Point-vortex stability under the influence of an external periodic flow*

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Abstract

We provide sufficient conditions for the stability of the particle advection around a fixed vortex in a two-dimensional ideal fluid under the action of a periodic background flow. The proof relies on the identification of closed invariant curves around the origin by means of Moser's *Invariant Curve Theorem*.

1 Introduction

The advection of a passive particle in a two-dimensional ideal fluid is ruled by a Hamiltonian system with the streamfunction playing the role of the Hamiltonian. In turn, the streamfunction is determined as the solution of the Poisson equation $-\Delta\Psi = \omega$, where ω is the vorticity of the fluid. A point vortex is a singularity (Dirac delta) of the vorticity of a two-dimensional ideal fluid, in such a way that the streamfunction of a vortex is just the fundamental solution or Green's function of the 2D Laplacian. This idea of vortex comes back to the seminal works of Helmholzt and Kirchoff in the XIXth century.Since then, vortex dynamics has been widely studied as a branch of Classical Mechanics, with close connections to Celestial Mechanics and the theory of Hamiltonian systems.

Vortices arise naturally in the atmosphere or in the ocean, or can be created artificially in stirred fluids. Its influence in the passive transport of particles plays an important role in a variety of disciplines related to Hydrodynamics and Geophysics. Here a word of caution is necessary, because the notion of point-vortex as a singularity of the vorticity of a 2D fluid is a highly idealized version of real-life vortices. From a physical point of

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view, a vortex is just a concentrated region of high vorticity, but at the moment the Fluids Dynamics community works with several (non-equivalent) mathematical definitions of vortex, see [5] for a summary of different interpretations available in the literature. Many definitions require a compact set of finite vorticity (vortex patch) rather than a singularity, in other cases the effect of a third spatial dimension is considered (filament or tubular vortices). We refer to [8, 5] for more information about the existing models.

In the absence of external fields, a point vortex placed in the origin induces the streamfunction $\Psi_0(x,y) = \frac{\Gamma}{4\pi}\ln(x^2+y^2)$, where Γ is the circulation or strength of the vortex. In the adequate unit, we can fix $\Gamma = 2\pi$ without loss of generality. The Hamiltonian system is of course integrable, with the particles rotating around the vortex in circular paths. We can say that the flow is stable around the vortex. The objective of this paper is to explore if this stability is preserved under the influence of an external superimposed non-stationary (periodic) streamfunction $p(t,x,y)$. Hence, we consider the perturbed Hamiltonian

$$\Psi(t,x,y) = \frac{1}{2}\ln(x^2+y^2) + p(t,x,y), \tag{1}$$

and the Hamiltonian system under study is

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\} \tag{2}$$

where $\mathcal{U}$ is some neighborhood of the origin.

Physically, system (2) models the passive advection of particles in a fluid subjected to the action of a steady vortex placed at the origin and a time-dependent background flow. This point of view of looking at the motion of single particles is known as the Lagrangian description for Fluid Dynamics, in contrast to the Eulerian perspective, which rather looks to the global properties of the flow.

Different aspects of the dynamics of advected particles in non-stationary flows have been studied in the literature, see for instance [2, 3, 6, 10, 13]. Numerical studies in related models [2, 3] strongly suggest the presence of stability (regular) islands around a point vortex. In spite of that, from a more analytical point of view, it seems necessary to state a precise mathematical notion of stability. To fill this gap, Section 2 is devoted to settle a rigorous definition of stability of a singularity on this context. We consider perpetual stability in the Lyapunov sense, meaning that a solution of (2) with small initial condition will remain close to the singularity forever. Once such definition is precisely stated, the main aim of this paper is to provide a simple sufficient condition for stability.

The rest of the paper is organized as follows. In Section 3, the main result is stated, where we identify a class of functions $p(t,x,y)$ for which the vortex singularity is stable. The model example would be a polynomial

$$p(t,x,y) = \sum_{4\leq h+k\leq N} \alpha_{h,k}(t)x^h y^k$$

with 1-periodic coefficients $\alpha_{h,k} \in \mathcal{C}(\mathbb{R}/\mathbb{Z})$ and $\alpha_{h,4-h} \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$. The proof is presented in Section 4. The first step, performed in Subsection 4.1 is to invert the role of the origin and infinite by the Kelvin transform. Also, we have to consider symplectic polar coordinates in order to preserve the symplectic structure of the system. In Subsection 4.2, we pass to complex formulation by considering the system as complex-valued. This formalism is necessary to apply the *Invariant Curve Theorem* in the version presented in Section 32 of [12]. Subsections 4.3 and 4.4 are devoted to prove that the Poincaré map is well-defined for a suitable domain. Some required estimates have been detailed in the attached Appendix. Finally, Subsection 4.5 proves the *intersection property* for the Poincaré map, therefore the *Invariant Curve Theorem* provides the existence of a sequence of invariant Jordan curves near infinite. After inversion of the Kelvin transform, such invariant curves surround the vortex, acting as flux barriers and giving rise to the desired stability. Following the suggestions of an anonymous referee, we have included a final section devoted to conclusions and remarks oriented to future work. Some numerical experiments are displayed as well.

2 The stability of a singularity

Assume that Ω is an open subset of $\mathbb{R}^d$ with $d \geq 2$ and let $\xi_* \in \Omega$ be a fixed point. We consider the punctured region $\Omega_* = \Omega \setminus \{\xi_*\}$ and a continuous vector field

$$\begin{aligned} X : \mathbb{R} \times \Omega_* &\longrightarrow \mathbb{R}^d \\ (t,\xi) &\longmapsto X(t,\xi). \end{aligned}$$

From now on we assume that there is global uniqueness for each initial value problem of the type

$$\begin{cases} \dot{\xi} = X(t,\xi), \\ \xi(0) = \xi_0, \qquad \xi_0 \in \Omega_*. \end{cases}$$

The corresponding solution will be denoted by $\xi(t,\xi_0)$.

The point $\xi = \xi_*$ will be interpreted as a singularity of the equation.

Definition 1. The singularity ξ_* will be called *stable* (in a perpetual sense) if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\xi_0 \in \Omega_*$ with

$$|\xi_0 - \xi_*| < \delta$$

then $\xi(t,\xi_0)$ is well defined everywhere and

$$|\xi(t,\xi_0) - \xi_*| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}.$$

Example 1. The model example is the unperturbed system (2) with $p \equiv 0$. Taking $z = \begin{pmatrix} x \\ y \end{pmatrix}$, it can be written as

$$\dot{z} = \frac{1}{|z|^2} J\, z \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The explicit solution is

$$z(t, z_0) = \mathcal{R}\left[\frac{t}{|z_0|^2}\right] z_0, \quad \text{where} \quad \mathcal{R}[\theta] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The stability of the point $z_* = 0$ is clear because $|z(t, z_0)| = |z_0|$. In contrast, the derivative satisfies

$$|\dot{z}(t, z_0)| = \frac{1}{|z_0|} \to \infty \quad \text{as} \quad z_0 \to z_*.$$

The previous example was autonomous but our definition also works in time-dependent situations. We will restrict to periodic vector fields satisfying

$$X(t + T, \xi) = X(t, \xi) \quad \text{for some} \quad T > 0.$$

In this case, we can define the Poincaré map

$$\begin{array}{rcl} \mathcal{P} : \mathcal{E} \subset \Omega_* & \longrightarrow & \Omega_* \\ \xi_0 & \longmapsto & \mathcal{P}(\xi_0) = \xi(T, \xi_0) \end{array}$$

where $\mathcal{E} = \{\xi_0 \in \Omega_* : \ \xi(t, \xi_0) \text{ is well defined on } [0, T]\}$.

