Periodic and quasi-periodic motions of a relativistic particle under a central force field*

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Abstract

We consider a relativistic particle under the action of a time-periodic central force field in the plane. When it is attractive at a given level there are many subharmonic and quasiperiodic motions.

1 Introduction and main results

Vector fields of the form
$$x \neq 0 \mapsto f(|x|)\frac{x}{|x|}$$
are called 'central force fields' and have a great importance in Mechanics from the very beginning of this discipline in the seventeenth century. The force field is attractive if $f(r) < 0$ for every r and repulsive if the opposite inequality holds. For instance, letting $f(r) = c/r^2$ for some constant c one obtains, if $c < 0$, the gravitational force field created by a point mass fixed at the origin (known as the Kepler problem), which is attractive. If, on the contrary, the constant c is positive we get the Coulombian force field created by an electrical charged particle fixed at the origin when acting on a free charged particle of the same sign (magnetic interaction between the charges is not considered).

These force fields are autonomous, i.e., they depend on the position but not directly on time. However, already Newton [19] in his study of Kepler's second law considered the motion of a particle subjected to a periodic sequence of discrete time impulses. On the other hand, many problems involving gravitating bodies with variable mass have been considered in Celestial Mechanics, being the best known of them the Gylden-Meshcherskii problem, that can be regarded as a Kepler problem with variable masses
$$\ddot{x} = -M(t)\frac{x}{|x|^3},$$

*Partially supported by project MTM2011-23652, Ministerio de Economía e Innovación, Spain

where $M(t) = G(m_1(t)+m_2(t))$, G is the gravitational constant and $m_1(t), m_2(t)$ are the masses of the bodies. Originally, the Gylden-Meshcherskii problem was proposed to explain the secular acceleration observed in the Moon's longitude, but nowadays it is used to describe a variety of phenomena including the evolution of binary stars, dynamics of particles around pulsating stars and many others (see [3, 7, 20, 21] and the references therein). In a different line of research, the Newtonian motion of a particle under a central force field which may depend periodically on time has been recently studied by A. Fonda and coworkers [10, 11, 12].

When dealing with particles moving at speed close to that of light it may be important to take into account the relativistic effects. Relativistic Dynamics is theoretically founded in the context of Special Relativity (see for instance [13, Chapter 33]), and the relativistic Kepler or Coulomb problem has been considered in previous works [1, 4, 18]. However, it seems that more general non-autonomous central force fields in this context are rather unexplored. When the mass of our particle at rest and the speed of light are normalized to one, we are led to consider the following family of second-order systems in the plane:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,|x|)\frac{x}{|x|}\,, \qquad\qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

Here, $f : \mathbb{R}\times]0,+\infty[\to \mathbb{R}$, $f = f(t,r)$ is assumed to be continuous and T-periodic in the time variable t. Notice however that it may be singular at $r = 0$. Solutions of (1) are understood in a classical sense, i.e., a C^2 function $x : \mathbb{R} \to \mathbb{R}^2$ is a solution provided that

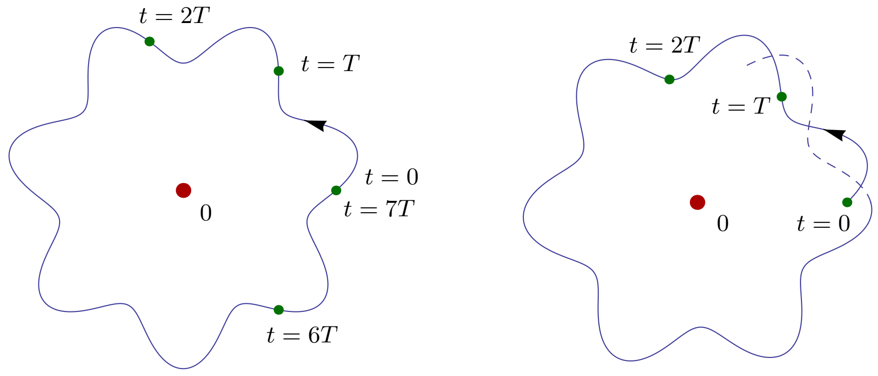
$$x(t) \neq 0, \qquad |\dot{x}(t)| < 1\,, \qquad\qquad t \in \mathbb{R},$$

and the equality (1) holds pointwise.

In this paper we shall be interested in a certain class of solutions including the T-periodic ones but also some subharmonic and quasi-periodic solutions. To introduce this class it will be convenient to use polar coordinates and rewrite (in complex notation) every continuous function $x : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\} \equiv \mathbb{C} \setminus \{0\}$ as $x(t) = r(t)e^{i\theta(t)}$, where $r(t) = |x(t)|$ and $\theta : \mathbb{R} \to \mathbb{R}$ is some continuous determination of the argument function along x. We shall say that x is T-*radially periodic* if $r(t)$ is T-periodic and there exists some number $\omega \in \mathbb{R}$ such that $\theta(t) - \omega t$ is T-periodic. In this case, the number $\omega = \frac{\theta(T)-\theta(0)}{T}$ can be interpreted on the average angular speed of x and will be denoted by $\omega = \operatorname{rot} x$.

For instance, the T-radially periodic function $x : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ is T- periodic if and only if $\operatorname{rot} x$ is an integer multiple of $2\pi/T$. If $\operatorname{rot} x = (m/n)\,(2\pi/T)$ for some relatively prime integers $m \neq 0 \neq n$ then x will be subharmonic with minimal period nT. Particularly significant is the case in which $m = 1$ and $\operatorname{rot} x = (1/n)\,(2\pi/T)$; then, our particle will complete a turn around the origin on each time interval $[0, nT]$. See Figure 1.

