A mechanical counterexample to KAM theory with low regularity

Stefano Marò

Dipartimento di Matematica - Università di Torino

Via Carlo Alberto 10, 10123 Torino - Italy

e-mail: stefano.maro@unito.it

Abstract

We give a mechanical example concerning the fact that some regularity is necessary in KAM theory. We consider the model given by the vertical bouncing motion of a ball on a periodically moving plate. Denoting with f the motion of the plate, some variants of Moser invariant curve theorem apply if $\dot{f}$ is small in norm C^5 and every motion has bounded velocity. This is not possible if the function f is only C^1. Indeed we construct a function $f \in C^1$ with arbitrary small derivative in norm C^0 for which a motion with unbounded velocity exists.

1 Introduction

Moser invariant curve theorem [6] is of fundamental importance to study the stability of the solutions of Hamiltonian systems [7, 11]. It deals with the existence of invariant curves for some diffeomorphisms of the cylinder that are "close" enough to an integrable twist map. More precisely, the map

$$(1) \qquad \begin{cases} \theta_1 = \theta_0 + \alpha(r_0) + R_1(\theta_0, r_0) \\ r_1 = r_0 + R_2(\theta_0, r_0) \end{cases}$$

with $\alpha' > 0$ has invariant curves if it possesses the intersection property and

$$(2) \qquad ||R_1||_{C^{333}(\mathcal{C})} + ||R_2||_{C^{333}(\mathcal{C})} < \epsilon$$

for ϵ sufficiently small. The request on the regularity was very strong and suddenly arose the question whether such regularity was in fact needed or not. Takens [12] gave a first counterexample in class C^1 and successively Herman [2] improved giving another counterexample in class $C^{3-\varepsilon}$ where ε is a small positive constant. Recently, Wang [13] proved a related result for Hamiltonian with d degrees of freedom. All these results aimed the search for optimal regularity in KAM theory and the perturbation is found between the whole class of symplectic diffeomorphisms or Hamiltonian flows.

Our purpose is different, indeed we are going to construct an example in class C^0 that comes from a mechanical model. The model describes the vertical motion of a bouncing ball on a moving plate. The plate is moving in the vertical direction as a 1-periodic function f and the gravity force is acting on the ball. Moreover we suppose that the bounces are elastic and do not affect the movement of the plate. This is a very simple mechanical model with interesting dynamics and has been considered by several authors. See [4, 1, 10, 3] and references therein for an insight. The motion of the ball can be described by an exact symplectic twist map that is close to the integrable twist map if the velocity is small [5]. So a direct application of Moser theorem shows that if the velocity of the plate $\dot{f}$ is small in norm C^{333} then invariant curves exist. It means that the velocity of the ball is always bounded. The smallness of $\dot{f}$ is essential for the boundedness of the velocity. Indeed Pustyl'nikov [9] proved that if $\dot{f}$ is sufficiently large then there exist motions of the ball with unbounded velocity. We are going to prove that some regularity is needed as well. Precisely given δ arbitrary small, we are going to construct a concrete function $f \in C^1(\mathbb{R}/\mathbb{Z})$ with $\sup|\dot{f}| \leq \delta$ such that the corresponding model admits a motion of the ball with unbounded velocity. So invariant curves cannot exist and Moser's theorem cannot hold in this context. Our function is not C^2 and this is consistent with Moser theorem leaving open the question on the optimal regularity for this model to have motions with bounded velocity. On this line we refer to the work of Zharnitky [14] to see a similar result on the Fermi-Ulam ping-pong model. Our idea of constructing the function is different to Zharnitky's one: we start from the result of Pustyl'nikov on the unbounded motion. He constructed an orbit that in the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ becomes a fixed point. Our idea will be to look at N-cycles in the same torus. This will lead to weaker conditions on the generation of unbounded orbits that, after some technical work, will allow to construct the function f and the corresponding unbounded orbit.