The standard theorem on continuous dependence implies that $\mathcal{E}$ is open. When ξ_* is a stable singularity there exists a neighborhood $\mathcal{O} \subset \mathbb{R}^d$ of ξ_* such that $\mathcal{O} \setminus \{\xi_*\}$ is contained in $\mathcal{E}$. Then it is natural to define $\mathcal{P}(\xi_*) = \xi_*$ so that $\mathcal{P}$ is extended to a one-to-one and continuous map from $\mathcal{E} \cup \{\xi_*\}$ into Ω. The standard notion of stable fixed point of a map is now applicable. A stable singularity $\xi = \xi_*$ will produce a stable fixed point of the Poincaré map. This means that for each neighborhood $\mathcal{V}$ of ξ_* there exists another neighborhood $\mathcal{W} \subset \mathcal{E} \cup \{\xi_*\}$ such that all the iterates $\mathcal{P}^n(\mathcal{W})$ are well defined and satisfy $\mathcal{P}^n(\mathcal{W}) \subset \mathcal{V}$, $n \in \mathbb{Z}$.

The next example will show that the stability of ξ_* as a fixed point of $\mathcal{P}$ is not sufficient to guarantee the stability of $\xi = \xi_*$ as a singularity of the differential equation.

Example 2. The equation for $z \in \mathbb{R}^2 \setminus \{0\}$,

$$\dot{z} = \sin(2t) \frac{z}{|z|}$$

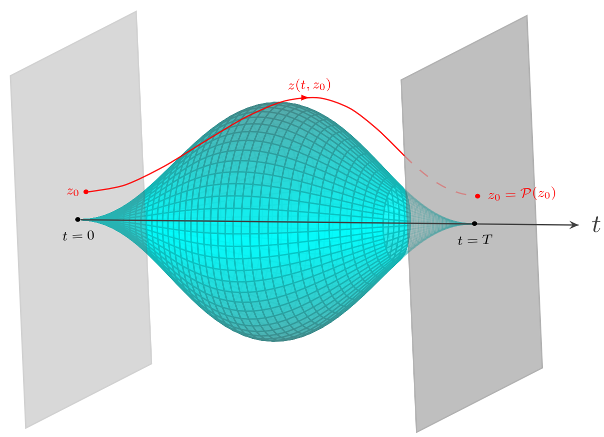

Figure 1: The singularity evolution over a period forces the solution to get away from $z_* = 0$.

is T-periodic with $T = \pi$ and

$$z(t, z_0) = \left(|z_0| + \sin^2 t\right) \frac{z_0}{|z_0|}, \qquad z_0 \in \mathbb{R}^2 \setminus \{0\}.$$

We observe that $\mathcal{P}$ is the identity and so $z_* = 0$ is stable as a fixed point of $\mathcal{P}$. On the other hand,

$$\left| z\left(\frac{T}{2}, z_0\right) - z_* \right| = \left| z\left(\frac{T}{2}, z_0\right) \right| = |z_0| + 1 > 1$$

and so $z_* = 0$ is not stable as a singularity (see Fig.1).

The reader who is familiar with the theory of the stability of the equilibrium will notice the sharp contrast between equilibria and singularities. In the second case continuous dependence may be lost. For this reason we make the following definition.

Definition 2. We say that there is *continuous dependence around the singularity* $\xi = \xi_*$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$0 < |\xi_0 - \xi_*| < \delta$$

then $\xi(t, \xi_0)$ is well defined on $[0, T]$ and

$$|\xi(t, \xi_0) - \xi_*| < \varepsilon \quad \text{for each} \quad t \in [0, T]\,.$$

The following result is a direct consequence of the previous definitions.

Proposition 1. *In the previous setting assume that there is continuous dependence around the singularity $\xi = \xi_*$ and moreover ξ_* is stable as a fixed point of $\mathcal{P}$. Then, $\xi = \xi_*$ is a stable singularity.*

3 Main result

For our purpose we will consider the complexification of the variables x and y. Given $\varepsilon > 0$, consider the disk $\mathbb{D}_\varepsilon = \{x \in \mathbb{C} : |x| < \varepsilon\}$. A function

$$\begin{array}{rcl} p : \mathbb{R} \times \mathbb{D}_\varepsilon^2 & \longrightarrow & \mathbb{C} \\ (t, x, y) & \longmapsto & p(t, x, y) \end{array}$$

belongs to the class $\mathcal{H}_\varepsilon$ if it satisfies the conditions below:

i) p is continuous.

ii) $p(t + 1, x, y) = p(t, x, y)$.

iii) p extends a real-valued function; that is,

$$p(t, x, y) \in \mathbb{R} \quad \text{if} \quad x, y \in \mathbb{D}_\varepsilon \cap \mathbb{R}.$$

iv) For each $t \in \mathbb{R}$,

$$(x, y) \in \mathbb{D}_\varepsilon^2 \longmapsto p(t, x, y) \in \mathbb{C}$$

is a holomorphic function in two variables.

v) $\|p\|_\infty := \sup \left\{ |p(t, x, y)| \ : \ (t, x, y) \in \mathbb{R} \times \mathbb{D}_\varepsilon^2 \right\} < \infty.$

Example 3. A simple example of $p(t, x, y)$ is a polynomial

$$p(t, x, y) = \sum_{h+k \leq N} \alpha_{h,k}(t) x^h y^k$$

with $\alpha_{h,k} \in \mathcal{C}(\mathbb{R}/\mathbb{Z})$.

Definition 3. Given a function p in $\mathcal{H}_\varepsilon$ we say that the origin is a *zero of order* N if for each $t \in \mathbb{R}$,

$$p(t, 0, 0) = 0 \quad \text{and} \quad \frac{\partial^{h+k}}{\partial x^h \partial y^k} p(t, 0, 0) = 0 \quad \text{when} \quad h + k < N.$$

Now, we are ready to state the main result of the paper.