Finally, if $\frac{\operatorname{rot} x}{2\pi/T}$ is irrational (and $|x|$ is not constant) then x will not be periodic of any period and instead will be quasi-periodic with two frequencies $\omega_1 = \frac{2\pi}{T}$; $\omega_2 = \operatorname{rot} x$. This is easy to check, since $x(t) = r(t)e^{i\theta(t)}$ can



$\operatorname{rot}(x) = (1/7)(2\pi/T)$ $\qquad$ $\operatorname{rot}(x) \neq (1/n)(2\pi/T)$

Figure 1: The rotation number

be decomposed on the product of the T-periodic $r(t)e^{i(\theta(t)-\operatorname{rot}(x)t)}$ and the $2\pi/\operatorname{rot} x-$periodic $e^{i\operatorname{rot}(x)t}$.

When our force field is repulsive, i.e.,

$$f(t,r) > 0 \qquad \text{for } (t,r) \in \mathbb{R}\times]0,+\infty[,$$

then (1) does not have T-radially periodic solutions, as it can be easily checked by integrating on the time interval $[0,T]$ the scalar product of both sides of (1) with x. However, if our force field is autonomous and attractive at some level $r_* > 0$, i.e.

$$f(r_*) < 0\,,$$

then there is a solution of (1) rotating on the circumference of radius r_* at a constant angular speed. Indeed, straight-forward computations show that $x(t) = r_* e^{i\omega t}$ is a solution if and only if

$$|\omega| = \frac{\sqrt{2}}{r_*\sqrt{1+\sqrt{1+\left(\frac{2}{r_* f(r_*)}\right)^2}}}.$$

Our first main result asserts that when $f = f(t,r)$ is allowed to depend periodically on time, this circular solution leaves its place to a radially periodic solution.

Theorem 1 *Assume the existence of $r_* > 0$ such that $f(t,r_*) < 0$ for every time t. Then there exists some T-radially periodic solution $x_* = x_*(t)$ of system* (1) *with $\min_{t\in\mathbb{R}} |x_*(t)| = r_*$.*

Under the conditions of Theorem 1, system (1) actually has infinitely many T-radially periodic solutions, not only because its rotational symmetry implies that the rotated curve $e^{i\tau}x(t)$ is a solution whenever $x(t)$ is a solution (these solutions may be considered geometrically equal) but also because if the continuous force field is attractive at a given level then it is also attractive at other levels nearby. Actually, our proof will show the existence of continuous branches of geometrically different T-radially periodic solutions and it will lead us to the second main result of this paper.

Theorem 2 *Under the assumptions of Theorem 1 there exists $\omega_* > 0$ with the following property: for any $\omega \in]-\omega_*, \omega_*[\setminus\{0\}$ there is some T-radially periodic solution $x_\omega = x_\omega(t)$ of (1) with $\min_{t\in\mathbb{R}} |x_\omega(t)| \geq r_*$ and $\mathrm{rot}(x_\omega) = \omega$.*

In particular, taking $\omega = \frac{2\pi}{nT}$ for some big natural number n we find the existence of subharmonic solutions having a large multiple of T as its minimal period. On the other hand, letting $\omega = \frac{2\pi}{T}s$ for some irrational s we deduce the existence of an infinite number of quasi-periodic orbits of our equation.

We point out that Theorems 1-2 apply, for instance, to the relativistic version of the Gylden-Meshcherskii problem

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = -M(t)\frac{x}{|x|^3}\,, \qquad\qquad x \in \mathbb{R}^2 \setminus \{0\},$$

provided that the continuous function $M : \mathbb{R} \to \mathbb{R}$ is periodic and positive. Under this condition Theorem 1 ensures the existence, for every $r_* > 0$, of a T-radially periodic solution $x = x(t)$ with $\min_{t\in\mathbb{R}} |x(t)| = r_*$.

Theorems 1-2 resemble the results in [10, 11, 12] for the Newtonian case. However, there are also important differences between them. For instance, in [12, Theorem 1.1] the force field was assumed to be sublinear (and attractive) at infinity in order to avoid resonance, requirements which are not needed in this paper. On the other hand, in [12] the T-radially periodic solutions could be continued, not only over the region where the force fields is attractive, but all the way up to the origin. This property cannot be directly translated to the relativistic problem which occupies us here because the speed of light imposes bounds on the oscillation of solutions, and T-radially periodic solutions may not exist on some region where we do not have any assumption on f.

We also point out that Theorems 1-2 do not hold for Newtonian systems $\ddot{x} = f(t, |x|)\frac{x}{|x|}$, and therefore we are identifying a genuine relativistic effect. An example is provided by the Mathieu equation in the plane,

$$\ddot{x} = (\cos t - \lambda)x\,, \qquad x \in \mathbb{R}^2\,,$$

where $\lambda \in \mathbb{R}$ is a parameter. Then $x = x(t)$ is a solution if and only if each of its components solves the corresponding one-dimensional Mathieu equation. It is well-known (see Theorems 1.3.1, 2.3.1 and 2.5.1 of [8]) that there exists an infinite sequence $I_1, I_2, I_3, ...$ of pairwise disjoint open intervals (usually called *intervals of instability*), none of which is empty, and such that all solutions of the Mathieu equation are unbounded whenever λ belongs to any of these intervals. This sequence of intervals is divergent in the sense that $\sup I_n \to +\infty$ as $n \to +\infty$; in particular there are numbers $\lambda > 1$ in the set $\cup_n I_n$. For such values λ the force field is attractive but there are no T-radially periodic solutions.

