{S}^3$.* As the fibers of the Hopf fibration are geodesics of $\mathbb{S}^3$, it turns out that $h^{-1}(c)$ has, in general, the topology of a cylinder. This is the reason why these surfaces $h^{-1}(c)$ are called *Hopf cylinders*.

Moreover, if the chosen regular curve c in $\mathbb{S}^2$ is closed, the resulting Hopf cylinder $h^{-1}(c)$ is compact and has the topology of a torus. Thus, it is named a *Hopf torus*. Moreover, if c is embedded, the resulting Hopf cylinder (or torus) is embedded. This provides a large family of flat tori and complete flat cylinders in $\mathbb{S}^3$, some of which are actually embedded. Moreover, they can be explicitly calculated once we know the curve c in $\mathbb{S}^2$.

The simplest choice of the curve in $\mathbb{S}^2$ is when c is a circle. Then $h^{-1}(c)$ is congruent to a *product torus* $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}^3$. All these tori are commonly known as *Clifford tori*, even though in many papers the Clifford torus corresponds to the minimal case $r = 1/\sqrt{2}$.

Once here, let us try to figure out the general situation of flat surfaces in $\mathbb{S}^3$, by understanding the behavior of the asymptotic curves of Hopf cylinders. As we know, any Hopf cylinder is foliated by great circles, which are actually the fibers of the Hopf fibration. It can be seen that these great circles are actually asymptotic curves of the Hopf cylinder. Now, at any point of the surface there must be another asymptotic curve. If $h^{-1}(c)$ is congruent to a product torus, this second asymptotic curve is also a great circle in $\mathbb{S}^3$, for every point. If the curve c has non-constant curvature, it can be shown that this second asymptotic curve has torsion $\tau = \pm 1$ at points where its curvature does not vanish. Moreover, the angle function ω in (2.1) of a Hopf cylinder depends only on one variable (u or v). Furthermore, ω is constant exactly when $h^{-1}(c)$ is congruent to a piece of a Clifford torus. We will see next how these considerations extend to the general case of flat surfaces in $\mathbb{S}^3$.

2.4 The Bianchi-Spivak construction of flat surfaces in $\mathbb{S}^3$

The process that follows is a reformulation following [Spi] of a 19th century work by Bianchi [Bia]. It will show that a flat surface in $\mathbb{S}^3$ can be locally recovered by means of the two asymptotic curves that exist at an arbitrary point of the surface.

First of all, observe that the existence of T-coordinates around any point of a flat surface in $\mathbb{S}^3$ implies the existence of two different smooth families of asymptotic curves around any point: the u-curves and the v-curves. But now, as $K_{\text{ext}} = -1$, the Beltrami-Enneper theorem in $\mathbb{S}^3$ (see [Spi]) proves that any asymptotic curve must have (at its regular points, i.e. point with non-vanishing curvature) torsion $\tau = \pm 1$. Moreover, if the elements of one of the two asymptotic families have torsion 1, the elements of the other family must have torsion -1 (at regular points).

Using then these facts as well as the Lie group structure of $\mathbb{S}^3$ discussed above, one can arrive as in [Spi] to the fundamental property of the asymptotic curves of flat surfaces in $\mathbb{S}^3$: *two asymptotic curves of a flat surface in* $\mathbb{S}^3$ *that belong to the same asymptotic family only differ by a (left or right) translation in* $\mathbb{S}^3$. In particular, around any point of a flat surface in $\mathbb{S}^3$ there only exist at most two non-congruent asymptotic curves.

This last result is the key to produce a general way to construct locally flat surfaces in $\mathbb{S}^3$: let $a(u)$ and $b(v)$ be two regular curves in $\mathbb{S}^3$ parametrized by arc-length, and such that a (resp. b) has non-vanishing curvature and torsion 1 (resp. -1). Assume moreover that $a(0) = b(0) = \mathbf{1}$ (the unit quaternion, i.e. the north pole of $\mathbb{S}^3$), and that $a'(0)$ and $b'(0)$ are not collinear. Then

$$f(u, v) = a(u)\, b(v) : I \times J \to \mathbb{S}^3$$

is a flat surface in $\mathbb{S}^3$ passing through $\mathbf{1}$, and whose asymptotic curves at this point are precisely $a(u)$ and $b(v)$. This process also works if one of the curves (or both) is a geodesic of $\mathbb{S}^3$.

Moreover, it also turns out that *essentially* any flat surface in $\mathbb{S}^3$ can be locally recovered by this process (up to a rigid motion of $\mathbb{S}^3$ ensuring that the initial conditions are fulfilled). Here *essentially* means that we need to impose that the curvature of the asymptotic curves is nowhere zero, or vanishes identically.

It might be interesting to remark additionally that if a is a curve in $\mathbb{S}^3$ with torsion 1, then $\bar{a}$ is a curve with torsion -1.

Up to now, this process is local. Nevertheless, Spivak observed in [Spi] that the same construction algorithm works globally under a completeness assumption on the surface. So, it was Spivak's merit to recover from oblivion the old Bianchi's results, and to formulate them globally, thus stepping the path for the development of the theory that has taken place over the last 20 years. In addition, Spivak also raised several questions and problems that guided the first steps of the research on flat surfaces in $\mathbb{S}^3$, and even some of them remain unsolved at the present time.

2.5 The Kitagawa representation

The Bianchi-Spivak construction algorithm has two basic problems. First, it is not able to treat in a diligent way the situation in which an asymptotic curve has non-isolated points with zero curvature. And second, it is not clear at first how to compute explicitly curves in $\mathbb{S}^3$ of torsion ± 1, so it can be difficult to use the method as it stands in order to investigate properties of flat surfaces in $\mathbb{S}^3$.

In 1988 Kitagawa was able to settle both matters in [Kit1], what showed him the way of giving an answer to some of the questions posed by Spivak.

First of all, Kitagawa observed that the condition for a curve α in $\mathbb{S}^3$ to have torsion ± 1 at points with curvature $\kappa \neq 0$ can be formulated in an equivalent way that does not need a discussion on the points where $\kappa = 0$. More specifically, the above condition is equivalent to the existence of some $\xi_0 \in \mathbb{S}^2$ such that $\langle \alpha', \alpha\, \xi_0 \rangle = 0$ (for torsion 1) or $\langle \alpha', \xi_0\, \alpha \rangle = 0$ (for torsion -1).

Secondly, Kitagawa was able to give a geometric construction that allows the construction of curves with torsion ± 1 without solving the corresponding differential equation. To explain it, let us consider the unit tangent bundle $U\mathbb{S}^2$ of $\mathbb{S}^2$, that can be seen as

$$U\mathbb{S}^2 = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle x, y \rangle = 0\}.$$

Then, given $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}$ and $\mathbf{i}$, we can define the map $\pi : \mathbb{S}^3 \to U\mathbb{S}^2$ given by

$$\pi(x) = (\mathrm{Ad}(x)\mathbf{i}, \mathrm{Ad}(x)\xi_0)\,.$$

This map is a double cover with $\pi(x) = \pi(-x)$ for every $x \in \mathbb{S}^3$. With this, the key observation by Kitagawa was the following one: let c denote a regular curve in $\mathbb{S}^2$ with tangent indicatrix c^*, and define $\widehat{c} := (c, c^*)$, with values in $U\mathbb{S}^2$. Thus, there is a regular curve a in $\mathbb{S}^3$ (unique up to an initial condition) such that $\pi(a) = \widehat{c}$. In particular, $h(a) = c$. Then c' is collinear with $\mathrm{Ad}(a)\xi_0$, and consequently we get

$$\langle a', a\, \xi_0 \rangle = 0. \tag{2.2}$$

But this means, as we saw above, that the curve a in $\mathbb{S}^3$ has torsion 1 at points with non-vanishing curvature. And conversely, any regular curve a in $\mathbb{S}^3$ verifying (2.2) can be constructed by the above process.

This construction also tells that $a(u)$ must necessarily be an asymptotic curve of the Hopf cylinder $h^{-1}(c)$, belonging to the *non-trivial* asymptotic family, that is, the one which is not made up by the fibers (great circles) of the Hopf fibration.

Given a curve c in $\mathbb{S}^2$, Kitagawa called the *asymptotic lift* of c to any of these non-trivial asymptotic curves of $h^{-1}(c)$. This concept is well defined just taking into account that two such asymptotic curves only differ by a left or right translation in $\mathbb{S}^3$, due to the Bianchi-Spivak results.