Theorem 1. *Assume that $p \in \mathcal{H}_\varepsilon$ and the origin is a zero of order 4. In addition, the functions*

$$\alpha_h(t) = \frac{\partial^4}{\partial x^h \partial y^{4-h}} p(t, 0, 0) \tag{3}$$

are of class $\mathcal{C}^1$ for each h, $0 \leq h \leq 4$. Then, the origin of system (2) *is stable in the sense of Definition 1.*

4 Proof of the main result

4.1 Some transformations: from zero to infinity

The system (2) is defined on some neighborhood $\mathcal{U}$ of the origin. It will be convenient to transform it into another system defined in a neighborhood of infinity. To do this we employ the Kelvin transform:

$$\begin{array}{rcl}\kappa : \mathbb{R}^2 \setminus \{0\} & \longrightarrow & \mathbb{R}^2 \setminus \{0\} \\ (u,v) & \longmapsto & \kappa(u,v) = (x,y)\end{array}$$

where

$$x = \frac{u}{u^2+v^2}\,, \qquad y = -\frac{v}{u^2+v^2}\,.$$

This map is an analytic involution ($\kappa^2 = \mathrm{Id}$) satisfying

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{\left(u^2+v^2\right)^2}\,\mathrm{d}u \wedge \mathrm{d}v. \tag{4}$$

Hence it is not a canonical map. To remain in a Hamiltonian framework we will introduce a new symplectic structure in the phase space. First we recall some well known facts, see [9] for more details. Let Ω be our phase space, an open subset of $\mathbb{R}^2$. We consider the symplectic form

$$\hat{\omega}_z = \Phi\left(z\right)\mathrm{d}q \wedge \mathrm{d}p$$

where $z = (q,p) \in \Omega$ and $\Phi : \Omega \longrightarrow \mathbb{R}$ is a smooth and positive function. Given $H \in \mathcal{C}^\infty(\Omega)$, the Hamiltonian system associated to the triplet $(\Omega, \hat{\omega}, H)$ is

$$\dot{z} = \frac{1}{\Phi(z)}\,J\,\nabla H(z) \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

Let $\omega = \mathrm{d}x \wedge \mathrm{d}y$ be the standard symplectic form in $\mathbb{R}^2$, the identity (4) implies that κ is a symplectic diffeomorphism between $\left(\mathbb{R}^2 \setminus \{0\}, \hat{\omega}\right)$ and $\left(\mathbb{R}^2 \setminus \{0\}, \omega\right)$. The previous remark can be applied with

$$\Phi\left(u,v\right) = \frac{1}{\left(u^2+v^2\right)^2}$$

and so the system (2) is transformed into

$$\left\{\begin{array}{l} \dot{u} = \left(u^2+v^2\right)^2 \partial_v \hat{\Psi}(t,u,v) \\ \\ \dot{v} = -\left(u^2+v^2\right)^2 \partial_u \hat{\Psi}(t,u,v) \end{array}\right. \quad \text{with} \quad \hat{\Psi} = \Psi \circ \kappa. \tag{6}$$

This system is defined in $\kappa(\mathcal{U})$, a neighborhood of infinity. Next we pass to symplectic polar coordinates, defined by the diffeomorphism ($\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$)

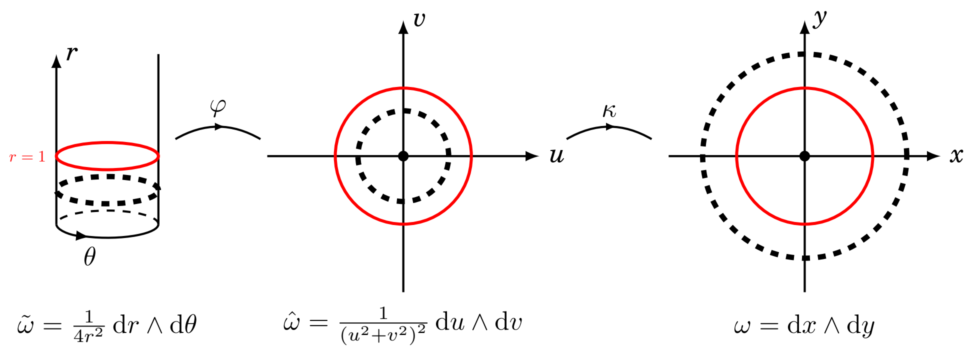

Figure 2: The transformations κ, φ and related symplectic forms.

$$
\begin{array}{rcl}
\varphi :]0,\infty[\times \mathbb{T} & \longrightarrow & \mathbb{R}^2 \setminus \{0\} \\
(r,\theta) & \longmapsto & \varphi(r,\theta) = (u,v)
\end{array}
$$

where

$$
u = \sqrt{2r}\cos\theta, \qquad v = \sqrt{2r}\sin\theta.
$$

We observe that

$$
\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{\left(u^2+v^2\right)^2}\,\mathrm{d}u \wedge \mathrm{d}v = \frac{1}{4r^2}\mathrm{d}r \wedge \mathrm{d}\theta,
$$

and so $\kappa\circ\varphi$ is a symplectic diffeomorphism from $\Omega =]0,\infty[\times\mathbb{T}$ with the form $\tilde{\omega} = \frac{1}{4r^2}\,\mathrm{d}r \wedge \mathrm{d}\theta$ onto $\left(\mathbb{R}^2 \setminus \{0\}, \omega\right)$. Now the original system is transformed into

$$
\left\{
\begin{array}{l}
\dot{r} = 4r^2\,\partial_\theta H(t,r,\theta) \\
\\
\dot{\theta} = -4r^2\,\partial_r H(t,r,\theta)
\end{array}
\right.
\quad \text{with} \quad H = \Psi\circ\kappa\circ\varphi. \tag{7}
$$

Thus $H(t,r,\theta) = -\frac{1}{2}\ln(2r) + h(t,r,\theta)$ with $h(t,r,\theta) = p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right)$.

The previous transformations and the different symplectic forms are illustrated in Fig. 2.

4.2 The complexified system

System (7) can be expressed in the form

$$
\left\{
\begin{array}{l}
\dot{r} = F(t,r,\theta) \\
\dot{\theta} = 2r + G(t,r,\theta)
\end{array}
\right. \tag{8}
$$

where

$$
F(t,r,\theta) = -2\sqrt{2}\,r^{3/2}\left(\mathrm{p}_1\sin\theta + \mathrm{p}_2\cos\theta\right),
$$

$$G(t,r,\theta) = \sqrt{2}\, r^{1/2}\,(\mathrm{p}_1 \cos\theta - \mathrm{p}_2 \sin\theta)$$

with

$$\mathrm{p}_1 = \partial_x p \left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right), \qquad \mathrm{p}_2 = \partial_y p \left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right).$$

This is a periodic planar system defined with coordinates $r > 0$ and $\theta \in \mathbb{R}$ or on a cylinder $]0,\infty[\, \times \mathbb{T}$. As we have announced in Section 3, for convenience we embed it into the complex system. This is understood in the following sense: the equations are the same but the unknowns $r = r(t)$ and $\theta = \theta(t)$ will be complex valued, the independent variable t will remain in $\mathbb{R}$.

Remark. To avoid multi-valued functions, $w = \sqrt{z}$ will be interpreted as the holomorphic function defined on the half plane $\mathrm{Re}(z) > 0$ which extends the positive square root of positive real numbers. Also the function $z^{3/2} = z\sqrt{z}$ is holomorphic in $\mathrm{Re}\,(z) > 0$.