Finally, we notice that Theorems 1-2 also become false if our particle is restricted to move on a line instead of the plane. In this case, (1) becomes

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-|\dot{r}|^2}}\right) = f(t,r)\,, \qquad r > 0\,.$$

If the force field is globally attractive, integration of both sides of the equation shows that no solution can be periodic.

This paper is structured as follows. In Section 2 the second order system is written as a suitable first order system with a more convenient structure. Section 3 takes advantage of such structure to find a-priori bounds for the eventual solutions. That, in turn, will enable us to construct a modified problem sharing some solutions with the original one and use Leray-Schauder degree continuation arguments; this will be done in Section 4. We refer to [6, 15] for the definition and fundamental properties of Leray-Schauder topological degree.

2 From a second order system to a first order system in the plane

Through this paper the function $f : \mathbb{R}\times]0,+\infty[\to \mathbb{R}$ will always be assumed T-periodic in time and continuous. We start with the following observation: the radial symmetry of equation (1) can be exploited to reduce its order. Indeed, introducing polar coordinates $x(t) = r(t)e^{i\theta(t)}$ in (1) we arrive to the first order system

$$\begin{cases} \frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}}\right) - \frac{r\dot{\theta}^2}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}} = f(t,r), \\ \frac{\dot{r}\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}} + \frac{d}{dt}\left(\frac{r\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}}\right) = 0. \end{cases} \tag{2}$$

Multiplying the second equation by r we see that

$$\frac{d}{dt}\left(\frac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}}\right) = 0\,.$$

This is a conservation law: if $x(t) = r(t)e^{i\theta(t)}$ is a solution of (1), then

$$\mu = \frac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}}$$

does not depend on t. We can get a physical insight of this quantity by identifying $\mathbb{R}^2$ with the coordinate plane $\{z = 0\}$ of $\mathbb{R}^3$ and observing that

$$\left(\frac{1}{\sqrt{1-|\dot{x}(t)|^2}}\right) x(t) \wedge \dot{x}(t) = \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix}.$$

The scalar factor $1/\sqrt{1-|\dot{x}(t)|^2}$ being the relativistic mass of our particle, we may call μ the *angular momentum* of the solution $x(t)$. At this moment it is convenient to introduce the relativistic *linear momentum* as a new dependent variable:

$$p = \frac{\langle x, \dot{x}\rangle}{\sqrt{1-|\dot{x}|^2}} = \frac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}}.$$

The (nonlinear algebraic) system

$$\frac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}} = \mu, \qquad \frac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}} = p\,,$$

can be easily solved in the variables $\dot{r}, \dot{\theta}$, to get

$$\dot{r} = \frac{rp}{\sqrt{\mu^2+r^2+r^2p^2}}\,, \qquad \dot{\theta} = \frac{\mu}{r\sqrt{\mu^2+r^2+r^2p^2}}\,. \tag{3}$$

Thus, the second term on the left side of the first equation of (2) reads

$$\frac{r\dot{\theta}^2}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}} = \left(\frac{\dot{\theta}}{r}\right)\mu = \frac{\mu^2}{r^2\sqrt{\mu^2+r^2+r^2p^2}}\,.$$

Combining the first equations of systems (2), (3) one arrives to a first order system defined on the half plane $\{r > 0\}$ and depending on the parameter μ:

$$\dot{r} = \frac{rp}{\sqrt{\mu^2+r^2+r^2p^2}}\,, \qquad \dot{p} = \frac{\mu^2}{r^2\sqrt{\mu^2+r^2+r^2p^2}} + f(t,r)\,. \tag{HS}$$

An important aspect to notice now is that this process is reversible, i.e., if $(r(t), p(t); \mu)$ is a solution to (HS) and we let $\theta = \theta(t)$ be any primitive of $\dfrac{\mu}{r(t)\sqrt{\mu^2+r^2(t)+r^2(t)p^2(t)}}$, then $x(t) = r(t)e^{i\theta(t)}$ is a solution of (1) with angular momentum μ (different choices of this primitive correspond to the solution $x(t)$ being rotated). Furthermore, $x(t)$ is T-radially periodic if and only if $r(t)$ and $p(t)$ are both T-periodic, and the rotation number can be computed from r, p and μ:

$$\mathrm{rot}(r, p; \mu) = \frac{\mu}{T}\int_0^T \frac{dt}{r(t)\sqrt{\mu^2+r^2(t)+r^2(t)p^2(t)}}\,. \tag{4}$$

Thus, finding T-radially periodic solutions $x(t)$ of (1) becomes equivalent to finding T-periodic solutions $(r(t), p(t); \mu)$ of (HS). This fact will allow us to use system (HS) in order to study equation (1).

3 Some a-priori bounds

In Relativistic Mechanics, particles cannot travel faster than light (which in our model is assumed to be 1). This basic physical principle implies the existence of bounds on the variation of T-radially periodic solutions of (1), both in the angular and the radial components. More precisely:

Lemma 1 *Let $(r,p;\mu)$ be a T-periodic solution of (HS). Then,*

(a) $\max_{t\in\mathbb{R}} r(t) - \min_{t\in\mathbb{R}} r(t) < T$.