Using the above results, Kitagawa [Kit1] was able to give a more general and handier method to construct flat surfaces in $\mathbb{S}^3$. Let us expose in detail this method following [GaMi1]:

Theorem 1 (Kitagawa's representation) *Let $c_1(u), c_2(v)$ be two regular curves in $\mathbb{S}^2$, with $c_i(0) = \mathbf{i}$, $c_i'(0) = \xi_0$, for some $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}, \mathbf{i}$, and that verify the condition*

$$k_1(u) \neq k_2(v) \qquad \text{for all } u, v,$$

where here k_1, k_2 are the geodesic curvatures of $c_1(u)$ and $c_2(v)$, respectively. Let $\pi : \mathbb{S}^3 \to U\mathbb{S}^2$ be the double cover given by

$$\pi(x) = (\mathrm{Ad}(x)\mathbf{i}, \mathrm{Ad}(x)\xi_0)\,,$$

consider $a_1(u), a_2(v)$ *two curves in* $\mathbb{S}^3$ *parametrized by arclength and satisfying* $\pi(a_i) = (c_i, c_i'/||c_i'||)$*, and define*

$$\begin{aligned} \Phi(u,v) &= a_1(u)\,\bar{a_2}(v) \\ N(u,v) &= a_1(u)\,\xi_0\,\bar{a_2}(v). \end{aligned}$$

on a rectangle R *in the* u, v*-plane. If* Σ *is a simply connected surface and* $\Psi : \Sigma \to \Psi(\Sigma) = R$ *is an immersion, then* $f = \Phi \circ \Psi$ *is a flat surface in* $\mathbb{S}^3$ *with unit normal* $N \circ \Psi$*. In that case* Ψ *is a coordinate Tschebyscheff immersion, and the angle function of this flat surface is*

$$\omega(u,v) = \cot^{-1}(k_1(u)) - \cot^{-1}(k_2(v)).$$

Conversely, every analytic flat surface in $\mathbb{S}^3$ *is constructed in this way for some* ξ_0*.*

There are several remarks that should be made regarding this result. First of all, the hypothesis of analyticity for the converse is essential, as there are examples of flat surfaces in $\mathbb{S}^3$ with three mutually non-congruent asymptotic curves [GaMi1]. Nevertheless, the converse always works locally, and also for complete flat surfaces with bounded mean curvature. Moreover, in these cases the map Ψ can be assumed to be injective, and thus (u, v) constitute globally defined T-parameters.

2.6 Fundamental global results

M. Spivak raised in [Spi] several questions regarding the geometry of flat surfaces and flat tori in $\mathbb{S}^3$. It is no surprise that the attempts of answering these questions have produced the basis for the global development of the theory.

The classification of flat tori: this is a problem posed by S.T. Yau [Yau], that was solved by Kitagawa [Kit1] and Weiner [Wei1] from two different perspectives. In [Kit1] Kitagawa used its representation theorem to prove that *the asymptotic curves of a flat torus in* $\mathbb{S}^3$ *are periodic*, thus answering a question by Spivak. This result showed that any flat tori is generated by the construction process exposed above if the two regular curves c_1, c_2 in $\mathbb{S}^2$ are closed.

An alternative classification was given by Weiner. It can be shown that the generalized Gauss map $\mathcal{G} : \Sigma \to G_{2,4} \equiv \mathbb{S}^2 \times \mathbb{S}^2$ into the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ of a flat torus in $\mathbb{S}^3$ is the product of two closed curves $\gamma_1 \times \gamma_2 \subset \mathbb{S}^2 \times \mathbb{S}^2$. In [Wei1] Weiner gave a necessary and sufficient condition for the curves γ_i that describes exactly when $\gamma_1 \times \gamma_2$ is the Gauss map of a flat torus in $\mathbb{S}^3$.

Embedded flat tori in $\mathbb{S}^3$: It is known that a Hopf torus $h^{-1}(c)$ is embedded if and only if its generating curve c in $\mathbb{S}^2$ is embedded. The embeddedness condition for a general flat torus in $\mathbb{S}^3$ was considered in [Kit3] and [DaSh]. More specifically, in [DaSh] it was obtained a structure theorem for embedded flat tori in $\mathbb{S}^3$, in terms of a certain topological condition on the curves c_i in $\mathbb{S}^2$ of the Kitagawa representation theorem. It was also proved in [Kit3, DaSh] that embedded flat tori in $\mathbb{S}^3$ have antipodal symmetry.

Non-orientable flat surfaces in $\mathbb{S}^3$: In [Spi] Spivak posed the problem of studying the non-orientable flat surfaces in $\mathbb{S}^3$. In response to this problem, Kitagawa showed in [Kit1] that *any complete flat surface in $\mathbb{S}^3$ is orientable*. Nevertheless, the existence of non-orientable (non-complete) flat surfaces in $\mathbb{S}^3$ was still unclear. In [GaMi1] it was shown that *any real analytic flat surface in $\mathbb{S}^3$ is orientable*, which contrasts with the $\mathbb{R}^3$ situation. Moreover, this condition cannot be weakened to smoothness, as there are examples of flat Möbius strips in $\mathbb{S}^3$, constructed in [GaMi1].

2.7 Open problems

One of the features of the theory of flat surfaces in $\mathbb{S}^3$ is the existence of very basic questions that have not been answered up to now, and whose solution is likely to be quite complicated. This is one of the main points that make the theory interesting. We will expose in this last part of the section just the most relevant ones.

Existence of an isometric embedding from $\mathbb{R}^2$ into $\mathbb{S}^3$: this is surely the biggest open problem in the theory. A complete flat surface has the topology of a plane, a cylinder, a torus, a Möbius strip or a Klein bottle. Of these, the two non-orientable cases are impossible if the flat surface is isometrically immersed in $\mathbb{S}^3$, as we saw before. Moreover, it is known that there exist both embedded flat tori (like the Clifford tori) and complete embedded flat cylinders in $\mathbb{S}^3$ [DaSh]. However, the existence of a complete embedded simply connected flat surface in $\mathbb{S}^3$ is unknown. This problem was first posed by Spivak [Spi] using a slightly different formulation. In [DaSh] it was conjectured that the problem has a negative answer, i.e. *the Euclidean plane $\mathbb{R}^2$ cannot be isometrically embedded into $\mathbb{S}^3$*.

Rigidity of Clifford tori: The rigidity problem is a fundamental topic in submanifold theory. It asks whether two different isometric immersions of a Riemannian manifold M^n into another Riemannian manifold N^{n+p} must necessarily differ just by an isometry of the ambient space N^{n+p}. If this is the case, it is said that M^n is *rigid* in N^{n+p}. As the simplest flat surfaces in $\mathbb{S}^3$ are the Clifford tori $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$, it is quite natural to ask if these tori are rigid in $\mathbb{S}^3$.

The problem can also be interpreted in an interesting alternative way. A flat torus $\mathbb{T}$ can be identified with a parallelogram with identified opposite edges. The case where this parallelogram is actually a rectangle corresponds to the case in which $\mathbb{T}$ is (intrinsically) isometric to a Clifford torus. In this case, we will call $\mathbb{T}$ a *rectangular flat torus*. So, the rigidity problem for the Clifford tori can be formulated as: *is an isometric immersion of a rectangular flat torus into $\mathbb{S}^3$ necessarily congruent to a product torus $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$?*.

This problem has been an attractive one among specialists, and some natural conditions under which Clifford tori are rigid have been achieved (see for instance [EKW, Kit2, Kit3, Kit5]). Nevertheless, the original rigidity question remains unanswered.

The space of isometric immersions of a flat torus: as we exposed above, the flat tori in $\mathbb{S}^3$ have been classified in [Kit1] in terms of two curves in $\mathbb{S}^2$ satisfying some compatibility conditions, and in [Wei1] in terms of their Gauss maps. However, the following natural classification problem has not been settled: *given an abstract flat torus $\mathbb{T}$, which is the space of isometric immersions of $\mathbb{T}$ into $\mathbb{S}^3$?*

Flat surfaces with singularities: In recent years there has been an increasing interest on surfaces with a certain type of admissible singularities, called *fronts*. In particular, flat fronts in $\mathbb{R}^3$ and $\mathbb{H}^3$ have been studied in detail in [GaMi4, GMM, KUY, KRSUY, KRUY, MuUm, Roi, SUY]. It seems an interesting question to investigate how flat fronts in $\mathbb{S}^3$ behave. The first step into this direction was given in [GaMi1], where it was shown that flat fronts in $\mathbb{S}^3$ are an important tool in the problem of classifying (regular) isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$.

3 The non-characteristic Cauchy problem

Let $\beta : I \subseteq \mathbb{R} \to \mathbb{S}^3$ and $V : I \subseteq \mathbb{R} \to \mathbb{S}^3$ denote smooth curves in $\mathbb{S}^3$ such that

$$\langle \beta'(s), \beta'(s) \rangle \neq 0, \quad \text{and in addition} \quad \langle \beta(s), V(s) \rangle = \langle \beta'(s), V(s) \rangle = 0, \tag{3.1}$$

for all $s \in I$. Note that, in particular, the condition $\langle \beta(s), V(s) \rangle = 0$ means that $V(s) \in T_{\beta(s)}\mathbb{S}^3$.

The geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ consists in finding all the flat surfaces in $\mathbb{S}^3$ that pass through $\beta(s)$ and whose unit normal field along this curve is $V(s)$.

Observe that this formulation is the restriction of the general geometric Cauchy problem posed in Section 1 to the case of flat surfaces in $\mathbb{S}^3$. Here, we are prescribing the unit normal along the curve, which is obviously equivalent to prescribing the distribution of tangent planes along this curve.

Definition 2 *We will refer to the pair of curves $\{\beta(s), V(s)\}$ satisfying (3.1) as the* Cauchy data *or* initial data *of the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$.*

We will say that the Cauchy data are non-characteristic *if $\langle \beta'(s), V'(s) \rangle \neq 0$, $\forall s \in I$.*

Remark 3 *If S is a flat surface in $\mathbb{S}^3$ which is a solution to the Cauchy problem for the initial data $\{\beta(s), V(s)\}$, then these data will be non-characteristic exactly when $\beta'(s)$ never points at an asymptotic direction of S at $\beta(s)$ (see Subsection 2.1).*

Let us also note that, from (2.1), the asymptotic directions $\{\partial_u, \partial_v\}$ of the surface are the characteristic directions of the associated wave equation $\omega_{uv} = 0$.

