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EFFECTIVE COMPUTATION OF THE GELFAND–KIRILLOV DIMENSION FOR MODULES OVER POINCARÉ-BIRKHOFF-WITT ALGEBRAS WITH NON QUADRATIC RELATIONS

JOSÉ L. BUESO, J. GÓMEZ TORRECILLAS, AND F. J. LOBILLO

ABSTRACT. We give an algorithm to compute the Gelfand–Kirillov dimension of finitely generated modules over algebras like universal enveloping algebras of finite–dimensional Lie algebras, Weyl algebras, quantized Weyl algebras, quantum affine spaces, differential operators algebras, the possitive part of quantized universal enveloping algebras of semisimple Lie algebras and the quantized $U_q(\mathfrak{sl}_3(\Bbbk))$.

INTRODUCTION

We give an algorithm to compute the Gelfand-Kirillov dimension of a given finitely generated left module over a $\Bbbk$-algebra R, where $\Bbbk$ is a (commutative) field. Of course, this cannot be done for any algebra R, so that we will restrict ourselves to Poincaré-Birkhoff-Witt algebras (cf. [5, 6]). These are associative algebras generated over $\Bbbk$ by finitely many elements $x_1, \ldots, x_n$ in such a way that the standard monomials $X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ form a basis of R as $\Bbbk$-vector space. Moreover, the size of the semi-commutators $x_j x_i - q_{ij} x_i x_j$ are controlled by some good ordering in $\mathbb{N}^n$ (see Definition 1.4 for the details). A classical example of this situation occurs when R is the universal enveloping algebra of a Lie $\Bbbk$-algebra with basis $x_1, \ldots, x_n$. An algorithm to computing Gelfand-Kirillov dimension for cyclic modules in this case was obtained in [7]. This effective method depends on the existence of a standard filtration on R with a commutative polynomial ring as associated graded algebra. In fact, the essential point there is that the commutators $[x_i, x_j]$ are linear polynomials in $x_1, \ldots, x_n$. Therefore, this approach does not work for algebras with relations of bigger degree. For PBW algebras with quadratic relations (i.e., when the semi-commutators among the generators $x_1, \ldots, x_n$ are polynomials of total degree at most 2), we included a first version of our algorithm in [5, 6], but it works only for cyclic modules and does not apply to important examples of algebras like Sigurdsson's iterated differential operators algebras [19], the positive part of the quantized enveloping algebra of a finite-dimensional semi-simple Lie algebra or the quantized enveloping algebra of $\mathfrak{sl}(3)$. The problem here is that the variables that appear in the Poincaré-Birkhoff-Witt basis have non quadratic relations. We solve this in the following pages.

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Our algorithm works for modules of the form R/L, where R is a PBW algebra in the sense of [6] with respect to an almost graded order (see Definition 2.1), and L is a left ideal of R. The central idea is the effective assignment of a set of exponents $\mathrm{Exp}(L)$ to L by using Gröbner bases and then recognize the Gelfand-Kirillov dimension of R/L as certain computable dimension of $\mathrm{Exp}(L)$ (Theorem 2.5).

1. PBW ALGEBRAS

1.1. We will assume that R is generated as $\Bbbk$-algebra by finitely many elements $x_1, \ldots, x_n$. For $\alpha \in \mathbb{N}^n$, consider the *standard monomial* $X^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The algebra R is a *multivariate polynomial* $\Bbbk$*-algebra in* $x_1, \ldots, x_n$ if the set $\mathcal{B} = \{X^\alpha \mid \alpha \in \mathbb{N}^n\}$ is a basis of R as $\Bbbk$-vector space. This means that every element f of R has a unique *standard representation* as

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \tag{1}$$

1.2. We begin with $\mathbb{N}^n$ and its additive monoid structure. Consider an *admissible* order $\leq$ in $\mathbb{N}^n$, i.e., $\leq$ is a total order satisfying the following conditions:

(1) $0 = (0, \ldots, 0) \leq \alpha$ for every $\alpha \in \mathbb{N}^n$.
(2) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$ for every $\gamma \in \mathbb{N}^n$.

Dickson's Lemma (see, e.g., [2, Corollary 4.48]) gives that $\mathbb{N}^n$ is well ordered by $\leq$.

