ON THE TOPOLOGY OF TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW

F. MARTÍN, A. SAVAS-HALILAJ, AND K. SMOCZYK

Abstract. In the present article we obtain classification results and topological obstructions for the existence of translating solitons of the mean curvature flow in euclidean space.

1. Introduction

An oriented smooth hypersurface $f : M^m \to \mathbb{R}^{m+1}$ is called *translating soliton* (or a *translator* for short) of the mean curvature flow if its mean curvature vector field $\vec{H}$ satisfies

$$\vec{H} = \mathrm{v}^{\perp}, \tag{1.1}$$

where $\mathrm{v} \in \mathbb{R}^{m+1}$ is a fixed unit length vector and $\mathrm{v}^{\perp}$ stands for the orthogonal projection of v onto the normal bundle of the immersion f. Translating solitons are important in the singularity theory of the mean curvature flow since they often occur as Type-II singularities. On the other hand they also form interesting examples of precise solutions of the flow since the smooth family of immersions $F : M^m \times \mathbb{R} \to \mathbb{R}^{m+1}$, $F(x,t) := f(x) + t\mathrm{v}$, evolves, up to some tangential diffeomorphisms, by its mean curvature. If one chooses a smooth unit length normal vector field ξ along f, then equation (1.1) may be expressed in terms of scalar quantities. More precisely, equation (1.1) is equivalent to

$$H := -\langle \vec{H}, \xi \rangle = -\langle \mathrm{v}, \xi \rangle, \tag{1.2}$$

where here H is the *scalar mean curvature* of f. Since equation (1.1) is invariant under isometries one may, without loss of generality, always assume that the velocity vector v is given by $\mathrm{v} = \mathrm{e}_{m+1}$, where $\{\mathrm{e}_1, \dots, \mathrm{e}_{m+1}\}$ denotes the standard orthonormal basis of $\mathbb{R}^{m+1}$.

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Translating solitons of the euclidean space $\mathbb{R}^{m+1}$ are closely related to minimal hypersurfaces. In fact, translators can be regarded as minimal hypersurfaces of $(\mathbb{R}^{m+1}, \mathrm{G})$ where G is a Riemannian metric conformal to the usual inner product of $\mathbb{R}^{m+1}$ (for more details see Section 3). However, at present there is no general method to construct examples of translating solitons. Even in the 2-dimensional case there is no useful Weierstraß type representation known to exist for translators, like there is for minimal surfaces in $\mathbb{R}^3$. Moreover, although it is believed that there exists an abundance of translators, there are only a very few available examples of complete translating solitons in the euclidean space $\mathbb{R}^{m+1}$. For instance, any minimal hypersurface of $\mathbb{R}^{m+1}$ tangent to the translating direction v is a translating soliton (however, in this case $H \equiv 0$ implies that the translator actually does not move at all). The euclidean product $\Gamma \times \mathbb{R}^{m-1}$, where Γ is the *grim reaper* in $\mathbb{R}^2$ represented by the immersion $f : (-\pi/2, \pi/2) \to \mathbb{R}^2$ given by

$$f(x) = 1 - \log\cos x,$$

gives rise again to a translating soliton in the direction of e_{m+1}. More generally, any translator in the direction of e_{m+1} which is a Riemannian product of a planar curve and an euclidean space $\mathbb{R}^{m-1}$ can be obtained from this example by a suitable combination of a rotation and a dilation. Each of these translators will be called a *grim hyperplane*.

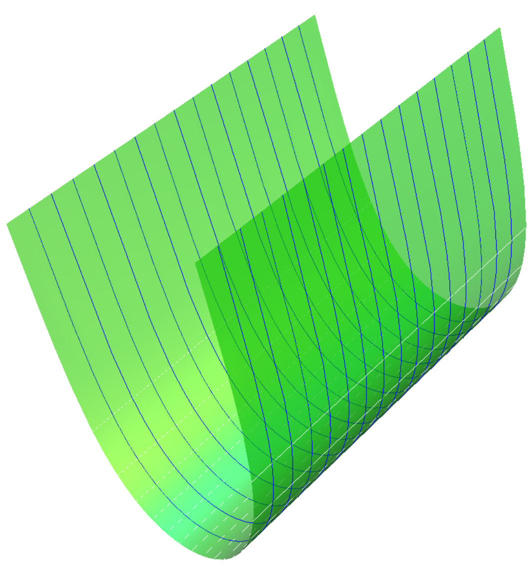

FIGURE 1. A grim hyperplane

Another way to construct complete translators is by rotating special curves around the translating axis. From this procedure one gets the rotational symmetric *translating paraboloid* and the *translating catenoids* which are unique up to rigid motions (see the examples in subsection 2.2d). We would like to point out that it is not known if there are examples of complete translating solitons with finite non-zero genus.

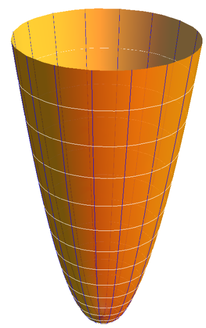

FIGURE 2. A translating paraboloid.

Our goal is to classify, under suitable conditions, translating solitons and to obtain topological obstructions for the existence of translating solitons with finite non-zero genus in $\mathbb{R}^3$. Using the Alexandrov's reflection principle we prove a uniqueness theorem for complete embedded translating solitons with a single end that are asymptotic to a translating paraboloid. More precisely we show the following.

Theorem A. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a complete embedded translating soliton of the mean curvature flow with a single end that is smoothly asymptotic to a translating paraboloid. Then the hypersurface $M := f(M^m)$ is a translating paraboloid.*

We give the following characterization of the grim hyperplanes.

Theorem B. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a translating soliton which is not a minimal hypersurface. Then $f(M^m)$ is a grim hyperplane if and only if the function $|A|^2H^{-2}$ attains a local maximum on the open set $M^m - \{H = 0\}$.*

As an immediate consequence of the above theorem we prove that a translating soliton with zero scalar curvature either coincides with a grim hyperplane or with a minimal hypersurface tangential to the translating direction.

The rest of the paper is devoted to translators in $\mathbb{R}^3$. We focus on the study of the distribution of the Gauß map of a complete translating soliton. In particular we investigate how the Gauß image affects the genus of a complete translating soliton in $\mathbb{R}^3$. For instance, we would like to mention that it is perhaps true that *any complete translating soliton in* $\mathbb{R}^3$ *whose Gauß map omits the north pole must have genus zero*. In the next theorem we give a partial answer to this question.

Theorem C. *Let $f : M \to \mathbb{R}^3$ be a complete translator, where M has finite topology. Suppose that the following conditions are satisfied.*

(a) *Each end is either bounded from above or from below.*
(b) *The mean curvature satisfies $H > -1$ everywhere and there exists a compact subset $C \subset M$ and a positive real number ε such that $H^2 < 1 - \varepsilon$ on $M - C$.*

Then the genus g of M satisfies $g \leq 1$. If f is an embedding, then $g = 0$ and thus M is a planar domain.

Remark 1.1. From equation (1.2) the scalar mean curvature H of a translator always satisfies the inequality

$$-1 \leq H = -\langle \mathrm{v}, \xi \rangle \leq 1.$$

Hence, the assumption $H > -1$ means that the Gauß map ξ is omitting the north pole of $\mathbb{S}^2$. Moreover, there are plenty of examples of complete embedded surfaces with genus $g \geq 1$ in $\mathbb{R}^3$ whose Gauß map is not onto.

We prove that within the class of translating solitons for which the Gauß map omits the north pole (or more generally the direction of translation) it holds that a translator is mean convex, if it is mean convex outside a compact subset. More precisely, we show the following.

Theorem D. *Let $f : M^2 \to \mathbb{R}^3$ be a translating soliton whose mean curvature satisfies $H > -1$. Suppose that $H \geq 0$ outside a compact subset of M. Then either $M = f(M^2)$ is part of a flat plane or $H > 0$ on all of M^2. If, in addition, M is complete and embedded, then it is a graph and so it has genus zero.*

The paper is organized as follows. In Section 2, after setting up the notation and computing the basic equations that translators satisfy, we prove Theorem B. In Section 3 we prove Theorem A and in the last Section 4 we prove Theorems C and D.

2. Translating solitons

2.1. **Local formulas.** In this subsection we will derive and summarize the most relevant equations related to translators. Let $f : M^m \to \mathbb{R}^{m+1}$ be an immersion and $\mathrm{g} = f^*\langle\cdot,\cdot\rangle$ the induced metric. Denote by D the Levi-Civita connection of $\mathbb{R}^{m+1}$ and by ∇ the Levi-Civita connection of g. The *second fundamental form* $\vec{A}$ of f is

$$\vec{A}(v,w) := D_{\mathrm{d}f(v)}\,\mathrm{d}f(w) - \mathrm{d}f\big(\nabla_v w\big),$$

where v, w are tangent vectors on M. The *mean curvature vector field* $\vec{H}$ of f is defined by

$$\vec{H} := \operatorname{trace}_{\mathrm{g}} \vec{A}.$$

Let now ξ be a local unit vector field normal along f. The symmetric bilinear form A given by

$$A(v,w) := -\langle \vec{A}(v,w), \xi\rangle,$$

where $v, w \in TM^m$, is called the *scalar second fundamental from* of f. The *scalar mean curvature* H is defined as the trace of A with respect to g. Suppose now that f is a translating soliton, that is

$$\vec{H} = \mathrm{v}^{\perp}$$

where $\mathrm{v} = (0,\dots,0,1) \in \mathbb{R}^{m+1}$ and where $\mathrm{v}^{\perp}$ is the orthogonal projection of v onto the normal bundle of f. The orthogonal projection of v onto the tangent bundle of f will be denoted by $\mathrm{v}^{\top}$. Let us introduce the height function $u : M^m \to \mathbb{R}$, given by

$$u := \langle f, \mathrm{v}\rangle. \tag{2.1}$$

In the next lemma we give some important relations between the mean curvature H and the height function u.

Lemma 2.1. *The following equations hold on any translating hypersurface in* $\mathbb{R}^{m+1}$.

(a) $\nabla u = \mathrm{v}^{\top}$,
(b) $|\nabla u|^2 = 1 - H^2$,
(c) $\nabla^2 u = HA$,
(d) $\Delta u + |\nabla u|^2 - 1 = 0$,
(e) $\langle \nabla H, \cdot\,\rangle = -A(\nabla u, \cdot\,)$,
(f) $\Delta H + H|A|^2 + \langle \nabla H, \nabla u\rangle = 0$,
(g) $\operatorname{Ric}(\nabla u, \nabla u) = -|\nabla H|^2 - H\langle \nabla H, \nabla u\rangle$,
(h) $\Delta|A|^2 - 2|\nabla A|^2 + \langle \nabla|A|^2, \nabla u\rangle + 2|A|^4 = 0$.

Proof. Let $\{e_1, \dots, e_m\}$ be an orthonormal frame defined on an open neighborhood of M^m.