Before solving the Cauchy problem for flat surfaces, let us point out a couple of basic algebraic facts regarding the quaternionic model for $\mathbb{S}^3$ (see Subsection 2.2). Given quaternions x, y it is immediate to check that

$$\langle x, y \rangle = \mathrm{Re}(x\overline{y})$$

where $\mathrm{Re}(x_1, x_2, x_3, x_4) = x_1$ is the real part of the quaternion. Now, since $\mathbb{S}^2 \equiv \mathbb{S}^3 \cap \{x_1 = 0\}$ we have that

$$\langle x, y \rangle = 0 \Longleftrightarrow x\overline{y} \in \mathbb{S}^2. \tag{3.2}$$

We are now ready to solve the non-characteristic Cauchy problem for flat surfaces in $\mathbb{S}^3$.

Theorem 4 (A geometric D'Alembert Formula) *Given $\{\beta, V\}$ non-characteristic initial data, the associated geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ has a solution. Moreover:*

i) The solution is unique in the following sense: If S_1 and S_2 are two solutions to the Cauchy problem, then they agree on an open set $U \subseteq S_1 \cap S_2$ containing $\beta(I)$.

ii) In addition, this unique solution can be explicitly constructed as follows: we can assume, up to a rigid motion, that there exists $s_0 \in I$ such that $\beta(s_0) = \mathbf{1}$ and $V(s_0) = \xi_0$ for a certain $\xi_0 \in \mathbb{S}^2$ satisfying $\xi_0 \perp \mathbf{i}$. Thus, we define

$$\gamma_1(s) := V(s)\,\overline{\beta(s)}, \qquad \gamma_2(s) := \overline{\beta(s)}\,V(s), \tag{3.3}$$

which are regular curves in $\mathbb{S}^2$, and take $u(s)$ and $v(s)$ as

$$u(s) := \frac{1}{2}\int_{s_0}^{s} \sqrt{\langle \gamma_1', \gamma_1' \rangle}, \qquad v(s) := \frac{1}{2}\int_{s_0}^{s} \sqrt{\langle \gamma_2', \gamma_2' \rangle}, \tag{3.4}$$

which are diffeomorphisms onto their images $J_1 = u(I)$, $J_2 = v(I)$. Let h_{ξ_0} be the skew Hopf fibration given by $h_{\xi_0}(x) = x\xi_0\overline{x}$ for all $x \in \mathbb{S}^3$, and let us take $\delta_i(s)$, $i = 1, 2$, regular curves in $\mathbb{S}^3$ such that $h_{\xi_0}(\delta_i(s)) = \gamma_i(s)$ and $\delta_i(s_0) = \mathbf{1}$. Now we define

$$\widetilde{a}_i(s) := \delta_i(s)(\cos\theta_i(s)\,\mathbf{1} + \sin\theta_i(s)\,\xi_0), \qquad i = 1, 2, \tag{3.5}$$

where

$$\theta_i(s) = -\int_{s_0}^{s} \langle \delta_i'(r), \delta_i(r)\,\xi_0 \rangle dr,$$

which are regular curves with respective arclength parameters $u(s)$ (for $i = 1$) and $v(s)$ (for $i = 2$). Finally, take

$$a(u) := \widetilde{a_1}(s(u)), \qquad b(v) := \overline{\widetilde{a_2}(s(v))}. \tag{3.6}$$

Then

$$f(u, v) := a(u)\,b(v) : J_1 \times J_2 \subset \mathbb{R}^2 \to \mathbb{S}^3 \tag{3.7}$$

is the desired unique (in the above sense) solution.

Remark 5 *It follows immediately from the above construction process that the solution to the Cauchy problem for non-characteristic analytic initial data must necessarily be real analytic in an open set containing the curve.*

Proof: Let us start by checking the **uniqueness** of the solution. Let $f : \Sigma^2 \to \mathbb{S}^3$ be an immersed flat surface solution to the Cauchy problem. We will see that f is completely determined by the initial data in a neighborhood of the curve $\beta(s)$, and so it is unique in the above sense.

First, observe that we can assume that Σ is simply connected by restricting the parameter domain if necessary. Thus, there exists a Tschebyscheff coordinate immersion $\psi : \Sigma^2 \to u, v$-plane.

Since f is an immersion, there exists a curve $\Gamma(s)$ in Σ^2 such that $f(\Gamma(s)) = \beta(s)$. Then we define $(u(s), v(s)) := \psi(\Gamma(s))$.

As $\{\beta, V\}$ are non-characteristic data, i.e. $\langle \beta', V' \rangle \neq 0$, we see that β' never points at an asymptotic direction. This means that $u'(s)$ and $v'(s)$ never vanish. That implies that $(u(s), v(s))$ is simultaneously a graph with respect to the two coordinate axes in the u, v-plane. In particular, it has no self-intersections. Hence, ψ is injective on $\Gamma(s)$ and we can assume that the restriction of ψ to Σ^2 is also injective (by shrinking Σ^2 if necessary). With all of this, we can choose Σ^2 small enough to admit global Tschebyscheff coordinates, and so we can see our surface as an immersion $f(u, v) : D \subset \mathbb{R}^2 \to \mathbb{S}^3$ with respect to Tschebyscheff parameters (u, v), with Gauss map $N(u, v) : D \subset \mathbb{R}^2 \to \mathbb{S}^3$ and such that $f(u(s), v(s)) = \beta(s)$ and $N(u(s), v(s)) = V(s)$ for a certain regular curve $(u(s), v(s)) : I \to D$ with $u'(s) \neq 0$ and $v'(s) \neq 0$.

In order to simplify and without loss of generality, we will also assume that

$$(u(s_0), v(s_0)) = (0, 0)$$

for a certain $s_0 \in I$, that $f(0, 0) = \mathbf{1} \equiv (1, 0, 0, 0)$ and that $N(0, 0) = \xi_0$ is orthogonal to $\mathbf{1}$ and $\mathbf{i}$.

As we have seen in Subsection 2.5, there exist unit speed regular curves $a(u), b(v)$ in $\mathbb{S}^3$ such that

$$f(u, v) = a(u)\, b(v), \qquad N(u, v) = a(u)\, \xi_0\, b(v)$$

with $\langle a'(u), a(u)\, \xi_0 \rangle = \langle b'(v), \xi_0\, b(v) \rangle = 0$ and $a(0) = b(0) = \mathbf{1}$.

Hence

$$\begin{aligned} \beta(s) &= f(u(s), v(s)) = a(u(s))\, b(v(s)), \\ V(s) &= N(u(s), v(s)) = a(u(s))\, \xi_0\, b(v(s)). \end{aligned}$$

Taking into account the properties about quaternions mentioned in Subsection 2.2, we easily get

$$V(s)\, \overline{\beta(s)} = a(u(s))\, \xi_0\, b(v(s))\, \overline{b(v(s))}\, \overline{a(u(s))} = a(u(s))\, \xi_0\, \overline{a(u(s))}$$

and similarly $\overline{\beta(s)}\, V(s) = \overline{b(v(s))}\, \xi_0\, b(v(s))$. Thus, we define

$$\begin{aligned} \gamma_1(s) &:= V(s)\, \overline{\beta(s)} = a(u(s))\, \xi_0\, \overline{a(u(s))} \\ \gamma_2(s) &:= \overline{\beta(s)}\, V(s) = \overline{b(v(s))}\, \xi_0\, b(v(s)) \end{aligned}$$

which by (3.2) are curves in $\mathbb{S}^2$, since $\langle \beta(s), V(s) \rangle = 0$.

Next, we will prove that $a(u(s))$ and $b(v(s))$ can be expressed in terms of γ_1 and γ_2, and therefore in terms of the initial data. To do that we will check that all the quantities $a(u)$, $b(v)$, $u(s)$ and $v(s)$ can be expressed in terms of γ_1 and γ_2.

Firstly, let us see that $u(s)$ (resp. $v(s)$) is one half of the arc-length of $\gamma_1(s)$ (resp. $\gamma_2(s)$). In fact, bearing in mind that the metric of $\mathbb{S}^3$ is bi-invariant for the quaternions product, we have

$$\langle \gamma_1'(s), \gamma_1'(s) \rangle = 2u'(s)^2 \left(\langle a'(u(s)), a'(u(s)) \rangle + \langle a'(u(s))\, \xi_0\, \overline{a(u(s))}, a(u(s))\, \xi_0\, \overline{a'(u(s))} \rangle \right).$$

Observe that $u(s)$ is the arc-length of the curve a, and so $\langle a'(u(s)), a'(u(s))\rangle = 1$. Moreover, from (3.2) we see that $a(u(s))\,\xi_0\,\overline{a'(u(s))} \in \mathbb{S}^2$, because $\langle a'(u(s)), a(u(s))\,\xi_0\rangle = 0$. Thereby, we have

$$\begin{aligned}\langle a'(u(s))\,\xi_0\,\overline{a(u(s))}, a(u(s))\,\xi_0\,\overline{a'(u(s))}\rangle &= \langle a'(u(s))\,\xi_0\,\overline{a(u(s))}, -\overline{a(u(s))\,\xi_0\,\overline{a'(u(s))}}\rangle \\ &= \langle a'(u(s))\,\xi_0\,\overline{a(u(s))}, a'(u(s))\,\xi_0\,\overline{a(u(s))}\rangle \\ &= \langle a'(u(s)), a'(u(s))\rangle = 1.\end{aligned}$$

With all of this, $\langle \gamma_1'(s), \gamma_1'(s)\rangle = 4u'(s)^2$. Analogously, $\langle \gamma_2'(s), \gamma_2'(s)\rangle = 4v'(s)^2$.