(b) $|\operatorname{rot}(r,p;\mu)| \leq \dfrac{1}{\min_{t\in\mathbb{R}} r(t)}$.

Proof. The first equation of (HS) immediately implies that $|\dot r(t)| < 1$ for any $t \in \mathbb{R}$, from where (a) follows. On the other hand, (4) gives

$$|\operatorname{rot}(r,p;\mu)| \leq \frac{1}{T}\int_0^T \frac{dt}{r(t)} \leq \frac{1}{\min_{t\in\mathbb{R}} r(t)}\,,$$

showing (b).

■

Fix now constants $0 < a < b$. It will be convenient to consider the set

$$\Sigma(a,b) := \Big\{T\text{-periodic solutions } (r,p;\mu) \text{ of (HS) with } a \leq r(t) \leq b\ \forall t \in \mathbb{R}\Big\}\,. \tag{5}$$

After identifying solutions of (HS) and (1) this is just the set of T-radially periodic solutions living on the closed annulus $\{x \in \mathbb{R}^2 : a \leq |x| \leq b\}$. The definition immediately implies that the set $\{r : (r,p;\mu) \in \Sigma(a,b)\}$ is uniformly bounded; in the result below we show the boundedness of $\Sigma(a,b)$ in the remaining components:

Lemma 2 *There are constants $P, M > 0$ (depending only on f, a, b) such that*

$$|p(t)| < P \quad \forall t \in \mathbb{R}\,, \qquad\qquad |\mu| < M\,,$$

for every $(r,p;\mu) \in \Sigma(a,b)$.

Proof. We first check the existence of uniform bounds for the linear momentum p. With this aim we choose any constant

$$P > \max\left\{0, -T \min_{(t,r)\in\mathbb{R}\times[a,b]} f(t,r)\right\}\,. \tag{6}$$

We assume that $\Sigma(a,b) \neq \emptyset$ and choose some element $(r,p;\mu) \in \Sigma(a,b)$. The second equation of (HS) implies

$$\dot p(t) > -P/T\,, \qquad\qquad t \in \mathbb{R}\,.$$

Such a one-sided estimation implies bounds for the total oscillation of p. This can be shown by choosing times $t_0 < t_1 < t_0+T$ with $p(t_0) = \max\{p(t) : t \in \mathbb{R}\}$ and $p(t_1) = \min\{p(t) : t \in \mathbb{R}\}$; then,

$$\max_{t\in\mathbb{R}} p(t) - \min_{t\in\mathbb{R}} p(t) = -\int_{t_0}^{t_1} \dot{p}(t)dt \le (P/T)(t_1 - t_0) < P\,.$$

On the other hand, r being periodic, it follows from the first equation of (HS) that p vanishes somewhere. Thus, $\max_{t\in\mathbb{R}} p(t) \ge 0 \ge \min_{t\in\mathbb{R}} p(t)$ and $|\max_{t\in\mathbb{R}} p(t)|, |\min_{t\in\mathbb{R}} p(t)| < P$. It proves the part of the statement concerning the uniform bounds for p.

With the aim of finding bounds for the angular momentum μ we fix some constant P as in (6) and pick some element $(r, p; \mu) \in \Sigma(a, b)$. Then,

$$r(t)^2 \le b^2\,, \quad p(t)^2 \le P^2\,, \qquad\qquad t \in \mathbb{R}\,,$$

and from the second equation of (HS) we deduce

$$\dot{p}(t) \ge \frac{\mu^2}{b^2\sqrt{\mu^2 + b^2 + b^2P^2}} - P/T\,, \qquad\qquad t \in \mathbb{R}\,.$$

The constants b, P, T being fixed, the right-hand side above diverges to $+\infty$ as $|\mu| \to \infty$; in particular, it becomes positive for $|\mu|$ big enough. However p is periodic and consequently its derivative should vanish somewhere. It provides the desired bounds for $|\mu|$ and concludes the proof. ■

The argument above does not only show the existence of the bounds P, M, but actually may be used to obtain explicit formulas for them. Being given by (6) the expression for P is particularly simple; this fact will be used later in the paper.

The main assumption of Theorems 1-2 is the attractiveness of our force field at a given level r_*, i.e.

$$f(t, r_*) < 0 \text{ for any } t \in \mathbb{R}. \tag{7}$$

Under this condition one easily checks that no T-periodic solution $(r, p; \mu)$ of (HS) with angular momentum $\mu = 0$ can satisfy $\min_{t\in\mathbb{R}} r(t) = r_*$. Remembering (4) we may equivalently reformulate this fact as: $\mathrm{rot}(r, p; \mu) > 0$ for any T-periodic solution $(r, p; \mu)$ of (HS) with $\min_{t\in\mathbb{R}} r(t) = r_*$. It suggests a partial converse of Lemma 1(b) and the second assertion of Lemma 2 on the existence of nonzero lower bounds for the rotation number of these solutions. This is the content of the following result:

Lemma 3 *Assume (7). Then there exists some $\omega_* > 0$ (depending only on r_* and f) such that $|\mathrm{rot}(r, p; \mu)| \ge \omega_*$ for any T-periodic solution $(r, p; \mu)$ of (HS) with $\min_{t\in\mathbb{R}} |x(t)| = r_*$.*

