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Symplectic Methods in ODEs

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Prefazione

Ricordo benissimo il giorno in cui chiesi ad Anna Capietto un consiglio su come e dove poter fare una buona tesi di dottorato su equazioni differenziali e sistemi dinamici. La sua risposta fu netta ed immediata: "Rafael Ortega, Granada". Ora posso dire che non ci fu consiglio più saggio di quello. Ho potuto così conoscere una persona straordinaria che mi ha dato molto dal punto di vista umano e professionale. Imparare Matematica da Rafael Ortega è stata una fortuna ed un onore. Questa tesi è in gran parte merito suo e le parole non bastano ad esprimergli tutta la mia gratitudine. Posso comunque sottolineare la sua immensa disponibilità e pazienza (molta pazienza) con cui mi ha sempre accolto e guidato in questo percorso.

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Contents

Introduction

This thesis is concerned with the qualitative study of some mechanical models and is a survey of the results we have in [44, 43, 42]. More precisely we are going to study some models whose behaviour can be described through a so called "exact symplectic twist" map. This kind of maps appear in the qualitative study of Hamiltonian systems, impact problems, billiards, geodesics on surfaces and many other cases. We concentrate on the applications to Hamiltonian systems and impact problems. Even if these models show a continuous motion, to understand the dynamics it is sufficient to study the behaviour of a discrete map. In the case of Hamiltonian systems it is well known that the so-called Poincaré map is a powerful tool for the qualitative study. On the other hand, impact dynamics can be described through a discrete map. Many qualitative properties of the corresponding map, have a translation to qualitative properties of the model itself.

The study of general maps is very complicated and usually one can get little informations. So we will restrict our study to some class of maps. First we assume that the map is a diffeomorphism defined on the cylinder. This is reflected by the assumption of some regularity and periodicity in the model. Moreover we will assume that the maps is exact symplectic. This condition imply the preservation of the area. We will see that it is satisfied by a large class of Hamiltonian systems and is typical in impact problems. So, the tool of our work will be an exact symplcetic map f defined on the cylinder $\mathbb{T}\times\mathbb{R}$ with coordinates (θ, r). Here θ is considered as an angle and we denote $f(\theta, r) = (\theta_1, r_1)$. We also suppose that the twist condition

$$\frac{\partial \theta_1}{\partial r} > 0 \tag{1}$$

is satisfied. One can understand the meaning of the twist condition through the following example. Suppose that we are considering the motion of a pendulum. Let (θ, r) represent, respectively, the angular position and the velocity at an initial time, say $t = 0$. Analogously, let (θ_1, r_1) represent the position and the velocity at time $t = 1$. Then the twist condition means that the displacement $\theta_1 - \theta$ increases as the initial velocity r increases. Exact symplectic twist diffeomorphisms of the cylinder have been widely studied in literature and many general results exist. See [21, 50, 48] for the general theory and [18, 36, 55, 71, 76] for some applications. The main feature of such maps is the fact that they can be expressed through a function $h = h(\theta, \theta_1)$ that acts as the Hamiltonian function in Hamiltonian systems. Precisely we have that the diffeomorphism f can be implicitly written as

$$\begin{cases} r = -\partial_1 h(\theta, \theta_1) \\ r_1 = \partial_2 h(\theta, \theta_1) \end{cases}$$

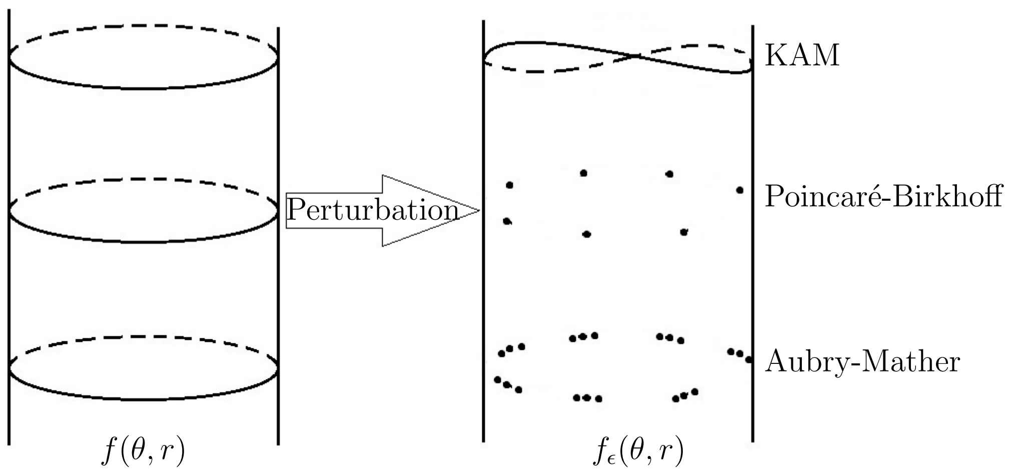
and the function h is called generating function. In this thesis we need to concentrate on three theorems that can be considered as milestones of this topic: The Poincaré-Birkhoff fixed point theorem, the KAM theory and the Aubry-Mather theory. To be precise, one has to cite the fact the the twist condition in the Poincaré-Birkhoff theorem has to be understood in an other sense than (1). We will precise it later on. Before going into the details, let us introduce the three theorems in an informal way. Consider the twist integrable map

$$f : \begin{cases} \theta_1 = \theta + \phi(r) \\ r_1 = r. \end{cases} \tag{2}$$

Suppose the map to be defined in $\mathbb{T} \times [a, b]$ and the function ϕ to be smooth in $[a, b]$ with $\phi' > 0$. It is easily seen that the map is exact symplectic and twist. The dynamics is very simple. Every circle $r = r_*$ is invariant under the action of f. Moreover, the dynamics on every such curve is described by the so-called rotation number ω defined as

$$\omega = \lim_{n\to\infty} \frac{\theta_n}{n} \tag{3}$$

where $(\theta_n, r_n) = f^n(\theta, r)$. In this case we simply have $\omega = \phi(r_*)$. If it is rational, say $\omega = p/q$ then all orbits are periodic in the sense that $\theta_{n+q} = \theta_n + p$. If ω is irrational then every orbit is quasi periodic with frequencies ω and 1. Now consider a small perturbation of f in the class of exact symplectic maps and call it f_ϵ. We have that many invariant sets of f are preserved. This is a delicate theory, as it depends on the arithmetic properties of ω. For any compact interval $[\phi_-, \phi_+]$ with $\phi(a) < \phi_- < \phi_+ < \phi(b)$, and for any rational $\omega \in [\phi_-, \phi_+]$ there exist at least two periodic orbits with rotation number ω. This a consequence of the Poincaré-Birkhoff theorem. If ω is irrational then we have two cases. Either the invariant curve of f persists and every orbit on it is quasi-periodic with frequencies ω and 1, or the minimal invariant set become a Cantor set. The dynamics on a Cantor set is of Denjoy type with rotation number ω [48]. The first case is studied by KAM theory, while the second is considered by Aubry-Mather theory.