Let us take now the skew Hopf fibration h_{ξ_0} given by $h_{\xi_0}(x) = x\,\xi_0\,\overline{x}$ for all $x \in \mathbb{S}^3$. Then $h_{\xi_0}(a(u(s))) = \gamma_1(s)$ and $h_{\xi_0}(\overline{b(v(s))}) = \gamma_2(s)$. Moreover, $\delta(s) := a(u(s))$ verifies

$$\langle \delta', \delta\,\xi_0\rangle = 0 \quad \text{(by (3.2))}, \qquad \text{and} \qquad h_{\xi_0}(\delta(s)) = \gamma_1(s), \tag{3.8}$$

and it is uniquely determined by these conditions and $\delta_1(0) = \mathbf{1}$. Indeed, assume that $\delta^*(s)$ verifies (3.8) and $\delta^*(0) = \mathbf{1}$. Since the fibers of a Hopf fibration are great circles of $\mathbb{S}^3$, there exists a function $\theta(s)$ such that

$$\delta(s) = \delta^*(s)(\cos\theta(s)\,\mathbf{1} + \sin\theta(s)\,\xi_0),$$

and so $0 = \langle \delta'(s), \delta(s)\,\xi_0\rangle = \theta'(s)$. This means that $\theta(s) = \theta(0) = 0$ and $\delta^*(s) = \delta(s)$. So, $a(u(s)) = \delta(s)$ is uniquely determined by the curve $\gamma_1(s)$. A similar argument shows that $b(v(s))$ is uniquely determined by $\gamma_2(s)$.

In conclusion, $f(u,v) = a(u)b(v)$ is univocally determined by the initial data β, V in a neighborhood of the curve $\beta(s)$. This proves the uniqueness (in the desired way) of the solution to the Cauchy problem.

Remark 6 *Let $\beta(s), V(s) : I \to \mathbb{S}^3$ be non-characteristic Cauchy data, and take $\Sigma \subset \mathbb{S}^3$ a local solution to the Cauchy problem, which must be unique in the above sense. Then, given $\varphi : J \to I$ a diffeomorphism , the pair $(\beta \circ \varphi)(t), (V \circ \varphi)(t) : J \to \mathbb{S}^3$ are also non-characteristic Cauchy data, and Σ is a solution (in fact, the only one) to the corresponding Cauchy problem. In particular, this shows that the solution to the Cauchy problem for non-characteristic data $\{\beta, V\}$ is locally unique and does not depend on the chosen parameter.*

Let us prove now the **existence** of the solution by means of the constructive process described in the statement of the theorem.

So, let us start with non-characteristic initial data β, V. We can assume, up to a rigid motion, that there exists $s_0 \in I$ such that $\beta(s_0) = \mathbf{1}$ and $V(s_0) = \xi_0$ for a certain $\xi_0 \in \mathbb{S}^2$ satisfying $\xi_0 \perp \mathbf{i}$.

Let us define γ_i, $i = 1, 2$, as in (3.3), which are curves in $\mathbb{S}^2$ by (3.2). Moreover, since $\{\beta(s), V(s)\}$ are non-characteristic data, the curves γ_i are regular. In fact, let us see that $\gamma_1'(s) = 0$ implies that $\langle \beta'(s), V'(s)\rangle = 0$ (we can argue analogously for γ_2).

If $\gamma_1'(s) = 0$ for some $s \in I$, then $V'(s)\,\overline{\beta(s)} = -V(s)\,\overline{\beta'(s)}$ and so $V'(s) = -V(s)\,\overline{\beta'(s)}\,\beta(s)$. On the other hand, from (3.1) and (3.2) it follows that

$$\overline{V(s)\,\overline{\beta'(s)}} = -V(s)\,\overline{\beta'(s)} \qquad \text{and} \qquad \overline{\overline{\beta(s)}\,V(s)} = -\overline{\beta(s)}\,V(s).$$

With all of this, we have

$$\begin{aligned} \beta'(s)\,\overline{V'(s)} &= -\beta'(s)\,\overline{\beta(s)}\,\beta'(s)\,\overline{V(s)} = \beta'(s)\,\overline{\beta(s)}\,V(s)\,\overline{\beta'(s)} = -\beta'(s)\,\overline{V(s)}\,\beta(s)\,\overline{\beta'(s)} \\ &= V(s)\,\overline{\beta'(s)}\,\beta(s)\,\overline{\beta'(s)} = -V'(s)\,\overline{\beta'(s)} = -\overline{\beta'(s)\,\overline{V'(s)}} \end{aligned}$$

which means that $\beta'(s)\,\overline{V'(s)} \in \mathbb{S}^2$ or, equivalently, $\langle \beta'(s), V'(s)\rangle = 0$. Since this is not possible, we deduce that $\gamma_1'(s)$ does not vanish for any $s \in I$.

Next we take $u(s)$ and $v(s)$ as in (3.4), which are diffeomorphisms onto their respective images $J_1 = u(I)$ and $J_2 = v(I)$, because γ_i, $i = 1, 2$, are regular curves. We also take $\widetilde{a}_i(s) : I \to \mathbb{S}^3$, $i = 1, 2$, as in (3.5). It is easy to check that

$$\langle \widetilde{a}_i'(s), \widetilde{a}_i(s)\,\xi_0\rangle = 0. \tag{3.9}$$

Moreover, using that $\gamma_i(s) = \widetilde{a}_i(s)\,\xi_0\,\overline{\widetilde{a}_i(s)}$ we have

$$\langle \gamma_i'(s), \gamma_i'(s)\rangle = 4\langle \widetilde{a}_i'(s), \widetilde{a}_i'(s)\rangle$$

and so, from (3.4), we get

$$\langle \widetilde{a}_1'(s), \widetilde{a}_1'(s)\rangle = u'(s)^2, \qquad \langle \widetilde{a}_2'(s), \widetilde{a}_2'(s)\rangle = v'(s)^2.$$

In particular, $\widetilde{a}_1(s)$ (resp. $\widetilde{a}_2(s)$) is a regular curve with arc-parameter $u(s) : I \to J_1$ (resp. $v(s) : I \to J_2$). Then, we define $a(u)$ and $b(v)$ as in (3.6), which are curves parametrized by arc-length and satisfy that

$$a(0) = \mathbf{1} = b(0).$$

Besides, from (3.9) it follows that

$$\langle a'(u), a(u)\,\xi_0\rangle = 0 = \langle b'(v), \xi_0\,b(v)\rangle. \tag{3.10}$$

Let us define the parametrization $f(u, v)$ as in (3.7). First, let us check that this parametrization passes through the curve $\beta(s)$, that is, $f(u(s), v(s)) = \beta(s)$. In order to do that, we define the function $g(s) := \overline{\beta(s)}\,a(u(s))$. Observe that $g(s_0) = \mathbf{1}$ and

$$\begin{aligned} h_{\xi_0}(g(s)) = g(s)\,\xi_0\,\overline{g(s)} &= \overline{\beta(s)}\,a(u(s))\,\xi_0\,\overline{a(u(s))}\,\beta(s) = \overline{\beta(s)}\,\gamma_1(s)\,\beta(s) \\ &= \overline{\beta(s)}\,V(s)\,\overline{\beta(s)}\,\beta(s) = \overline{\beta(s)}\,V(s) = \gamma_2(s) \end{aligned}$$

where we have used (3.3). In addition, using (3.10), one gets

$$\begin{aligned} \langle g'(s), g(s)\,\xi_0\rangle &= \langle \overline{\beta'(s)}\,a(u(s)), \overline{\beta(s)}\,a(u(s))\,\xi_0\rangle = \langle \overline{\beta'(s)}, \overline{\beta(s)}\,a(u(s))\,\xi_0\,\overline{a(u(s))}\rangle \\ &= \langle \overline{\beta'(s)}, \overline{\beta(s)}\,\gamma_1(s)\rangle = \langle \overline{\beta'(s)}, \overline{\beta(s)}\,V(s)\,\overline{\beta(s)}\rangle \\ &= -\langle \overline{\beta'(s)}, \overline{\beta(s)}\,\beta(s)\,\overline{V(s)}\rangle = -\langle \overline{\beta'(s)}, \overline{V(s)}\rangle = 0. \end{aligned}$$

From the argument used before to prove the uniqueness of the solution, it becomes clear that $\widetilde{a}_2(s)$ is the only curve in $\mathbb{S}^3$ such that $\widetilde{a}_2(s_0) = \mathbf{1}$, $h_{\xi_0}(\widetilde{a}_2(s)) = \gamma_2(s)$ and $\langle \widetilde{a}_2'(s), \widetilde{a}_2(s)\,\xi_0\rangle = 0$, and so $g(s) = \widetilde{a}_2(s) = \overline{b(v(s))}$, that is,

$$\beta(s) = a(u(s))\,b(v(s)) = f(u(s), v(s)). \tag{3.11}$$

Let us define now

$$N(u,v) = a(u)\,\xi_0\,b(v) : J_1 \times J_2 \to \mathbb{S}^3.$$

Since $\xi_0 \perp \mathbf{1}$, it follows that $\langle f, N\rangle = 0$. Moreover, from (3.10) one gets

$$\langle f_u, N\rangle = 0 = \langle f_v, N\rangle \tag{3.12}$$

and so $N(u,v)$ is the unit normal vector field of the parametrization $f(u,v)$ at its regular points. Note that $N(u,v)$ agrees with $V(s)$ along $\beta(s)$, since

$$\begin{aligned} N(u(s), v(s)) &= a(u(s))\,\xi_0\,b(v(s)) = a(u(s))\,\xi_0\,\overline{a(u(s))}\,a(u(s))\,b(v(s)) \\ &= \gamma_1(s)\,\beta(s) = V(s)\,\overline{\beta(s)}\,\beta(s) = V(s). \end{aligned} \tag{3.13}$$

Thus, $f(u,v)$ fulfils the initial data.