From now on, $r_* > 0$ and $\Delta_* > 0$ are two fixed numbers satisfying

$$\frac{\cosh(\Delta_*)}{\sqrt{2r_*}} < \frac{\varepsilon}{2}. \tag{9}$$

If we define the domain

$$\mathcal{D} = \left\{(r,\theta) \in \mathbb{C}^2 : \mathrm{Re}\,(r) > r_*, \quad |\mathrm{Im}\,(\theta)| < \Delta_*\right\} \tag{10}$$

then the inequalities

$$\left|\frac{\cos\theta}{\sqrt{2r}}\right| < \frac{\varepsilon}{2}, \qquad \left|\frac{\sin\theta}{\sqrt{2r}}\right| < \frac{\varepsilon}{2}$$

holds for any point lying on $\mathcal{D}$. In consequence $F, G : \mathbb{R} \times \mathcal{D} \longrightarrow \mathbb{C}$ are continuous functions which are 1-periodic in t and such that $F(t,\cdot,\cdot), G(t,\cdot,\cdot)$ are holomorphic for each $t \in \mathbb{R}$.

4.3 Some preliminary estimates

The main aim of Subsections 4.3 and 4.4 is to prove that, under the assumptions of Theorem 1, the Poincaré map is well-defined in a suitable open set. In this subsection we replace the main assumption of Theorem 1 by a stronger condition. More concretely, let us assume that the origin is a zero of order 5 of the function $p \in \mathcal{H}_\varepsilon$. In particular, the functions α_h vanish and so the assumptions of Theorem 1 are satisfied. The proof under the general assumptions will be presented later in Subsection 4.4.

Using Lemma 3 of Appendix A, we have the estimate

$$|\partial p\,(t,x,y)| \le C(|x|^4 + |y|^4) \quad \text{if} \quad |x| < \frac{\varepsilon}{2},\ |y| < \frac{\varepsilon}{2}. \tag{11}$$

Here $\partial = \partial_x$ or ∂_y and C only depends upon $\|p\|_\infty$ and ε. Then, from the definition of F and G, we have

$$|F(t,r,\theta)| + |2r|\,|G(t,r,\theta)| \le C_1 \frac{e^{5|\mathrm{Im}\,(\theta)|}}{|r|^{1/2}} \tag{12}$$

for any $t \in \mathbb{R}$ and $(r,\theta) \in \mathcal{D}$. Here C_1 is a constant depending upon $r_*, \Delta_*, \varepsilon$ and C.

Next, we introduce a family of domains in $\mathbb{C}^2$ whose geometry is well adapted to our differential equation.

Given real numbers a, b, R, Δ such that

$$b - a = 1 \quad \text{and} \quad R > 0,\ \ \Delta > 0,$$

we define

$$\mathcal{C}_\Delta = \{\theta \in \mathbb{C} \,:\, |\mathrm{Im}\,(\theta)| < \Delta\}$$

and

$$[a,b]_R = \{r \in \mathbb{C} \,:\, \mathrm{dist}\,(r,[a,b]) < R\}$$

where $\mathrm{dist}\,(z,\mathcal{K})$ is the distance from a point z to a compact set $\mathcal{K} \subset \mathbb{C}$. The set $\mathcal{C}_\Delta$ is a horizontal strip and $[a,b]_R$ has the shape of a stadium. We consider the domain in $\mathbb{C}^2$

$$\Omega = [a,b]_R \times \mathcal{C}_\Delta.$$

Sometimes, to emphasize the dependence on R and Δ, we will write $\Omega(R,\Delta)$. Obviously this set also depends of the interval $[a,b]$ but this dependence will not be made explicit.

The domain $\Omega(R,\Delta)$ is contained in $\mathcal{D}$ defined by (10) as soon as

$$\Delta < \Delta_* \qquad \text{and} \qquad a - R > r_*. \tag{13}$$

This is important to be sure that the complexified system (8) is well defined on $\Omega(R,\Delta)$.

For $0 < \rho < R,\ 0 < \delta < \Delta$ it is clear that $\Omega(\rho,\delta)$ is contained in $\Omega(R,\Delta)$. In the next result we will prove that the solutions starting at $\Omega(\rho,\delta)$ do not exit $\Omega(R,\Delta)$ in one period (see Fig.3). This will require some smallness on the parameters ρ and δ, and a large enough.

Lemma 1. *Let us assume that the origin is a zero of order* 5 *of the function* $p \in \mathcal{H}_\varepsilon$. *Assume that* $R > \rho > 0,\ \Delta > \delta > 0$ *are given numbers satisfying* (13)*. In addition,*

$$\delta + 2\rho < \Delta. \tag{14}$$

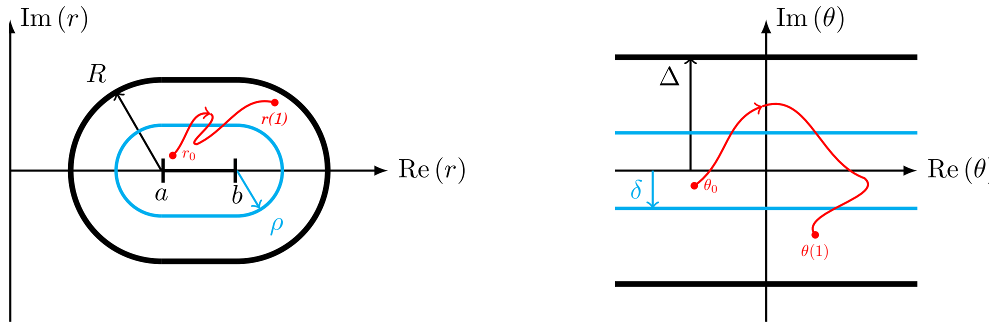

Figure 3: Domains $[a,b]_R$, $\mathcal{C}_\Delta$, subdomains $[a,b]_\rho$, $\mathcal{C}_\delta$ and solution $(r(t),\theta(t))$ with $t\in[0,1]$.

Then there exists $a_*>R$ *such that if* $a>a_*$ *the solution of* (8) *with initial condition* $(r_0,\theta_0)\in\Omega(\rho,\delta)$ *is well defined on* $t\in[0,1]$. *Moreover,*

$$(r(t),\theta(t))\in\Omega(R,\Delta)$$

and

$$|r(t)-r_0|+|\theta(t)-\theta_0-2r_0t|\le\frac{Ke^{5\Delta}}{|r_0|^{1/2}}\quad\text{if}\quad t\in[0,1].\tag{15}$$

Proof. Let $[0,\tau]$ be a compact sub-interval of $[0,1]$ where the solution $(r(t),\theta(t))$ is well defined and remains in $\Omega(R,\Delta)$. The geometry of $[a,b]_R$ implies that

$$|r(t)|\ge a-R\quad\text{if}\quad t\in[0,\tau].$$

From the first equation in (8) and the estimate (12)

$$|r(t)-r_0|\le\int_0^t|\dot r(s)|\,\mathrm{d}s\le\frac{C_1e^{5\Delta}}{(a-R)^{1/2}},\qquad t\in[0,\tau].$$

Note that for a large enough we can assume

$$\frac{C_1e^{5\Delta}}{(a-R)^{1/2}}<R-\rho.$$

This inequality guarantees that $r(t)$ does not touch the boundary of $[a,b]_R$. Also,

$$|r(t)|\ge|r_0|-|r(t)-r_0|\ge|r_0|-R,\qquad t\in[0,\tau].$$

After increasing a we can assume that $a>\rho+R$ and $|r_0|-R$ is positive.