2 Statement of the problem

We are concerned with the problem studying the motion of a bouncing ball on a moving plate. We assume that the impacts do not affect the motion of the racket that is supposed to be described by a function $f \in C^1(\mathbb{R}/\mathbb{Z})$. The fact that both the liner momentum and the energy are preserved, allows to describe the motions through the following map

$$P_f : \begin{cases} t_1 = t_0 + \frac{2}{g} v_0 - \frac{2}{g} f[t_1, t_0] \\ v_1 = v_0 + 2\dot{f}(t_1) - 2f[t_1, t_0] \end{cases} \tag{3}$$

where

$$f[t_1, t_0] = \frac{f(t_1 - f(t_0))}{t_1 - t_0}.$$

Here the coordinate t represents the impact time. The coordinate v represents the velocity of the ball immediately after the impact. This is the formulation considered by Pustil'nikov in [9]. Another approach based on differential equations was considered by Kunze and Ortega [4] and leads to a map that is equivalent to (3) [5]. The map is implicit and is well defined for $v > \bar{v}$ for some $\bar{v}$ sufficiently large. Moreover, by the periodicity of the function f, the coordinate t can be seen as an angle. So the map P_f is defined on the half cylinder $\mathbb{T} \times (\bar{v}, +\infty)$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $f \in C^6$, one can consider the strip $\Sigma_a = \mathbb{T} \times [a, a+k]$ with $a > \bar{v}$ and k sufficiently large, and notice that a simple application of Moser invariant curve theorem [6] in the form [8] gives the existence of an invariant curve of P_f in Σ_a if

$$||\dot{f}||_{C^5[0,1]} \leq \delta \tag{4}$$

for some δ sufficiently small. Invariant curves act as barriers so that repeating the argument for $a \to +\infty$ one can prove that if condition (4) is satisfied then every orbit (t_n^*, v_n^*) of P_f is such that

$$\sup_{n \in \mathbb{Z}} v_n^* < \infty.$$

This result depends on the regularity of f. More precisely we shall prove the following result

Theorem 1. *For every $0 < \delta < \frac{g}{4}$ there exist $f \in C^1(\mathbb{R}/\mathbb{Z})$ and an initial condition (t_0^*, v_0^*) such that:*

1. $||\dot{f}||_{C^0[0,1]} \leq \delta$,

2. *the orbit of* P_f *with initial condition* (t_0^*, v_0^*) *satisfies*

$$t_{n+N}^* = t_n^* + \sigma_n, \quad \sigma_n \in \mathbb{N}$$

$$v_{n+N}^* = v_n^* + \frac{g}{2}V \quad \text{for some } V \in \mathbb{N} \setminus \{0\}$$

for every $n \in \mathbb{N}$.

3 Unbounded orbits

In this section we are going to construct unbounded orbits for the map P_f. We will obtain some intricate conditions that generalize Pustil'nikov result. The fundamental observation is that the map P_f shares some obits with a generalized standard map. More precisely, if $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$ is a complete orbit satisfying

$$f(t_n^*) = f(t_0^*) \quad \text{for every } n \in \mathbb{Z} \tag{5}$$

then $f[t_n^*, t_{n-1}^*] = 0$ for every $n \in \mathbb{Z}$ and $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$ becomes a complete orbit for the generalized standard map

$$GS : \begin{cases} t_1 = t_0 + \frac{2}{g}v_0 \\ v_1 = v_0 + 2\dot{f}(t_1) \end{cases} \tag{6}$$

Clearly the converse is also true, if $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$ is a complete orbit of GS with $v_n > \bar{v}$ for every n and satisfying condition (5) then it is also an orbit for P. This fact will be crucial in the following. We start constructing unbounded orbits for GS.

Lemma 1. *Let* $t_0^* < t_1^*$ *be real numbers and let* $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$ *be the orbit of the map* GS *with initial conditions* $t_0 = t_0^*, v_0 = v_0^* = g(t_1^* - t_0^*)/2$. *Suppose that there exist* $N, W, V \in \mathbb{N} \setminus \{0\}$ *such that*

1. $N(t_1^* - t_0^*) + \frac{4}{g}\sum_{k=1}^{N-1}(N-k)\dot{f}(t_k^*) = W$,

2. $\frac{4}{g}\sum_{k=0}^{N-1}\dot{f}(t_k^*) = V$,

Then

$$t^*_{n+N} = t^*_n + \sigma_n, \quad \sigma_n \in \mathbb{N}$$

$$v^*_{n+N} = v^*_n + \frac{g}{2}V.$$

Moreover, there exists $T > 0$ such that if $t^*_1 - t^*_0 > T$ then $v^*_n > \bar{v}$ for every $n \geq 0$.