Note that, from (2.1), $f(u,v)$ is regular at (u_0, v_0) if and only if $\sin\omega(u_0, v_0) \neq 0$, being $\sin\omega = -\langle f_u, N_v\rangle$. But along $\beta(s) = f(u(s), v(s))$ we have, bearing in mind that $u(s)$ and $v(s)$ are regular curves, that

$$\sin\omega(u(s), v(s)) = \frac{\langle \beta'(s), V'(s)\rangle}{-2u'(s)v'(s)} \neq 0,$$

and so $f(u,v)$ is regular in an open set containing $\beta(s)$.

Finally, let us see that f is indeed a flat surface. A simple way to do this is to check that $a'(0)$ and $b'(0)$ are not collinear, and then to use the Bianchi-Spivak representation theorem. For that, since $\{\beta(s), V(s)\}$ are non-characteristic data, we have

$$\begin{aligned} 0 \neq\ & \langle \beta'(s), V'(s)\rangle = u'(s)v'(s)\left(\langle a'(u(s))\,b(v(s)), a(u(s))\,\xi_0\,b'(v(s))\rangle\right. \\ & \left.+\langle a(u(s))\,b'(v(s)), a'(u(s))\,\xi_0\,b(v(s))\rangle\right). \end{aligned}$$

Then, since both $\overline{a(u(s))}\,a'(u(s))$ and $b'(v(s))\,\overline{b(v(s))}$ lie in $\mathbb{S}^2$, we get

$$\begin{aligned} \langle a'(u(s))\,b(v(s)), a(u(s))\,\xi_0\,b'(v(s))\rangle &= -\langle \xi_0\,\overline{a(u(s))}\,a'(u(s)), b'(v(s))\,\overline{b(v(s))}\rangle \\ &= -\langle \overline{\xi_0\,\overline{a(u(s))}\,a'(u(s))}, \overline{b'(v(s))\,\overline{b(v(s))}}\rangle \\ &= \langle \overline{a(u(s))}\,a'(u(s))\,\xi_0\,, b'(v(s))\,\overline{b(v(s))}\rangle \\ &= -\langle \overline{a(u(s))}\,a'(u(s)), b'(v(s))\,\overline{b(v(s))}\,\xi_0\rangle \\ &= \langle a(u(s))\,b'(v(s)), a'(u(s))\,\xi_0\,b(v(s))\rangle \end{aligned}$$

and so, as $u'(s)$ and $v'(s)$ do not vanish for any $s \in I$,

$$\langle a'(u(s))\,b(v(s)), a(u(s))\,\xi_0\,b'(v(s))\rangle \neq 0. \tag{3.14}$$

In particular, one obtains easily from (3.14) that the unit vectors $a'(u(s))\,b(v(s))$ and $a(u(s))\,b'(v(s))$ are not collinear for any $s \in I$. Hence,

$$|\langle \overline{a(u(s))}\,a'(u(s)), b'(v(s))\,\overline{b(v(s))}\rangle| = |\langle a'(u(s))\,b(v(s)), a(u(s))\,b'(v(s))\rangle| < 1,$$

that is, the unit vectors $\overline{a(u(s))}\,a'(u(s))$ and $b'(v(s))\,\overline{b(v(s))}$ are not collinear either, and our claim follows from evaluating both vectors at s_0. This completes the proof.

□

4 The characteristic case

Let $\Sigma \subset \mathbb{S}^3$ be a flat surface which is a solution to the geometric Cauchy problem for the initial data $\beta(s), V(s) : I \to \mathbb{S}^3$. Next we are going to consider the characteristic case, that is, we will assume that there are points $s \in I$ where $\langle \beta'(s), V'(s)\rangle = 0$.

This situation means a great difference with respect to the non-characteristic case treated in the previous section. For instance, if $\langle \beta'(s), V'(s)\rangle$ vanishes identically, then $\beta(s)$ is an asymptotic curve and $V(s)$ is completely determined by $\beta(s)$. However, we have not any knowledge about the other asymptotic curve at each point $\beta(s)$, which can adopt a very general shape. Consequently, under this assumption we do not have uniqueness of the solution in a neighborhood of $\beta(s)$ anymore. Moreover, we cannot control the shape of the surface around $\beta(s)$ just from the knowledge of this curve.

This case where $\langle \beta'(s), V'(s)\rangle \equiv 0$ is, therefore, a degenerate case which we will exclude. Thus, from now on we will assume that $\langle \beta'(s), V'(s)\rangle$ only vanishes at isolated points.

Even under this more restrictive assumption, it is important to point out that neither we should expect uniqueness of the solution, nor an explicit control of the surface around $\beta(s)$. Indeed, let c_1, c_2 be two regular curves in $\mathbb{S}^2$ which agree in an open interval, and denote by $\Sigma_i = h^{-1}(c_i)$, $i = 1, 2$, their respective Hopf cylinders (see Figure 1).

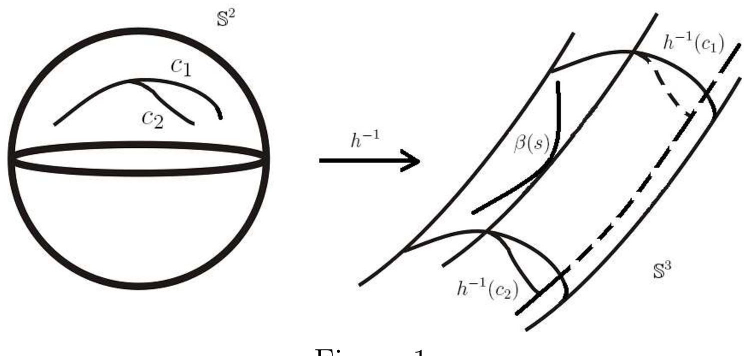

Figure 1.

Observe that both surfaces have the same initial data on the highlighted curve, but they do not coincide in a neighborhood of such curve. In this example it is clear that the problem relies in the fact that the curve is tangent to a fiber, which is always an asymptotic curve of the Hopf cylinder. Anyway, the figure above also suggests that we can have a local control (as well as uniqueness) at least at one side of the asymptotic curve. Our aim in this section is to show that this is precisely what happens for arbitrary characteristic initial data, and not just for the example above.

To start with, let us carry out a local analysis of the initial data of a flat surface at characteristic points. Thus, let us take $s_0 \in I$ such that $\langle \beta'(s_0), V'(s_0)\rangle = 0$ and $\langle \beta'(s), V'(s)\rangle \neq 0$ around s_0. As we saw in Remark 3, this means that $\beta'(s_0)$ points at one of the two asymptotic directions of the flat surface Σ at $\beta(s_0)$.

Definition 7 *Let $\beta(s), V(s)$ be initial data, and take $s_0 \in I$ such that $\langle \beta'(s_0), V'(s_0)\rangle = 0$ and $\langle \beta'(s), V'(s)\rangle \neq 0$ around s_0. We will say that s_0 is a* characteristic point*:*

- of type I *if $\langle \beta'(s), V'(s)\rangle$ does not change sign around s_0.*
- of type II *if $\langle \beta'(s), V'(s)\rangle$ changes sign around s_0.*

Let us analyze this notion from a geometrical point of view. If we take $\{u, v\}$ Tschebyscheff parameters of the flat surface Σ around $\beta(s_0)$, then $\beta(s) = f(u(s), v(s))$ for a certain curve $(u(s), v(s))$ in the u, v-plane. Thus, from (2.1) we get

$$\langle \beta'(s), V'(s)\rangle = -2u'(s)v'(s)\sin\omega(u(s), v(s)), \qquad 0 < \omega < \pi.$$

Since $\sin\omega > 0$, the condition $\langle \beta'(s_0), V'(s_0)\rangle = 0$ is equivalent to $u'(s_0) = 0$ or $v'(s_0) = 0$ (but recall that these two conditions cannot hold simultaneously). Consequently, $\langle \beta'(s), V'(s)\rangle$ changes sign around s_0 if and only if so does $u'(s)$ (if $\beta'(s)$ points at the asymptotic direction ∂_v) or $v'(s)$ (if $\beta'(s)$ points at the asymptotic direction ∂_u). Bearing in mind that the asymptotic curves are precisely $u = \text{const}$ and $v = \text{const}$, we can characterize the characteristic points as follows:

Lemma 8 *Let Σ denote a solution to the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ for the initial data $\beta(s), V(s)$. Then, a characteristic point s_0 of the data is of type II if, and only if, $\beta(s)$ stays (locally) in Σ at one side of the asymptotic curve to which it is tangent at $\beta(s_0)$.*

If s_0 is a characteristic point of type II in the conditions of the above lemma, we will call the *(local) positive side* of Σ at $\beta(s_0)$ to the one where the curve is (locally) contained.