Proof. In view of (4), Lemma 1(a) and the first assertion of Lemma 2 (take $a = r_*, b = r_* + T$) we may equivalently show the existence of some $m > 0$ such that $|\mu| \geq m$ for any $(r, p; \mu) \in \Sigma(r_*, r_* + T)$ with $\min_{t\in\mathbb{R}} r(t) = r_*$. We define

$$m := -r_*^2 \max_{t\in\mathbb{R}} f(t, r_*) > 0 \,.$$

We use a contradiction argument and assume instead the existence of some $(r, p; \mu) \in \Sigma(r_*, r_* + T)$ with $\min_{t\in\mathbb{R}} r(t) = r_* = r(t_*)$ for some $t_* \in \mathbb{R}$ and $-m < \mu < m$. Thus $\dot{r}(t_*) = 0$, and the second equation of (HS) implies

$$\dot{p}(t_*) \leq \frac{|\mu|}{r_*^2} + f(t_*, r_*) < \frac{m}{r_*^2} + f(t_*, r_*) \leq 0 \,.$$

We deduce the existence of some $\epsilon > 0$ such that $p(t) > 0$ if $t_* - \epsilon < t < t_*$ and $p(t) < 0$ if $t_* < t < t_* + \epsilon$. The first equation of system (HS) then implies

$$\dot{r}(t) > 0 \text{ if } t_* - \epsilon < t < t_* \,, \qquad\qquad \dot{r}(t) < 0 \text{ if } t_* < t < t_* + \epsilon \,,$$

contradicting the fact that r attains its minimum at r_*. It concludes the proof. ■

We point out that the comparison of this result with Lemma 1(b) implies that the constant ω_* cannot be greater than $1/r_*$. Before closing this Section we consider the opposite case in which our force field is repulsive at some level $\ell > 0$, i.e.,

$$f(t, \ell) > 0 \text{ for any } t \in \mathbb{R}. \tag{8}$$

Lemma 4 *Under this condition,* $\max_{t\in\mathbb{R}} r(t) \neq \ell$ *for any* T*-periodic solution* $(r, p; \mu)$ *of (HS).*

Proof. Using a contradiction argument we assume that $(r, p; \mu)$ is a T-periodic solution of (HS) with $\max_{t\in\mathbb{R}} r(t) = r(t_0) = \ell$ for some $t_0 \in \mathbb{R}$. Then $\dot{r}(t_0) = 0$, and the first equation of (HS) implies $p(t_0) = 0$. On the other hand, from the second equation of the system we see that $\dot{p}(t_0) > 0$ and we deduce the existence of some $\epsilon > 0$ such that $p(t) > 0$ if $t_0 < t < t_0 + \epsilon$. Going back to the first equation, $\dot{r}(t) > 0$ on $]t_0, t_0 + \epsilon[$, which is not possible since r attains its global maximum at t_0. The proof is complete. ■

4 Continuation of solutions with nonzero degree

Theorems 1-2 were formulated in terms of equation (1). However, the translation to system (HS) is straightforward. The equivalent versions read:

Theorem 1bis. Assume the existence of $r_* > 0$ such that $f(t, r_*) < 0$ for every time t. Then there exists some T-periodic solution $(r, p; \mu)$ of (HS) with $\min_{t\in\mathbb{R}} r(t) = r_*$.

Theorem 2bis. Under the assumptions of Theorem 1bis above there exists $\omega_* > 0$ with the following property: for any $\omega \in]-\omega_*, \omega_*[\setminus\{0\}$ there is some T-periodic solution $(r_\omega, p_\omega; \mu_\omega)$ of (HS) with $\min_{t\in\mathbb{R}} r_\omega(t) \geq r_*$ and $\mathrm{rot}(x_\omega, p_\omega; \mu_\omega) = \omega$.

This Section is devoted to prove these results. A suitable functional framework for our problem is provided by the Banach space $Y := C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$ of continuous and T-periodic functions with values on the plane. At some moments it will be convenient to represent the elements $y \in Y$ by their components $y = (r, p)$, where $r, p \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$. For instance, the natural domain of the system (HS) is the set

$$\Omega := \Big\{(r, p; \mu) \in Y \times \mathbb{R} : r(t) > 0 \ \forall t \in \mathbb{R}\Big\},$$

and further examples of subsets of $Y \times \mathbb{R}$ are the sets $\Sigma(a, b)$ which we already defined in (5).

A key step towards the proof of Theorems 1bis-2bis is the result which occupies us now. Under the assumption that our vector field is attractive and repulsive at two different levels we obtain connected families of solutions lying between them:

Proposition 1 *Assume the existence of levels $0 < r_* < \ell$ with (7)-(8). Then there is a connected set $\mathcal{C} \subset \Sigma(r_*, \ell)$ with*

$$\mathcal{C} \cap (Y \times \{0\}) \neq \emptyset \neq \mathcal{C} \cap \Big\{(r, p; \mu) \in \Omega : \min_{t\in\mathbb{R}} r(t) = r_*\Big\}. \tag{9}$$

Here, the adjective 'connected' refers to the inherited topology from $Y \times \mathbb{R}$. We postpone the proof of Proposition 1 to the end of the paper; at this moment let us see how it can be used to obtain Theorems 1bis-2bis.