Proof. Notice that from (6) we obtain the following expression for the n-th iterate:

$$v_n = v_0 + 2\sum_{k=1}^{n} \dot{f}(t_k) \tag{7}$$

$$t_n = t_0 + \frac{2}{g}nv_0 + \frac{4}{g}\sum_{k=1}^{n-1}(n-k)\dot{f}(t_k). \tag{8}$$

We claim that for every $j \in \mathbb{N}$, there exists $\sigma_j \in \mathbb{N}$ such that

$$t^*_{N+j} = t^*_j + \sigma_j. \tag{9}$$

Let us prove it by induction on j. The fact that $v^*_0 = g(t^*_1 - t^*_0)/2$ and the hypothesis, together with (8) give the first step for $j = 0$ with $\sigma_0 = W$. Notice that by periodicity we have also $\dot{f}(t^*_N) = \dot{f}(t^*_0)$.
Now suppose that $t^*_{N+i} = t^*_i + \sigma_i$ for every $i < j$. Using (6) we have

$$\begin{aligned} t^*_{N+j} &= t^*_{N+j-1} + \frac{2}{g}v^*_{N+j-1} = t^*_{j-1} + \sigma_{j-1} + \frac{2}{g}[v^*_{j-1} + 2\sum_{k=0}^{N-1}\dot{f}(t^*_{k+j})] = \\ &(t^*_{j-1} + \frac{2}{g}v^*_{j-1}) + \sigma_{j-1} + \frac{4}{g}\sum_{k=0}^{N-1}\dot{f}(t^*_{k+j}) = t^*_j + \sigma_{j-1} + \frac{4}{g}\sum_{k=0}^{N-1}\dot{f}(t^*_{k+j}). \end{aligned} \tag{10}$$

We just have to prove that the last term is an integer. Notice that for every k, there exist $d \in \mathbb{N}$ and $r \in \{0, \ldots, N-1\}$ such that $k + j = Nd + r$. Moreover, the fact that $k \in \{0, \ldots, N-1\}$ implies that $N(d-1) + r < j$. This allows to use the inductive hypothesis several times and get

$$t^*_{k+j} = t^*_{Nd+r} = t^*_{N+N(d-1)+r} = t^*_{N(d-1)+r} + \sigma_{N(d-1)+r} = \cdots = t^*_r + \sigma,$$

where $\sigma \in \mathbb{N}$. Moreover, from the definition, we have that r takes all the values in $\{0, \dots, N-1\}$ as k goes from 0 to $N-1$. Finally we have

$$\frac{4}{g}\sum_{k=0}^{N-1} \dot{f}(t^*_{k+j}) = \frac{4}{g}\sum_{r=0}^{N-1} \dot{f}(t^*_r) = V$$

and we conclude by hypothesis.
This allows us to write, from (7),

$$v^*_{N+n} = v^*_n + 2\sum_{k=n+1}^{n+N} \dot{f}(t^*_k) = v^*_n + 2\sum_{k=0}^{N-1} \dot{f}(t^*_k) = v^*_n + \frac{g}{2}V.$$

Finally, once more from (7) we have the last assertion remembering that $v^*_0 = g(t^*_1 - t^*_0)/2$ and $\dot{f}$ is bounded. $\square$

Remark 1. *This result has a well-known geometrical interpretation. The map GS satisfies*

$$GS(t_0 + 1, v_0) = GS(t_0, v_0) + (1, 0)$$

$$GS(t_0, v_0 + \frac{g}{2}) = GS(t_0, v_0) + (1, \frac{g}{2}).$$

It means that GS induces a map on the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\frac{g}{2}\mathbb{Z}$ and the orbit $(t^*_n, v^*_n)_{n\in\mathbb{Z}}$ becomes an N-cycle on this torus.