(a) Differentiating u with respect to e_i we get

$$e_i u = \langle \mathrm{d}f(e_i), \mathrm{v}\rangle.$$

Therefore,

$$\nabla u = \mathrm{v}^\top.$$

(b) Since v has unit length, we obtain the crucial identity

$$1 = |\mathrm{v}|^2 = |\mathrm{v}^\perp|^2 + |\mathrm{v}^\top|^2 = H^2 + |\nabla u|^2.$$

(c) Differentiating ∇u once more, we deduce that

$$\begin{aligned}\nabla^2 u(e_i, e_j) &= e_i e_j u - \langle \nabla u, \nabla_{e_i} e_j\rangle \\ &= e_i \langle \mathrm{d}f(e_j), \mathrm{v}\rangle - \langle \mathrm{d}f(\nabla_{e_i} e_j), \mathrm{v}\rangle \\ &= \langle \vec{A}(e_i, e_j), \mathrm{v}\rangle \\ &= HA(e_i, e_j).\end{aligned}$$

(d) From the last formula giving the Hessian of u, we see that

$$\Delta u = H^2 = 1 - |\nabla u|^2.$$

(e) Differentiating H with respect to the direction e_i, we get

$$\begin{aligned}\langle \nabla H, e_i\rangle &= -e_i\langle \mathrm{v}, \xi\rangle = -\langle \mathrm{v}, \mathrm{d}\xi(e_i)\rangle = -\langle \mathrm{v}^\top, \mathrm{d}\xi(e_i)\rangle \\ &= -A(\nabla u, e_i).\end{aligned}$$

(f) Differentiating once more the gradient of H and using the Codazzi equation, we have

$$\begin{aligned}\nabla^2 H(e_i, e_j) &= -\sum_{k=1}^m (\nabla_{e_k} A)(e_i, e_j) e_k u \\ &\quad -\sum_{k=1}^m A(e_k, e_j)\nabla^2 u(e_i, e_k) \\ &= -\sum_{k=1}^m (\nabla_{e_k} A)(e_i, e_j) e_k u \\ &\quad -H\sum_{k=1}^m A(e_k, e_j) A(e_i, e_k).\end{aligned}$$

Therefore,

$$\Delta H = -\sum_{k=1}^m (e_k H)(e_k u) - H|A|^2 = -\langle \nabla H, \nabla u\rangle - H|A|^2.$$

(g) From the relation (e), we get

$$|\nabla H|^2 = A^{[2]}(\nabla u, \nabla u)$$

where $A^{[2]}$ is the symmetric 2-tensor given by

$$A^{[2]}(v, w) = \sum_{i=1}^{m} A(v, e_i)A(e_i, w),$$

for any $v, w \in TM^m$. Again from the relation (e), we have

$$H\langle\nabla H, \nabla u\rangle = -HA(\nabla u, \nabla u).$$

Consequently,

$$\big(HA - A^{[2]}\big)(\nabla u, \nabla u) = -|\nabla H|^2 - H\langle\nabla H, \nabla u\rangle.$$

By Gauß' equation, the left hand side of the above equation equals the Ricci curvature of the induced metric applied to ∇u. Therefore,

$$\mathrm{Ric}(\nabla u, \nabla u) = -|\nabla H|^2 - H\langle\nabla H, \nabla u\rangle.$$

(h) From Simons' formula [Sim68], we have that

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 \;\;&=\;\; |\nabla A|^2 - |A|^4 + \sum_{i,j=1}^{m} A(e_i, e_j)\nabla^2 H(e_i, e_j)\\ &\quad + H\sum_{i,j,k=1}^{m} A(e_i, e_j)A(e_j, e_k)A(e_k, e_i).\end{aligned}$$

Bearing in mind the formula which relates the Hessian of H with A and u, we get the desired formula.

This completes the proof of the lemma. □

Remark 2.2. From Lemma 2.1 (d) it follows that the height function u does not admit any local maxima. In particular the manifold M^m cannot be compact, a fact that intuitively is clear. Moreover, Lemma 2.1 (b) and (e) imply that the critical sets $\mathrm{Crit}(H)$, $\mathrm{Crit}(u)$ of the mean curvature and the height function on a translating soliton satisfy

$$\begin{aligned}\mathrm{Crit}(u) \;\;&=\;\; \{x \in M : H^2(x) = 1\}\\ \mathrm{Crit}(u) \;\;&\subset\;\; \mathrm{Crit}(H),\\ \mathrm{Crit}(H){-}\mathrm{Crit}(u) \;\;&\subset\;\; M_{\mathrm{reg}} \cap \{x \in M : \det A = 0\},\end{aligned}$$

where $M_{\mathrm{reg}} := M^m{-}\mathrm{Crit}(u)$ denotes the *regular* part of M^m.

2.2. **Examples.** We will expose here some examples of translators in the euclidean space.

(a) All solutions $f : (-\pi/2, \pi/2) \to \mathbb{R}^2$ of (1.1) in the plane $\mathbb{R}^2$ are of the form $f(x) = (x, c - \log\cos x)$, where c is a real constant. The

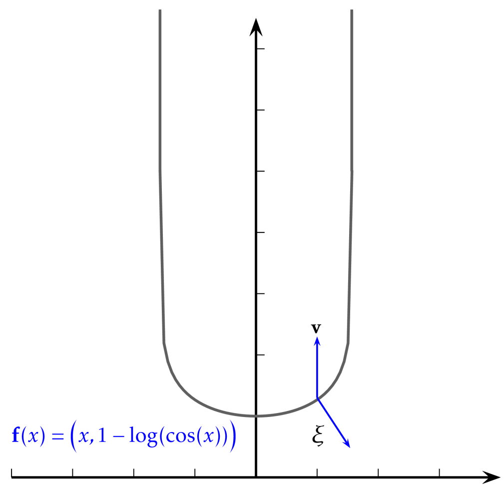

FIGURE 3. Grim reaper

curve f is known as the *grim reaper*. These curves are geodesics of the Riemannian metric

$$\mathrm{G}_p := e^{2\langle p, \mathrm{v}\rangle}\langle \cdot\,, \cdot\rangle,$$

on $\mathbb{R}^2$. Any *grim hyperplane* in $\mathbb{R}^{m+1}$ consists of a suitable rotation and dilation of the orthogonal product of the grim reaper with an euclidean factor $\mathbb{R}^{m-1}$. These examples are mean convex. In fact, these examples have only one non-zero principal curvature.

(b) Suppose that f is a translator which is also minimal. Then v must be tangential to the translator. Consequently, the only minimal translating surfaces in $\mathbb{R}^3$ are the flat planes which are tangential to the vector v.

(c) Altschuler and Wu [AW94] evolved graphs by mean curvature flow defined over compact convex domains Ω in $\mathbb{R}^2$ with prescribed contact angle to the boundary $\partial\Omega$. They were able to prove that solutions converge to translating solitons that are neither convex nor rotationally symmetric. Moreover, they showed the existence of complete, rotationally symmetric translators.

(d) It is shown by Altschuler and Wu [AW94] and Clutterbuck, Schnürer and Schulze in [CSS07] that there do exist an entire rotationally symmetric, strictly convex graphical translator, $u : \mathbb{R}^m \to \mathbb{R}$, $m \geq 2$, having the following asymptotic expansion at infinity

$$u(x) = \frac{|x|^2}{2(m-1)} - \frac{1}{2}\log|x|^2 + O\left(\frac{1}{|x|}\right),$$

see for example [CSS07, Lemma 2.2, p. 286]. This solution is called *translating paraboloid*. If one assumes rotational symmetry around the v axis, then the equation describing the translators reduces to an ODE. As was shown by Clutterbuck, Schnürer and Schulze [CSS07], such complete rotationally symmetric translating solitons coincide (up to translations) either with the rotational symmetric

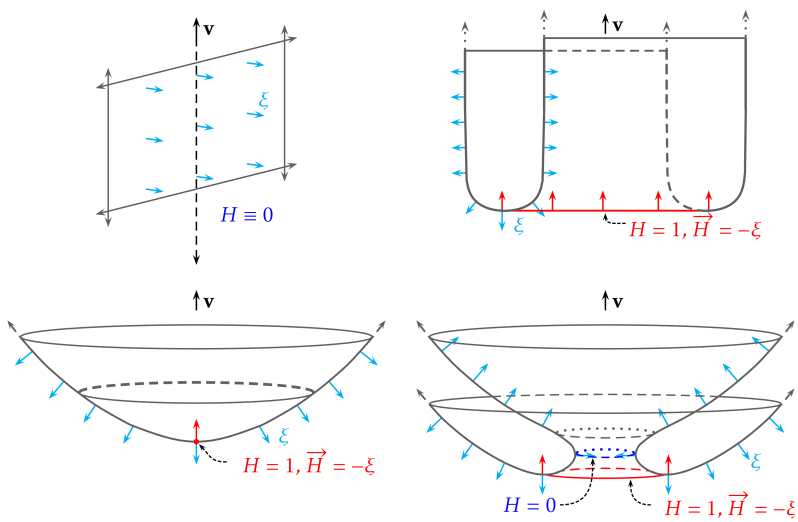

FIGURE 4. A plane tangential to v, a grim plane, a translating paraboloid and a translating catenoid.

translating paraboloid or with the rotationally symmetric *translating catenoids* which can be seen as the desingularization of two paraboloids connected by a small neck of some radius.

(e) Halldorsson [Hal13] proved the existence of helicoidal type translators. Nguyen [Ngu09, Ngu13] desingularized the intersection of a grim reaper and a plane by a Scherk's minimal surface in $\mathbb{R}^3$, and obtained a complete embedded translator of infinite genus. It is not known if there are complete translating solitons with finite non-trivial topology.

(f) Wang [Wan11] studied graphical translators in $\mathbb{R}^{m+1}$. He proved that for any m there exist complete convex graphical translating solitons, defined in strip regions, which are not rotationally symmetric. For $m \geq 3$, Wang proved that there are entire convex graphical translating solitons. On the other hand, Wang proved that any entire convex graphical translating soliton in $\mathbb{R}^3$ must be rotationally symmetric in an appropriate coordinate system. It is still an open problem, if *any entire graphical translating soliton in* $\mathbb{R}^3$ *(not necessarily convex) is rotationally symmetric.*

Remark 2.3. We would like to mention that Shahriyari [Sha13] proved non-existence of complete translating graphs over bounded domains of the euclidean space $\mathbb{R}^2$.