Proof of Theorem 1bis. Choose some constant $\ell > r_* + T + 1$ and fix some continuous and 2π-periodic in time function $\tilde{f} : \mathbb{R}\times]0 + \infty[\to \mathbb{R}$, $\tilde{f} = \tilde{f}(t, r)$, satisfying

$$\tilde{f}(t, r) = f(t, r) \text{ if } 0 < r \leq \ell - 1\,, \qquad \tilde{f}(t, r) > 0 \text{ if } r \geq \ell\,. \tag{10}$$

We apply Proposition 1 to the Hamiltonian system $(\widetilde{\mathrm{HS}})$ obtained after replacing f by $\tilde{f}$ in (HS). We obtain in particular the existence of a solution $(r, p; \mu)$ of $(\widetilde{\mathrm{HS}})$ with $\min_{t\in\mathbb{R}} r(t) = r_*$. Lemma 1(a) states that $\max_{t\in\mathbb{R}} r(t) < r_* + T < \ell - 1$ and in view of the first part of (10) we deduce that $(r, p; \mu)$ is a solution of (HS). It completes the proof. ∎

It is clear from (HS) that whenever $(r, p; \mu)$ is a solution of this system $(r, p; -\mu)$ is another one; furthermore, $\mathrm{rot}(r, p; -\mu) = -\mathrm{rot}(r, p; \mu)$. This fact, which has its roots in the complex conjugate $\bar{x}(t) = r(t)e^{-i\theta(t)}$ of any solution $x(t) = r(t)e^{i\theta(t)}$ of (1) being again a solution, has the following consequence: in order to establish Theorem 2 it suffices to check the existence of some $\omega_* > 0$

with the property that for any $\omega \in]0, \omega_*[$ there is some T-periodic solution $(r, p; \mu)$ of (HS) with $\min_{t\in\mathbb{R}} r(t) \geq r_*$ and $|\operatorname{rot}(r, p; \mu)| = \omega$. It will be our goal next.

Proof of Theorem 2bis. Choose some constant $\ell > r_* + T + 1$, fix $\tilde{f}$ as in (10) and consider the associated Hamiltonian system ($\widetilde{\text{HS}}$). Applying again Proposition 1 we deduce the existence of a connected set $\mathcal{C} \subset \Omega$ of solutions of ($\widetilde{\text{HS}}$) with

$$r_* \leq r(t) \leq \ell \quad \text{for any } t \in \mathbb{R} \text{ and } (r, p; \mu) \in \mathcal{C}\,,$$

and satisfying (9). Lemma 1(a) implies that all elements of

$$\Gamma := \{(r, p; \mu) \in \mathcal{C} : \min_{t\in\mathbb{R}} r(t) \leq \ell - T - 1\}$$

are solutions of (HS). Choose now some constant $\omega_* > 0$ as given by Lemma 3. By (9),

$$\{|\operatorname{rot}(r, p; \mu)| : (r, p; \mu) \in \mathcal{C}\} \supset [0, \omega_*]\,,$$

while, by lemma 1(b),

$$\Big\{|\operatorname{rot}(r, p; \mu)| : (r, p; \mu) \in \mathcal{C}\backslash\Gamma\Big\} \subset \left[0, \frac{1}{\ell - T - 1}\right]\,,$$

and we deduce that

$$\Big\{|\operatorname{rot}(r, p; \mu)| : (r, p; \mu) \in \Gamma\Big\} \supset \left[\frac{1}{\ell - T - 1}, \omega_*\right]\,.$$

Thus, for any $\omega \in [1/(\ell - T - 1), \omega_*]$ there is some solution $(r, p; \mu)$ of (HS) with $\min_{t\in\mathbb{R}} r(t) \geq r_*$ and $|\operatorname{rot}(r, p; \mu)| = \omega$. The result follows because here ℓ is any number greater than $r_* + T + 1$. ■

At this moment it only remains to show Proposition 1. With this aim we first rewrite system (HS) in an abstract form. The two-dimensional subspace of Y made of constant functions is naturally identified with $\mathbb{R}^2$; we use this identification to construct two linear projections $\Pi, Q : Y \to Y$ on this subspace:

$$\Pi y := y(0) = y(T)\,, \qquad\qquad Qy := \frac{1}{T}\int_0^T y(t)dt\,.$$

For any $y \in \ker Q$ we denote by Ky to the primitive of y vanishing at times $t = 0, T$, and the linear operator $K : \ker Q \to \ker \Pi$ defined in this way is compact. The Nemytskii operator $N : \Omega \to Y$ is defined by

$$N[r, p; \mu] := \left(\frac{rp}{\sqrt{\mu^2 + r^2 + r^2p^2}}\,,\ \frac{\mu^2}{r^2\sqrt{\mu^2 + r^2 + r^2p^2}} + f(t, r)\right)\,.$$

For each value of the parameter μ, system (HS) may now be rewritten as a fixed point equation on Ω:

$$y = F[y; \mu]\,,$$

the (nonlinear) operator $F : \Omega \to Y$ being given by

$$F[y;\mu] := \Pi y + QN[y;\mu] + K(I-Q)N[y;\mu]\,,$$

(we denote by I to the identity map on Y). We point out that F is completely continuous, i.e., it is continuous and maps bounded subsets of $Y \times \mathbb{R}$ whose closure is contained in Ω into relatively compact subsets of Y.