We shall use this lemma to find unbounded orbit for the original map P. This is the aim of the following

Proposition 1. *Consider a function $f \in C^1(\mathbb{R}/\mathbb{Z})$ and a sequence $(t^*_n)_{n\in\mathbb{N}}$. Suppose that there exist $N, W, V \in \mathbb{N}\setminus\{0\}$ such that*

1. $t^*_N - t^*_0 = W$,
2. $\frac{4}{g}\dot{f}(t^*_0) + (t^*_N - t^*_{N-1}) - (t^*_1 - t^*_0) = V$,
3. $f(t^*_0) = f(t^*_1) = \cdots = f(t^*_{N-1})$,
4. $\dot{f}(t^*_k) = \frac{g}{4}(t^*_{k+1} - 2t^*_k + t^*_{k-1})$ *for* $1 \le k \le N-1$.

Then if we define $v^*_{n+1} = v_n + 2\dot{f}(t_{n+1})$ *and* $v_0 = \frac{g(t_1 - t_0)}{2}$ *we have that there exists an orbit* $(\tau^*_n, \nu^*_n)_{n\in\mathbb{N}}$ *of* P_f *such that* $(\tau^*_n, \nu^*_n) = (t^*_n, v^*_n)$ *for* $0 \le n \le N$ *and*

$$\tau^*_{n+N} = \tau^*_n + \sigma_n, \quad \sigma_n \in \mathbb{N}$$

$$\nu^*_{n+N} = \nu^*_n + \frac{g}{2}V.$$

Moreover, there exists $T > 0$ *such that if* $t^*_1 - t^*_0 > T$ *then* $v^*_n > \bar{v}$ *for every* $n \ge 0$.

Proof. First of all it is not difficult to prove that condition 3. and 4. imply that (t^*_n, v^*_n) is a partial orbit of P_f for $0 \le n \le N$. Notice that we get the case $n = N$ using condition 1 and the periodicity of f.
So, to prove our result, it is sufficient to prove that hypothesis 1,2 and 4 allows to apply lemma 1. Indeed from hypothesis 3 and the generalized periodicity of the sequence (t_n) we have that condition (5) holds and we can repeat the discussion of the beginning of this section.

So let us start proving that hypothesis 2 and 4 allows to recover condition 2 in lemma 1. We just have to verify that

$$(t^*_N - t^*_{N-1}) - (t^*_1 - t^*_0) = \frac{4}{g}\sum_{k=1}^{N-1} \dot{f}(t^*_k)$$

and, remembering hypothesis 4 is sufficient to prove that

$$(t^*_N - t^*_{N-1}) - (t^*_1 - t^*_0) = \sum_{k=1}^{N-1} T_k. \tag{11}$$

Here, for brevity, we have denoted

$$T_k = t^*_{k+1} - 2t^*_k + t^*_{k-1}. \tag{12}$$

Let us prove (11) it by induction on N. It is easily verified the basic case $N = 1$. So suppose as induction hypothesis (11) to be true. Using it we have

$$\sum_{k=1}^{N} T_k = \sum_{k=1}^{N-1} T_k + t^*_{N+1} - 2t^*_N + t^*_{N-1} = (t^*_{N+1} - t^*_N) - (t^*_1 - t^*_0). \tag{13}$$

So (11) is proved and condition 2 of lemma 1 is recovered. To get condition 1 of lemma 1 notice that, from hypothesis 1 we have

$$\begin{aligned} W = t_N^* - t_0^* &= t_N^* - t_0^* + N(t_1^* - t_0^*) - N(t_1^* - t_0^*) \\ &= N(t_1^* - t_0^*) + (N-1)t_0^* - Nt_1^* + t_N^*. \end{aligned} \tag{14}$$