Observe that if $\{\beta(s), V(s)\}$ are initial data such that the characteristic points are all of type I, then both $u(s)$ and $v(s)$ are strictly monotone, because their derivatives do not change sign and only vanish at isolated points. In particular, this implies that the proof of uniqueness in the non-characteristic case can be translated to this situation. So, we have:

Proposition 9 *Let $\{\beta(s), V(s)\}$ be initial data such that their (isolated) characteristic points are all of type I. Then the solution to the geometric Cauchy problem for such data, if it exists, is unique in the sense of Theorem 4.*

When there exist characteristic points of type II, the situation changes completely. Indeed, let $f(u,v) : J_1 \times J_2 \to \mathbb{S}^3$ be a local Tschebyscheff parametrization of a flat surface around a point $p \in \mathbb{S}^3$, and let $\beta(s), V(s) : I \to \mathbb{S}^3$ be initial data on the surface such that $\beta(s) = f(u(s), v(s))$, $\beta(s_0) = p$ and s_0 is a characteristic point of type II. Under these assumptions, either $u'(s_0) = 0$ or $v'(s_0) = 0$. Let us suppose, without loss of generality, that $u'(s_0) = 0$; in other words, we assume that $\beta'(s_0)$ points at the asymptotic direction ∂_v.

Since s_0 is a characteristic point of type II, then $u(s)$ has a local maximum or minimum at s_0, let us say, a maximum. If we put $(u(s_0), v(s_0)) = (u_0, v_0)$, this means that $u(s) \leq u_0$ around s_0. Therefore, the positive side of the characteristic point $\beta(s_0)$ is $\{f(u,v) : u \leq u_0, (u,v) \in B_\epsilon(u_0, v_0)\}$ for ϵ small enough (see Figure 2).

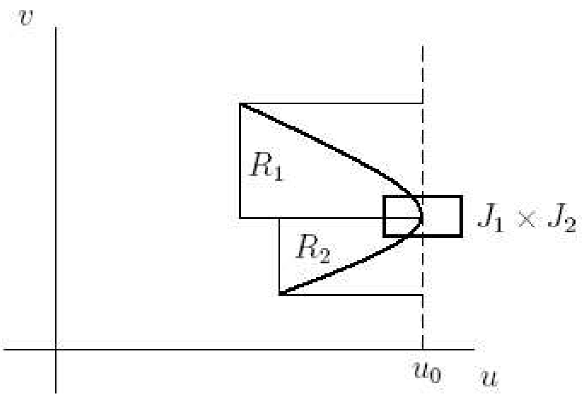

Figure 2.

Let us denote by $a(u)$ and $b(v)$ the two asymptotic curves passing through $p = \beta(s_0)$. Then, it is clear that $b(v)$ can be recovered around p in terms of $\beta(s)$ and $V(s)$ following the constructive process described in Theorem 4, because $v'(s_0) \neq 0$ in this case. In addition, since $\{\beta(s), V(s)\}$ are non-characteristic data in the subintervals $(s_0 - \varepsilon, s_0)$ and $(s_0, s_0 + \varepsilon)$, we see again from Theorem 4 that $a(u)$ is totally determined by $\beta(s)$ and $V(s)$ in such subintervals for $u \leq u_0$.

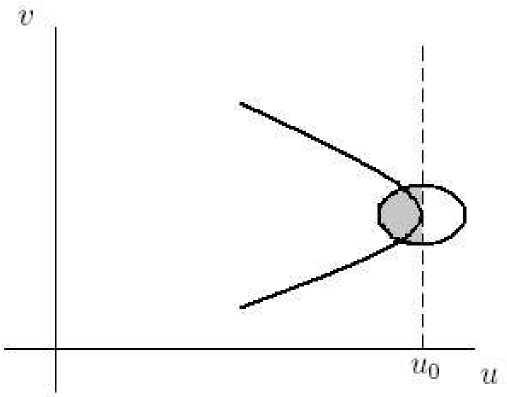

Figure 3.

In the Figure above, the surface $f(u,v) : J_1 \times J_2 \to \mathbb{S}^3$ is determined in the region $J_1 \cap \{u : u \leq u_0\}$ since, as we proved in the non-characteristic case, we can control the surface in the regions R_1 and R_2.

From the preceding analysis we can conclude:

- The Cauchy problem does not have unique solution in the sense of Theorem 4. Indeed, let us take

$$\widetilde{a}(u) = \begin{cases} a(u) & \text{if } u \in (u_0 - \varepsilon, u_0] \\ \widehat{a}(u) & \text{if } u \in [u_0, u_0 + \varepsilon) \end{cases}$$

  where $\widehat{a}(u) : (u_0 - \varepsilon, u_0 + \varepsilon) \to \mathbb{S}^3$ is a curve parametrized by arclength, different from $a(u)$, and such as $\langle \widehat{a}', \widehat{a}\ \xi_0 \rangle \equiv 0$. Observe that the flat surface $\widetilde{\Sigma}$ determined by $\widetilde{a}(u)$ and $b(v)$ agrees with Σ wherever $u \leq u_0$, and so both surfaces have $\{\beta(s), V(s)\}$ as initial data. However, it becomes clear that $\widetilde{\Sigma}$ and Σ do not agree in a neighborhood of $\beta(s)$, because they are different when $u > u_0$. In other words, the initial data $\{\beta(s), V(s)\}$ are not enough to describe the surface in $(J_1 \times J_2) \cap \{(u,v) : u > u_0\}$.

- Although the data $\{\beta(s), V(s)\}$ do not determine the surface around p, they do in the region $(J_1 \times J_2) \cap \{(u,v) : u \leq u_0\}$. Observe that $f(u_0, v)$ is the asymptotic curve of the flat surface passing through p to which $\beta(s)$ is tangent at s_0. Hence, we can recover the surface in terms of $\{\beta(s), V(s)\}$ exactly for the positive side of the characteristic point p. In particular, at a characteristic point p of type II we have uniqueness for the Cauchy problem just on an open set for which p is a boundary point (so, p does not belong to this open set)

These comments along with Proposition 9 allow us to state:

Theorem 10 (General uniqueness theorem) *Let $\{\beta(s), V(s)\}$ be initial data with isolated characteristic points and Σ_1, Σ_2 two flat surfaces in $\mathbb{S}^3$ which are solutions to the corresponding geometric Cauchy problem. Then Σ_1 and Σ_2 agree on a connected open set which contains all points of $\beta(s)$ that are not characteristic points of type II.*

Next we will study the **existence** of the solution to the geometric Cauchy problem when $\{\beta(s), V(s)\}$ are initial data with isolated characteristic points. In general, as we will see, in this case there is no flat surface in $\mathbb{S}^3$ which solves the problem. Anyway, we will discuss next under which additional assumptions on the initial data we can guarantee such existence.

We will start by analyzing the case where $\langle \beta'(s), V'(s)\rangle$ does not change its sign, that is, where all characteristic points are of type I. Following the proof of the existence part in Theorem 4, we realize that the first difference is that the curves γ_i, $i = 1, 2$, defined as in (3.3), may be singular at characteristic points, i.e. $\gamma_i'(s) = 0$ at those points. Nonetheless, on a flat surface, the traces of the curves γ_i are just the projection via the skew Hopf fibration $h_{\xi_0} : \mathbb{S}^3 \to \mathbb{S}^2$ of the asymptotic curves of the surface. Thus, they must be regular curves when parametrized by arclength.

Hence, we shall impose on the initial data the (necessary) additional restriction that the curves $\gamma_i(s)$ in (3.3) are regular when parametrized by arclength (equivalently, that the images $\gamma_i(I) \subset \mathbb{S}^2$ are regular curves).

With this hypothesis, the proof of the existence part in Theorem 4 works in this context almost unchanged. We explain the key details next.

First, as the characteristic points of β, V are isolated, the functions $u(s), v(s)$ in (3.4) are injective C^1 functions, although they are not diffeomorphisms. But now, by hypothesis, the parametrized curves $\gamma_1(u)$ and $\gamma_2(v)$ are regular. Hence, so are the asymptotic lifts $a(u), b(v)$ constructed in the existence part of Theorem 4, and characterized by $h_{\xi_0}(a(u)) = \gamma_1(u)$, $\langle a', a\,\xi_0\rangle = 0$, and $\langle a'(u), a'(u)\rangle = 1$ (analogously for $b(v)$ in terms of $\gamma_2(v)$). Therefore, we may define the smooth map $f(u, v) = a(u)b(v)$.

It can then be checked, as in Theorem 4, that (3.11), (3.12) and (3.13) hold, and consequently $f(u, v)$ is a flat surface passing through the initial data. However, it is no longer true in general that $f(u, v)$ is regular around $\beta(s)$. Indeed, as we saw in the proof of Theorem 4, the regularity at a point $s_0 \in I$ is equivalent to the condition $\sin\omega(u(s_0), v(s_0)) \neq 0$, being

$$\langle \beta'(s_0), V'(s_0)\rangle = -2u'(s_0)v'(s_0) \sin\omega(u(s_0), v(s_0)).$$

Therefore, $\sin\omega$ does not vanish along $\beta(s)$ if, and only if, for every $s_0 \in I$ the limit

$$\lim_{s\to s_0} \frac{\langle \beta'(s), V'(s)\rangle}{2u'(s)v'(s)}$$

exists and is non-zero. Observe that this property trivially holds for non-characteristic points. Moreover, this property cannot hold if $\langle \beta', V'\rangle$ changes sign, since $u'(s), v'(s)$ are always non-negative by definition. Thus, the hypothesis that $\langle \beta', V'\rangle$ does not change sign is necessary here.