The Poincaré-Birkhoff theorem [65, 13] gives the existence of two fixed points of an area preserving homeomorphism f defined on a compact strip $\Sigma = \mathbb{T} \times [a, b]$ of the cylinder. The strip Σ is supposed to be invariant. Also the boundaries are invariant and rotated in opposite directions under the action of f. More precisely the following condition is required

$$\begin{aligned} \theta_1 - \theta < 0 \quad \text{on } \mathbb{T} \times \{a\} \\ \theta_1 - \theta > 0 \quad \text{on } \mathbb{T} \times \{b\}. \end{aligned} \tag{4}$$

This condition is called boundary twist condition. In many applications one has to deal with homeomophisms for which the invariance condition of the strip is not guaranteed. So a generalization to a non-invariant strip is needed. This a delicate point and we refer to the beginning of Section 2.1 for an historical introduction to the theorem and its variants. In this thesis we concentrate on Franks version [20]. He was able to remove the hypothesis of invariance of the strip. This purpose is reached supposing the map f being an exact symplectic diffeomorphism. Franks proof uses deep and abstract tools from differential geometry. Based on his work, we will give a complete and detailed proof of the theorem that can also be understood by people that are not familiar with differential geometry.

Once one has found fixed points, questions about its stability arise. We will prove, using a result of Ortega [62], that, if the diffeomorphism f is analytic, then at least on of the fixed points is unstable. If one also assume that the twist condition is satisfied, then other informations on the nature of the fixed points can be deduced.

The main purpose of the KAM theory is to investigate what is preserved when an integrable system is perturbed. Kolmogorov [30] and Arnold [3] studied the case in the framework of Hamiltonian systems while Moser concentrated on the discrete version. In this thesis we will use Moser approach, precisely we study his invariant curve theorem [54]. Moser considers small perturbations of system (2) and gives sufficient conditions that guarantee the preservation of some invariant curve. Invariant curves are particularly important as they act as barriers. This fact is crucial when one deals with question of stability or boundedness. In Section 2.2 we will give a brief historical introduction to KAM theory and state Moser invariant curve theorem in one of its recent versions.

KAM theory does not consider the case of a large perturbation of the integrable map. Neither gives any information on the invariant curves that are not preserved. The works of Aubry-LeDaeron [4] and Mather [46] give an answer to this questions. Precisely, Mather, as in the Poincaré-Birkhoff theorem, considers an area preserving diffeomorphism of the strip Σ leaving invariant the strip itself. The great difference with the Poincaré-Birkhoff theorem is that Mather supposed the map to be twist in the sense of formula (1) and not only on the boundaries as in (4). With these hypothesis he was able to prove, for every ω in the twist interval, the existence of an invariant set with rotation number ω defined as in (3). This invariant set are called Aubry-Mather sets. The arithmetic properties of ω determine the structure of the Aubry-Mather set and the dynamics on it. If ω is rational then the minimal invariant set is made of a finite number of points and the orbit is periodic. If ω is irrational then we have two cases: either the minimal invariant set is a curve and every orbit is quasi-periodic with frequencies ω and 1 or the minimal invariant set is a Cantor set and the dynamics on it is of Denjoy type. Notice that, as in the Poincaré-Birkhoff theorem, these results hold in the case of the invariance of the strip and one would need not to check this condition. This is possible considering the work of Mather [48] in which the author considers exact symplectic twist diffeomorphism of the infinite cylinder $\mathbb{T} \times \mathbb{R}$ and gets the existence of Aubry-Mather sets for every rotation number $\omega \in \mathbb{R}$. Actually Mather's result is more general. In fact he proves it not only for a single exact symplectic twist diffemorphism but for a finite composition of them. Notice that this result is not trivial, as the composition of twist maps, in general is not a twist map. This result was possible using the expression of the diffeomorphism through the generating function. At this moment two remarks have to be done. First it is worth mentioning that Mather proved his two results [46, 48] with two different techniques and the results are formally different. We will show explicitly that, actually, they are equivalent. Secondly, to prove the result on the infinite cylinder, the technical hypothesis of an infinite twist at infinity was considered. Precisely, one needs that $\theta_1 - \theta \to \pm\infty$ as $r \to \pm\infty$. In many applications this condition is not satisfied and it is not clear if Mather results still hold in the case of a finite twist at infinity. We will prove that adding the hypothesis of the existence a sequence of invariant curves approaching both the top and the bottom of the cylinder we can get Mather's conclusions also in the case of a finite twist at infinity. In the case of a single map it follows applying iteratively the result in [46]. The case of composition of twist maps is more delicate and we will need a careful work on the generating functions to get the result. Section 2.3 is dedicated to the study of the Aubry-Mather theory and its variants.

The second part of the thesis is concerned with the application of these theorems. First we will consider planar Hamiltonian systems. We remember that we mean systems of ODEs that can be written in the form

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(t; q, p) \\ \dot{p} = -\frac{\partial H}{\partial q}(t; q, p) \end{cases} \tag{5}$$

where $H = H(t; q, p)$ is a continuous real valued function differentiable in the variables q and p. It is called Hamiltonian function. Moreover $q = q(t)$ and $p = p(t)$ are the variables and $\dot{q}$ and $\dot{p}$ stand respectively for $\frac{d}{dt}q(t)$ and $\frac{d}{dt}p(t)$. We will be dealing with an Hamiltonian of the form

$$H(t; q, p) = \mathcal{H}(t; q, p) + f(t)q \tag{6}$$

for a continuous function f. We will impose the following periodicity assumption

$$\mathcal{H}(t; q + 1, p) = \mathcal{H}(t; q, p). \tag{7}$$

Given the initial conditions

$$\begin{cases} q(0) = \theta \\ p(0) = r \end{cases} \tag{8}$$

we can define the *time-$\bar{t}$ map* $\Pi_{\bar{t}} : \mathbb{R}^2 \to \mathbb{R}^2$ with components

$$\begin{cases} \theta_1 = q(\bar{t}; \theta, r) \\ r_1 = p(\bar{t}; \theta, r). \end{cases} \tag{9}$$

This map is defined on the cylinder and is exact symplectic if and only if

$$\int_0^{\bar{t}} f(t)dt = 0.$$

The choice of the time $\bar{t}$ is fundamental if one wants to describe the qualitative properties of the solutions. If the periodicity condition

$$\mathcal{H}(t + T; q, p) = \mathcal{H}(t; q, p), \quad f(t + T) = f(t). \tag{10}$$

is introduced then the corresponding time-T map describes the qualitative properties of the solutions. This map is also called Poincaré map. In section 3.1 we made a survey of the properties of the Poincaré map for our Hamiltonian systems.

As a concrete example we consider the differential equation

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + a\sin x = f(t), \tag{11}$$

where $a > 0$ is a parameter. Moreover we are going to suppose that f is a T-periodic continuous function such that

$$\int_0^T f(t)dt = 0. \tag{12}$$

This is usually called equation of the forced relativistic pendulum. Notice that it can be written from the pendulum equation

$$\ddot{x} + a\sin x = f(t)$$

substituting the velocity $\dot{x}$ with the relativistic term

$$\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}. \tag{13}$$

Despite the name, equation (11) cannot be considered the equation of motion in a relativistic framework. Indeed it is not invariant under Lorentz transformation. So this model has been considered as a middle way between the classical and the relativistic mechanics. See [6, 29, 28, 73, 27, 40, 51, 53, 72, 74] for related models involving the relativistic operator. We are going to show that solutions of (11) share some properties that are typical in restricted relativity. To show better our results, let us consider the autonomous case $f \equiv 0$. A standard study gives the phase portrait in figure 1. One can easily realize that the condition $|\dot{x}(t)| < 1$ has to be imposed. It means that the velocity is bounded. Actually the more restrictive condition

$$\sup_{t\in\mathbb{R}} |\dot{x}(t)| < 1 \tag{14}$$

holds. It means that the velocity is uniformly bounded and this is in accordance with the axiom of restricted relativity. We will refer to it as the relativistic effect. Furthermore, we have two constant solutions given by two fixed points (up to translations), one stable, the other unstable. When the energy increases, more periodic solutions appear and tend to a heteroclinc.