With this information we go back to the first equation of (8) to obtain the new estimate

$$|r(t)-r_0|\le\frac{C_1e^{5\Delta}}{(|r_0|-R)^{1/2}},\qquad t\in[0,\tau].\tag{16}$$

Now, let us consider the second equation of (8). After integrating from 0 to t, some simple manipulations lead to

$$\theta(t) = \theta_0 + 2r_0 t + 2\int_0^t [r(s) - r_0]\,\mathrm{d}s + \int_0^t G(s, r(s), \theta(s))\,\mathrm{d}s.$$

Using the inequalities (12) and (16),

$$|\theta(t) - \theta_0 - 2r_0 t| \leq \frac{2C_1 e^{5\Delta}}{(|r_0| - R)^{1/2}} + \frac{C_1 e^{5\Delta}}{2\,(|r_0| - R)^{3/2}}, \qquad t \in [0, \tau].$$

It is the time to employ the condition (14). It allows to find a_* large enough so that

$$\delta + 2\rho + \frac{2C_1 e^{5\Delta}}{(a - \rho - R)^{1/2}} + \frac{C_1 e^{5\Delta}}{2\,(a - \rho - R)^{3/2}} < \Delta \quad \text{if} \quad a > a_*.$$

Then,

$$|\mathrm{Im}\,(\theta(t))| \leq |\mathrm{Im}\,(\theta_0)| + 2\,|\mathrm{Im}\,(r_0)| + |\theta(t) - \theta_0 - 2r_0 t| < \Delta, \qquad t \in [0, \tau].$$

We conclude that $\theta(t)$ cannot touch the boundary of $\mathcal{C}_\Delta$.

The previous discussions imply that the solution cannot touch the boundary of $\Omega(R, \Delta)$. In particular $(r(t), \theta(t))$ is well defined on $[0, 1]$ and we can take $\tau = 1$. The estimate (15) is a consequence of the above estimates because $(|r_0| - R)^{1/2}$ and $|r_0|^{1/2}$ are of the same order as $|r_0| \to \infty$.

□

4.4 Estimates for the general case

As commented in the previous subsection, our intention now is to prove Lemma 1 under the general assumptions of our main result. Then, let us assume that the function p is in the conditions of Theorem 1 and go back to the setting for the complexified system (8) proposed in Subsection 4.2. The constants r_* and Δ_* satisfying (9) are determined in the same way. The first difference appears with the estimate (12) that can be replaced by

$$|F(t, r, \theta)| + |2r|\,|G(t, r, \theta)| \leq C_2 e^{4|\mathrm{Im}\,(\theta)|} \tag{17}$$

for any $t \in \mathbb{R}$ and $(r, \theta) \in \mathcal{D}$. Note that the origin is now a zero of order 4. The estimate (17) does not seem to provide enough information in order to obtain a result in the line of Lemma 1.

The new idea will be to split the function F as

$$F(t, r, \theta) = F_*(t, \theta) + \tilde{F}(t, r, \theta) \tag{18}$$

where $\tilde{F}$ satisfies an estimate of the type (12) and F_* can be averaged. To describe this splitting in precise terms we write

$$p(t, x, y) = T(t, x, y) + \tilde{p}(t, x, y)$$

where T is the Taylor polynomial of degree 4. In consequence the origin is a zero of order 5 for $\tilde{p}$ and we can assume that the first derivatives $\partial\tilde{p}(t,x,y)$ satisfy (11).

Going back to the equations in (7) we observe that

$$F(t,r,\theta)=4r^2\partial_\theta\left[p\left(t,\frac{\cos\theta}{\sqrt{2r}},\frac{-\sin\theta}{\sqrt{2r}}\right)\right].$$

The homogeneity of T allows to write F in the form (18) with

$$F_*(t,\theta)=\partial_\theta\left[T\left(t,\cos\theta,-\sin\theta\right)\right],$$

$$\tilde{F}(t,r,\theta)=4r^2\partial_\theta\left[\tilde{p}\left(t,\frac{\cos\theta}{\sqrt{2r}},\frac{-\sin\theta}{\sqrt{2r}}\right)\right].$$

To average F_*, we will apply Lemma 4 (see Appendix A) for the polynomial of degree $N=4$

$$q(t,x,y)=-y\frac{\partial T}{\partial x}(t,x,-y)-x\frac{\partial T}{\partial y}(t,x,-y).$$

Then, $T(t,\cos\theta,-\sin\theta)$ is a periodic primitive in the variable θ of the function $q(t,\cos\theta,\sin\theta)$ and so the condition (21) holds. The assumption (3) also plays a role since we need to know that q has coefficients that are $\mathcal{C}^1$ in the variable t.

We are ready to prove that Lemma 1 is also valid in the assumptions of Theorem 1.

Lemma 2. *Under the hypotheses of Theorem 1, the conclusion of Lemma 1 holds.*

Proof. Let $[0,\tau]$ be a compact sub-interval of $[0,1]$ where the solution $(r(t),\theta(t))$ is well defined and remains in $\Omega(R,\Delta)$. Then $r(t)$ and r_0 belong to $[a,b]_R$ and so

$$|r(t)-r_0|\leq d \quad \text{if} \quad t\in[0,\tau],$$

where $d=1+2R$ is the diameter of $[a,b]_R$. In particular, $|r(t)|\geq|r_0|-d$ and we will impose $a-R>d$ to guarantee that $|r_0|-d$ is positive.

Integrating on the second equation of (8), we deduce that

$$|\theta(t)-\theta_0-2r_0t|\leq 2d+\frac{C_2e^{4\Delta}}{2(|r_0|-d)}.$$

Here we have employed (17) and the above estimates on $r(t)$.

