STRUCTURAL APPROACH TO THE GEOMETRY OF GRAVITY

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ABSTRACT

The fundamental mathematical tools used by General Relativity to explain and to handle gravity are the geometrical structures. Different theories of gravity try to separate the geometry into independent compounds to promote the understanding about physical interpretation of geometric variables. The theory of G-structures of higher order is possibly the more natural framework for studying the interrelations involved among the relevant structures: pseudo-Riemannian metrics, volume forms, conformal metrics, linear connections, projective structures, Weyl geometries, etc. With this formalism, we give a unified description of these geometrical structures and, finally, we try to clarify the relationships among them.

1 Motivation

General relativity (GR) is a physical theory, which is heavily based on differential geometry. The space-time of general relativity is described by a 4-dimensional manifold with a *Lorentzian metric field*. The GR theory put the matter on space-time, being mainly represented by curves in the manifold or by the overall stress-energy tensor.

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The law of inertia in the space-time is translated into a *projective structure* on the manifold, which is provided by the geodesics of the metric in keeping with the equivalence principle. Furthermore, the space-time in GR is a dynamical entity because the metric field is subject to the Einstein field equations, which almost equate Ricci curvature with stress-energy of matter.

Other main structures are the *volume form*, that is used to get action functionals by integration over the manifold, and the *Lorentzian conformal structure*, that gives an account of light speed invariance. All of them can be explained with the use of the same mathematical tools.

2 Frame bundles

A differentiable manifold M is a set of points with the property that we can cover with the charts of an atlas. Indeed, the *primordial structure* of M is a C^∞ n-dimensional maximal atlas $\mathcal{A}$.

The *bundle of r-frames* $\mathcal{F}^r M$ is a quotient set over $\mathcal{A}$. Every class-point, an r-frame, collect the charts with equal origin of coordinates which produce identical r-th order Taylor series expansion of differentiable functions [5, 8].

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An r-frame is an r-jet at 0 of inverses of charts of M; two charts are in the same r-jet if they have the same partial derivatives up to r-th order at the origin of coordinates. Every $\mathcal{F}^r M$ is naturally equipped with a *principal bundle* structure with group, say, G^r_n.

The group of the *bundle of 1-frames* is $\mathrm{GL}(n,\mathbb{R}) \cong \mathrm{G}^1_n$. Its natural representation on $\mathbb{R}^n$ gives an *associated bundle* coinciding with the tangent bundle TM. In the end, we *identify $\mathcal{F}^1 M$ with the linear frame bundle, LM*. Other representations of G^1_n on subspaces of the tensorial algebra over $\mathbb{R}^n$ give associated bundles whose sections are mathematical tensor fields.

The *bundle of 2-frames $\mathcal{F}^2 M$* is somehow more complicated. Every 2-frame is characterized by a *torsion-free transversal n-subspace* $H_l \subset T_l\mathcal{F}^1 M$. It happens that the chart's first partial derivatives fix $l \in \mathcal{F}^1 M$ and the second partial derivatives give the 'inclination' of that n-subspace. The group G^2_n *is isomorphic to* $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n$, the semidirect product with S^2_n being the additive group of symmetric bilinear maps of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$; and the multiplication rule given by $(a,s)(b,t) := (ab, b^{-1}s(b,b)+t)$, for $a,b \in \mathrm{G}^1_n$, $s,t \in \mathrm{S}^2_n$.

Let $\mathfrak{g}$ denote the Lie algebra of $G \subset \mathrm{G}^1_n$, then the *first prolongation of* $\mathfrak{g}$ is defined by $\mathfrak{g}_1 := \mathrm{S}^2_n \cap L(\mathbb{R}^n, \mathfrak{g})$. We obtain that $G \rtimes \mathfrak{g}_1$ *is a subgroup of* $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n \cong \mathrm{G}^2_n$.

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3 1st order G-structures

We define an *r-th order G-structure* of M as a *reduction* of $\mathcal{F}^r M$ to a subgroup $G \subset \mathrm{G}^r_n$. First order G-structures are just called G-structures. Let us see some of them.

Let us define a *volume on M* as a G-structure, with $G = \mathrm{SL}^\pm_n := \{a \in \mathrm{G}^1_n \colon |\det(a)| = 1\}$. If M is orientable, a volume on M has two components: two $\mathrm{SL}(n,\mathbb{R})$-structures for two equal, except sign, *volume n-forms*. For a general M, a volume corresponds to an *odd type n-form*.