2.3. **The tangency principle.** A basic tool employed in the proof of one of the main theorems is the tangency principle. According to this principle two different translating solitons cannot "touch" each other at one interior or boundary point. More precisely,

Theorem 2.4. *Let Σ_1 and Σ_2 be m-dimensional embedded translating solitons with boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$ in the euclidean space $\mathbb{R}^{m+1}$.*

(a) (**Interior principle**) *Suppose that there exists a common point x in the interior of Σ_1 and Σ_2 where the corresponding tangent spaces coincide and that Σ_1 lies at one side of Σ_2. Then $\Sigma_1 \equiv \Sigma_2$.*

(b) (**Boundary principle**) *Suppose that $\partial\Sigma_1$ and $\partial\Sigma_2$ lie on the same hyperplane Π of $\mathbb{R}^{m+1}$ and the intersection of Σ_1, Σ_2 with Π is transversal. Assume that Σ_1 lies at one side of Σ_2 and that there exists a common point of $\partial\Sigma_1$ and $\partial\Sigma_2$ where Σ_1 and Σ_2 have the same tangent space. Then $\Sigma_1 \equiv \Sigma_2$.*

Proof. It is well known that translating solitons can be considered as minimal hypersurfaces in the conformally changed metric

$$\mathrm{G}_p := e^{\frac{2}{m}\langle p, \mathrm{v}\rangle}\langle \cdot\,,\cdot\rangle$$

on $\mathbb{R}^{m+1}$ (see for example [Ilm94]). Therefore, translators are real analytic hypersurfaces. Hence, if two translators coincide in an open neighborhood they should coincide everywhere. The theorem follows now from the interior and boundary tangency principle for embedded minimal hypersurfaces in a Riemannian manifold. For a nice exposition of these principles we recommend the beautiful paper of Eschenburg [Esc89, Theorem 1 and Theorem 1a]. $\square$

2.4. **Global characterizations.** We shall conclude this section with some characterizations of translators.

Theorem B. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a translating soliton which is not a minimal hypersurface. Then $f(M^m)$ is a grim hyperplane if and only if the function $|A|^2 H^{-2}$ attains a local maximum on $M^m - \{H = 0\}$.*

Proof. Inspired by ideas developed by Huisken [Hui93], consider the smooth function $h : U \to \mathbb{R}$, $U := M - \{H = 0\}$, given by $h = |A|^2 H^{-2}$. Notice at first that

$$\nabla h = \frac{\nabla|A|^2}{H^2} - 2h\frac{\nabla H}{H},$$

provided $H \neq 0$. Let us now relate the Laplacian of the function h with the quantities H, A and u. We have,

$$\begin{aligned}\Delta h &= H^{-2}\Delta|A|^2 + |A|^2\Delta H^{-2} + 2\langle\nabla|A|^2, \nabla H^{-2}\rangle \\ &= H^{-2}\Delta|A|^2 - 2H^{-3}|A|^2\Delta H \\ &\quad +6H^{-4}|A|^2|\nabla H|^2 - 4H^{-3}\langle\nabla|A|^2, \nabla H\rangle.\end{aligned}$$

By virtue of Lemma 2.1, we get

$$\Delta h + H^{-1}\langle\nabla h, H\nabla u + 2\nabla H\rangle - 2H^{-4}Q^2 = 0,$$

where

$$\begin{aligned}Q^2 : &= H^2|\nabla A|^2 + |A|^2|\nabla H|^2 - H\langle\nabla H, \nabla|A|^2\rangle \\ &= \big|H\nabla A - \nabla H \otimes A\big|^2.\end{aligned}$$

Since h attains a local maximum, by the strong maximum principle we obtain now that h is constant and Q^2 is identically zero. Thus, for any triple of indices i, j, k, it holds

$$H(\nabla_{e_i}A)(e_j, e_k) - e_i(H)A(e_j, e_k) = 0, \tag{2.2}$$

where here $\{e_1, \ldots, e_m\}$ is a local orthonormal frame in the tangent bundle of the hypersurface. From the last identity and the Codazzi equation we see that

$$e_i(H)A(e_j, e_k) - e_j(H)A(e_i, e_k) = 0. \tag{2.3}$$

for any triple of indices i, j, k.

Case 1. Assume at first that H is constant. Since by assumption f is not minimal, this constant is non-zero. Then, from equation (2.2) it follows that $|\nabla A| = 0$. Then, all the principal curvatures of f are constant. Due to a theorem of Lawson [Law69, Theorem 4], it follows that $f(M^m)$ is locally isometric to a round sphere or to a product of a round sphere with a euclidean factor. However, none of these examples are translators and consequently this situation cannot happen.

Case 2. Assume now that there is a simply connected neighborhood V where $|\nabla H|$ is not zero. In the neighborhood V, choose the frame

$$\big\{e_1 := \nabla H/|\nabla H|, e_2, \dots, e_m\big\}.$$

Then, from equation (2.3), we obtain $A(e_j, e_k) = 0$, for any $k \geq 1$ and $j \geq 2$. Therefore, f has only one non-zero principal curvature. Denote now by $\mathscr{D} : M^m \to TM^m$,

$$\mathscr{D}(x) := \{v \in T_xM^m : A(v, \cdot\,) = 0\},$$

the nullity distribution and by $\mathscr{D}^\perp := \operatorname{span}\{e_1\}$ its orthogonal complement. It is well known that the distribution $\mathscr{D}$ is smooth and integrable. Moreover, $\mathscr{D}$ is an autoparallel distribution, that is for any $X, Y \in \mathscr{D}$ it follows that $\nabla_X Y \in \mathscr{D}$. The integral submanifolds of $\mathscr{D}$ are totally geodesic in M^m and their images via the immersion f are totally geodesic submanifolds of $\mathbb{R}^{m+1}$. Furthermore, the Gauß map of the immersion f is constant along the leaves of $\mathscr{D}$. A classical reference for the proofs of these facts is the paper of Ferus [Fer70, Lemma 2, p. 311].

We claim now that $\mathscr{D}^\perp$ is also autoparallel and its integral curves are geodesics of M. Indeed, fix a point x_0 and set $e = e_1(x_0)$. Extend the vector e by parallel transport to a vector field along the integral curve $\gamma : (-\varepsilon, \varepsilon) \to M^m$ of e_1 passing through x_0. Let $B : TM^m \to TM^m$ be the Weingarten operator associated to A. From the equation (2.2), we obtain that

$$(\nabla_{e_1} B)v = e_1(H)v,$$

for any v in TM^m. A straightforward computation yields

$$\begin{aligned} \frac{d}{dt}\left(|(B - H \cdot \mathrm{Id})e|^2\right) &= 2\langle(\nabla_{e_1} B)e - e_1(H)e, Be - He\rangle \\ &= 2\langle e_1(H)e - e_1(H)e, Be - He\rangle \\ &= 0. \end{aligned}$$

Therefore $e = e_1$ is a parallel vector field along the integral curves of e_1. Thus,

$$\nabla_{e_1} e_1 = 0$$

and so the orthogonal complement $\mathscr{D}^\perp$ of $\mathscr{D}$ is autoparallel in TM^m.

Observe now that

$$TM^m = \mathscr{D} \oplus \mathscr{D}^\perp.$$

We claim now that both the above distributions are parallel. Indeed, let $X, Y \in \mathscr{D}$ and $Z \in \mathscr{D}^{\perp}$. Then,

$$0 = X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Hence, for any $X \in TM^m$ and $Z \in \mathscr{D}^{\perp}$ we have that $\nabla_X Z \in \mathscr{D}^{\perp}$. This means that the distribution $\mathscr{D}^{\perp}$ is actually parallel. Similarly, one can show that the nullity distribution $\mathscr{D}$ is also parallel. Hence, from the de Rham decomposition theorem, the hypersurface $f(M^m)$ splits as the Cartesian product of a plane curve Γ with a euclidean factor $\mathbb{R}^{m-1}$. Obviously, this planar curve Γ must be a grim reaper. Consequently, $f(M^m)$ contains an open neighborhood that is part of a grim hyperplane. Because of the real analyticity, according to the analytic continuation theorem [Sam78, Theorem 1, p.213] we deduce that $f(M^m)$ should coincide everywhere with a grim hyperplane. This completes the proof. □

Corollary 2.5. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a translating soliton with zero scalar curvature. Then either $f(M^m)$ is a grim hyperplane or $f(M^m)$ is a totally geodesic hyperplane tangent to* v*.*

Proof. Note that under our assumptions,

$$|A|^2 = H^2 - \mathrm{scal} = H^2.$$

Hence, if H is identically zero then $|A|^2$ is identically zero and $f(M^m)$ must be a totally geodesic hypersurface tangent to v. Suppose now that there is a point where H is not zero. In this case, there is an open neighborhood where the function $|A|^2H^{-2}$ is well-defined and equals 1. Consequently, the above theorem implies that $f(M^m)$ coincides with a grim hyperplane. □

Similarly we can prove the following:

Corollary 2.6. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a translating soliton with $H > 0$ and* $\mathrm{scal} \geq 0$*. Then either $f(M^m)$ is a grim hyperplane or* $\mathrm{scal} > 0$ *everywhere.*

Theorem 2.7. *Let $f : M^m \to \mathbb{R}^{m+1}$ be a weakly-convex translator. If there is a point where the Gauß-Kronecker curvature vanishes, then the Gauß-Kronecker curvature vanishes everywhere.*

Proof. We have that

$$\Delta A + \langle \nabla A, \nabla u\rangle + |A|^2 A = 0.$$

By the assumptions A is a non-negative symmetric 2-tensor. If there is a point where the smallest principal curvature vanishes, from the strong elliptic maximum principle for tensors (see for example [Ham86] or [SHS14, Section 2]), we get that the smallest principal curvature of f vanishes everywhere. □

3. Uniqueness of the translating paraboloid

The aim of this section is to show that a complete embedded translating soliton of the mean curvature flow with a single end which is asymptotic to the rotationally symmetric translating paraboloid, must be a translating paraboloid. The proof exploits the method of moving planes which was first introduced by Alexandrov [Ale56] for the investigation of compact hypersurfaces with constant mean curvature in a euclidean space. However, our approach follows ideas developed by Schoen [Sch84] where he applied the method of moving planes to minimal hypersurfaces of the euclidean space.

3.1. **A uniqueness theorem.**

Before stating and proving the main result of this section we have to introduce some notation and definitions. Denote by $\mathfrak{p}:\mathbb{R}^{m+1}\to\Pi$ the orthogonal projection to the plane

$$\Pi:=\{(x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}:x_1=0\},$$

that is

$$\mathfrak{p}(x_1,\dots,x_{m+1}):=(0,x_2,\dots,x_{m+1}).$$

Definition 3.1. *Let A and B be two arbitrary subsets of $\mathbb{R}^{m+1}$. We say that the set A is on the right hand side of B and write $A\geq B$ if and only if for every point $x\in\Pi$ for which*

$$\mathfrak{p}^{-1}(x)\cap A\neq\emptyset \quad \textit{and} \quad \mathfrak{p}^{-1}(x)\cap B\neq\emptyset,$$

we have that

$$\inf\big[x_1\{\mathfrak{p}^{-1}(x)\cap A\}\big]\geq\sup\big[x_1\{\mathfrak{p}^{-1}(x)\cap B\}\big],$$

where here $x_1\{P\}$ denotes the x_1-component of the point $P\in\mathbb{R}^{m+1}$.