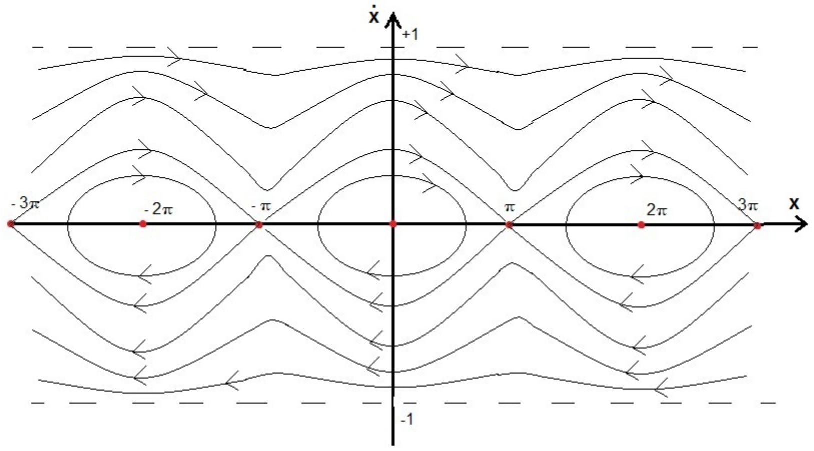


Figure 1: The autonomous case

For high values of the energy, the so called running solutions appear. These solutions are characterized by the quantity

$$\omega = \lim_{t\to\infty} \frac{x(t)}{t}.$$

The renormalized quantity $\frac{\omega}{2\pi}$ is usually known as rotation number. It comes from the relativistic effect that we can only have $\omega \in (-1, 1)$. Moreover, for every possible ω there exists one level of energy whose solutions have rotation number $\frac{\omega}{2\pi}$.

When one considers the forced case (11) the condition $|\dot{x}(t)| < 1$ is obvious, while condition (14) is no longer immediate. We will prove that assuming condition (12) the result it is still true. To prove it we will write the equation in the Hamiltonian form through the change of variable

$$p = \frac{\dot{x}}{\sqrt{1-\dot{x}^2}}.$$

The variable p has a precise meaning in restricted relativity as represents the momentum. We will show that Moser invariant curve theorem applies to the Poincaré map of the associated Hamiltonian systems. This proves that

$$\sup_{t\in\mathbb{R}} |p(t)| < \infty$$

that actually is equivalent to condition (14). This result is the relativistic counterpart of a known result concerning the classical pendulum [76]. Usually to apply Moser's theorem one is lead to many cumbersome estimates in some norm C^k. We will use a refined version of the theorem of differentiability with respect to parameters to make these estimates much easier. Moreover we prove that condition (12) is essential for this result. If condition (12) is not satisfied we can construct solutions with unbounded momentum.

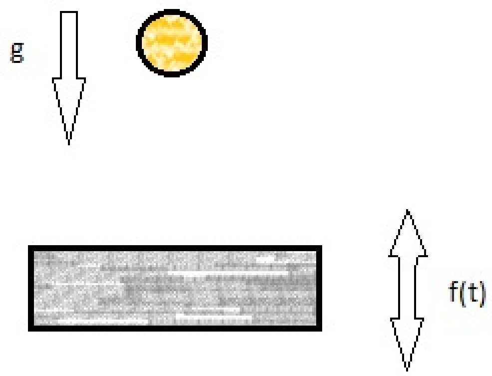
Once we have proved that motions have bounded momentum we will investigate what more of the autonomous case is still preserved. Another consequence of Moser invariant curve theorem is that one can find invariant curves with diophantine rotation number ω. Comparing with the autonomous case, one has that the curves whose rotation number satisfies the diophantine condition are preserved. These solutions are quasi-periodic with frequencies $\frac{2\pi}{T}$ and ω.

To get solutions with other rotation numbers, we will apply the Poincaré-Birkhoff fixed point theorem to prove that at least two fixed points of the Poincaré map exist. These correspond to periodic solutions of the equations and one can see it as the analogous of the constant solutions of the autonomous case. Similar results have been obtained also with other techniques, both topological and variational [8, 7, 74, 75, 12, 19, 41]. A simple change of variable allows to prove the existence of at least two solutions with rotation number $\frac{N}{T}$ with $N \in \mathbb{Z}$ and such that $|\frac{N}{T}| < 1$. Comparing with the autonomous case, this means that the curves corresponding to running solutions with rotation number $\frac{N}{T}$ brakes down but still two solutions (corresponding to two points of the invariant curve) persist. Moreover, using the analyticity of the equation we can prove that at least one of these solutions is unstable.

As one can imagine we want solutions for every possible rotation number. We will fill the missing numbers using Aubry-Mather theory. A first attempt could be use the theory in [46]. In principle, to apply this theory we need to known that the Poincaré map has twist. In the paper [44] it was shown that it does not hold unless a restriction on the parameters is imposed, namely the condition $a \leq \frac{\pi^2}{T^2}$ is necessary. Since we want to obtain results for arbitrary parameters we will use the version for finite composition of twist maps. The Poincaré map Π of equation (11) can be seen as a finite composition $\Pi = f_1 \circ \cdots \circ f_N$ where every f_i is a "small-time-map" that is twist without any restriction. To apply the result in [48] we need to check that the twist of each map f_i goes to infinity as the action goes to infinity. The relativistic effect prevents the velocity from being too large and this makes impossible to satisfy this assumption of large twist. For this reason Mather's theorem cannot be applied directly. The presence of small twist also prevents Moser approach [57] from holding. So we will have to use our version of the theory using the fact that we also have the existence of invariant curves. With this modified theorem we can produce, for every $\omega \in (-1, 1)$ a solution with rotation number $\frac{\omega}{2\pi}$. In this way we partially recover the solutions found with

Moser theorem and the Poincaré-Birkhoff theorem and we fill the missing rotation numbers.

Another example of application of the abstract tool are the impact problems. As an example we consider the model of a ball bouncing on a racket that is moving in the vertical direction according to a given periodic function $f(t)$. Moreover, we suppose that the gravity force is acting on the ball. The mass of the ball is supposed to be small in comparison to the mass of the racket, so that the impacts do not affect the motion of the racket itself.



This model can be formulated in terms of continuous or discrete dynamics. Actually, we can follow the continuous motion of the ball or just look at the sequence of impact times and velocities (t_n, w_n) produced by successive impacts. The first approach leads to a differential equation and the second to a map $S(t_n, w_n) = (t_{n+1}, w_{n+1})$ in the plane. In this context an unbounded motion has to be understood in the sense of the possibility of speeding up the ball, i.e. finding an orbit (t_n^*, w_n^*) of S such that $\lim_{n\to+\infty} w_n^* = +\infty$ or, equivalently $\lim_{n\to+\infty}(t_{n+1}^* - t_n^*) = +\infty$.
This model was first considered by Pustil'nikov, who proved the existence of an uncountable set of unbounded orbits. Such orbits exist if the velocity of the racket is sufficiently large since, for small velocity, Moser invariant curve theorem applies and all the motions are with bounded velocity. See [67, 66] and the references therein. More recently it has been studied in [69]. We also mention the paper by Dolgopyat [17] dealing with non-gravitational forces and the paper by Kunze and Ortega [34] concerning a non-periodic function $f(t)$.