The expression of N is particularly simple when $\mu = 0$; indeed

$$N[r,p;0] := \left(\frac{p}{\sqrt{1+p^2}}\,,\ f(t,r) \right)\,,$$

and one easily checks that (r,p) is a fixed point of $F[\cdot;0]$ if and only if r is a solution of the one-dimensional equation

$$\frac{d}{dt}\left(\frac{\dot r}{\sqrt{1-\dot r^2}}\right) = f(t,r)\,, \qquad\qquad r > 0\,, \tag{11}$$

and $p = \dot r/\sqrt{1-\dot r^2}$. Assumptions (7)-(8) may be reinterpretated now by saying that $r(t) \equiv r_*$ and $r(t) \equiv \ell$ are, respectively, lower and upper solutions for the periodic problem associated to (11). We choose some constant P as in (6) with $a = r_*$, $b = \ell$, and the first assertion of Lemma 2 implies that every fixed point (r,p) of $F[\cdot,0]$ with $r_* < r(t) < \ell$ for every $t \in \mathbb{R}$ belongs to the open bounded set

$$U := \Big\{ (r,p) \in Y : r_* < r(t) < \ell,\ |p(t)| < P\ \forall t \in \mathbb{R} \Big\}\,.$$

Before going into the proof of Proposition 1 we compute the Leray-Schauder fixed-point degree of $F[\cdot,0]$ on U. This will open the door to the use of continuation techniques.

Lemma 5 *Assume the conditions of Proposition 1 and let the set U be chosen as above. Then $F[\cdot,0]$ does not have fixed points on ∂U and*

$$\deg_{LS}\big(I - F[\cdot,0], U, 0\big) = -1\,.$$

Proof. We first check that $F[y,0] \neq y$ for any $y \in \partial U$. Indeed, this set can be divided into three (nondisjoint) subsets: the set of elements $(r,p) \in Y$ such that $\min_{t\in\mathbb{R}} r(t) = r_*$, the set of elements $(r,p) \in Y$ for which $\max_{t\in\mathbb{R}} r(t) = \ell$, and the set of elements $\{(r,p) \in Y$ satisfying that $\max_{t\in\mathbb{R}} |p(t)| = P$. The fact that $F[0,\cdot]$ does not have fixed points on each of them is a direct consequence, respectively, of Lemma 3, Lemma 4 and the choice of P.

In order to check the statement on the degree we consider the 'homotopy to the averaged nonlinearity':

$$H : [0,1] \times \bar U \to Y\,, \qquad\qquad H[\lambda;y] := \Pi y + QN[y;0] + K(I-Q)N_\lambda[y]\,,$$

where $N_\lambda[r,p] := \big(p/\sqrt{1+p^2}, (1-\lambda)f(t,r)\big)$. Observe that H is completely continuous and $H[0;y] = F[y;0]$. On the other hand, $H[\lambda;r,p] = (r,p)$ if and only if it solves the system

$$\dot{r} = \frac{p}{\sqrt{1+p^2}}\,, \qquad \dot{p} = (1-\lambda)f(t,r) + \frac{\lambda}{T}\int_0^T f(s,r(s))ds\,. \tag{12}$$

Integrating both sides of the second equation on the time interval $[0,T]$ we see that $\int_0^T f(t,r(t))dt = 0$. Thus, any solution $(\lambda; r, p)$ of (12) must also satisfy

$$\dot{r} = \frac{p}{\sqrt{1+p^2}}\,, \qquad \dot{p} = (1-\lambda)f(t,r)\,.$$

A similar argument to the one already used to show that $F[\cdot;0]$ does not have fixed points on ∂U proves now that $H[\lambda;y] \neq y$ for any $\lambda \in [0,1[$ and $y \in \partial U$. When $\lambda = 1$ system (12) becomes

$$\dot{r} = \frac{p}{\sqrt{1+p^2}}\,, \qquad \dot{p} = \frac{1}{T}\int_0^T f(s,r(s))ds\,,$$

and we see that $H[1,\cdot]$ does not have fixed points on ∂U either. It means that H is an admissible homotopy and

$$\deg_{LS}\big(I - F[\cdot;0], U, 0\big) = \deg_{LS}\big(I - H[0;\cdot], U, 0\big) = \deg_{LS}\big(I - H[1;\cdot], U, 0\big)\,. \tag{13}$$

One easily checks that the image of $\bar{U}$ by $H[1,\cdot]$ is contained in the subspace $C(\mathbb{R}/T\mathbb{Z},\mathbb{R}) \times \mathbb{R}$ of curves $(r,p) \in Y$ for which $p(t) \equiv \bar{p}$ is constant. In its turn, the image of $\bar{U} \cap \big(C(\mathbb{R}/T\mathbb{Z},\mathbb{R}) \times \mathbb{R}\big)$ is contained in the subspace $\mathbb{R}^2$ of constant curves. Combining (13) with Theorem 8.7 in page 59 of [6] we see that

$$\deg_{LS}\big(I - F[\cdot;0], U, 0\big) = \deg_{B}\big(I_{\mathbb{R}^2} - H[1;\cdot]\big|_{\mathbb{R}^2}, U \cap \mathbb{R}^2, 0\big)\,.$$

where $\deg_B$ denotes the Brouwer degree. Observe that $U \cap \mathbb{R}^2 =]r_*, \ell[\times] - P, P[$ and

$$\big(I_{\mathbb{R}^2} - H[1;\cdot]\big|_{\mathbb{R}^2}\big)(\bar{r},\bar{p}) = -\left(\frac{\bar{p}}{\sqrt{1+\bar{p}^2}}, \frac{1}{T}\int_0^T f(t,\bar{r})\,dt\right)\,.$$