1.3. Consider R a multivariate polynomial $\Bbbk$-algebra (see 1.1). As every element $f \in R$ can be uniquely expressed by its standard representation, we define the *Newton diagram* of f by

$$\mathcal{N}(f) = \{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}$$

and the *exponent* of f by

$$\exp(f) = \max \mathcal{N}(f)$$

To avoid irrelevant hypothesis in our statements, define $\exp(0) = -\infty$, where $-\infty \notin \mathbb{N}^n$. We assume that $-\infty < \alpha$ and $-\infty + \alpha = -\infty$ for every $\alpha \in \mathbb{N}^n$. The *leading coefficient* and *leading monomial* of f are

$$\mathrm{lc}(f) = c_{\exp(f),f} \qquad \text{and} \qquad \mathrm{lm}(f) = \mathrm{lc}(f) X^{\exp(f)}.$$

Definition 1.4. The algebra R is called a *Poincaré-Birkhoff-Witt (PBW algebra, for short)* if there is an admissible order $\leq$ such that $\exp(fg) = \exp(f) + \exp(f)$ for every $f, g \in R$. By [6] or [5], this is equivalent to the existence of nonzero scalars q_{ij} in $\Bbbk$ such that

$$x_j x_i = q_{ji} x_i x_j + \sum_{\gamma < \epsilon_i + \epsilon_j} c_\gamma X^\gamma \tag{2}$$

where $e_i = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$. These algebras are called polynomial rings of solvable type in [13].

Proposition 1.5. *A PBW algebra is a filtered structure in the sense of* [17]. *Moreover, as PBW algebras are noetherian (see* [6, Corollary 2.9]*) the procedures in* [17] *finish in a finite number of steps.*

2. Gelfand–Kirillov dimension.

Fix a commutative field $\Bbbk$ and assume that R is a $\Bbbk$-algebra. While we introduce some notation, we recall the notion of *Gelfand-Kirillov dimension* of finitely generated $\Bbbk$-algebras (details in [16, 14, 15]). If V is a finite-dimensional vector subspace of R such that $1 \in V$ and n is a positive integer, then V^n denotes the vector subspace of R generated by all n-fold products $v_1 \cdots v_n$ where $v_i \in V$. It is understood that $V^0 = \Bbbk$. It is clear that $V^n \subseteq V^{n+1}$ for all n. Now, assume that R is finitely generated as $\Bbbk$-algebra by V. The Gelfand-Kirillov dimension of R ($\mathrm{GKdim}(R)$ for short) measures the rate of growth of the *Hilbert function* $HF_V(n) = \dim_{\Bbbk} V^n$. In fact, $\mathrm{GKdim}(R)$ is the infimun of the real numbers r such that $HF_V(n) \leq n^r$ for $n \gg 0$. It is known that this value is independent of the choice of V. The GK dimension of a finitely generated left R-module M is given by the growth of the function $HF_{V,U} = \dim_{\Bbbk}(V^n U)$, where U is any finite-dimensional vector subspace of M which generates M as left R-module. This dimension can be expressed by the following formula.

$$\mathrm{GKdim}(M) = \gamma(HF_{V,U}(n))$$

where $\gamma(f) = \limsup \log_n(f(n))$ for any function f from $\mathbb{N}$ to $\mathbb{N}$ (see [16, 8.1.5, 8.1.6].

In this section we are going to develop an algorithm to compute the Gelfand–Kirillov dimension of cyclic left R-modules in the case that R is a $\Bbbk$-algebra. All admissible orders does not allow the computation of GK dimension. We need some finiteness condition. Let $\omega \in (\mathbb{R}^+)^n$ be a n-vector of positive real components. For any $\alpha \in \mathbb{N}^n$ we put

$$\langle \alpha, \omega \rangle = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n.$$

Definition 2.1. An admissible order $\leq$ on $\mathbb{N}^n$ is called *almost graded* if there exists $\omega \in (\mathbb{R}^+)^n$ such that $\alpha \leq \beta$ implies $\langle \alpha, \omega \rangle \leq \langle \beta, \omega \rangle$. The vector ω is called weight vector.

Example 2.2. Let $\leq$ be the lexicographical order over $\mathbb{N}^n$ and $\omega \in (\mathbb{R}^+)^n$. Easy computations show that the order $\leq_\omega$ defined by

$$\alpha \leq_\omega \beta \iff \begin{cases} \langle \alpha, \omega \rangle < \langle \beta, \omega \rangle & \text{or} \\ \langle \alpha, \omega \rangle = \langle \beta, \omega \rangle \text{ and } \alpha \leq \beta \end{cases}$$

is an almost graded admissible order. The same procedure can be applied to any admissible order $\leq$.