From *bundle theory* [4], $\mathrm{SL}^\pm_n$-structures are the sections of the *associated bundle* to $\mathcal{F}^1 M$ and the left action of G^1_n on $\mathrm{G}^1_n/\mathrm{SL}^\pm_n$. This is the *volume bundle, $\mathcal{V}M$*. Furthermore, the sections of $\mathcal{V}M$ correspond to *equivariant functions* f of $\mathcal{F}^1 M$ to $\mathrm{G}^1_n/\mathrm{SL}^\pm_n \simeq \mathrm{H}_n := \{k\, I_n \,:\, k > 0\} \simeq \mathbb{R}^+$, verifying $f(la) = |\det a|^{\frac{-1}{n}} f(l)$, $\forall a \in \mathrm{G}^1_n$. We have the bijections:

$$\text{Volumes on } M \longleftrightarrow \mathrm{Sec}\,\mathcal{V}M \longleftrightarrow C^\infty_{\text{equi}}(\mathcal{F}^1 M, \mathbb{R}^+)$$

Analogous bijective diagram can be obtained for every reduction of a principal bundle.

The Lie algebra of $\mathrm{SL}^\pm_n$ is $\mathfrak{sl}(n,\mathbb{R})$; and its first prolongation is $\mathfrak{sl}(n,\mathbb{R})_1 = \{s \in \mathrm{S}^2_n \colon \sum_k s^k_{ik} = 0\}$.

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A *pseudo-Riemannian metric field* is a G-structure, with $G = \mathrm{O}_{q,n\text{-}q} := \left\{a \in \mathrm{G}^1_n \colon a^t \eta a = \eta \equiv \begin{pmatrix} -I_q & 0 \\ 0 & I_{n\text{-}q} \end{pmatrix}\right\}$. As above, we obtain bijections between the metrics and the sections of the associated bundle with typical fiber $\mathrm{G}^1_n/\mathrm{O}_{q,n\text{-}q}$, and also with equivariant functions of $\mathcal{F}^1 M$ in $\mathrm{G}^1_n/\mathrm{O}_{q,n\text{-}q}$ (see [4]). The first prolongation of the Lie algebra $\mathfrak{o}_{q,n\text{-}q}$ is $\mathfrak{o}_{q,n\text{-}q\;1} = 0$; a consequence of this fact is the uniqueness of *Levi-Civita connection*.

A *pseudo-Riemannian conformal structure* is a G-structure, with $G = \mathrm{CO}_{q,n\text{-}q} := \mathrm{O}_{q,n\text{-}q} \cdot \mathrm{H}_n$ (direct product). This definition is equivalent to consider a class of metrics related by a positive factor. In the Lorentzian case ($q=1$), conformal structure is characterized by the *field of null cones*. The first prolongation of the Lie algebra $\mathfrak{co}_{q,n\text{-}q}$ is $\mathfrak{co}_{q,n\text{-}q\;1} = \{s \in \mathrm{S}^2_n \colon s^i_{jk} = \delta^i_j \mu_k + \delta^i_k \mu_j - \eta^{il}\eta_{jk}\mu_l,\ \mu_i \in \mathbb{R}\} \simeq \mathbb{R}^{n*}$.

Volumes on M and conformal structures are *extensions* of pseudo-Riemannian metrics because of the inclusion of $\mathrm{O}_{q,n\text{-}q}$ in $\mathrm{SL}^\pm_n$ and $\mathrm{CO}_{q,n\text{-}q}$. Reciprocally:

Theorem 1 *A pseudo-Riemannian metric field on M is given by a pseudo-Riemannian conformal structure and a volume on M.*

This statement is proved in [7] by the facts that $\mathrm{G}^1_n = \mathrm{SL}^\pm_n \cdot \mathrm{CO}_{q,n\text{-}q}$. and $\mathrm{O}_{q,n\text{-}q} = \mathrm{SL}^\pm_n \cap \mathrm{CO}_{q,n\text{-}q}$; these imply that volume and conformal bundles intersect in $\mathrm{O}_{q,n\text{-}q}$-structures.