Note that "$\geq$" is not a well ordered relation since there are subsets of $\mathbb{R}^{m+1}$ that cannot be related to each other. However, the relation "$\geq$" is an order for graphs over Π.

Consider now the family of planes $\{\Pi(t)\}_{t\geq0}$ given by

$$\Pi(t):=\{(x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}:x_1=t\}.$$

Given a subset A of $\mathbb{R}^{m+1}$ let us also define the following subsets:

$$\begin{aligned}
\delta_t(A): &= \{(x_1,\dots,x_{m+1})\in A \;:\; x_1=t\}=A\cap\Pi(t),\\
A_+(t): &= \{(x_1,\dots,x_{m+1})\in A \;:\; x_1\geq t\},\\
A_-(t): &= \{(x_1,\dots,x_{m+1})\in A \;:\; x_1\leq t\},\\
A_+^*(t): &= \{(2t-x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}:(x_1,\dots,x_{m+1})\in A_+(t)\},\\
A_-^*(t): &= \{(2t-x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}:(x_1,\dots,x_{m+1})\in A_-(t)\},\\
Z_t: &= \{(x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}:x_{m+1}>t\}.
\end{aligned}$$

Note that $A_+(t)$ are elements of A that are on the right hand side of

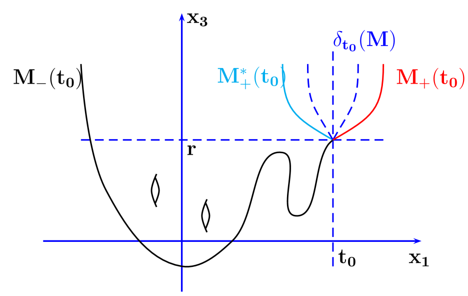

FIGURE 5

the plane $\Pi(t)$ and $A_-(t)$ are those elements of A that belong to the left hand side of $\Pi(t)$. The subset $A_+^*(t)$ is the reflection of $A_+(t)$ with respect to the plane $\Pi(t)$ while $A_-^*(t)$ stands for the reflection of $A_-(t)$ with respect to $\Pi(t)$ (see Figure 5).

Theorem A. *Let $f: M^m\to\mathbb{R}^{m+1}$ be a complete embedded translating soliton of the mean curvature flow with a single end that is smoothly asymptotic to a translating paraboloid. Then the hypersurface $M=f(M^m)$ is a translating paraboloid.*

Proof. For the sake of intuition we will present the proof for $m=2$. The arguments for the general case are analog. From our assumptions it follows that there exists a positive real number r such that $M-B(0,r)$ can be written as the graph of a function

$$g(x_1,x_2)=\frac{1}{2}\big(x_1^2+x_2^2\big)-\frac{1}{2}\log\big(x_1^2+x_2^2\big)+O\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right),\qquad(3.1)$$

where here $B(0,r)$ stands for the open euclidean ball of $\mathbb{R}^3$ which is centered at the origin and has radius r. Consider now the vectors

$$v_\vartheta:=(\cos\vartheta,\sin\vartheta,0),$$

where here $\vartheta \in [0, 2\pi)$. Our goal is to show that for any angle ϑ the translator M is symmetric with respect to the plane perpendicular to v_ϑ passing through the origin of $\mathbb{R}^3$. Since the rotations around the x_3-axis preserve the property of being a translating soliton, we deduce that it suffices to prove the symmetry only along the plane Π. To this end, consider the set

$$\mathcal{A} := \big\{ t \in [0, +\infty) : M_+(t) \text{ is a graph over } \Pi \text{ and } M_+^*(t) \geq M_-(t) \big\}.$$

The proof will be finished if we can show that 0 is contained in the set $\mathcal{A}$. This will be achieved by proving that $\mathcal{A}$ is a non-empty open and closed subset of the interval $[0, +\infty)$. The proof of this fact will be concluded by several claims.

Claim 1. *The set $\mathcal{A}$ is not empty. Moreover if $s \in \mathcal{A}$, then $[s, \infty) \subset \mathcal{A}$.*

We will exploit the asymptotic behavior of the end of the translator to prove the existence of t_1. Indeed, choose the radius r sufficiently large. In fact, one can choose r so large such that the part of the translator $M \cap Z_r$ is a graph over the x_1x_2-plane. We conclude the proof of the claim in three steps.

Step 1: Because M has a single end which is asymptotic to a translating paraboloid it follows that $M_+(t)$ has only one unbounded component. Indeed, if there were an compact component then from the tangency principle in the interior we would get that M is flat, which contradicts our assumptions. Hence, for any positive number t the set $M_+(t)$ is connected.

Step 2: Since by assumption the end of M is smoothly asymptotic to a translating paraboloid we deduce that there exists a number $t' > r$, large enough, such that $\frac{\partial g}{\partial x_1}(x_1, x_2) > 0$ for any $x_1 > t'$. Consequently $\langle \xi, \mathrm{e}_1 \rangle > 0$ for any point of $M_+(t')$, where $\xi : M \to \mathbb{S}^2$ stands for the Gauss map of M. Taking into account that M is embedded, $M_+(t')$ is connected and $M_+(t') \cup \mathfrak{p}(M_+(t'))$ bounds a domain of $\mathbb{R}^3$, it follows that $M_+(t')$ is a graph over Π.

Step 3: For fixed $t > t'$ we can represent $M_+^*(t) \cap Z_r$ as the graph over the x_1x_2-plane of the function g_t given by the expression

$$\begin{aligned} g_t(x_1, x_2) &= \frac{1}{2}\big\{(2t - x_1)^2 + x_2^2\big\} \\ &\quad -\frac{1}{2}\log\big\{(2t - x_1)^2 + x_2^2\big\} + O\left(\frac{1}{\sqrt{(2t - x_1)^2 + x_2^2}}\right). \end{aligned}$$

Comparing the functions g_t and g we get that

$$\begin{aligned}
g_t(x_1,x_2) \quad &- \quad g(x_1,x_2) \\
&= 2t(t-x_1) - \frac{1}{2}\log\left\{\frac{4t(t-x_1)}{x_1^2+x_2^2}+1\right\} \\
&\quad + O\left(\frac{1}{\sqrt{(2t-x_1)^2+x_2^2}}\right) - O\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right) \\
&\geq 2t(t-x_1) - \frac{1}{2}\log\left\{\frac{4t(t-x_1)}{x_1^2+x_2^2}+1\right\} \\
&\quad - \frac{C}{\sqrt{(2t-x_1)^2+x_2^2}} - \frac{C}{\sqrt{x_1^2+x_2^2}},
\end{aligned}$$

where C is a positive constant. Recall that the above relation holds for $x_1^2+x_2^2 > r^2$ and $x_1 \leq t$. Moreover, note that

$$\log\left\{\frac{4t(t-x_1)}{x_1^2+x_2^2}+1\right\} \leq \frac{4t(t-x_1)}{x_1^2+x_2^2},$$

and

$$\begin{aligned}
(2t-x_1)^2+x_2^2 &= (t+t-x_1)^2+x_2^2 \\
&> t^2+(t-x_1)^2+x_2^2 \\
&> r^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_t(x_1,x_2)-g(x_1,x_2) &> \frac{2t(t-x_1)(x_1^2+x_2^2-1)}{x_1^2+x_2^2} - \frac{2C}{r} \\
&> \frac{2r(t-x_1)(r^2-1)-2rC}{r^2}.
\end{aligned}$$

Our aim is to compare $M_+^*(t)$ with $M_-(t)$ following the definition given in relation (3.1). Choose a positive constant a, not depending on r. Then, for $t-x_1>a$ and r large enough we have

$$g_t(x_1,x_2)-g(x_1,x_2) > \frac{2ra(r^2-1)-2rC}{r^2} > 0. \tag{3.2}$$

Then from inequality (3.2) we have that

$$\begin{aligned}
&\Big[M_+^*(t'+a)\cap\big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1\leq t'\big\}\Big] \\
&\qquad \geq \Big[M_-(t'+a)\cap\big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1\leq t'\big\}\Big].
\end{aligned}$$

Moreover, from the fact that $M_+(t')$ is a graph over Π we deduce that

$$\Big[M_+^*(t'+a)\cap\big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : t'\le x_1\le t'+a\big\}\Big] \\ \ge \Big[M_-(t'+a)\cap\big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : t'\le x_1\le t'+a\big\}\Big].$$

Hence, $[t'+a,+\infty)\subset\mathcal{A}$. It is straightforward to check that if $s\in\mathcal{A}$ then $[s,+\infty)\subset\mathcal{A}$. This concludes the proof of the claim.

Claim 2. *The set $\mathcal{A}$ is a closed subset of the interval $[0,+\infty)$.*

Suppose that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of points in $\mathcal{A}$ converging to t_0. We have to prove that t_0 is contained in $\mathcal{A}$. Indeed, at first notice that according to Claim 1 we have that $(t_0,+\infty)\subset\mathcal{A}$. Our goal is to prove that $M_+(t_0)$ can be written as the graph of the coordinate function x_1 and that $M_+^*(t_0)\ge M_-(t_0)$. Let us suppose at first that the graphical condition is violated. That is there are points $P=(p_1,p_2,p_3)$ and $Q=(q_1,p_2,p_3)$ in $M_+(t_0)$ such that $q_1>p_1$. Then we must have $p_1=t_0$ since $s\in\mathcal{A}$ for any $s>t_0$ (see Figure 6 below). We will show that this situation cannot occur.

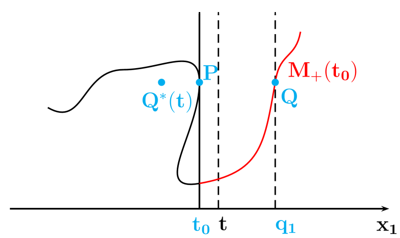

FIGURE 6

Indeed, take

$$t=\frac{q_1+3t_0}{4}>t_0.$$

But then it follows that $M_+^*(t)$ is not in the right hand side of $M_-(t)$, which contradicts the fact that $t\in\mathcal{A}$. Consequently, the set $M_+(t_0)$ can be represented as a graph over the plane Π. Moreover, because of the continuity and the graphical condition we get

$$M_+^*(t_0)\ge M_-(t_0).$$

Hence, $t_0\in\mathcal{A}$ and this completes the proof of the claim.

Claim 3. *The minimum of the set $\mathcal{A}$ is 0. In particular, $\mathcal{A}=[0,+\infty)$.*

We argue again in this proof by contradiction. Suppose to the contrary that $s_0 := \min \mathcal{A} > 0$. Then we will show that there exists a positive number ε such that $s_0 - \varepsilon \in \mathcal{A}$, which will be the contradiction.