We will prove that bounded motions with remarkable qualitative characteristics are possible as well. Indeed we will prove that for every real and sufficiently big number α there exists an invariant set of orbits with bounded velocity. Moreover, the mean time between the bounces coincides with α and orbits with the same mean time between the bounces can be ordered. If $\alpha = p/q$ is rational, the invariant set contains periodic points of period q that correspond to p-periodic motions with q bounces in a period. Between two consecutive of them there is an heteroclinic orbit. If α is irrational, then we have an alternative: either an invariant curve corresponding to quasi-periodic motions in the classical sense with frequencies $(1, \alpha)$ or the invariant set is a Cantor. In the last case the motion is not quasi-periodic in the classical sense but displays a dynamic of Denjoy-type.

To prove the result we use the classical theory of Aubry-Mather. Precisely, after replacing the velocity by the energy, the map $(t_n, E_n) \mapsto (t_{n+1}, E_{n+1})$ becomes symplectic and has an associated variational principle. This means that the sequence of successive impact times (t_n) satisfies the second order difference equation

$$\partial_2 h(t_{n-1}, t_n) + \partial_1 h(t_n, t_{n+1}) = 0$$

where $h = h(t_0, t_1)$ is the so-called generating function. A nice feature of this model is that the function h can be computed explicitly. This was done in [34] and we will employ it. The general theory asks for generating function defined in the whole plane (t_0, t_1), while our function h is only defined in an half-plane $t_1 - t_0 \geq k$. Then we have to extend h to the whole plane preserving the condition $\partial_{t_0 t_1} h \leq \epsilon < 0$, that is crucial in Aubry-Mather theory. To achieve this we use a trick based on the D'Alambert formula for the wave equation

$$\frac{\partial^2 h}{\partial t_0 \partial t_1} = p(t_0, t_1).$$

The use of the Aubry-Mather theory gives more informations on the orbits.

In principle, invariant curves can appear. They would act as barriers and exclude unbounded motion. We will find some simple conditions on the function $f(t)$ implying that there are unbounded motion and this excludes the presence of invariant curves. There are already results on this line due to Pustyl'nikov and our results will be an extension of those in [67]. The main idea in [67] was to find an orbit $\{(t_n^*, w_n^*)\}$ in the plane time/velocity satisfying

$$t_{n+1}^* = t_n^* + \sigma_n, \qquad w_{n+1}^* = w_n^* + V$$

where σ_n and V belongs to $\mathbb{N} \setminus \{0\}$. Then

$$\lim_{n \to +\infty} w_n = +\infty$$

and the orbit is unbounded (in velocity or energy). In the torus $\mathbb{T} \times \mathbb{T}$ this orbit becomes a fixed point. Our idea will be to look at N-cycles in the torus. This means orbits $\{(t_n^*, w_n^*)\}$ satisfying

$$t_{n+N}^* = t_n^* + \sigma_n, \qquad w_{n+N}^* = w_n^* + V$$

for some $N \geq 2$.

Once we know that bounded and unbounded motions coexist, it is natural to ask for the size of the escape set. This is a very difficult question and all we can say is that the explicit orbit $\{(t_n^*, w_n^*)\}$ is immersed on a one-dimensional continuum of unbounded orbits. This is again based on an observation by Pustyl'nikov: the stable manifold around $\{(t_n^*, v_n^*)\}$ is composed by unbounded orbits. We extend the analysis to the case of cycles and present all the details of the proof.

Chapter 1

Basic notions on maps in the cylinder

In this introductory chapter we are going to recall some basic facts on maps in the cylinder. Most of the material comes from [35].

Consider the strip $\Sigma = \mathbb{R} \times [a, b]$ with $-\infty \leq a < b \leq +\infty$. We will work with a C^k-embedding $f : \Sigma \to \mathbb{R}^2$, with coordinates

$$f(\theta, r) = (\Theta(\theta, r), R(\theta, r))$$

where we suppose that Θ and R are C^k functions satisfying

$$\Theta(\theta + 1, r) = \Theta(\theta, r) + 1, \quad R(\theta + 1, r) = R(\theta, r).$$

Notice that this generalized periodicity allows to say that the map is the lift to $\mathbb{R}^2$ of the corresponding map $\bar{f} : \bar{\Sigma} \to \mathscr{C}$ where $\bar{\Sigma} = \mathbb{T} \times [a, b]$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathscr{C} = \mathbb{T} \times \mathbb{R}$; in this case the variable θ is understood as an angle. We will denote by f the function and by Σ the domain in both cases we are working in the plane or in the cylinder. We will use the notations $\bar{f}$ and $\bar{\Sigma}$ in cases of possible confusion. We remember that saying that f is an embedding means that it is a C^k-diffeomorphism on the image. A C^0-diffeomorphism will be understood as a homeomorphism. We will also suppose that f is isotopic to the inclusion. It means that there exists a function $H(\lambda, x)$ that is a homeomorphism for every $\lambda \in [0, 1]$ and such that $H(0, x) = f(x)$ and $H(1, x) = x$. The class of these maps will be indicated by $\mathcal{E}^k(\Sigma)$.
Throughout the text we will consider also finite compositions of maps $f_i \in \mathcal{E}^k(\mathbb{R}^2)$. We will let $F = f_1 \circ \cdots \circ f_N$ and use the notation

$$\begin{aligned} F(\theta, r) &= (\Theta(\theta, r), R(\theta, r)) \\ f_i(\theta, r) &= (\Theta^{(i)}(\theta, r), R^{(i)}(\theta, r)) \end{aligned} \tag{1.1}$$

for the coordinates. Sometimes, for brevity, we will use the notation

$$\theta_1 = \Theta(\theta, r), \quad r_1 = R(\theta, r).$$

We have the following

Lemma 1. *If f_1 and f_2 belong to $\mathcal{E}^k(\mathbb{R}^2)$, then $F = f_1 \circ f_2$ belongs to $\mathcal{E}^k(\mathbb{R}^2)$*

Proof. First of all F is a C^k-diffeomorfism onto the image being composition of C^k-diffeomorphisms onto the images. Moreover

$$\begin{aligned} \Theta(\theta + 1, r) &= \Theta^{(2)}(\Theta^{(1)}(\theta + 1, r), R^{(1)}(\theta + 1, r)) \\ &= \Theta^{(2)}(\Theta^{(1)}(\theta, r) + 1, R^{(1)}(\theta, r)) = \Theta(\theta, r) + 1 \end{aligned} \tag{1.2}$$

and analogously $R(\theta + 1, r) = R(\theta, r)$. Finally we conclude remembering that the composition of maps isotopic to the inclusion are still isotopic to the inclusion. □