So, once again using hypothesis 4 we are done if we can prove that

$$(N-1)t_0^* - Nt_1^* + t_N^* = \sum_{k=1}^{N-1} [T_k(N-k)] \tag{15}$$

where T_k is defined by (12). Let us prove it by induction on N. It is easily verified the basic case $N = 1$. So suppose as induction hypothesis (15) to be true. Simple computations give

$$\begin{aligned} \sum_{k=1}^{N} [T_k(N+1-k)] &= \sum_{k=1}^{N-1} [T_k(N+1-k)] + T_N \\ &= \sum_{k=1}^{N-1} [T_k(N-k)] + \sum_{k=1}^{N-1} T_k + T_N. \end{aligned} \tag{16}$$

Using the inductive hypothesis and the definition of T_N we get

$$\sum_{k=1}^{N} [T_k(N+1-k)] = (N-1)t_0^* - Nt_1^* - t_N^* + t_{N+1}^* + t_{N-1}^* + \sum_{k=1}^{N-1} T_k.$$

Now we can use (11) and get

$$\sum_{k=1}^{N} [T_k(N+1-k)] = Nt_0^* - (N+1)t_1^* + t_{N+1}^*.$$

So we can recover also condition 1 in lemma 1 and conclude the proof. □

4 Proof of theorem 1

Proposition 1 says that we can decide if a finite sequence $(t_n, v_n)_{0 \le n < N}$ "generates" an unbounded orbit of P. We want to use it to produce unbounded orbits. The idea is to first construct the sequence and then construct the function f in such a way that proposition 1 is applicable. The next lemma deals with the construction of such sequence.

Lemma 2. *For every $\delta \in (0, \frac{g}{4})$ there exist three positive integers N, W, V and an increasing sequence $(t_n)_{0 \le n \le N}$ such that the following holds.*

1. $t_N - t_0 = W$,
2. $\frac{4}{g}\eta + (t_N - t_{N-1}) - (t_1 - t_0) = V$ *for some* $0 < \eta \le \delta$,
3. $\frac{g}{4}(t_{n+1} - 2t_n + t_{n-1}) = \delta$ *for* $1 \le n \le N-1$.

Proof. We construct the sequence (t_n) for $0 \le n \le N$ for some N to be fixed later. So let $t_0 = 0$ and t_1 positive to be fixed later. Define, for every $0 \le n \le N-1$ the increasing sequence

$$t_{n+1} = \frac{4}{g}\delta + 2t_n - t_{n-1} \tag{17}$$

so that condition 3. is satisfied with the equality. Now let us adjust the constants t_1, N, W, V and η in order to satisfy conditions 1. and 2. To do it let us start by noticing that letting $t_0 = 0$, the following formula holds for every $n \ge 0$ and $t_1 > 0$:

$$t_n = \frac{n(n-1)}{2}\frac{4}{g}\delta + nt_1. \tag{18}$$

We use it to rewrite condition 1. as

$$Nt_1 + N(N-1)\frac{2\delta}{g} = W \tag{19}$$

and condition 2. as

$$\frac{4}{g}\eta + (N-1)\frac{4}{g}\delta = V \tag{20}$$

Let us find $N, V, W \in \mathbb{N} \setminus \{0\}$, $t_1 > 0$ and $0 < \eta \le \delta$ such that (19) and (20) are satisfied. Let us start with (20). Fix $V = 1$ so that (20) is equivalent to

$$\eta = \frac{g}{4} - (N-1)\delta. \tag{21}$$

Let us impose $0 < \frac{g}{4} - (N-1)\delta \le \delta$ that is equivalent to

$$\frac{g}{4\delta} \le N < \frac{g}{4\delta} + 1.$$

It is sure that there exists $N > 1$ satisfying this condition. Using such N we can define η through (21). Now we can pass to (19). We have

$$t_1 = \frac{W}{N} - N(N-1)\frac{2\delta}{g}.$$

If we chose $W = N^2(N-1)$ we can conclude as

$$t_1 = N(N-1)(1 - \frac{2\delta}{g}) > 0.$$

□

The following proposition will allow to construct the function f.