Step 1: We will show at first that there exists a positive constant $\varepsilon_1 < s_0$ such that $M_+(s_0 - \varepsilon_1)$ is a graph. Indeed, as in the proof of Step 1 of Claim 1, we deduce that there exists a positive number α which is sufficiently bigger than r and such that

$$\mathrm{dist}_{\mathbb{R}^3}\Big[\xi\big\{M_+(s_0) \cap Z_\alpha\big\}, \Pi\Big] > 0. \tag{3.3}$$

From (3.3) it follows that there is $\varepsilon_0 > 0$ such that $M_+(s_0 - \varepsilon_0) \cap Z_\alpha$ can be represented as a graph over the plane Π and furthermore

$$M_+^*(s_0 - \varepsilon_0) \cap Z_\alpha \geq M_-(s_0 - \varepsilon_0) \cap Z_\alpha. \tag{3.4}$$

Consider now the compact set

$$\mathcal{K} := M \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq \alpha\}.$$

Taking into account that $s_0 \in \mathcal{A}$, we deduce that $\mathcal{K}_+(s_0)$ is a graph over the plane Π. At first notice that there is no point in $\mathcal{K}_+(s_0)$ with normal vector included in the plane Π. Indeed, if this was true then from the tangency principle at the boundary we would get that M is symmetric around a plane parallel to Π which contradicts the assumption on the end of M. Consequently,

$$\xi\big\{\mathcal{K}_+(s_0)\big\} \cap \Pi = \emptyset.$$

Because the set $\mathcal{K}_+(s_0)$ is compact, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ small enough such that, for all $t \in [s_0 - \varepsilon_1, s_0]$,

$$\xi\big\{\mathcal{K}_+(t)\big\} \cap \Pi = \emptyset,$$

Because of this fact and because of the compactness we deduce that the set $\mathcal{K}_+(t)$ can be represented as graph Π for every $t \in [s_0 - \varepsilon_1, s_0]$. Consequently, $M_+(t)$ is a graph over the plane Π, for all $t \geq s_0 - \varepsilon_1$. Hence, the first step of the plan is finished.

Step 2: Now we will conclude the plan by proving that there exist a positive constant $\varepsilon_2 < \varepsilon_1$ such that $M_+^*(s_0 - \varepsilon_2) \geq M_-(s_0 - \varepsilon_2)$.

First notice that

$$\big(M_+^*(t) \cap M_-(t) \cap \mathcal{K}\big) - \delta_t(M) \subset \mathcal{K}_-(s_0 - \varepsilon_1), \tag{3.5}$$

for all $t \geq s_0 - \varepsilon_1$. Next, because $M_+^*(s_0) \geq M_-(s_0)$, we have that

$$M_+^*(s_0) \cap M_-(s_0) = \delta_{s_0}(M).$$

We will show now that there exists a positive number $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$M^*_+(t) \cap M_-(t) \cap \mathcal{K} = \delta_t(M) \cap \mathcal{K},$$

for all $t \geq s_0 - \varepsilon_2$. Arguing indirectly, suppose that this is not true. Then, there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to s_0 such that

$$\big(M^*_+(t_n) \cap M_-(t_n) \cap \mathcal{K}\big) - \delta_{t_n}(M) \neq \emptyset.$$

For each natural n denote by $P_n = (p^n_1, p^n_2, p^n_3)$ a point in the above set. From (3.5) we deduce that for each natural number n it holds that

$$p^n_1 \leq s_0 - \varepsilon_1.$$

By the compactness of $\mathcal{K}$, without loss of generality, we may assume that the sequence $\{P_n\}_{n\in\mathbb{N}}$ converges to a point $P_0 = (p^0_1, p^0_2, p^0_3) \in \mathcal{K}$. By continuity we have that $p^0_1 \leq s_0 - \varepsilon_1$. But

$$P_0 \in M^*_+(s_0) \cap M_-(s_0) \cap \mathcal{K} = \delta_{s_0}(M) \cap \mathcal{K},$$

which is absurd. Thus, combining the above results for the parts of M above and below $\{x_3 = \alpha\}$, we get that for given $t \in (s_0 - \varepsilon_2, s_0]$ it holds

$$M^*_+(t) \cap M_-(t) = \delta_t(M).$$

Then, for any fixed $t \geq s_0 - \varepsilon_2$ a continuity argument gives that

$$M^*_+(t) \geq M_-(t),$$

and the second and final step of the plan is completed.

Hence, there exists a positive constant $\varepsilon < s_0$ such that $s_0 - \varepsilon \in \mathcal{A}$. This contradicts the assumption that s_0 is a minimum. Consequently, $s_0 = 0$ and the proof is finished.

From Claim 3 we get that $M^*_+(0) \geq M_-(0)$. A symmetric argument yields that $M^*_-(0) \leq M_+(0)$. Therefore, $M_-(0) \geq M^*_+(0)$ and so $M^*_+(0) = M_-(0)$ and so M is symmetric with respect to the plane Π. This completes the proof of the theorem. □

Remark 3.2. Using similar ideas to those applied in Theorem 3.1 we can deduce some results about the asymptotic behavior of entire graphical translating solitons in $\mathbb{R}^{m+1}$.

(a) Let M an entire graphical translating soliton satisfying the growth condition

$$|u(x)| \leq C\,|x|^\alpha,$$

for all $x \in \mathbb{R}^m - B(0, r)$. Then $\alpha \geq 2$. In order to prove this fact, we proceed again by contradiction. Suppose to the contrary that there exists an entire graphical translator in the euclidean space $\mathbb{R}^{m+1}$ satisfying the above growth condition with $\alpha < 2$. Let X be the translating paraboloid of Example 2.2 (d) and translate it vertically until the surfaces X and M do not intersect. Note that this is possible because we are assuming $\alpha < 2$ and X has the asymptotic behavior described in (3.1). Then, the paraboloid X will move vertically downwards until there is a first point of contact with the surface M. This first contact can not occur at infinity, because we are assuming $\alpha < 2$. Then, the tangency principle implies that M should coincide with a translated copy of X. But this is absurd, because X is asymptotic to the graph over

$$g(x) = \frac{1}{2}|x|^2 - \log|x| + O\big(|x|^{-1}\big)$$

at infinity.

(b) Exchanging the role of M and X in the above argument, then we can prove that: Suppose that M is an entire graphical translator satisfying the following growth condition

$$|u(x)| \geq C\,|x|^{\alpha},$$

for all $x \in \mathbb{R}^m - B(0,r)$. Then, $\alpha \leq 2$.

(c) Using again the tangency maximum principle we can show that there are no complete and embedded translators that are contained in the solid half-cylinder

$$\mathscr{C} := \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \;:\; x_1^2 + \cdots + x_m^2 \leq r^2,\ x_{m+1} > 0\}.$$

(d) The reason that the mean curvature flow of compact hypersurfaces in $\mathbb{R}^{m+1}$ form singularities is the following comparison principle: *Suppose that M_1 and M_2 are two compact Riemannian manifolds of dimension m and let $f : M_1 \to \mathbb{R}^{m+1}$, $g : M_2 \to \mathbb{R}^{m+1}$ disjoint isometric immersions. Then also the solutions f_t and g_t of the mean curvature flow remains disjoint.* We would like to point out that the compactness assumption cannot be relaxed with that of completeness. Indeed, take as $f : M_1 \to \mathbb{R}^3$ be the unit euclidean sphere and as $g : M_2 \to \mathbb{R}^3$ a complete minimal surface lying inside the unit ball. Such examples were first constructed by Nadirashvili [Nad96]. Obviously f and g do not have intersection points. However, under the mean curvature flow, f shrinks to a point in finite time while g remains stationary.

4. The two-dimensional case

In this section we will investigate 2-dimensional translating solitons lying in the two dimensional case. We will study the Gauß image of such surfaces and will obtain several classification results as well as topological obstructions for the existence of translating solitons of the mean curvature flow in the euclidean space $\mathbb{R}^3$.

4.1. **Gauß image of punctured Riemann surfaces.** It is a well known fact that any smooth oriented compact surface is diffeomorphic either to a sphere or to a torus with g holes. The Euler characteristic of such compact surface Σ_g depends only on the genus g and is given by the formula

$$\mathcal{X}(\Sigma_g) = 2 - 2g.$$

This classification can be extended also to compact surfaces Σ with boundary. Note that from compactness, the boundary $\partial\Sigma$ has a finite number k of components. Moreover, each boundary component of Σ is a connected compact 1-manifold, that is a circle. Now, if we take k closed discs and glue the boundary of the i-th disc to the i-th component of the boundary of Σ, we obtain a smooth compact surface which we will denote with the letter Σ^*. It turns out that the topological type of a compact surface with boundary Σ depends only on the number k of its boundary components and the topological type of the surface Σ^*. More precisely, Σ is diffeomorphic to $\Sigma_{g,k}$, where

$$\Sigma_{g,k} = \Sigma_g - \{p_1, \dots, p_k\}$$

is a punctured Riemann surface given by a closed Riemann surface Σ_g of genus g with k points $p_1, \dots, p_k \in \Sigma_g$ removed. The Euler characteristic of a compact surface with boundary is defined exactly in the same way as in the case of a compact surface without boundary. It follows that,

$$\mathcal{X}(\Sigma_{g,k}) = \mathcal{X}(\Sigma) = \mathcal{X}(\Sigma_g) - k = 2 - 2g - k.$$

The *genus of a compact surface* Σ *with boundary* is defined to be the genus of the compact surface $\Sigma_g = \Sigma^*$.

From now we will focus on Riemann surfaces of the form

$$\Sigma_{g,k} := \Sigma_g - \{p_1, \dots, p_k\},$$

where Σ_g is a compact Riemann surface of genus g and $p_1, \dots, p_k \in \Sigma_g$ are k distinct punctures. Recall that an *end* of the surface $\Sigma_{g,k}$ is a diffeomorphism $\tilde{\varphi}_j : D \to \Sigma_g$ of the closed unit disc D to Σ_g such that $\tilde{\varphi}_j(0) = p_j$ for one index $j \in \{1, \dots, k\}$ and

$$\tilde{\varphi}_j(D) \cap \{p_1, \dots, p_k\} = \{p_j\}.$$

We will denote by $\varphi_j : D^* \to \Sigma_{g,k}$ the maps

$$\varphi_j := \left(\tilde{\varphi}_j\right)_{|D^*},$$

where

$$D^* := \{x \in \mathbb{R}^2 : 0 < |x| \leq 1\}$$

denotes the punctured unit disc. The surface $\Sigma_{g,k}$ will be called a *planar domain*, if $g = 0$ and $k \geq 1$, that is if $\Sigma_{g,k}$ is a punctured sphere.

In the following lemma we present a result which will allow in the rest of the paper a surgery argument for cylindrical ends.