1.1 Symplectic vs exact symplectic maps

A map in $\mathcal{E}^1(\Sigma)$ is said *symplectic* if it preserves the symplctic form $\omega = d\theta \wedge dr$, that is

$$d\Theta \wedge dR = d\theta \wedge dr$$

on Σ. Notice that this condition can be reformulated as

$$\det f' = 1 \quad \text{on } \Sigma$$

that is the classical definition of area-preserving map. This definition has a well known interpretation in measure theory: for each measurable set Ω of Σ, its image $\Omega_1 = f(\Omega)$ is measurable and $\mu(\Omega) = \mu(\Omega_1)$.
We will be interested in a stronger property than being symplectic. A map in $\mathcal{E}^1(\Sigma)$ is said *exact symplectic* if the differential form

$$\alpha = R d\Theta - r d\theta$$

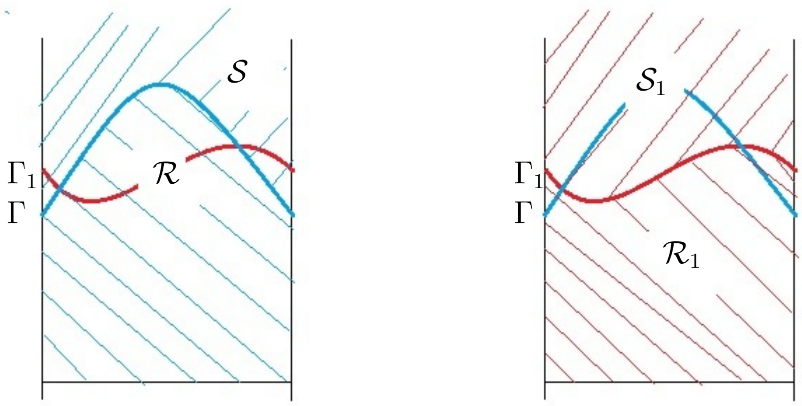
is exact in the cylinder, that means that there exists a $C^2(\overline{\Sigma})$ function $V(\theta, r)$, 1-periodic in θ, such that

$$dV = R d\Theta - r d\theta. \tag{1.3}$$

This definition needs some comments. We can give a geometrical interpretation. First of all, by the theory of differential forms, we have that f is exact symplectic if and only if for every closed path c in $\bar{\Sigma}$ we have

$$\int_c \gamma = \int_{f(c)} \gamma \tag{1.4}$$

where $\gamma = rd\theta$. Consider a C^1 non-contractible Jordan curve Γ. Being f an embedding, its image $\Gamma_1 = f(\Gamma)$ is still a Jordan curve in the same conditions. We can suppose that both curves lie in $\mathbb{T} \times (r_0, +\infty)$ for some r_0. Let $\mathcal{R}$ and $\mathcal{R}_1$ be the bounded components of $\mathbb{T} \times (r_0, +\infty) \setminus \Gamma$ and $\mathbb{T} \times (r_0, +\infty) \setminus \Gamma_1$ respectively. Moreover, let us call $\mathcal{S}$ and $\mathcal{S}_1$ their complements in $\mathbb{T} \times (r_0, +\infty)$.



Noticing that $\omega = d\gamma$ the Stoke's theorem gives

$$\begin{aligned}\int_{\mathcal{R}} \omega = \int_{\partial \mathcal{R}} \gamma = \int_{r=r_0} \gamma + \int_{\Gamma} \gamma = \int_{r=r_0} \gamma + \int_{\Gamma_1} \gamma \\ = \int_{\partial \mathcal{R}_1} \gamma = \int_{\mathcal{R}_1} \omega.\end{aligned} \tag{1.5}$$

It means that $\mathcal{R}$ and $\mathcal{R}_1$ have the same area. Using this characterization, it is easy to see that the notions of symplectic and exact symplectic are not equivalent. The map

$$\begin{cases} \theta_1 = \theta \\ r_1 = r + 1 \end{cases} \tag{1.6}$$

is a translation upwards that is clearly symplectic but cannot satisfy the geometric characterization that we gave. Anyway an exact symplectic map is always symplectic. Indeed we have that $d\alpha = -d\Theta \wedge dR + d\theta \wedge dr$ so that α is closed if and only if f is symplectic. An exact form on the cylinder is always closed but the converse is not true as the cylinder is not simply connected. This is why we have just a one-way implication between the two notions we are considering. Now we are going to give a criteria to decide whether the diffeomorphism f is exact symplectic or not. It is based on the following criteria for the exactness of the differential form

$$\beta = A(\theta, r)d\theta + B(\theta, r)dr.$$

To decide if it is exact on the cylinder or not, one just have to test the integral

$$\int_c \beta$$

on the curve $\mathbb{T} \times \{r_*\}$ for some $r_* \in [a, b]$ and prove that it vanishes. This translates in the condition

$$\int_0^1 A(\theta, r_*) d\theta = 0.$$

In our case the differential form is $Rd\Theta - rd\theta$ and, applying this criteria we have that the diffeomorphism f is exact symplectic if and only if there exists $r_* \in [a, b]$ such that

$$\int_0^1 R(\theta, r_*) \frac{\partial \Theta}{\partial \theta}(\theta, r_*) d\theta = r_*. \tag{1.7}$$

Concerning the composition we have

Lemma 2. *If f_1 and f_2 are symplectic, then $F = f_1 \circ f_2$ is symplectic. If f_1 and f_2 are exact symplectic, then $F = f_1 \circ f_2$ is exact symplectic.*

Proof. The first assertion comes directly from the fact that

$$\det F' = \det(f_1' \circ f_2) \det f_2' = 1.$$

The second from

$$\begin{aligned} Rd\Theta - rd\theta &= R^{(2)}(\Theta^{(1)}, R^{(1)}) d\Theta^{(2)}(\Theta^{(1)}, R^{(1)}) - R^{(1)} d\Theta^{(1)} + R^{(1)} d\Theta^{(1)} - rd\theta \\ &= dV^{(2)}(\Theta^{(1)}, R^{(1)}) + dV^{(1)}(\theta, r) = d[V^{(2)}(\Theta^{(1)}, R^{(1)}) + V^{(1)}(\theta, r)]. \end{aligned} \tag{1.8}$$

□

These two concepts are important because they are related to the concept of intersection property that we are going to define using the same notation used so far. We say that a map f has the *intersection property* if for every non-contractible regular Jordan curve $\Gamma \subset \Sigma$,

$$\Gamma \cap \Gamma_1 \neq \emptyset.$$

The *strong intersection property* will hold if

$$\Gamma_1 \cap \mathcal{R} \neq \emptyset \quad \text{and} \quad \Gamma \cap \mathcal{R}_1 \neq \emptyset$$

unless $\Gamma = \Gamma_1$.
These two properties are not equivalent, for example the following map

$$\begin{cases} \theta_1 = \theta \\ r_1 = r + \sin^2 2\pi\theta \end{cases}$$

has the intersection property but not the strong intersection property. In fact $r_1 \geq r$ and $r_1 = r$ if and only if $\theta = k/2$ for $k \in \mathbb{Z}$. Nevertheless the strong intersection property implies the intersection property. More precisely we have

Lemma 3. *If a map $f : \Sigma \to \mathscr{C}$ has the strong intersection property, then for every non-contractible Jordan curve $\Gamma \subset \Sigma$ we have that the set $\Gamma \cap \Gamma_1$ has at least two points.*

Proof. The case $\Gamma = \Gamma_1$ is trivial. In the other case the strong intersection property implies that $\Gamma \cap \mathcal{R}_1 \neq \emptyset$. Moreover, $\Gamma \cap \mathcal{S}_1 \neq \emptyset$. Indeed, suppose by contradiction that Γ lies in $\mathcal{R}_1$ and notice that in this case $\Gamma_1 \cap \mathcal{R} = \emptyset$. So Γ and Γ_1 must intersect in a point p. Moreover, as Γ and Γ_1 are non-contractible Jordan curves, they must intersect in another point $p_1 \neq p$. □

If we suppose that the map f belongs to $\mathcal{E}^1(\Sigma)$ and is exact symplectic or symplectic, we can deduce something on the intersection properties.