2.3. For any ideal $E \subseteq \mathbb{N}^n$ and any weight vector $\omega \in (\mathbb{R}^+)^n$, we define the ω-*Hilbert function* of E as

$$HF_E^\omega(s) = \mathrm{card}\{\alpha \in \mathbb{N}^n \setminus E \mid \langle \alpha, \omega \rangle \leq s\}.$$

Lemma 2.4. *Let $E \subseteq \mathbb{N}^n$ be an ideal and let $\omega, \omega' \in (\mathbb{R}^+)^n$. Then $\gamma(HF_E^\omega) = \gamma(HF_E^{\omega'})$.*

This number is known as the *dimension* of E, i.e., $\dim(E) = \gamma(HF_E^\omega)$ for some (and then all) $\omega \in (\mathbb{R}^+)^n$. Our algorithm to computing the Gelfand-Kirillov dimension is based on the following theorem, whose proof is postponed to the end of the section.

Theorem 2.5. *Let R be a PBW $\Bbbk$-algebra with respect an almost graded order over $\mathbb{N}^n$ with weight vector ω. Let $L \subseteq R$ be a left ideal. Then*

$$\mathrm{GKdim}\left(\frac{R}{L}\right) = \dim(\mathrm{Exp}(L))$$

Next, we will prove Theorem 2.5. We split the proof in several Lemmas. Fix $V = \Bbbk x_1 + \cdots + \Bbbk x_n$ as generating subspace of R as $\Bbbk$-algebra and let U be the vector subspace of R/L spanned by $1 + L$.

Lemma 2.6. *With the previous assumptions,*

$$\mathrm{card}\{\alpha \in \mathbb{N}^n \setminus \mathrm{Exp}(L) \mid \langle \alpha, \omega \rangle \leq s\} = \dim_\Bbbk\{f + L \mid \langle \exp(f), \omega \rangle \leq s\}$$

Lemma 2.7. *Let $w = \max\{\omega_1, \ldots, \omega_n\}$. Then*

$$V^k U \subseteq \{f + L \mid \langle \exp(f), \omega \rangle \leq wk\}.$$

Lemma 2.8. *There exists $a \in \mathbb{N}$ such that*

$$\{f + L \mid \langle \exp(f), \omega \rangle \leq k\} \subseteq V^{ak} U$$

Proof of Theorem 2.5. Using the lemmas 2.7, 2.8 and [16, 8.1.7], we have

$$\mathrm{GKdim}\left(\frac{R}{L}\right) = \gamma(HF_{V,U}) = \gamma(F^\omega)$$

where $F^\omega(k) = \dim_\Bbbk\{f + L \mid \langle \exp(f), \omega \rangle \leq k\}$. The statement follows from lemma 2.6 and 2.3 □

3. Examples

Theorem 2.5 can be obviously applied to cyclic left R-modules in the case $R = U(\mathfrak{g})$, the universal enveloping algebra of a f.d. Lie algebra $\mathfrak{g}$, or for $R = \mathcal{O}_{\underline{q}}(\Bbbk^n)$, the multi-parameter quantum coordinate algebra of the n-dimensional affine space over $\Bbbk$. More generally, our method covers Berger's q-enveloping algebras [3]. In all these cases we can take $\omega = (1, \ldots, 1)$ as weight vector.

A wide class of quantum groups are PBW algebras with respect to an almost graded order. In fact, we have the following algorithmic method to find the weight vector ω for 'homothetic' iterated Ore extensions.

Example 3.1. ([9, Example 3.3]) Let $R = k[x_1][x_2;\sigma_2,\delta_2]\dots[x_n;\sigma_n,\delta_n]$ be an iterated Ore extension of $\mathbb{k}$. Assume that $\sigma_j(x_i) = q_{ij}x_i$, for every $i < j \leq p$, where the q_{ij}'s are nonzero scalars in $\mathbb{k}$. The set $B = \{X^\alpha; \alpha \in \mathbb{N}^n\}$ is a $\mathbb{k}$-basis of R. We will construct a weight vector $\omega = (w_1,\dots,w_n)$. Set $w_1 = 1$ and define w_{12} as the degree in x_1 of the polynomial $\delta_2(x_1)$. Put $w_2 = \max\{1, w_{12}\}$. Suppose we have defined $w_1,\dots,w_{j-1}$ for $j \geq 2$. For $k = 1,\dots,j-1$, set $w_{kj} = \max\{\alpha_1 w_1 + \dots + \alpha_{j-1}w_{j-1};\ \alpha \in \mathcal{N}(\delta_j(x_k))\}$ and choose $w_j = \max\{1, w_{kj} - w_i;\ 1 \leq i,k \leq j-1\}$. Endow $\mathbb{N}^n$ with the admissible order $\leq_\omega$ (see 2.2). By [9, Example 3.3], R is a PBW algebra with respect to $\leq_\omega$.