Lemma 4.1 (Spherical cap lemma)**.** *Let $\alpha_t : \mathbb{S}^1 \to \mathbb{R}^2$, $t \in [0, 1]$, be a smooth family of closed embedded curves and let d_0 denote the diameter of α_0. Define the cylinder $\Theta : \mathbb{S}^1 \times [0, 1] \to \mathbb{R}^3$ given by*

$$\Theta(s, t) := (\alpha_t(s), t).$$

Then for any $\sigma > 0$ there exists an $\varepsilon \in (0, \sigma)$ and a smooth embedding $C : D \to \mathbb{R}^3$ of the closed unit disc D such that:

(a) *For all $x \in D$ with $|x| \in [1 - \varepsilon, 1]$, it holds*
$$C(x) = \Theta(x/|x|, 1 - |x|).$$
(b) *The height function u given by $u = \langle C, \mathrm{e}_3\rangle$ satisfies $0 \leq u \leq \sigma$. Moreover $u(0) = \sigma$ and the critical set of u is $\mathrm{Crit}(u) = \{0\}$.*
(c) *The diameter d_C of the embedded disc satisfies*
$$\mathrm{d}_C \leq 2\sigma + \mathrm{d}_0\,.$$
(d) *The Gauß curvature K at $C(0)$ is positive.*

Proof. At first let us state some basic facts that will be used in the proof of the lemma.

Fact 1. Let $p \in \mathbb{R}^2$ be an arbitrary point in the interior of the curve α_0 such that

$$|\alpha_0(s) - p| \leq \frac{\mathrm{d}_0}{2}$$

for all $s \in \mathbb{S}^1$. For $\sigma > 0$ we can find an $\tilde{\varepsilon} \in (0, \min\{1, \sigma\})$ such that for all $t \in [0, \tilde{\varepsilon}]$ it holds

$$|\alpha_t(s) - p| \leq \sigma + \frac{\mathrm{d}_0}{2}.$$

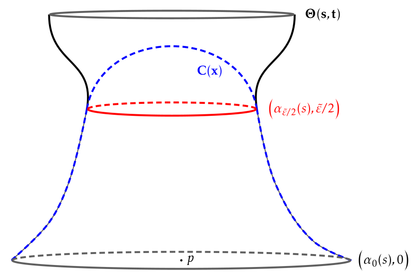

FIGURE 7. Adding a spherical cap to a cylinder

Fact 2. According to a well-known theorem of Grayson [Gra87], the curve shortening flow provides a smooth isotopy of a closed embedded curve γ_0 to a single point $q \in \mathbb{R}^2$ in the interior of γ_0 by smooth embedded curves. Moreover, all evolved curves satisfy the inequality

$$\max_{\mathbb{S}^1} |\gamma_\tau - q| \leq \max_{\mathbb{S}^1} |\gamma_0 - q|.$$

Furthermore, the above inequality is still valid, if one blows up the solutions homothetically around q by a time dependent factor so that the length of the evolving curve is fixed. Under this rescaling the curves become circular in the limit.

Starting with the curve $\gamma_0 := \alpha_{\tilde\varepsilon/2}$, from Fact 1 and Fact 2 we see that there exists a smooth isotopy $\beta_t : \mathbb{S}^1 \to \mathbb{R}$, $t \in [0,1]$, such that:

- $\beta_t = \alpha_t$ for $t \in [0, \tilde\varepsilon/2]$,
- $\beta_t(s) = p + (1-t)e^{is}$, for $(s,t) \in \mathbb{S}^1 \times [1 - \tilde\varepsilon/2, 1]$,
- $\beta_t : \mathbb{S}^1 \to \mathbb{R}^2$ is a closed embedded curve for all $t \in [0,1)$ with
$$\max_{\mathbb{S}^1} |\beta_t - p| \leq \sigma + \mathrm{d}_0 /2$$
for all $t \in [0,1]$.

Choose a smooth function $\phi : [0,1] \to \mathbb{R}$ with the following properties

- $\phi(t) = t$ for $t \in [0, \tilde\varepsilon/2]$,
- $\phi(t) = \sqrt{\sigma^2 - (1-t)^2}$ for $t \in [1 - \tilde\varepsilon/2, 1]$,
- $\phi'(t) > 0$ for $t \in [0,1)$.

Let us now define the map $C : D \to \mathbb{R}^2$ by

$$C(x) = \big(\beta_{1-|x|}(x/|x|), \phi(1-|x|)\big).$$

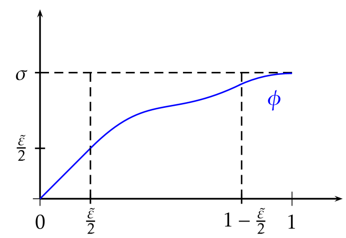

FIGURE 8. Smoothing function

If we set $\varepsilon := \tilde{\varepsilon}/2$, then for any $x \in D$ with $|x| \in [1-\varepsilon, 1]$ we get

$$\begin{aligned} C(x) &= \big(\beta_{1-|x|}(x/|x|), \phi(1-|x|)\big) \\ &= \big(\alpha_{1-|x|}(x/|x|), 1-|x|\big) \\ &= \Theta\big(x/|x|, 1-|x|\big). \end{aligned}$$

The last equality proves assertion (a) of the Lemma. Since the height function is given by

$$u(x) = \langle C(x), \mathrm{e}_3 \rangle = \phi(1-|x|),$$

from the properties of ϕ we immediately get (b). Finally, from the construction of the curves β_t, we obtain that C is contained in the ball of radius $\sigma + \mathrm{d}_0/2$ with center at p. Thus the diameter d_C of the cap is bounded by $2\sigma + \mathrm{d}_0$, which implies assertion (c) of the lemma. That the Gauß curvature at the top is strictly positive, follows from the fact that C coincides with the portion of a round sphere close to the top. This proves assertion (d) and completes the proof of the lemma. $\square$

In the following theorem we present a general theorem concerning the Gauß image of a complete surface of the euclidean space $\mathbb{R}^3$.

Theorem 4.2. *Let $f : \Sigma_{g,k} = \Sigma_g - \{p_1, \dots, p_k\} \to \mathbb{R}^3$ be a complete immersion of a punctured Riemann surface. Suppose $\mathrm{v} \in \mathbb{S}^2$ is a fixed unit vector and let $u := \langle f, \mathrm{v} \rangle$ denote the height function of the surface with respect to the direction v. Suppose that for each puncture p_j, $1 \le j \le k$, the following two conditions holds:*

(a) *There exists a constant $\varepsilon > 0$ such that $\liminf_{x \to p_j} |\nabla u(x)| \ge \varepsilon$.*
(b) *The interval $\big(\liminf_{x \to p_j} u(x), \limsup_{x \to p_j} u(x)\big)$ does not coincide with the real line.*

Then either the image of the Gauß map $\xi : \Sigma_{g,k} \to \mathbb{S}^2$ contains the pair $\{\mathrm{v}, -\mathrm{v}\}$ or the genus satisfies $g \le 1$. If in addition f is an embedding, *then either the image of ξ contains $\{\mathrm{v}, -\mathrm{v}\}$ or $g = 0$ and thus $\Sigma_{g,k}$ is a planar domain. Moreover, in all cases we have*

$$\limsup_{x \to p_j} |u(x)| = +\infty$$

for all punctures p_j.

Proof. The idea of the proof is to glue spherical caps along each end and to apply degree theory to obtain informations about the image of the Gauß map. We divide the proof into several steps:

Step 1. To do the surgery, at first we need some information about the nature and the behavior of u along the ends.

Claim: *For each puncture $p \in \{p_1, \dots, p_k\}$ of $\Sigma_{g,k}$ there exists $\varepsilon > 0$ and a closed embedded curve $\alpha \subset \Sigma_{g,k}$ such that:*

(i) *The coordinate function u is constant along the curve α,*

(ii) *The closure of one of the connected components V of $\Sigma_{g,k} - \alpha$ within Σ_g is diffeomorphic to the closed unit disc $D \subset \mathbb{R}^2$ such that $\overline{V} \cap \{p_1, \dots, p_k\} = \{p\}$ and $\inf_V |\nabla u| \geq \varepsilon$.*

Indeed, from the assumption (a) of the theorem it is clear that (ii) can be fulfilled. That is, there exists an open set $U \in \Sigma_{g,k}$ such that:

- $\overline{U}$ is diffeomorphic to D,
- $\overline{U} \cap \{p_1, \dots, p_k\} = \{p\}$,
- $\inf_U |\nabla u| \geq \varepsilon$ for a positive constant $\varepsilon > 0$.

Hence it suffices to prove that U contains a closed level set curve of u which encloses p in its interior, i.e. a curve like α_1 in Figure 9. Let us denote by γ_0 the boundary of U, that is $\gamma_0 := \partial U$, and set

$$u_0^+ := \max_{x \in \gamma_0} u(x), \quad u_0^- := \min_{x \in \gamma_0} u(x), \quad \delta_0 := u_0^+ - u_0^-.$$

Let x be a point in U and

$$\mathrm{d}_x := \operatorname{dist}(x, \gamma_0)$$

its distance from the boundary curve γ_0. Denote by β_x the flow line of $\nabla u / |\nabla u|$ with $\beta_x(0) = x$. Since the vector field $\nabla u / |\nabla u|$ is well defined in U and because $\operatorname{dist}(x, \gamma_0) = \mathrm{d}_x$, the flow line can at least be parametrized over the interval $[0, \mathrm{d}_x]$ until it reaches the boundary curve γ_0 (if it reaches γ_0 at all). We compute

$$\begin{aligned} |u(\beta_x(\mathrm{d}_x)) - u(x)| &= \left| \int_0^{\mathrm{d}_x} \frac{d}{dt}\,(u(\beta_x(t))dt \right| \\ &= \left| \int_0^{\mathrm{d}_x} \langle \nabla u \circ \beta_x(t), \frac{d}{dt}\,\beta_x(t)\rangle dt \right| \\ &= \left| \int_0^{\mathrm{d}_x} |\nabla u \circ \beta_x(t)| dt \right| \\ &= \int_0^{\mathrm{d}_x} |\nabla u \circ \beta_x(t)| dt \\ &\geq \varepsilon\, \mathrm{d}_x\,. \end{aligned}$$

Since the surface $\Sigma_{g,k}$ is complete, the Hopf-Rinow Theorem implies that there exists a point $q \in U$ such that d_x is arbitrarily large. So, the last inequality shows

$$\limsup_{x \to p} |u(x)| = +\infty.$$

In particular, if we choose $x \in U$ such that $\mathrm{d}_x > \delta_0/\varepsilon$, then we see that there must exist another point $q \in U$ with $u(q) \notin [u_0^-, u_0^+]$. Let α be the connected component of the level set $u^{-1}(u(q)) \cap (U \cup \gamma_0)$. Since $\nabla u \neq 0$ on $U \cup \gamma_0$, α must be an embedded regular curve. Thus, one of the following cases holds (cf. Figure 9).