Defining $\beta(t)=\theta(t)-\hat{\theta}_0-2r_0t$, where $\hat{\theta}_0-\theta_0\in 2\pi\mathbb{Z}$ and $\hat{\theta}_0\in[0,2\pi[$. The previous estimate can be interpreted as a bound of $\|\beta\|_\infty$. To get a bound of $\|\dot{\beta}\|_\infty$ we observe that

$$\dot{\beta}(t)=\dot{\theta}(t)-2r_0=2\left(r(t)-r_0\right)+G(t,r(t),\theta(t)).$$

Then

$$\|\dot{\beta}\|_\infty \leq 2d + \frac{C_2 e^{4\Delta}}{2(|r_0| - d)}.$$

Lemma 4 in Appendix A can be applied to deduce that

$$\left| \int_0^t F_*(s, \theta(s)) \mathrm{d}s \right| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|}.$$

Going back to the first equation in (8) and taking into account the splitting (18) we obtain

$$|r(t) - r_0| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|} + \frac{C_1 e^{5\Delta}}{(|r_0| - d)^{1/2}}, \qquad t \in [0, \tau].$$

At this point we have applied that from (12), $\tilde{F}$ satisfies

$$\left| \tilde{F}(t, r, \theta) \right| \leq \frac{C_1 e^{5|\mathrm{Im}\,(\theta)|}}{|r|^{1/2}} \quad \text{if} \quad t \in [0, \tau] \quad \text{and} \quad (r, \theta) \in \mathcal{D}.$$

Going back to (8) with the improved estimate for $|r(t) - r_0|$, one obtains

$$|\theta(t) - \theta_0 - 2r_0 t| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|} + \frac{C_1 e^{5\Delta}}{(|r_0| - d)^{1/2}} + \frac{C_2 e^{4\Delta}}{2(|r_0| - d)}. \tag{19}$$

The rest of the proof follows along the lines of Lemma 1. □

4.5 Existence of invariant curves

Let us fix numbers $R > \rho > 0$ and $\Delta > \delta > 0$ in the conditions of Lemma 1. The corresponding numbers a_* and K will be also fixed.

The Poincaré map associated to the system (8) is given by the formula

$$\mathcal{P}(r(0), \theta(0)) = (r(1), \theta(1))$$

where $(r(t), \theta(t))$ is a solution defined on $t \in [0, 1]$. From the previous section we know that $\mathcal{P}$ is well defined on the open set

$$\mathcal{E}_{\rho,\delta} = \left\{ (r_0, \theta_0) \in \mathbb{C}^2 \;:\; \mathrm{dist}\,(r_0, [a_*, +\infty[) < \rho, \qquad |\mathrm{Im}\,(\theta)| < \delta \right\}.$$

Moreover $\mathcal{P}$ maps $\mathcal{E}_{\rho,\delta}$ into a subset of $\mathcal{E}_{R,\Delta}$. The standard theory for the Cauchy problem implies that

$$\mathcal{P} : \mathcal{E}_{\rho,\delta} \subset \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

is a holomorphic diffeomorphism from $\mathcal{E}_{\rho,\delta}$ onto $\mathcal{P}\left(\mathcal{E}_{\rho,\delta}\right)$. System (8) is 2π-periodic in θ and this property is inherited by $\mathcal{P}$,

$$\mathcal{P}(r_0, \theta_0 + 2\pi) = \mathcal{P}(r_0, \theta_0) + (0, 2\pi).$$

Moreover, $\mathcal{P}$ can be expressed as

$$\begin{cases} r_1 = r_0 + f(r_0, \theta_0) \\ \theta_1 = \theta_0 + 2r_0 + g(r_0, \theta_0) \end{cases}$$

where $f, g : \mathcal{E}_{\rho,\delta} \longrightarrow \mathbb{C}$ are holomorphic functions, 2π-periodic in θ and satisfying

$$|f(r_0, \theta_0)| + |g(r_0, \theta_0)| \leq \frac{Ke^{5\Delta}}{(a-\rho)^{1/2}} \tag{20}$$

when $a > a_*$ is such that $\mathrm{dist}\,(r_0, [a, +\infty[) < \rho$. Note that the bound $M(a) := \frac{Ke^{5\Delta}}{(a-\rho)^{1/2}} \to 0 \quad \text{as} \quad a \to +\infty$.

Let us consider the restriction of the Poincaré map to the real domain

$$\mathcal{P} : E_{\rho,\delta} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

where

$$E_{\rho,\delta} = \left\{ (r_0, \theta_0) \in \mathbb{R}^2 \,:\, r_0 > a_* - \rho \right\}.$$

We claim that this map has the *intersection property*. In the book by Siegel and Moser [12], this means that

$$\mathcal{P}(\Gamma_\phi) \cap \Gamma_\phi \neq \emptyset$$

when $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is any analytic and 2π-periodic function with $\phi(\theta) > a_* - \rho$ and Γ_ϕ is the corresponding graph,

$$\Gamma_\phi = \{ (\phi(\theta), \theta) \,:\, \theta \in \mathbb{R} \}.$$

In fact, we will prove that the map $\mathcal{P}$ has a stronger intersection property, with more topological flavour. Let us interpret system (8) as a system in the cylinder

$$\mathfrak{C} = \left\{ (r, \overline{\theta}) \,:\, r > a_* - \rho, \quad \overline{\theta} \in \mathbb{T} \right\}.$$

Similarly, $\mathcal{P}$ will be understood as an embedding of the cylinder

$$\overline{\mathcal{P}} : \mathfrak{C} \subset \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R} \times \mathbb{T}.$$

This means that $\overline{\mathcal{P}}$ is a homeomorphism from $\mathfrak{C}$ onto the open set $\overline{\mathcal{P}}(\mathfrak{C})$. We will prove that

$$\overline{\mathcal{P}}(\overline{\Gamma}) \cap \overline{\Gamma} \neq \emptyset$$

for any Jordan curve $\overline{\Gamma} \subset \mathfrak{C}$ that is not contractible. The key idea to prove this claim is to observe that $\overline{\mathcal{P}}$ preserves a finite measure on the cylinder, namely

$$\mu(\mathcal{A}) = \int\int_A \frac{1}{4r^2} \,\mathrm{d}\overline{\theta}\,\mathrm{d}r$$

for each measurable set $\mathcal{A} \subset \mathbb{R} \times \mathbb{T}$. This is a consequence of the Hamiltonian structure of (8) described in Subsection 4.1. Once we know that $\mu(\overline{\mathcal{P}}(\mathcal{A})) = \mu(\mathcal{A})$ for each $\mathcal{A}$, we can prove that $\overline{\mathcal{P}}$ has the intersection property.

Let $\overline{\Gamma} \subset \mathfrak{C}$ be a non-contractible Jordan curve. Then $(\mathbb{R}\times\mathbb{T})\setminus\overline{\Gamma}$ splits into two connected components R_+ and R_- (see Fig.4). Since $\overline{\mathcal{P}}$ is an embedding, the image $\overline{\Gamma}' = \overline{\mathcal{P}}(\overline{\Gamma})$ is also a non-contractible Jordan curve with $(\mathbb{R}\times\mathbb{T})\setminus \overline{\Gamma}' = R_+^{'} \cup R_-^{'}$. Assume by contradiction that $\overline{\Gamma}\cap\overline{\Gamma}' = \emptyset$, then either $R_+ \subset R_+^{'}$ or $R_+{}' \subset R_+$ and the inclusion is strict. This is impossible because R_+ and $R_+^{'}$ are open subsets of the cylinder with $\mu(R_+) = \mu(R_+^{'}) < \infty$.