Proposition 2. *Consider a pair of sequences* $(t_k)_{0\leq k\leq N}$ *and* $(D_k)_{1\leq k\leq N}$ *such that* $t_k \leq t_{k+1}$ *and* $0 \leq D_k \leq \delta$ *for some* $\delta > 0$. *Suppose that* $t_N - t_0 = W$ *for some* $W \in \mathbb{N}$ *and* $D_0 = D_N$. *Then there exists* $f \in C^1(\mathbb{R}/\mathbb{Z})$ *such that*

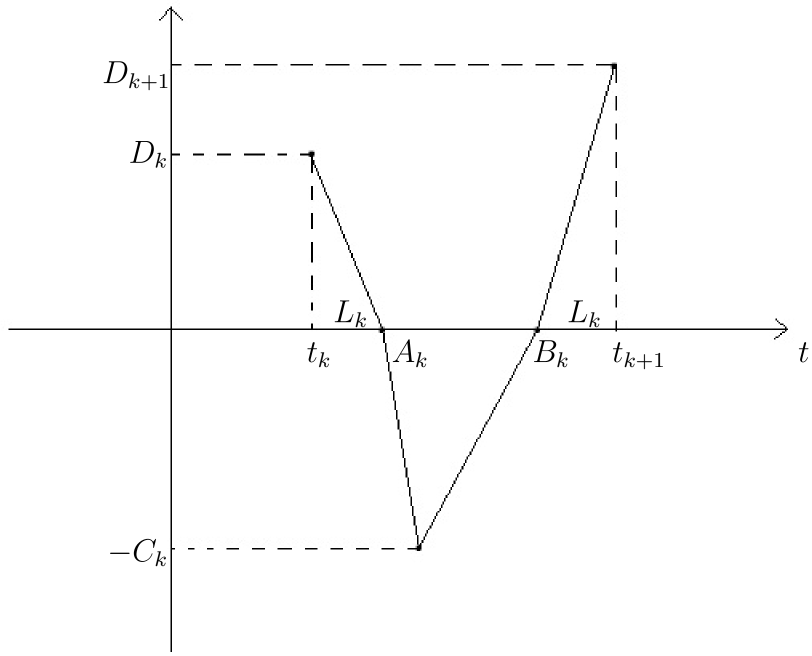
1. $f(t_0) = f(t_1) = \cdots = f(t_{N-1})$
2. $\dot{f}(t_k) = D_k$ *for* $1 \leq k \leq N$
3. $||\dot{f}||_{C^0[0,1]} \leq \delta$

Proof. To fix the ideas, suppose that $t_0 = 0$. Consider the new sequence $(t_k^1)_{0\leq k\leq N}$ defined as

$$\begin{cases} t_k^1 = t_k - [t_k] & \text{for } 0 \leq k \leq N-1 \\ t_N^1 = 1 \end{cases}$$

where $[x]$ represents the integer part of x. Morally, we are considering the sequence (t_k) modulo 1. Moreover, we can reorganize the sequence supposing it to be monotone non-decreasing. To be consistent we will reorganize also the sequence (D_k) following the permutation made on the sequence (t_k^1). Now for $t \in [0,1]$ consider the function $\zeta(t)$ being piecewise linear defined for $t_k^1 \leq t < t_{k+1}^1$, $0 \leq k < N$ as in figure. With reference to the figure, the points A_k and B_k are determined by the positive quantity $L_k < \frac{t_{k+1}-t_k}{2}$ and the constant C_k is such that $0 < C_k < \delta$. If we were able to get the signed area between t_k and t_{k+1} to be zero, we would get the thesis extending $\zeta(t)$ to the whole $\mathbb{R}$ by periodicity and letting

$$f(t) = \int_0^t \zeta(s)ds.$$



So let us see that it is possible to construct such function ζ finding suitable C_k and L_k. Instead of giving cumbersome computations, let us think geometrically with the use of the figure. The signed area between t_k and t_{k+1} is given by $L_k(D_k + D_{k+1}) - C_k(t_{k+1} - t_k - 2L_k)$. As we want it to be zero we get that

$$C_k = \frac{L_k(D_k + D_{k+1})}{t_{k+1} - t_k - 2L_k} > 0.$$

Remembering that we need $C_k < \delta$ we have that if we chose L_k such that

$$L_k < \frac{\delta(t_{k+1} - t_k)}{D_{k+1} + D_k + 2\delta}$$

we get the thesis. □

We are ready for the

Proof of theorem 1. Given δ, consider the sequence (t_k^*) coming from lemma 2 and the corresponding constants η and N. It comes from the proof that we have $t_0 = 0$ and $t_N = W \in \mathbb{N}$. Now consider the corresponding sequence (D_k) defined as

$$
\begin{aligned}
D_k =& \frac{g}{4}(t^*_{k+1} - 2t^*_k + t^*_{k-1}) \text{ for } 1 \leq k \leq N-1 \\
D_N =& D_0 = \eta.
\end{aligned} \tag{22}
$$