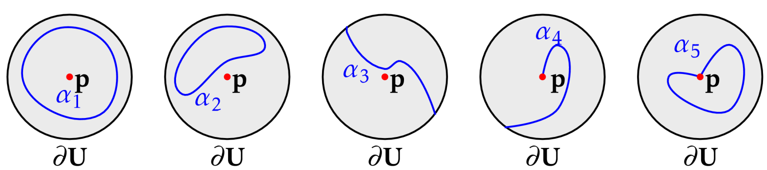

FIGURE 9. Possible types of level sets.

- (C1) $\alpha \subset U$ and p lies in the interior of α.
- (C2) $\alpha \subset U$ and p lies in the exterior of α.
- (C3) $\alpha \cap \gamma_0 \neq \emptyset$.
- (C4) Both ends of α connect to p.

The case (C2) cannot occur, because then the function u would admit a local extremum in the interior of α which in particular implies $\nabla u = 0$ there. Case (C3) is impossible since

$$u_{|\alpha} \equiv u(q) \notin [u_0^-, u_0^+].$$

So we only need to exclude the last case (C4). Choose a closed curve $\gamma_1 \subset U$ such that $q \in \gamma_1$ and $\alpha - \{q\} \in \operatorname{int}(\gamma_1)$ (cf. Figure 10). Similar as above define

$$u_1^+ := \max_{x \in \gamma_1} u(x), \quad u_1^- := \min_{x \in \gamma_1} u(x), \quad \delta_1 := u_1^+ - u_1^-.$$

Applying once more the Hopf-Rinow Theorem, we can find a point $\tilde{q} \in \alpha$ with $\operatorname{dist}(\tilde{q}, \gamma_1) > \delta_1/\varepsilon$. Computing as above we obtain for all points q' on the flow line $\beta_{\tilde{q}}$ of $\nabla u/|\nabla u|$ the estimate

$$|u(q') - u(q)| = |u(q') - u(\tilde{q})| \geq \varepsilon \operatorname{dist}(\tilde{q}, \gamma_1) > \delta_1.$$

From this estimate we deduce that the flow line $\beta_{\tilde{q}}$ can never intersect the curve γ_1. Thus $\beta_{\tilde{q}} \subset \operatorname{int}(\gamma_1)$ and since $|\nabla u| \geq \varepsilon$ this implies

$$\liminf_{\beta_{\tilde{q}}} u = -\infty \quad \text{and} \quad \limsup_{\beta_{\tilde{q}}} u = +\infty$$

which by assumption (b) is impossible. Hence α is a simple closed level

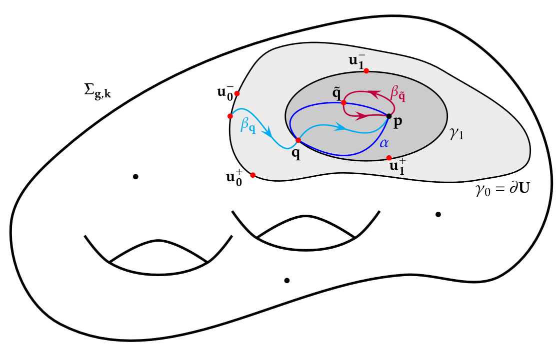

FIGURE 10. The level set α cannot tend to the puncture p, if it contains a point q that is too far away from γ_0.

curve enclosing p. Now let $V := \operatorname{int}(\gamma_1)$, this proves the claim.

Step 2. We will use the above mentioned results to proceed with the gluing of spherical caps along the ends of our surface. For each puncture $p_j \in \{p_1, \dots, p_k\}$ choose a closed level curve α_j and an open set V_j as in Step 1. From Morse Theory (see for example [Mil63]) it follows that all level curves $\tilde{\alpha}_j$ contained in V_j are isotopic. Since for each puncture exactly one of the conditions

$$\limsup_{x \to p_j} u(x) = +\infty \quad \text{or} \quad \liminf_{x \to p_j} u(x) = -\infty$$

holds, we can without loss of generality assume that $\lim_{x\to p_j} u(x) = \infty$ for any index $j \in \{1, \dots, l\}$ and $\liminf_{x\to p_j} u(x) = -\infty$ for any index $j \in \{l+1, \dots, k\}$, where $l \in \{0, \dots, k\}$, and that

$$\bigcup_{1\le j\le k} \alpha_j = |u|^{-1}(L)$$

for a large positive constant L. Let $E \subset \mathbb{R}^3$ be the 2-dimensional subspace perpendicular to v and denote by

$$P_{\pm L} := E \pm L\,\mathrm{v}$$

the affine planes parallel to E at distance L. Then

$$\bigcup_{1\le j\le l} \alpha_j \subset P_L \quad \text{and} \quad \bigcup_{l+1\le j\le k} \alpha_j \subset P_{-L}. \tag{4.1}$$

For each curve $\alpha_1, \dots, \alpha_l$ let us define

$$a_j := \max_{p\in\alpha_j} |p - L\,\mathrm{v}\,|$$

and for the curves $\alpha_{l+1}, \dots, \alpha_k$ we set

$$b_j := \max_{p\in\alpha_j} |p + L\,\mathrm{v}\,|.$$

It is then even possible to sort the curves in such a way that α_l is the outermost and α_1 the innermost curve in P_L, measured from the point $L\,\mathrm{v}$, i.e. so that $a_1 \le \dots \le a_l$. In the same way one can sort the curves $\alpha_{l+1} \le \dots \le \alpha_k$, so that $b_{l+1} \le \dots \le b_k$. If f is an embedding, the curves α_j do not intersect each other. This implies that the interior of the curve α_j cannot contain any of the curves $\alpha_{j'}$ for $1 \le j < j' \le l$ and for $l+1 \le j < j' \le k$. Using Morse Theory again we can replace (α_j, V_j) by the level curve $(\tilde\alpha_j, \tilde V_j)$ for the value $L + j - 1$, $j = 1, \dots, l$ and by the value $-L - j + (l+1)$ for $l+1 \le j \le k$. By Lemma 4.1 we can now do the following surgery. First we remove all ends, that is consider the set

$$M := f\left(\Sigma_{g,k} - \bigcup_{1\le j\le k} \tilde V_j\right).$$

To each end in the upper half space, i.e. for $1 \le j \le l$, we smoothly glue an upper spherical cap to M as described in Lemma 4.1 with some $\sigma > 0$. To each end in the lower half space, i.e. for $l+1 \le j \le k$, we smoothly glue a lower spherical cap with the same σ. Choosing σ sufficiently small we can guarantee that the result is still embedded, if that was the case for f. We obtain a smooth immersion (resp. an embedding if f is one) $F : \Sigma_g \to \mathbb{R}^3$ such that

$$F_{|\left(\Sigma_{g,k} - \bigcup_{1\le j\le k} \tilde V_j\right)} = f.$$

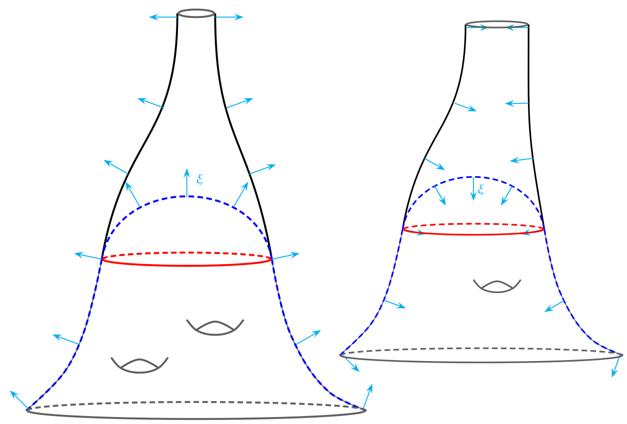

FIGURE 11. Gluing spherical caps to truncated ends.

Step 3. We use now degree theory to investigate the Gauß image. Suppose the Gauß map ξ of $f : \Sigma_{g,k} \to \mathbb{R}^3$ does not contain v or $-$v. Without loss of generality we assume that v is not attained, since the case that the Gauß map does not attain $-$v can be treated in the same way.

Denote by $\tilde{\xi}$ the Gauß map of F. Since $\xi^{-1}(\mathrm{v}) = \emptyset$ we see that the new Gauß map $\tilde{\xi}$ can attain the value v only at the poles of the added caps. By Lemma 4.1 the Gauß curvature at the poles is strictly positive so that v is a regular value of $\tilde{\xi}$.

It is well known (see for example [Hop83]) that the degree of the Gauß map $\tilde{\xi}$ of the compact surface Σ_g is equal to

$$\deg \tilde{\xi} = 1 - g.$$

On the other hand

$$\deg \tilde{\xi} = \sum_{\mathrm{p} \in \tilde{\xi}^{-1}(q)} \operatorname{sign} \det \mathrm{d}\tilde{\xi}(\mathrm{p}) = \sum_{\mathrm{p} \in \tilde{\xi}^{-1}(q)} \operatorname{sign} K(\mathrm{p}),$$

where q is an arbitrary regular value of $\tilde{\xi}$ (see for example [Mil65]). Take as q the value v of $\mathbb{S}^2$. Note that $\tilde{\xi}^{-1}(\mathrm{v})$ consists of points at the poles of the spherical caps that we added. Thus

$$\begin{aligned} 1 - g &= \{\text{number of poles where } \tilde{\xi} = \mathrm{v}\} \\ &\geq 0. \end{aligned}$$

Thus, the first assertion of the theorem is true. Let us now investigate the case where f is an embedding. Since in case of embedded surfaces the unit normal vector field $\tilde{\xi}$ can be chosen to be outward pointing, we can move a plane perpendicular to v from infinity by parallel transport until it touches the surface from above. So in this case there exists at least (and at most) one pole, where $\tilde{\xi} = \mathrm{v}$. Consequently,

$$\begin{aligned} 1 - g &= \{\text{number of poles where } \tilde{\xi} = \mathrm{v}\} \\ &\geq 1, \end{aligned}$$

which yields $g = 0$. This completes the proof of the theorem. $\square$

4.2. **Translating surfaces.** Suppose $f : M^2 \to \mathbb{R}^3$ is an immersion of an oriented manifold M as a translating surface in direction of v, where v shall denote the north pole of $\mathbb{S}^2$. Since the Gauß map $\xi : M^2 \to \mathbb{S}^2$ satisfies

$$H = -\langle \xi, \mathrm{v} \rangle,$$

we have $H \in [-1, 1]$ and we immediately observe that:

(a) The mean curvature H is strictly positive if and only if around each point of M^2 the hypersurface is a graph over a portion of the plane with normal v.
(b) The mean curvature H is strictly bigger than -1 if and only if the vector v is not contained in the Gauß image of f.

Theorem C. *Let $f : M \to \mathbb{R}^3$ be a complete translating soliton, where M has finite topology. Suppose that the following two conditions are satisfied.*

(a) *Each end is either bounded from above or from below.*
(b) *The mean curvature satisfies $H > -1$ everywhere and there exists a compact subset $C \subset M$ and a positive real number ε such that $H^2 < 1 - \varepsilon$ on $M - C$.*

Then the genus g of M satisfies $g \leq 1$. If f is an embedding, then $g = 0$ and thus M is a planar domain.