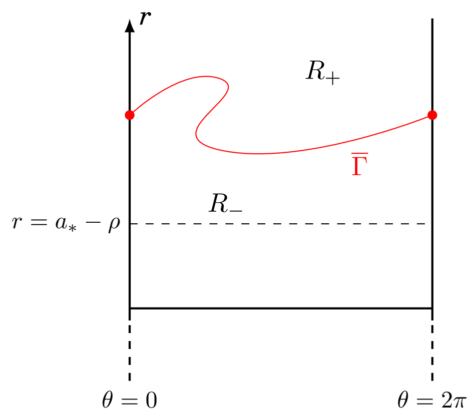

Figure 4: Splitting $(\mathbb{R}\times\mathbb{T})\setminus\overline{\Gamma}$ in connected components R_+, R_-.

Once we know that the intersection property holds we can go back to the estimate (20) in order to apply the *Invariant Curve Theorem* as stated in [12]. By taking a large enough we can assume that the map is in the conditions of the theorem in Section 32 of [12]. Then there exists an analytic and 2π-periodic function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ with $a \leq \psi(\theta) \leq a+1$ for each $\theta \in \mathbb{R}$ and such that Γ_ψ is invariant under $\mathcal{P}$; that is $\mathcal{P}(\Gamma_\psi) = \Gamma_\psi$. In this way, we deduce that the map $\overline{\mathcal{P}}$ has a family of invariant Jordan curves that are non-contractible and converge to $r = +\infty$.

4.6 Conclusion of the proof

We are ready to prove the stability of the singularity $x = y = 0$ for the original system (2). The Kelvin transform is a diffeomorphism of $\mathbb{R}^2 \setminus \{0\}$ sending neighborhoods of infinity into neighborhoods of the origin. Then we can transport solutions from (8) to (2) and so we know that the solutions of (2) are well defined in $[0,1]$ if the initial condition (x_0, y_0) belongs to some small punctured neighborhood of the origin. Let $\mathcal{P}_I$ be the Poincaré map associated to (2). In the previous section we found a family of invariant curves under $\mathcal{P}_{IV}$, the Poincaré map for (8).

The corresponding invariant curves under $\mathcal{P}_I$ are Jordan curves which are non-contractible in $\mathbb{R}^2 \setminus \{0\}$ and converge to the origin. The stability of the origin as a fixed point of $\mathcal{P}_I$ is now standard. According to Proposition 1 it remains to check that there is continuous dependence around the singularity $x = y = 0$. The successive changes of variables and the estimate in Lemma 1 and Lemma 2 lead to

$$x(t;x_0,y_0)^2 + y(t;x_0,y_0)^2 = \frac{1}{u(t)^2+v(t)^2} = \frac{1}{2r(t)}$$
$$\le \frac{1}{2\left(r_0 - \frac{K_1}{r_0^{1/2}}\right)} = \frac{x_0^2+y_0^2}{1-K_1\sqrt[3]{2(x_0^2+y_0^2)}}\,.$$

This inequality is valid if $(x_0, y_0) \neq (0,0)$ is small enough and $t \in [0,1]$. The continuous dependence of the origin follows.

5 Conclusions and final remarks.

This paper can be seen as a first step on the study of stable singularities in Hamiltonian systems, in analogy to stable equilibria. To this aim, we have used the basic point-vortex model as a canonical example of singularity. Our study of the complex valued version of the system is inspired by Morris approach in [7]. Most probably it is also possible to obtain similar results based only on real analysis. This should requires the use of a non-analytic version of the *Invariant Curve Theorem* together with long computations.

Of course, our initial standpoint admits several generalizations. In principle, it should not be too difficult to find similar results with a quasiperiodic perturbation by using the quasiperiodic version of twist theorem by Zharnitsky [14]. Extensions to 3D flows would be more involved because the Hamiltonian structure is lost, being necessary an extra dimension to recover it, or the use of alternative theorems for volume-preserving maps (see for instance [11]).

By using a stardard symbolic computation package, we have computed numerically the Poincaré map with different types of perturbations and it can be said that numerical experiments confirm the main result (see Fig. 5). However, the question if the sufficient condition of the fourth order of the perturbation is essential or not is unclear due to the limitations of standard numerical methods near the singularity. It remains as an intriguing an open problem.

A Appendix

This appendix is intended to present some technical estimates that are required for the proof of the main result. The first lemma states some bounds for functions $p \in \mathcal{H}_\varepsilon$ with a zero of order $N+1$.

Lemma 3. *Assume that $p \in \mathcal{H}_\varepsilon$ and the origin is a zero of order $N+1$. Then there exists a constant C, depending on $\|p\|_\infty$ and ε, such that*