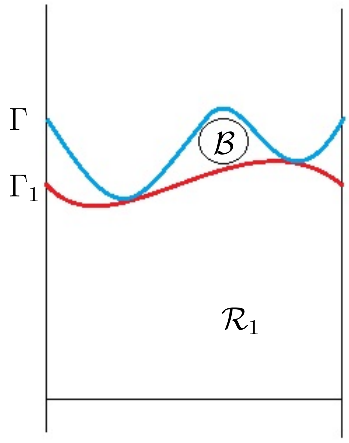
Lemma 4. *If a map $f \in \mathcal{E}^1(\Sigma)$ is exact symplectic, then it has the strong intersection property.*

Proof. Fix a non contractible regular Jordan curve Γ and consider the sets $\mathcal{R}$ and $\mathcal{R}_1$ previously defined. By the geometric interpretation of the exact symplectic condition we have that $\mu(\mathcal{R}) = \mu(\mathcal{R}_1)$ where μ is the Lebesgue measure. Suppose that $\Gamma \neq \Gamma_1$. If $\Gamma \cap \mathcal{R}_1 = \emptyset$, then we have that $\mathcal{R}_1 \subsetneq \mathcal{R}$. Suppose that we are in the first case being the second similar. Using the Jordan-Schonflies theorem one can find a ball $\mathcal{B}$ contained in $\mathcal{R}_1 \setminus \mathcal{R}$ (cfr figure).

So $\mathcal{R}$ and $\mathcal{R}_1$ cannot have the same area. We can get an analogous contradiction if $\Gamma_1 \cap \mathcal{R} = \emptyset$. □

Notice that if we suppose the map $f \in \mathcal{E}^1(\Sigma)$ only to be symplectic, the map (1.6) shows that the result is false. In fact the intersection property fails. We can arrange the result supposing some invariance property of the map f.

Lemma 5. *If a map $f \in \mathcal{E}^1(\Sigma)$ is symplectic and there exists a non-contractible Jordan curve Ξ such that $f(\Xi) = \Xi$, then it is exact symplectic and so it has the strong intersection property.*



Proof. Considering a non-contractible Jordan curve Γ, we can proceed as in the previous lemma considering the set $\mathcal{R}$ as the set defined by Γ and Ξ and the set $\mathcal{R}_1$ as the set defined by Γ_1 and Ξ. □

1.2 The twist condition and the generating function

A map $f \in \mathcal{E}^1(\Sigma)$ is said *twist* if

$$\frac{\partial \Theta}{\partial r} > 0 \quad \text{for every } (\theta, r) \in \Sigma.$$

If there exists $\beta > 0$ such that

$$\frac{\partial \Theta}{\partial r} > \beta \quad \text{for every } (\theta, r) \in \Sigma$$

then the map is β-twist. Heuristically, this condition says that if we fix a vertical segment, then the higher we start, the quicker we rotate. Note that, despite it is a formally easy definition, its consequences are far from being easy to study. To have an idea notice that the composition of twist map, in general, is not twist. As an example, consider the exact symplectic twist maps $f(\theta, r) = (\theta + r, r)$ and $g(\theta, r) = (\theta + r, r + \sin 2\pi(\theta + r))$. The composition $F = f \circ g$ has components $(\Theta(\theta, r), R(\theta, r))$ given by

$$\Theta(\theta, r) = \theta + 2r + \sin 2\pi(\theta + r), \quad R(\theta, r) = r + \sin 2\pi(\theta + r).$$

Notice that

$$\frac{\partial \Theta}{\partial r} = 2 + 2\pi \cos 2\pi(\theta + r)$$

and can assume both positive and negative values.
This twist condition is employed in solving the implicit function problem

$$\theta_1 = \Theta(\theta, r) \tag{1.9}$$

finding a function $\mathcal{R}(\theta, \theta_1)$. Notice that it is defined on the set

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \Theta(\theta, a) < \theta_1 < \Theta(\theta, b)\} \tag{1.10}$$

where

$$\Theta(\theta, a) = \lim_{r \downarrow a} \Theta(\theta, r) \quad \text{and} \quad \Theta(\theta, b) = \lim_{r \uparrow b} \Theta(\theta, r).$$

Notice that $\mathcal{R}$ is $C^1(\Omega)$ and that for every $\theta \in \mathbb{R}$

$$-\infty \leq \Theta(\theta, a) < \Theta(\theta, b) \leq +\infty.$$

We call the quantities $\Theta(\theta, a)$ and $\Theta(\theta, b)$ the *amount of twist at the ends.* So, supposing that the map f is also exact symplectic, we are ready to define its *generating function* as

$$h(\theta, \theta_1) = V(\theta, \mathcal{R}(\theta, \theta_1)), \quad (\theta, \theta_1) \in \Omega$$

where V comes from (1.3). We are going to study some properties of h. First of all notice that, by the uniqueness of the implicit function problem (1.9) and the generalized periodicity of the function Θ, we have that

$$\mathcal{R}(\theta + 1, \theta_1 + 1) = \mathcal{R}(\theta, \theta_1).$$

So it is easy verify that

$$h(\theta + 1, \theta_1 + 1) = h(\theta, \theta_1)$$

remembering the periodicity of V. Now let us compute the partial derivatives of h. To this aim we notice that condition (1.3) is equivalent to

$$V_\theta = R\Theta_\theta - r, \quad V_r = R\Theta_r$$

so that

$$\frac{\partial h}{\partial \theta} = V_\theta + V_r \mathcal{R}_\theta = R\theta_\theta - \mathcal{R} + R\Theta_r \mathcal{R}_\theta.$$

Differentiating (1.9) with respect to θ we get

$$\frac{\partial h}{\partial \theta}(\theta, \theta_1) = -\mathcal{R}(\theta, \theta_1).$$

Analogously, passing through the differentiation of (1.9) with respect to θ_1 we get

$$\frac{\partial h}{\partial \theta_1}(\theta, \theta_1) = R(\theta, \mathcal{R}(\theta, \theta_1)).$$

Notice that we have just proved that $h \in C^2(\Omega)$. Moreover, the map f, expressed by $\theta_1 = \Theta(\theta, r)$ and $r_1 = R(\theta, r)$ can be implicitly be defined by

$$\begin{cases} \partial_1 h(\theta, \theta_1) = -r \\ \partial_2 h(\theta, \theta_1) = r_1 \end{cases} \tag{1.11}$$

that justifies the name of generating function for h. The twist and the β-twist condition of the map f translate to the generating function respectively in

$$\partial_{12} h(\theta, \theta_1) < 0 \quad \text{and} \quad -\frac{1}{\beta} < \partial_{12} h(\theta, \theta_1) < 0$$

on Ω. This is true because, from implicit differentiation we have

$$\partial_{12} h(\theta, \theta_1) = \mathcal{R}_{\theta_1} = -\frac{1}{\Theta_r(\theta, \mathcal{R}(\theta, \theta_1))}.$$