Hence, we have the following:

Proposition 11 *Let $\{\beta(s), V(s)\}$ be initial data such that $\langle \beta'(s), V'(s)\rangle$ does not change sign and only vanishes at isolated points, and assume that the traces of the curves $\gamma_i(I) \subset \mathbb{S}^2$, $i = 1, 2$, in (3.3) are regular.*

Then there exists a flat surface solution to the associated geometric Cauchy problem if, and only if, for every characteristic point $s_0 \in I$ the limit

$$\lim_{s\to s_0} \frac{\langle \beta'(s), V'(s)\rangle}{2u'(s)v'(s)} \tag{4.1}$$

exists and is non-zero. Moreover, this flat surface can be constructed following the procedure exposed in Theorem 4.

This situation changes completely when $\langle \beta'(s), V'(s)\rangle$ changes sign, that is, when there exist characteristic points of type II. In order to see that, let us take $f(u,v) = a(u)\, b(v)$, $(u,v) \in J_1 \times J_2$, a flat surface in $\mathbb{S}^3$ and let $N(u,v) = a(u)\,\xi_0\, b(v)$ be its unit normal. Then

$$\beta(s) = f(s^2, s), \qquad V(s) = N(s^2, s)$$

are initial data which have at $s_0 = 0$ a characteristic point of type II.

Since $h_{\xi_0}(a(u(s))) = \gamma_1(s)$, we have

$$a(s^2)\,\xi_0\,\overline{a(s^2)} = \gamma_1(s) = V(s)\overline{\beta(s)}.$$

Therefore, we can conclude that the respective traces of $V(s)\overline{\beta(s)}$ for $(-\varepsilon, 0]$ and for $[0,\varepsilon)$ agree. This means that $\gamma_1(s)$ *comes back along its trace* after attaining the characteristic point s_0.

More generally, let us take $f(u,v)$ a flat surface in $\mathbb{S}^3$ and $\{\beta(s), V(s)\}$ initial data on $f(u,v)$ such that s_0 is a characteristic point of type II. Let us assume, without loss of generality, that $u'(s_0) = 0$, i.e., that $u(s)$ has a local extreme at s_0. Since

$$a(u(s))\,\xi_0\,\overline{a(u(s))} = V(s)\overline{\beta(s)} = \gamma_1(s),$$

it becomes clear that for $\varepsilon > 0$ small enough, the respective traces of $V(s)\overline{\beta(s)}$ in $[s_0, s_0+\varepsilon)$ and $(s_0-\varepsilon, s_0]$ agree. Hence, $\gamma_1(s)$ comes back along its trace after attaining the characteristic point s_0. This property is, therefore, a necessary condition for the existence of the flat surface around a characteristic point of type II.

Let us see now that it is also a sufficient condition. Indeed, let $\{\beta(s), V(s)\}$ be initial data and s_0 a characteristic point of type II such that the curve $\gamma_1(s) = V(s)\overline{\beta(s)}$ comes back along its trace in the sense above. Then, there exists $\varepsilon > 0$ such that the traces of γ_1 in $[s_0, s_0+\varepsilon)$ and $(s_0-\varepsilon, s_0]$ agree. Observe that we can solve the non-characteristic Cauchy problem in the intervals $I_1 = (s_0, s_0+\varepsilon)$ and $I_2 = (s_0-\varepsilon, s_0)$ to obtain, respectively, two flat surfaces

$$f_1(u_1, v) = a_1(u_1)b(v), \qquad f_2(u_2, v) = a_2(u_2)b(v).$$

Here, we are assuming that $\beta(s_0) = \mathbf{1}$ and that the parameters u_i, $i = 1,2$, are such that $a_i(0) = b(0) = \mathbf{1}$; in other words, u_i is the arc parameter of a_i taking $\beta(s_0)$ as a base point. In this situation we have:

- a_i is a lift of γ_1 in I_i for the fibration $h_{\xi_0}(x) = x\,\xi_0\,\overline{x}$. Moreover, a_i is the only lift of γ_1 satisfying that $\langle a_i', a_i\,\xi_0\rangle = 0$ and $a_i(0) = \mathbf{1}$.

- The traces of γ_1 in I_1 and I_2 agree.

Bearing all of this in mind, we can conclude that the traces of a_1 on I_1 and a_2 on I_2 coincide. Let us call $a(u)$ this curve, u being its arc parameter having $\beta(s_0)$ as a base point. Since u_i is the arc parameter of a_i with respect to that point, we get

$$u_1 = u_2 = u, \qquad a_1(u_1) = a_2(u_2) = a(u).$$

In particular, $f_1 = f_2$. Thus, we have constructed a flat surface $f(u, v) = a(u)\, b(v)$ passing through the initial data $\{\beta(s), V(s)\}$ on the intervals $(s_0 - \varepsilon, s_0)$ and $(s_0, s_0 + \varepsilon)$, but such that it does not contain in its interior the point $\beta(s_0)$. This happens because u is only defined on an interval of the form $(u_0, 0]$ or $[0, u_0)$.

Nonetheless, it is possible to extend $a(u)$ as a smooth curve $\widetilde{a} : (-u_0, u_0) \to \mathbb{S}^3$ (we are assuming $u_0 > 0$; we would define $\widetilde{a} : (u_0, -u_0) \to \mathbb{S}^3$ otherwise) satisfying $\langle \widetilde{a}', \widetilde{a}\, \xi_0 \rangle = 0$.

In this way, $f(u, v) = \widetilde{a}(u)b(v)$ is a solution of the geometric Cauchy problem for the initial data $\{\beta(s), V(s)\}$ in a neighborhood of the characteristic point of type II s_0.

So, in the end of this discussion we have:

Theorem 12 (General existence theorem) *Let $\{\beta(s), V(s)\}$ be initial data with isolated characteristic points. Then the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ with these initial data has a solution if, and only if:*

- *For every characteristic point s_0 of type I, the limit (4.1) exists and is not zero.*
- *For every characteristic point s_0 of type II, the curve $\gamma_1(s) = V(s)\overline{\beta}(s)$ or $\gamma_2(s) = \overline{\beta}(s)V(s)$ that is singular at s_0 comes back along its trace after attaining the characteristic point s_0, i.e. there exists $\varepsilon > 0$ such that the traces of γ_i on $(s_0, s_0 + \varepsilon)$ and $(s_0 - \varepsilon, s_0)$ agree.*

5 Applications

As a first application, we will regard the D'Alembert formula as a representation for flat surfaces in $\mathbb{S}^3$, and we will use it in order to analyze to what extent a symmetry in the Cauchy data induces a symmetry of the surface.

We will say that a rigid motion $\phi : \mathbb{S}^3 \to \mathbb{S}^3$ is a *symmetry* of the initial data $\beta, V : I \to \mathbb{S}^3$ if there exists a diffeomorphism $\varphi : I \to I$ such that $\phi(\beta(s)) = \beta(\varphi(s))$ and $\phi(V(s)) = V(\varphi(s))$ for every $s \in I$.

Theorem 13 (Generalized symmetry principle) *Let $\beta, V : I \to \mathbb{S}^3$ be Cauchy data with isolated characteristic points, and let $\phi : \mathbb{S}^3 \to \mathbb{S}^3$ denote a symmetry of the data.*

If Σ is a flat surface which is a solution to the corresponding Cauchy problem, then there exists a connected open subset Σ_1 of Σ containing all points of $\beta(s)$ that are not characteristic of type II, and such that $\phi(\Sigma_1) = \Sigma_1$.

Proof: The proof follows immediately using a standard argument (see [GaMi3] for instance), from the uniqueness in Theorem 4 and Remark 6.

□

This result concludes, in particular, that any symmetry of non-characteristic Cauchy data generates a symmetry of the resulting flat surface around the curve $\beta(s)$.

Next, we focus on the study of flat cylinders and flat Möbius strips in $\mathbb{S}^3$.

To start, we assert that given two points p, q on a flat surface Σ, there exists a smooth curve connecting p and q with (at most) isolated characteristic points. In particular, if Σ is a cylinder or a Möbius strip then given a point $p \in \Sigma$ there exits a closed non null-homotopic regular curve passing across p with isolated characteristic points.

In order to prove this assertion we locally parametrize Σ as $f(u, v)$ where $(u, v) \in I \times J$ are Tschebyscheff coordinates. Given $p_1 = f(u_1, v_1)$ and $p_2 = f(u_2, v_2)$ then it is easy to find a regular curve α in $I \times J$ joining (u_1, v_1) and (u_2, v_2) such that its tangent line is horizontal or vertical at a finite set of points. Or equivalently, the curve $\beta = f \circ \alpha$ is a smooth curve, with a finite set of characteristic points, connecting p_1 and p_2. Therefore, from a standard argument, there exists a regular curve with isolated characteristic points connecting any two points on Σ.