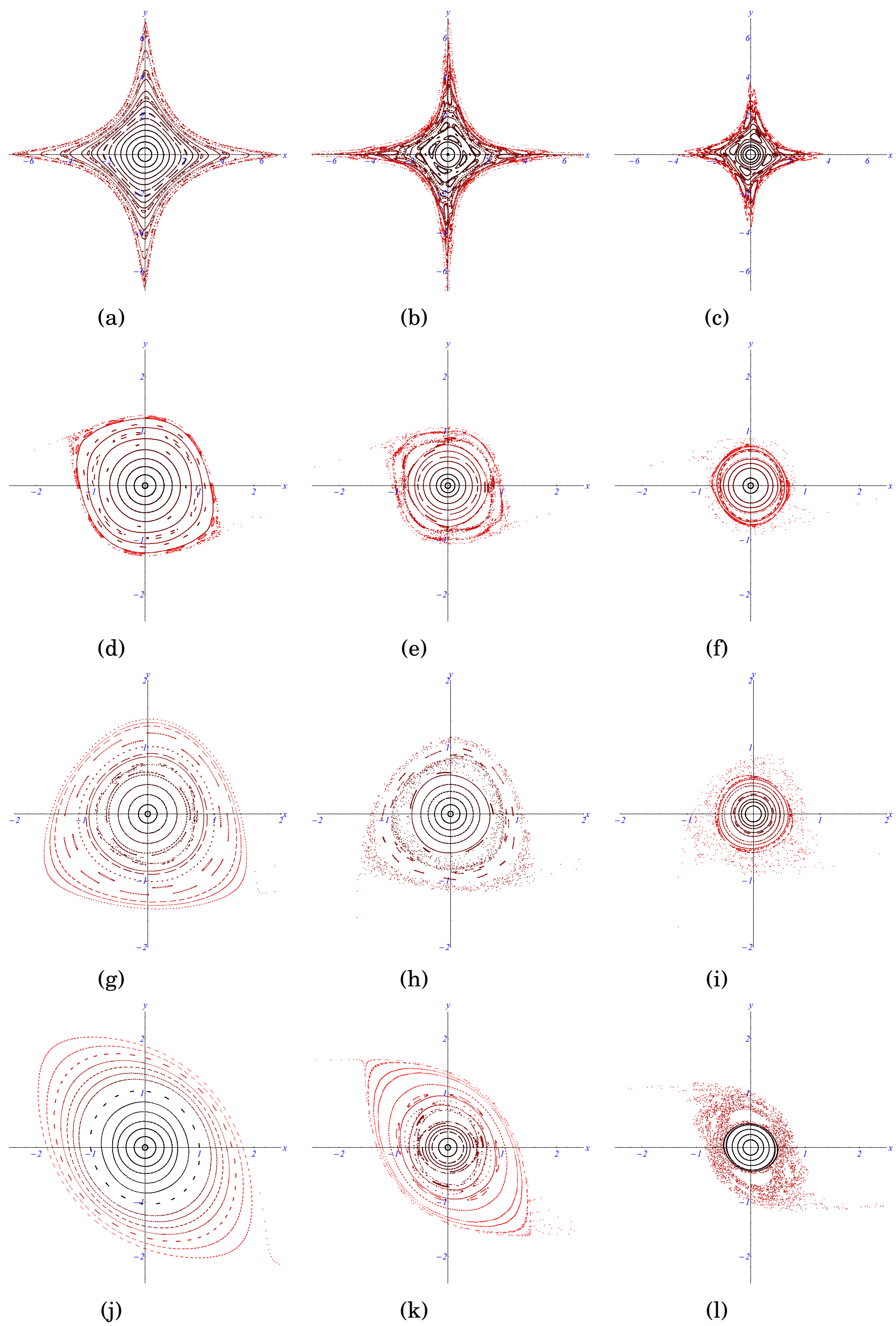

Figure 5: Poincaré sections of system (2) with two different choices of the perturbation $p(t, x, y)$. The first row corresponds to $p(t, x, y) = \epsilon(1 + \sin(2\pi t))x^2y^2$ with $\epsilon = 0.1, 0.4, 0.7$, respectively; the second corresponds to $p(t, x, y) = \epsilon(1 + \sin(2\pi t))x^3y$ with $\epsilon = 0.1, 0.2, 0.4$; the third one corresponds to $p(t, x, y) = \epsilon(1 + \sin(2\pi t))x^2y$ with $\epsilon = 0.1, 0.2, 0.4$ and the fourth row corresponds to $p(t, x, y) = \epsilon(1 + \sin(2\pi t))xy$ with $\epsilon = 0.1, 0.2, 0.4$.

I) $|p(t,x,y)| \le C\left(|x|^{N+1} + |y|^{N+1}\right)$.

II) $|\partial_x\, p(t,x,y)| + |\partial_y\, p(t,x,y)| \le C\left(|x|^{N} + |y|^{N}\right)$ if $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$.

Proof. From Cauchy estimates for functions of several complex variables it is easy to find an estimate

$$\left|\frac{\partial^{h+k}}{\partial\, x^h \partial\, y^k} p(t,x,y)\right| \le 2\frac{h!k!\, \|p\|_\infty}{(\varepsilon/2)^{h+k}} =: \widetilde{C}$$

valid if $h+k \le N+1$ and $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$.

The Taylor formula with integral remainder is valid for holomorphic functions (see Section 14 of Chapter VIII in [4]). Then, since the origin is a zero of order $N+1$, we have

$$p(t,x,y) = \left(\int_0^1 \frac{(1-\xi)^N}{N!} p^{(N+1)}(t,\xi x,\xi y)\,\mathrm{d}\xi\right)(x,y)^{(N+1)}$$

where $p^{(N+1)}(t,x,y)$ is the multilinear form associated to the derivative of order $N+1$ and $(x,y)^{(N+1)}$ stands for the repetition of (x,y) during $N+1$ times.

Then we can obtain the first estimate using the norm of a multilinear form. Note that C only depends on $\widetilde{C}$ and N. The second estimate can be obtained in the same way after applying Taylor formula to the functions $\partial_x\, p$ and $\partial_y\, p$. □

The second result is a lemma of Riemann-Lebesgue type.

Lemma 4. *Let $q(t,x,y)$ be a polynomial of degree N,*

$$q(t,x,y) = \sum_{j+h\le N} \alpha_{j,h}(t) x^j y^h$$

with $\alpha_{j,h}(t) \in \mathcal{C}^1(\mathbb{T})$. Assume in addition that for each t

$$\int_0^{2\pi} q(t,\cos\,\theta,\sin\,\theta)\,\mathrm{d}\theta = 0\,. \tag{21}$$

Let $\beta : [0,\tau] \longrightarrow \mathbb{C}$ be a $\mathcal{C}^1$ function defined on $[0,\tau] \subset [0,1]$ and $\Lambda > 0$. Then there exists $C_{RL} > 0$ such that

$$\left|\int_0^t q(s,\cos\,(\lambda s + \beta(s)),\sin\,(\lambda s + \beta(s)))\,\mathrm{d}s\right| \le \frac{C_{RL} e^{N\Lambda}}{|\lambda|}$$

if $t \in [0,\tau]$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\mathrm{Im}\ \lambda| < \Lambda$. Moreover, the constant C_{RL} only depends upon N, $\max_{j,h}\left[\|\alpha_{j,h}\|_\infty + \|\dot\alpha_{j,h}\|_\infty\right]$, $\|\beta\|_\infty$ and $\|\dot\beta\|_\infty$.

Proof. The function $q(t, \cos\theta, \sin\theta)$ has a finite Fourier expansion with respect to θ, say

$$q(t, \cos\theta, \sin\theta) = \sum_{|k|\le N} q_k(t) e^{ik\theta}.$$

The coefficients q_k can be expressed in terms of the functions $\alpha_{j,h}$ and belong to $\mathcal{C}^1(\mathbb{T})$,

$$q_k(t) = \frac{1}{2\pi}\int_0^{2\pi} q(t, \cos\theta, \sin\theta) e^{-ik\theta}\, \mathrm{d}\theta.$$

The condition (21) implies that $q_0(t)$ vanishes everywhere and so the integral $I(t)$ we want to estimate can be expressed as the sum

$$I(t) = \sum_{0<|k|\le N} I_k(t) \quad \text{with} \quad I_k(t) = \int_0^t q_k(s) e^{ik\beta(s)} e^{ik\lambda s}\, \mathrm{d}s.$$

Since we have excluded $k = 0$ these integrals can be estimated by a standard procedure in the theory of oscillatory integrals, see for instance [1]. After integrating by parts

$$I_k(t) = \frac{1}{ik\lambda}\Big[q_k(t) e^{ik\beta(t)} e^{ik\lambda t} - q_k(0) e^{ik\beta(0)} - \int_0^t \Big(q_k(s) e^{ik\beta(s)} \Big)' e^{ik\lambda s}\, \mathrm{d}s \Big].$$

Therefore, using that $1 \le e^{|k||\mathrm{Im}\ \lambda|\tau}$,

$$|I_k(t)| \le \frac{C_k}{|k|\,|\lambda|} e^{|k|\Lambda}$$

with

$$C_k = [2\, \|q_k\|_\infty + \|q_k'\|_\infty + |k|\, \|q_k\|_\infty\, \|\beta'\|_\infty]\, e^{|k|\|\beta\|_\infty}.$$

□

Remark. The condition (21) is essential. Consider the polynomial

$$q(t, x, y) = \sum_{j=1}^N \alpha_j(t)(x^2 + y^2)^j.$$

Then

$$I(t) = \sum_{j=1}^N \int_0^t \alpha_j(s)\, \mathrm{d}s$$

does not depend on λ and the conclusion of the lemma cannot hold.

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