Conversely, let α and β be two 1-periodic C^2 function of $\mathbb{R}$ and $\Omega = \{(\theta, \theta_1) : \alpha(\theta) + \theta < \theta_1 < \beta(\theta) + \theta\}$. Consider a $C^2(\Omega)$ function h such that $h(\theta + 1, \theta_1 + 1) = h(\theta, \theta_1)$ for every $(\theta, \theta_1) \in \Omega$. Suppose that

$$\partial_{12} h < 0. \tag{1.12}$$

So we can solve the implicit function problem

$$-r = \partial_1 h(\theta, \theta_1) \tag{1.13}$$

finding a C^1 function $\Theta(\theta, r)$ defined on $\Sigma = \mathbb{R} \times (-\sup_{\theta_1} \partial_1 h(\theta, \theta_1), -\inf_{\theta_1} \partial_1 h(\theta, \theta_1))$. Define for $(\theta, r) \in \Sigma$ the map f

$$\begin{cases} \theta_1 = \Theta(\theta, r) \\ r_1 = R(\theta, r) \end{cases} \tag{1.14}$$

where

$$R(\theta, r) = \partial_2 h(\theta, \Theta(\theta, r)).$$

By the uniqueness of the implicit function problem (1.13) and the periodicity of h we can see that

$$\Theta(\theta+1,r)=\Theta(\theta,r)+1, \quad R(\theta+1,r)=R(\theta,r)$$

so that f can be seen as map on the cylinder. Moreover, the condition $\partial_{12}h<0$ implies that f is one to one, so that $f\in\mathcal{E}^1(\Sigma)$. We also have an estimate on the amount of twist at the top and bottom of Σ, as $\alpha(\theta)<\Theta(\theta,r)-\theta<\beta(\theta)$. Moreover, the map f is exact symplectic: to verify it consider $V(\theta,r)=h(\theta,\Theta(\theta,r))$. One can see that $V(\theta+1,r)=V(\theta,r)$ and that condition (1.3) is satisfied. Furthermore, by implicit differentiation, we get

$$\frac{\partial\Theta}{\partial r}=-\frac{1}{\partial_{12}h(\theta,\Theta(\theta,r))}>0 \tag{1.15}$$

remembering that $\partial_{12}h<0$. So f is twist. Throughout the text we will be concerned with functions h satisfying a condition stronger than (1.12), precisely we will require the existence of $\delta>0$ such that

$$\partial_{12}h<-\delta.$$

Notice that from (1.15) we have that this correspond to a diffeomorphism such that

$$0<\frac{\partial\Theta}{\partial r}<\frac{1}{\delta}$$

i.e. we have a bound on the twist. Now the following question arise: how can we get a β-twist diffeomorphism? The answer once again comes from (1.15) as we deduce that the β-twist condition corresponds to

$$-\frac{1}{\beta}<\partial_{12}h(\theta,\theta_1)<0.$$

Summing up, we have the following relations between the twist $\frac{\partial\Theta}{\partial r}$ of an exact symplectic map $f\in\mathcal{E}^1(\Sigma)$ and the second derivative $\partial_{12}h$ of its generating function:

$$\begin{aligned}
\partial_{12}h<0 \text{ on } \Omega \Leftrightarrow 0<\frac{\partial\Theta}{\partial r} \text{ on } \Sigma,\\
\partial_{12}h<-\beta<0 \text{ on } \Omega \Leftrightarrow 0<\frac{\partial\Theta}{\partial r}<\frac{1}{\beta} \text{ on } \Sigma,\\
-\frac{1}{\beta}<\partial_{12}h<0 \text{ on } \Omega \Leftrightarrow 0<\beta<\frac{\partial\Theta}{\partial r} \text{ on } \Sigma,
\end{aligned} \tag{1.16}$$

where $\beta>0$ and Ω was defined in (1.10).

Chapter 2

Three symplectic theorems

In this chapter we are going to introduce and comment the main tools of our work, namely the Poicaré-Birkoff theorem, the Aubry-Mather theory ant the invariant curve theorem.

2.1 The Poincaré-Birkhoff theorem

In 1912, Poincaré [65], motivated by the study of the restricted three body problem, conjectured that an area preserving homeomorphism defined from an annulus onto itself that twisted the boundaries in opposite direction should have at least two fixed points. He proved the conjecture in several special cases but could not give a proof in the general case before his dead that occurred in the same year 1912. Birkhoff [9], one year later, was the first who gave a complete proof of the existence of one periodic point. In the same article he claimed to have proven the existence also of the second one. Anyway, as he too said in [10] the proof of the second part was incomplete. In 1925 [11] he published another proof of the existence of the second fixed point. However the proof was rather complicated and it was not accepted by many mathematicians. Subsequently, there were lots of attempts to adjust Birkhoff proof but every one was carrying some faults. We have to wait until 1977 when Brown and Neumann [13] published a complete and convincing proof of the Poincaré conjecture based on Birkhoff ideas.
To state the theorem we have to give a precise meaning to "twist the boundaries in opposite directions". To do so we will not consider the map defined in the annulus but we will consider its lift to $\mathbb{R}^2$.

Theorem 1 (Poincaré-Birkhoff Theorem)**.** *Consider $a > 0$ and let $f : \mathbb{R} \times [-a, a] \to \mathbb{R} \times [-a, a]$ be an area preserving homeomorphism isotopic to the identity with coordinates $f(\theta, r) = (\Theta(\theta, r), R(\theta, r))$ satisfying, for every $\theta \in$*

$\mathbb{R}$ *and* $r \in [-a, a]$

$$\Theta(\theta + 1, r) = \Theta(\theta, r) + 1, \quad R(\theta + 1, r) = R(\theta, r).$$

Suppose that

$$\begin{aligned} &\Theta(\theta, -a) - \theta < 0 \quad \textit{for every } \theta \in [0, 1] \\ &\Theta(\theta, a) - \theta > 0 \quad \textit{for every } \theta \in [0, 1]. \end{aligned} \tag{2.1}$$

Then f *has at least two fixed points* p_1 *and* p_2 *such that* $p_1 - p_2 \neq (k, 0)$ *for every* $k \in \mathbb{Z}$*.*

Hypothesis (2.1) is said *boundary twist condition.* Notice that by the periodicity assumptions on the coordinates we can say that f is defined on the cylinder $\mathbb{R} \times [-a, a]$ and then $f \in \mathcal{E}^0(\mathbb{R} \times [-a, a])$. Poincaré introduced the theorem dealing with maps in the annulus, we will talk about maps on the cylinder. The annulus is homeomorphic to a cylinder so that considering a map defined on the cylinder is equivalent to considering a map on the annulus. However, it is important to notice that the area element is $dr \wedge d\theta$ in the cylinder and $rdr \wedge d\theta$ in the annulus. This plays a role when dealing with area preserving maps in the sense that an area preserving map in the annulus is understood to preserve the differential form $rdr \wedge d\theta$. Once we have pointed out this we can suppose that there is no difference between maps in the cylinder and maps in the annulus.
Some remarks on the theorem are needed. The fact that f is an homeomorphism isotopic to the identity implies that the boundary circles $\mathbb{T} \times \{-a\}$ and $\mathbb{T} \times \{a\}$ are preserved by f. This hypothesis is essential, indeed consider the following homeomorphism defined for $r \in [-1, 1]$ by

$$\begin{cases} \theta_1 = \theta + r + \omega \\ r_1 = -r \end{cases} \tag{2.2}$$

with $\omega \notin \mathbb{Z}$. The homeomorphism satisfies all the hypothesis of the Poincaré-Birkhoff theorem apart from the preservation of the boundary. In fact the homemorphism is not isotopic to the identity. Notice that the homeomorphism cannot have fixed points. This observation will be crucial when we will drop the hypothesis of the invariance of the annulus.
With obvious changes the theorem can be stated considering a more general cylinder $\mathbb{R}/S\mathbb{Z} \times [a, b]$ for arbitrary $S > 0$ and $a < b$. Moreover we can consider the case in which the boundaries are not straight lines but graphs of continuous functions that are invariant under f. So we can consider the strip $\Sigma = \{\mu_1(\theta) \leq r \leq \mu_2(\theta)\}$ where μ_i are continuous 1-periodic functions

such that $\mu_1 < \mu_2$ for every θ. We suppose that the map $f : \Sigma \to \Sigma$ is area preserving and defined on the cylinder. Moreover, condition (2.1) translates into

$$\begin{aligned} &\Theta(\theta, \mu_1(\theta)) - \theta < 0 \quad \text{for every } \theta \in [0,1] \\ &\Theta(\theta, \mu_2(\theta)) - \theta > 0 \quad \text{for every } \theta \in [0,1]. \end{aligned} \tag{2.3}$$