Some examples of these iterated Ore extensions are the algebras $H(p,\lambda)$ defined in [1], which include the quantum coordinate algebras of $M_n(k)$; and the multiparameter quantized Weyl algebra $R = A_n^{Q,\Gamma}(k)$ from [12]. The iterated differential operator algebras of [19] are also covered, as well as the positive part of the quantized enveloping algebra of a finite-dimensional Lie algebra as defined by Drinfeld [8] and Jimbo [11] (see [18]).

Example 3.2. We want to consider now the quantized universal enveloping algebra $U_q(\mathfrak{sl}_3(\mathbb{k}))$. This $\mathbb{k}$-algebra has as $\mathbb{k}$-basis the monomials

$$f_{12}^{n_1} f_{13}^{n_2} f_{23}^{n_3} k_1^{\ell_1} k_2^{\ell_2} e_{12}^{m_1} e_{13}^{m_2} e_{23}^{m_3}$$

where $n_i, m_i \in \mathbb{N}$ and $\ell_i \in \mathbb{Z}$ (see [21, Theorem 1.1]). The finiteness conditions required to compute GK dimension do not allow the use of negative exponents. So we are going to define a new algebra generated by $f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2, e_{12}, e_{13}, e_{23}$ and relations obtained from [21, Section 3] using l_i as k_i^{-1}. This algebra is called $V_q(\mathfrak{sl}_3(\mathbb{k}))$. Using Bergman's Diamond Lemma [4], and with enough amount of paper, it is easy to see that the monomials

$$f_{12}^{n_1} f_{13}^{n_2} f_{23}^{n_3} k_1^{a_1} k_2^{a_2} l_1^{b_1} l_2^{b_2} e_{12}^{m_1} e_{13}^{m_2} e_{23}^{m_3}$$

form a $\mathbb{k}$-basis for $V_q(\mathfrak{sl}_3(\mathbb{k}))$ with $n_i, a_i, b_i, m_i \in \mathbb{N}$. Moreover, as proved in [9, Example 3.4], the vector $\omega = (7,10,11,1,1,1,1,7,10,11)$ is a weight vector which makes $V_q(\mathfrak{sl}_3(\mathbb{k}))$ a PBW algebra with respect to the almost graded order $\leq_\omega$ (see Example 2.2). As the elements $k_i l_i$ are central, we have

$$U_q(\mathfrak{sl}_3(\mathbb{k})) \cong \frac{V_q(\mathfrak{sl}_3(\mathbb{k}))}{I}$$

where I is the two-sided ideal generated (as left ideal!) by $k_1 l_1 - 1$ and $k_2 l_2 - 1$. This isomorphism allows us to compute GK dimension of finitely generated $U_q(\mathfrak{sl}_3(\mathbb{k}))$-modules through $V_q(\mathfrak{sl}_3(\mathbb{k}))$. In particular,

$$\mathrm{GKdim}(U_q(\mathfrak{sl}_3(\mathbb{k}))) = 8.$$

The same ideas can be exported to $U_q(\mathfrak{sl}_n(\mathbb{k}))$.

Remark 3.3. For examples like universal enveloping algebras of finite dimensional solvable Lie algebras [20] or coordinate algebras of quantum affine spaces and quantized Weyl algebras [10], the height of any prime ideal P can be computed as $\mathrm{GKdim}(R)-$

GKdim(R/P), so the computation of the height becomes effective. We can also compute the grade of a finitely generated module M as $j(M) = \text{GKdim}(R) - \text{GKdim}(M)$ for Auslander-Regular and Cohen-Macaulay PBW algebras. In particular, we can decide if a given finitely generated module over a Weyl algebra is holonomic.

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Dpto. de Álgebra, Facultad de Ciencias, Universidad de Granada, E18071 Granada, SPAIN

E-mail address: torrecil@ugr.es

E-mail address: jlobillo@ugr.es