Proof. By assumption the surface M^2 is diffeomorphic to a Riemann surface $\Sigma_{g,k} = \Sigma_g - \{p_1, \ldots, p_k\}$. Moreover, from assumption (b) it follows that

$$\limsup_{x \to p_j} H^2(x) \leq 1 - \varepsilon$$

for any of the punctures $p_j \in \{p_1, \dots, p_k\}$. Recall that the height function $u = \langle f, \mathrm{v}\rangle$ satisfies the equation

$$|\nabla u|^2 + H^2 = 1.$$

Since the mean curvature satisfies $H > -1$, we deduce that v does not belong to the Gauß image of f. Hence the result follows as a corollary of Theorem 4.2. ☐

Note that the grim hyperplane and the translating paraboloid are both mean convex, that is they satisfy the above condition (a). Moreover, the translating catenoid satisfies condition (b), that is its Gauß map omits the north pole. In the next theorem we prove that a translating soliton for which the Gauß map omits the north pole must be even strictly mean convex, if it is already mean convex outside some compact subset. Before, stating and proving this result, let us give the following useful lemma.

Lemma 4.3. *Let $\mathscr{C}$ denote the class of translating solitons $f : M^2 \to \mathbb{R}^3$ for which exists a constant $\varepsilon > 0$ such that its Gauß curvature satisfies $K \geq \varepsilon$ on the set $\{x \in M^2 : H(x) = -1\}$, where the set $\{x \in M^2 : H(x) = -1\} = \emptyset$ is allowed. If $f \in \mathscr{C}$ satisfies $H \geq 0$ outside a compact subset C of M^2, then either $H \equiv 0$ and $M = f(M^2)$ is isometric to a plane or $H > 0$ on all of M^2.*

Proof. Let $\lambda > 0$ be a constant to be chosen later and define the function $g : M^2 \to \mathbb{R}$ given by $g := He^{\lambda u}$. A straightforward computation yields

$$\begin{aligned}\nabla g &= e^{\lambda u}(\nabla H + \lambda H \nabla u),\\ \Delta g &= e^{\lambda u}\big\{\Delta H + 2\lambda\langle\nabla H, \nabla u\rangle + \lambda^2 H|\nabla u|^2 + \lambda H \Delta u\big\}.\end{aligned}$$

Now let $\mu := \inf_{M^2} g(x)$. We claim that $\mu \geq 0$. Suppose to the contrary that μ is negative. Since C is compact and g is non-negative on $M^2 - C$, the infimum is attained at some point $x_0 \in C$. Thus, $H(x_0) < 0$. Moreover at the point x_0 we have $\nabla g = 0$ and $\Delta g \geq 0$, that is

$$\nabla H = -\lambda H \nabla u$$

and

$$\begin{aligned}0 &\leq \Delta H + 2\lambda\langle\nabla H, \nabla u\rangle + \lambda^2 H|\nabla u|^2 + \lambda H\Delta u\\ &= -H^3 + 2HK - \lambda(2\lambda - 1)H|\nabla u|^2 + \lambda^2 H|\nabla u|^2 + \lambda H^3\\ &= -(1-\lambda)H^3 + \lambda(1-\lambda)H|\nabla u|^2 + 2HK\\ &= (1-\lambda)H\big\{\lambda|\nabla u|^2 - H^2\big\} + 2HK. \qquad (4.2)\end{aligned}$$

On the other hand, since always

$$K|\nabla u|^2 = -|\nabla H|^2 - H\langle \nabla H, \nabla u\rangle,$$

we get that at the point x_0 it holds

$$K|\nabla u|^2 = -\lambda^2 H^2|\nabla u|^2 + \lambda H^2|\nabla u|^2 = \lambda(1-\lambda)H^2|\nabla u|^2.$$

We distinguish now two cases:

Case 1. Suppose that $|\nabla u|(x_0) > 0$. Choose the parameter λ to take values in the interval $[1/2, 1)$. Then the Gauß curvature K must be positive at x_0 and in particular

$$K = \lambda(1-\lambda)H^2.$$

Substituting the above expression of K into the inequality (4.2) we deduce that

$$\begin{aligned} 0 &\leq (1-\lambda)H(-H^2 + \lambda|\nabla u|^2) + 2HK \\ &= (1-\lambda)H\big\{(2\lambda-1)H^2 + \lambda|\nabla u|^2\big\} \\ &< 0 \end{aligned}$$

which leads to a contradiction. Consequently, $\mu \geq 0$ and thus $H \geq 0$ on all of M. But then the strong elliptic maximum principle applied to Lemma 2.1 (f) gives either $H \equiv 0$ or $H > 0$ on M.

Case 2. Suppose now that $|\nabla u|(x_0) = 0$. Then, at the point x_0 we have $H = -1$. From the inequality (4.2) we obtain that at the point x_0, the following inequality is valid.

$$\lambda - 1 + 2K \leq 0.$$

Using the fact that $K \geq \varepsilon$ at x_0 we deduce that

$$\lambda - 1 + 2\varepsilon \leq 0. \tag{4.3}$$

On the other hand, notice that at x_0 we have

$$\varepsilon \leq K \leq H^2/4 = 1/4.$$

Therefore, for $\lambda \in (1-2\varepsilon, 1)$ equation (4.3) leads to a contradiction. Thus $H \geq 0$ everywhere and again by the strong maximum principle it follows that either $H \equiv 0$ or $H > 0$ everywhere.

It is obvious that it is possible to choose a parameter λ which works simultaneously for both cases. In the case where M is complete and properly embedded it turns out that it must be represented as a graph over a region of the plane E that is perpendicular to the translating direction v. Consequently, in this case the genus of M is zero. $\square$

As an immediate consequence of the above lemma we get the following:

Theorem D. *Let $f : M^2 \to \mathbb{R}^3$ be a translating soliton whose mean curvature satisfies $H > -1$. Suppose that $H \geq 0$ outside a compact subset of M^2. Then either $H \equiv 0$ and $M = f(M^2)$ is isometric to a plane or $H > 0$ on all of M^2. If, additionally, M is complete and embedded then it is a graph and so it has genus zero.*

Following the same strategy as in the proof of Lemma 4.3 we can show the following:

Lemma 4.4. *Let $f : M^2 \to \mathbb{R}^3$ be a translating soliton of the mean curvature flow. Suppose that*

(a) there exists a positive constant ε such that the Gauß curvature of f satisfies $K \geq \varepsilon$ on the set $\{x \in M^2 : H(x) = -1\}$,

(b) there exists a constant $\lambda \in (1-2\varepsilon, 1)$ such that the function $e^{2\lambda u}H^2$ is bounded.

(c) the sub-level sets $M_c := \{x \in M^2 : u(x) \leq c\}$, $c \in \mathbb{R}$, are compact.

Then, $H > 0$ on all of M^2.

Proof. Choose a positive constant $\delta > 0$ such that

$$\lambda - \frac{\delta}{2} > 1 - 2\varepsilon.$$

Because by assumption the function $H^2 e^{2\lambda u}$ is bounded, we deduce that there exists a positive constant C such that the following estimate holds

$$H^2 e^{(2\lambda-\delta)u} \leq Ce^{-\delta u}.$$

Because the height u is unbounded from above we get that $H^2 e^{(2\lambda-\delta)u}$ tends to 0 as u goes to infinity. Consider the function $g : M \to \mathbb{R}$ given by

$$g := He^{(\lambda-\delta/2)u}.$$

Suppose that there exists a point $x \in M^2$ where $H < 0$. Proceeding as in the proof of Lemma 4.3 we get a contradiction. Thus, either $H \equiv 0$ and $f(M^2)$ is a plane or $H > 0$. The first alternative is impossible because of assumption (c). Hence, H must be positive everywhere. ☐

We will conclude this section with some interesting formulas relating the Gauß curvature K of a translating soliton in $\mathbb{R}^3$ with its mean curvature H.

Lemma 4.5. *On a translating soliton in $\mathbb{R}^3$ the Gauß curvature K satisfies the following equations to which we will refer to as the (H,K)-formulas.*

$$\begin{aligned} K &= \Delta \log \sqrt{1+H^2} - \frac{2}{(1+H^2)^2}|\nabla H|^2 + \frac{H^4}{1+H^2} && (4.4)\\ &= \frac{H}{1+H^2}\Delta H - \frac{|\nabla H|^2}{1+H^2} + \frac{H^4}{1+H^2} && (4.5)\end{aligned}$$

Moreover, at each point where $H > 0$ we have

$$K = \frac{H^2}{1+H^2}\Delta \log(e^u H) = \frac{H^2}{1+H^2}\left(\Delta \log H + H^2\right). \tag{4.6}$$

Proof. From Lemma 2.1 (g), we have

$$K|\nabla u|^2 = \mathrm{Ric}(\nabla u, \nabla u) = -|\nabla H|^2 - H\langle \nabla H, \nabla u\rangle. \tag{4.7}$$

Taking into account the formula for ΔH in Lemma 2.1 (f) we compute

$$\begin{aligned} \Delta \log \sqrt{1+H^2} &= \frac{H}{1+H^2}\Delta H + \frac{1-H^2}{(1+H^2)^2}|\nabla H|^2 \\ &= -\frac{H}{1+H^2}\left(H|h|^2 + \langle \nabla H, \nabla u\rangle\right) \\ &\quad + \frac{1-H^2}{(1+H^2)^2}|\nabla H|^2. \end{aligned}$$

Combining the last equality with (4.7) and $|h|^2 = H^2 - 2K$ we get

$$\begin{aligned} \Delta \log \sqrt{1+H^2} &= -\frac{H^4}{1+H^2} + 2K\frac{H^2}{1+H^2} \\ &\quad + \frac{1}{1+H^2}\left(K|\nabla u|^2 + |\nabla H|^2\right) \\ &\quad + \frac{1-H^2}{(1+H^2)^2}|\nabla H|^2 \\ &= -\frac{H^4}{1+H^2} + K\frac{2H^2 + |\nabla u|^2}{1+H^2} \\ &\quad + \frac{2}{(1+H^2)^2}|\nabla H|^2 \\ &= -\frac{H^4}{1+H^2} + K + \frac{2}{(1+H^2)^2}|\nabla H|^2. \end{aligned}$$

This completes the proof of (4.4). Then (4.5) is just a reformulation of (4.4). Finally, if $H > 0$, then $\log H$ is well defined and (4.6) follows from $\Delta u = H^2$ and (4.5). □

Corollary 4.6. *Outside of the critical set of u, the vector field*

$$W = -\frac{\nabla H + H\nabla u}{|\nabla u|^2} = -\frac{\nabla(e^u H)}{e^u|\nabla u|^2}$$

is divergence free. Moreover,

$$K = \langle \nabla H, W\rangle = \operatorname{div}(HW).$$

Note that for the grim hyperplane with $\min u = 0$ we have $e^u H = 1$.

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