Theorem 14 (Classification of flat cylinders) *Let $\beta, V : \mathbb{R} \to \mathbb{S}^3$ denote T-periodic initial data with isolated characteristic points, and assume without loss of generality that 0 is non-characteristic. Let Σ denote a solution to the Cauchy problem for flat surfaces in $\mathbb{S}^3$ with the initial data $\beta, V : (-\varepsilon, T + \varepsilon) \to \mathbb{S}^3$, where $\varepsilon > 0$ is small enough. Then Σ has the topology of a cylinder in a neighborhood of $\beta(s)$.*

Conversely, every flat cylinder in $\mathbb{S}^3$ can be constructed in this way around any of its closed non null-homotopic regular curves with isolated characteristic points.

Proof: For $\varepsilon > 0$ small enough, the data β, V are non-characteristic on $I_1 = (-\varepsilon, \varepsilon)$, and also on $I_2 = (T-\varepsilon, T+\varepsilon)$. In addition, since Σ is a solution to the corresponding Cauchy problem on $(-\varepsilon, T + \varepsilon)$, given two subsets $\Sigma_1, \Sigma_2 \subset \Sigma$ such that Σ_i is a solution to the Cauchy problem on I_i, then Σ_1 and Σ_2 agree in a neighborhood of $\beta(s)$, $s \in I_1$. This is a consequence of the uniqueness part of Theorem 4. Therefore, Σ has the topology of a cylinder in a neighborhood of the whole $\beta(s)$.

The converse part of the result is clear bearing in mind that there exit closed non null-homotopic regular curves on Σ with isolated characteristic points.

□

In a similar way we can establish a characterization result for flat Möbius strips.

Theorem 15 (Classification of flat Möbius strips) *Let $\beta, V : \mathbb{R} \to \mathbb{S}^3$ denote initial data with isolated characteristic points, and assume without loss of generality that 0 is non-characteristic. Moreover, assume that $\beta(s)$ is T-periodic, and that $V(s)$ is T-antiperiodic, that is, $V(s+T) = -V(s)$ for every $s \in \mathbb{R}$. Let Σ denote a solution to the Cauchy problem for flat surfaces in $\mathbb{S}^3$ with the initial data $\beta, V : (-\varepsilon, T+\varepsilon) \to \mathbb{S}^3$, where*

$\varepsilon > 0$ *is small enough. Then* Σ *has the topology of a Möbius strip in a neighborhood of* $\beta(s)$.

Conversely, every flat Möbius strip in $\mathbb{S}^3$ *can be constructed in this way around any of its closed non null-homotopic smooth curves with isolated characteristic points.*

Proof: The proof follows as in the previous Theorem. It is only necessary to observe that, as above, given two subsets $\Sigma_1, \Sigma_2 \subset \Sigma$ such that Σ_i is a solution to the Cauchy data on I_i, then, from the generalized symmetry principle, Σ_1 and Σ_2 agree in a neighborhood of $\beta(s)$, $s \in I_1$, but they have different orientations. Therefore, Σ has the topology of a Möbius strip in a neighborhood of the whole $\beta(s)$.

□

Remark 16 *It is obvious from Theorem 15 that the tangent vector of a non null-homotopic regular curve on a flat Möbius strip in* $\mathbb{S}^3$ *points at an asymptotic direction at least once. This justifies the study developed in Section 4 of the Cauchy problem for flat surfaces in the presence of characteristic points.*

To end up the paper, we will analyze when a regular curve $\beta(s)$ can be a geodesic of a flat surface in $\mathbb{S}^3$. Let us assume that $\beta(s)$ is parametrized by arclength, and has non-vanishing curvature. It is then easy to check that if $\beta(s)$ is a geodesic of some flat surface in $\mathbb{S}^3$, then the unit normal $V(s)$ of the surface along the curve can be expressed in terms of $\beta(s)$ as

$$V(s) = \frac{\beta''(s) + \beta(s)}{||\beta''(s) + \beta(s)||} \tag{5.1}$$

up to the sign, which in the end simply means a change of orientation of the surface. Now, let $k_\beta(s)$ denote the geodesic curvature of $\beta(s)$ in $\mathbb{S}^3$. Then we have

$$\sqrt{-1 + ||\beta''(s)||^2} = k_\beta(s) = ||\beta''(s) + \beta(s)||$$

and so, if $V(s)$ is given by (5.1), we have

$$\langle \beta''((s)), V(s)\rangle = \frac{-1 + ||\beta''(s)||^2}{||\beta''(s) + \beta(s)||} = k_\beta(s) > 0.$$

Thus, the initial data β, V are non-characteristic. Therefore, as a direct consequence of Theorem 4 we have:

Corollary 17 *Let* $\beta(s)$ *be a regular curve in* $\mathbb{S}^3$ *parametrized by arclength, and let* $k_\beta(s) > 0$ *denote its geodesic curvature. Then there exists a unique flat surface in* $\mathbb{S}^3$ *containing* $\beta(s)$ *as a geodesic. This surface can be explicitly recovered via Theorem 4, taking* $V(s)$ *as in* (5.1).

Remark 18 *The case in which* β *is allowed to have isolated points with* $k_\beta(s) = 0$ *can be treated quite in the same way, using this time Theorem 10 and Theorem 12 instead of Theorem 4, to produce a more general version of Corollary 17. However, we omit the details, as the process is clear, and the final formulation of this general version is quite involved.*

A particular case that is of interest in this context is the study of Hopf cylinders. First let us observe the following fact.

Corollary 19 *Let $\{\beta(s), V(s)\}$ denote non-characteristic initial data. Then the solution to the geometric Cauchy problem for these data is a piece of a Hopf cylinder if and only if $V(s)\,\overline{\beta(s)}$ or $\overline{\beta(s)}\,V(s)$ parametrizes a piece of a great circle in $\mathbb{S}^2$.*

Proof: It is a known fact that a flat surface $f(u,v)$ in $\mathbb{S}^3$ is a piece of a Hopf cylinder if and only if its angle function $\omega(u,v)$ depends only on u or v. So, by the Kitagawa representation (Theorem 1), this is equivalent to ask that one of the curves c_i of $\mathbb{S}^2$ in that theorem has constant geodesic curvature. Equivalently, this means that the curve lies on a circle in $\mathbb{S}^2$, and so its tangent indicatrix $c_i^* = c_i'/||c_i'||$ lies on a great circle of $\mathbb{S}^2$. At last, again by Theorem 1, c_i^* is the projection via $h_{\xi_0} = x\,\xi_0\,\bar{x}$ of one of the asymptotic curves of the flat surface. Here, we have made the usual normalization: $\mathbf{1}$ is a point of the surface, and $\xi_0 \in \mathbb{S}^2$ is the unit normal of the surface at $\mathbf{1}$.

In other words, it holds that *a flat surface in $\mathbb{S}^3$ is a piece of a Hopf cylinder if, and only if, one of the two asymptotic curves passing through $\mathbf{1}$ projects via h_{ξ_0} to a great circle of $\mathbb{S}^2$*. From this fact and Theorem 4, the conclusion of the corollary is clear.

□

In the end, by putting together Corollaries 17 and 19, we can provide an alternative proof of a result by M. Barros.

Theorem 20 ([Bar]) *Let $\beta(s)$ be a regular curve in $\mathbb{S}^3$ parametrized by arclength, and with $k_\beta(s) > 0$. Then $\beta(s)$ is a geodesic of a Hopf cylinder if, and only if, $\beta(s)$ is a general helix, i.e. there exists a vector $\nu \in \mathbb{S}^2$ such that $\langle \nu\,\beta(s), \beta'(s)\rangle$ or $\langle \beta(s)\,\nu, \beta'(s)\rangle$ is constant.*

Proof: From Corollary 19, if $\beta(s)$ is a geodesic of a Hopf cylinder then $V(s)\,\overline{\beta(s)}$ or $\overline{\beta(s)}\,V(s)$ parametrizes a piece of a great circle in $\mathbb{S}^2$, where $V(s)$ is necessarily given as in (5.1). Let us suppose, for instance, that $V(s)\,\overline{\beta(s)}$ is a piece of great circle of $\mathbb{S}^2$. Then there exists $\nu \in \mathbb{S}^2$ such that $\langle V(s)\,\overline{\beta(s)}, \nu\rangle = 0$. Now, from the expression of $V(s)$ and bearing in mind that $\langle \nu, \mathbf{1}\rangle = 0$, this means that $\langle \nu\,\beta(s), \beta''(s)\rangle = 0$, or equivalently, that $\langle \nu\,\beta(s), \beta'(s)\rangle$ is constant. The argument is totally analogous if we suppose that $\overline{\beta(s)}\,V(s)$ is a piece of great circle of $\mathbb{S}^2$, obtaining in this case that $\langle \beta(s)\,\nu, \beta'(s)\rangle$ is constant.

Conversely, let us assume that there exists a vector $\nu \in \mathbb{S}^2$ such that $\langle \nu\,\beta(s), \beta'(s)\rangle$ is constant. Since $k_\beta > 0$, from Corollary 17 the curve β is geodesic of a flat surface, whose unit normal along β is given by (5.1). The desired result follows then immediately from Corollary 19, by noting that $\langle V(s)\,\overline{\beta(s)}, \nu\rangle = 0$ and so $V(s)\,\overline{\beta(s)}$ is a piece of great circle of $\mathbb{S}^2$.

□

We remark that Corollaries 17 and 19 can actually be seen as generalizations of Barros' result.

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The authors were partially supported by Ministerio de Educación y Ciencia Grant No. MTM2007-65249.