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4 2nd order G-structures

A linear connection on M involves a distribution of transversal n-subspaces, one in each $l \in \mathcal{F}^1 M$, which is invariant by the action of G^1_n. Thereby it follows that a *symmetric linear connection* (SLC) on M can be seen as the image of an *injective homomorphism* of $\mathcal{F}^1 M$ to $\mathcal{F}^2 M$ [5]; hence *an SLC is a second order G^1_n-structure*.

From *bundle theory* again, every SLC, ∇, is a section of the associated bundle to $\mathcal{F}^2 M$ and the action of G^2_n on $\mathrm{G}^2_n/\mathrm{G}^1_n \simeq \mathrm{S}^2_n$, and corresponds to an equivariant function $f^\nabla \colon \mathcal{F}^2 M \to \mathrm{S}^2_n$, verifying $f^\nabla(z(a,s)) = a^{-1} f^\nabla(z)(a,a) + s$.

A *projective structure* on M is a set of SLCs which has the same geodesics up to reparametrizations. It can be defined as a *$\mathrm{G}^1_n \rtimes \mathfrak{p}$-structure*, with $\mathfrak{p} := \{s \in \mathrm{S}^2_n \colon s^i_{jk} = \delta^i_j \mu_k + \mu_j \delta^i_k,\ \mu_i \in \mathbb{R}\} \simeq \mathbb{R}^{n*}$.

Given two SLCs, ∇, $\widehat{\nabla}$, we have $(f^\nabla - f^{\widehat{\nabla}})(z(a,s)) = a^{-1}(f^\nabla(z) - f^{\widehat{\nabla}}(z))(a,a)$; then this difference is projectable to a function $\rho \colon \mathcal{F}^1 M \to \mathrm{S}^2_n$ verifying $\rho(la) = a^{-1}\rho(l)(a,a)$, hence $\rho = (\rho^i_{jk})$ is a *tensor field*. Thereby, the difference of two SLCs, *reducible* from the same projective structure, is determined by the contraction $C(\rho) = (\rho^k_{ki})$, which is *an 1-form* on M.

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Let P be a G-structure on M; a *symmetric connection on P* (if there exists, [6]) is a distribution on P of torsion-free transversal n-subspaces, $H_l \subset T_l P \subset T_l \mathcal{F}^1 M$, for each $l \in P$; this distribution gives an injective homomorphism of P to $\mathcal{F}^2 M$, whose image is a second order G-structure; and also *by extension* an SLC on M.

Noteworthy examples of this are:

- The *Levi-Civita connection* of a pseudo-Riemannian metric is given by a *second order $\mathrm{O}_{q,n\text{-}q}$-structure*.
- An *equiaffine structure* on M is a SLC with a parallel volume; then, it is given as a *second order $\mathrm{SL}^\pm_n$-structure*.
- A *Weyl structure* is a conformal structure with a compatible SLC; then, it is given as a *second order $\mathrm{CO}_{q,n\text{-}q}$-structure* (see [1]).

If a G-structure P admits an SLC, it has as many SLCs as torsion-free transversal n-subspaces are in $T_l P$, for every $l \in P$. We have the following:

Theorem 2 *If P admits an SLC, the set P^2 of 2-frames, corresponding with torsion-free transversal n-subspaces included in TP, is a second order $G \rtimes \mathfrak{g}_1$-structure. The symmetric connections on P are in correspondence with the injective homomorphisms of P to P^2.*

The P^2 bundle will henceforth be called the (first) prolongation of P (see [8]).

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When a G-structure admits an SLC, it determines and is determined by its prolongation. Therefore, considering the second order, we can analyze the *intersection of geometrical structures*, like with the first order in Theorem 1.

Theorem 3 *A projective structure and a volume on M give an SLC belonging to the former and making the volume parallel.*

Hence, *a volume select a class of affine parametrizations* for the geodesics of a projective structure (see [7]). Contrarily, a projective structure and a prolonged conformal structure only intersect if they verify a compatibility condition (see [2]); in this case, getting a Weyl structure.

5 Concluding remarks

The geometrical structures described herein can be considered components of the *space-time geometry*. Indeed, the *causal set theory* separates the volume and conformal structure (for a review, see [3]); and Stachel proposes in [9] an approach, similar to the metric-affine variational principle, using conformal and projective structures. From the above results, *I suggest considering the volume as a set of independent dynamical variables*.

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