We are led to consider the functions $\varphi : [r_*, \ell] \to \mathbb{R}$, $\psi : [-P, P] \to \mathbb{R}$ defined by

$$\varphi(\bar{r}) = \frac{1}{T}\int_0^T f(t,\bar{r}), \quad \psi(\bar{p}) := \frac{\bar{p}}{\sqrt{1+\bar{p}^2}}\,,$$

and their cartesian product

$$\varphi \times \psi : [r_*, \ell] \times [-P, P] \to \mathbb{R}^2, \qquad (\bar{r},\bar{p}) \mapsto (\varphi(\bar{r}), \psi(\bar{p}))\,.$$

The usual properties of the degree imply

$$\deg_B\big(I_{\mathbb{R}^2} - H[1;\cdot]\big|_{\mathbb{R}^2}, U \cap \mathbb{R}^2, 0\big) = -\deg_B\big(\varphi \times \psi,]r_*, \ell[\times] - P, P[, 0\big) = -1\,,$$

since both φ and ψ change from negative to positive on their respective domains. It concludes the proof. ∎

Proof. [of Proposition 1] We consider the open set

$$V := \{(r,p) \in Y : r_* < r(t) < \ell \ \forall t \in \mathbb{R}\}\,,$$

and the restriction of F to $\bar{V} \times \mathbb{R} \subset \Omega$,

$$F : \bar{V} \times \mathbb{R} \to Y\,, \qquad\qquad (r,p;\mu) \mapsto F(r,p;\mu)\,.$$

which is completely continuous. Our choice of P implies that every fixed point $y \in \bar{V}$ of the section map $F[\cdot,0]$ belongs to U. On the other hand, Lemma 5 states that $\deg_{LS}(I - F[\cdot,0], U, 0) \neq 0$.

Under these conditions the classical Leray-Schauder continuation theorem ([14], see also [2, 5, 16, 17]) provides the existence of a connected set $\mathcal{C} \subset \Sigma(r_*,\ell)$ with $\mathcal{C} \cap (U \times \{0\}) \neq \emptyset$ and satisfying, either *(i):* $\mathcal{C}$ is unbounded in $V \times \mathbb{R}$, or *(ii):* $\mathcal{C} \cap (\partial V \times \mathbb{R}) \neq \emptyset$. In its turn *(ii)* may happen because *(ii$_a$):* $\mathcal{C} \cap \{(r,p;\mu) \in \bar{V} \times \mathbb{R} : \min_{t\in\mathbb{R}} = r_*\} \neq \emptyset$ or *(ii$_b$):* $\mathcal{C} \cap \{(r,p;\mu) \in \bar{V} \times \mathbb{R} : \max_{t\in\mathbb{R}} = \ell\} \neq \emptyset$. Representing the infinite-dimensional space $C(\mathbb{R}/T\mathbb{Z})$ in the one-dimensional ordinate axis and the set $\{r : r_* \leq r(t) \leq \ell \ \forall t \in \mathbb{R}\}$ as the interval $[r_*,\ell]$, these cases are depicted below.

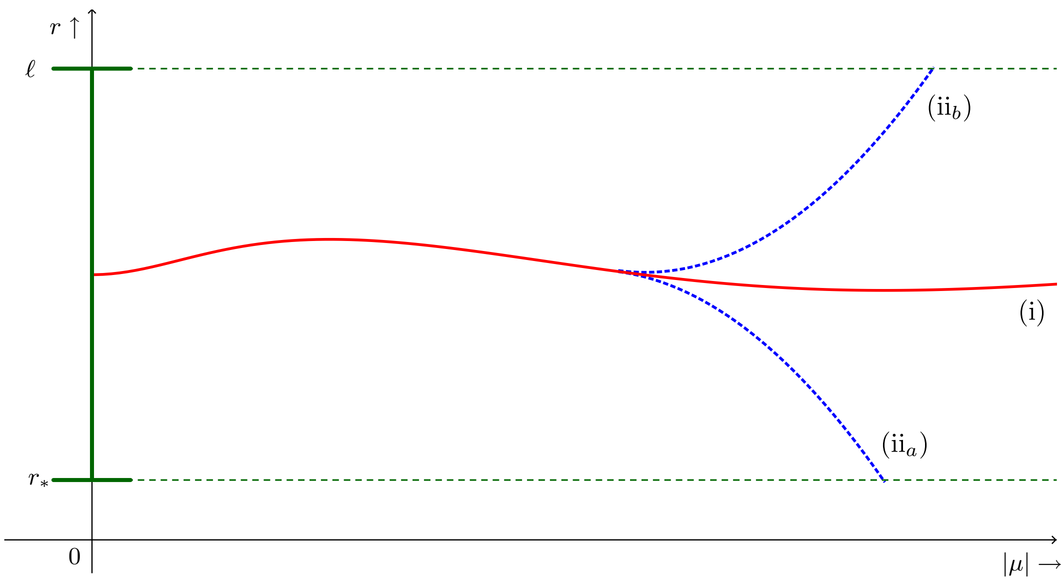


Figure 2: The possibilities for the connected set $\{(|\mu|, r) : (r,p;\mu) \in \mathcal{C}\}$.

Possibilities *(i)* and *(ii$_b$)* are ruled out by Lemmas 2 and 4 respectively. Consequently, *(ii$_a$)* states the existence of some element $(r,p;\mu) \in \mathcal{C}$ with $\min_{t\in\mathbb{R}} r(t) = r_*$. It completes the proof. ∎

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