From condition 2 and 3 in lemma 2 we have

$$0 \leq D_k \leq \delta$$

for every $0 \leq k \leq N-1$. So we can apply proposition 2 to the sequences $(t^*_k)_{0 \leq k \leq N-1}$ and $(D_k)_{0 \leq k \leq N}$ to get the corresponding function $\bar{f}$. So consider the corresponding map $P_{\bar{f}}$

$$
\begin{cases}
t_1 = t_0 + \frac{2}{g} v_0 - \frac{2}{g} \bar{f}[t_1, t_0] \\
v_1 = v_0 + 2\dot{\bar{f}}(t_1) - 2\bar{f}[t_1, t_0].
\end{cases} \tag{23}
$$

Let (τ^*_k, ν^*_k) the orbit with initial condition

$$(t_0, v_0) = (t^*_0, \frac{g(t^*_1 - t^*_0)}{2}).$$

Remembering conditions 1 and 2 of proposition 2 we have that $(\tau_k) = (t^*_k)$ and the corresponding sequence (t^*_k, v^*_k) is an orbit of $P_{\bar{f}}$ satisfying the hypothesis of proposition 1. Condition 3 of lemma 2 concludes the proof. $\square$

References

[1] D. Dolgopyat. Bouncing balls in non-linear potentials. *Discrete and Continuous Dynamical Systems*, 22:165–182, 2008.

[2] M.R. Herman. Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1. *Astérisque*, 103-104, 1983.

[3] P. J. Holmes. The dynamics of repeated impacts with a sinusoidally vibrating table. *J. Sound Vibration*, 84:173–189, 1982.

[4] M. Kunze and R. Ortega. Complete Orbits for Twist Maps on the Plane: Extensions and Applications. *J Dyn Diff Equat*, 23:405–423, 2011.

[5] S. Marò. Coexistence of bounded and unbounded motions in a bouncing ball model. *Nonlinearity*, 26:1439–1448, 2013.

[6] J.K. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, pages 1–20, 1962.

[7] J.K. Moser. *Stable and random motions in dynamical systems.* Princeton University Press, Princeton, N. J., 1973. With special emphasis on celestial mechanics, Hermann Weyl Lectures, the Institute for Advanced Study, Princeton, N. J, Annals of Mathematics Studies, No. 77.

[8] R. Ortega. Invariant curves of mappings with averaged small twist. *Adv. Nonlinear Stud*, 1:14–39, 2001.

[9] L.D. Pustyl'nikov. Existence of a set of positive measure of oscillating motions in a certain problem of dynamics. *Soviet. Math. Dokl.*, 13:94–97, 1972.

[10] D. Qian and P.J. Torres. Periodic motions of linear impact oscillators via the successor map. *SIAM J. Math. Anal.*, 36:1707–1725, 2005.

[11] C. L. Siegel and J. K. Moser. *Lectures on celestial mechanics.* Springer-Verlag, New York, 1971. Translation by Charles I. Kalme, Die Grundlehren der mathematischen Wissenschaften, Band 187.

[12] F. Takens. A C^1 counterexample to Moser's twist theorem. *Nederl. Akad. Wetensch. Proc. Ser. A* **74**=*Indag. Math.*, 33:378–386, 1971.

[13] L. Wang. Total destruction of Lagrangian tori. *J. Math. Anal. Appl.*, 410:827–836, 2014.

[14] V. Zharnitsky. Instability in Fermi-Ulam "ping-pong" problem. *Nonlinearity*, 11(6):1481–1487, 1998.