The functions μ_i correspond in the cylinder to non-contractible invariant Jordan curves parametrized by $(\theta, \mu_i(\theta))$. Consider the change of variables

$$V : \begin{cases} x = (\int_0^\theta \{\mu_2(s) - \mu_1(s)\}ds)(\int_0^1 \{\mu_2(s) - \mu_1(s)\}ds)^{-1} \\ y = (r - \mu_1(r))(\mu_2(r) - \mu_1(r))^{-1}. \end{cases} \tag{2.4}$$

Notice that it transforms the strip Σ into the set $A = \{(x, y) : 0 \le y \le 1\}$, $\det V'(\theta, r) = const$ and the periodicity condition $V(\theta+1, r) = V(\theta, r)+(1, 0)$ holds. So that the conjugated homeomorphism

$$g(x, y) = V \circ f \circ V^{-1}(x, y)$$

is defined on the cylinder and area preserving. Let us study the behaviour on the boundaries. Suppose that $y = 1$. So we have that $r = \mu_2(\theta)$ for some θ coming from the change of variable. Remembering the boundary twist condition (2.3) we have that $\theta_1 > \theta$ and $r_1 = \mu_2(\theta_1)$. Here we have used the notation $\theta_1 = \Theta(\theta, \mu_2(\theta))$. So

$$\begin{aligned} x_1 &= (\int_0^{\theta_1} \{\mu_2(s) - \mu_1(s)\}ds)(\int_0^1 \{\mu_2(s) - \mu_1(s)\}ds)^{-1} \\ &> (\int_0^{\theta} \{\mu_2(s) - \mu_1(s)\}ds)(\int_0^1 \{\mu_2(s) - \mu_1(s)\}ds)^{-1} = x. \end{aligned}$$

A similar argument holds for $y = 0$ so that all the conditions of the Poincaré-Birkhoff theorem are satisfied.
A serious problem occurs when we are trying to apply the Poincaré-Birkhoff theorem. In fact the hypothesis of the invariance of the strip Σ is hard to verify and not always guaranteed. Birkhoff was already aware of this problem as he proved in [11] the theorem supposing only the internal boundary of the annulus to be invariant while the external could vary. It is important that he needed to suppose that the image of the external boundary were star-shaped with respect to the origin. Both the star-shapedness of the image of the external boundary and the invariance of the internal one seem to be hard to verify in the applications. In 1983 Ding [16] proposed a proof in which he could drop the invariance of both the boundaries requiring the image of only

the internal boundary to be star-shaped. In his paper he conjectured that the condition requiring the inner boundary to be star-shaped depended on the proof he gave and was not necessary. An attempt of dropping this condition was given in [39]. Martins and Ureña [45] gave an example showing that one cannot assume both the boundaries not to be star-shaped. Subsequently Le Calvez and Wang [14] gave an example in which proved that also the image of the outer boundary has to be star-shaped, contradicting the result of Ding. There are two possibilities to overcome these problems. Rebelo [68] gave a convincing arrangement of Ding proposal and Franks [20] proved the theorem in the case of smooth diffeomorphisms. The smooth case has less difficulties because extensions are easier to do. Due to this facts we will concentrate in Franks version. He dealt with an embedding $f : \mathbb{R} \times [-a, a] \to \mathbb{R} \times [-b, b]$ with $b > a > 0$. We are going to state a slightly modified version of his theorem:

Theorem 2. *Consider $b > a > 0$ and let $f : \mathbb{R} \times [-a, a] \to \mathbb{R} \times [-b, b]$ be an exact symplectic diffeomorphism belonging to $\mathcal{E}^2$ such that $f(\mathbb{R} \times [-a, a]) \subset int(\mathbb{R} \times [-b, b])$. Suppose that there exists $\epsilon > 0$ such that*

$$\Theta(\theta, a) - \theta > \epsilon, \quad \theta \in [0, 1)$$
$$\Theta(\theta, -a) - \theta < -\epsilon, \quad \theta \in [0, 1).$$

Then f has at least two fixed points p_1 and p_2 in $\mathbb{R}\times[-a, a]$ such that $p_1-p_2 \neq (k, 0)$ for every $k \in \mathbb{Z}$.

Notice that having dropped the hypothesis of the invariance of both the boundaries we have to strengthen the area preserving condition, so that the diffeomorphism is supposed to be exact symplctic. This is clear in view of the following example. Consider the embedding $\mathbb{R} \times [-1, 1] \to \mathbb{R} \times [-2, 2]$

$$\begin{cases} \theta_1 = \theta + r \\ r_1 = r + 1/2. \end{cases} \tag{2.5}$$

It is clearly area preserving and satisfies the boundary twist condition. However it cannot have fixed points, and in fact it is not exact symplectic.
The proof of Franks theorem is quite complicated as he deals with general manifolds, so we are going to repeat it giving a more accessible proof restricting to the cylinder. During the proof we will use the following notation:

$$\begin{aligned} A &= \mathbb{R} \times [-a, a], \\ B &= \mathbb{R} \times [-b, b], \\ \bar{A} &= \mathbb{T} \times [-a, a], \\ \bar{B} &= \mathbb{T} \times [-b, b]. \end{aligned} \tag{2.6}$$

The strategy of the proof is to extend f to an homeomorphism g of that leaves the strip B invariant and the use the Poincaré-Birkhoff theorem. The periodicity of the components allows to consider the corresponding map $\bar{f}$: $\bar{A} \to \bar{B}$. At the beginning we will work in the annuli $\bar{A}$ and $\bar{B}$. We observe that in this case is not possible to apply the isotopy extension theorem as stated in [52, p.63] because $\bar{B}$ should be without boundary. Anyway, by the hypothesis $\bar{f}(\bar{A}) \subset int\bar{B}$ it vis possible to slightly modify it in order to extend $\bar{f}$ to $\bar{g}_1 : \bar{B} \to \bar{B}$ so that $\bar{g}_1$ is isotopic to the identity and restricted to a neighbourhood of $\partial\bar{B}$ is the identity (this is achieved in [26, Theorem 1.3, p.180]). Notice that $\bar{g}_1$ does not preserve the area out of $\bar{A}$.
Now choose a_0 slightly smaller than a and define the following subsets that we will use during the proof

$$\begin{aligned} A_0 &= \mathbb{R} \times [-a_0, a_0] \\ \bar{A}_0 &= \mathbb{T} \times [-a_0, a_0]. \end{aligned}$$

In order to apply the Poincaré-Birkhoff theorem and get the result, let us prove the following two lemmas:

Lemma 6. *It is possible to alter $\bar{g}_1 : \bar{B} \to \bar{B}$ finding a diffeomorphism $\bar{g}_2 : \bar{B} \to \bar{B}$ such that*

- *$\bar{g}_2$ is area preserving on $\bar{B}$,*
- *$\bar{g}_2|_{\bar{A}_0} = \bar{f}$,*
- *$\bar{g}_2|_{\partial\bar{B}} = Id$.*

Lemma 7. *It is possible to alter the lift $g_2 : B \to B$ finding $g : B \to B$ such that*

- *g is area preserving on B,*
- *g has no fixed points out of A,*
- *g satisfy the boundary twist condition (2.1) on B,*
- *$g = g_2$ on A.*

So theorem 2 will follow from the application of the Poincaré-Birkhoff theorem to g.

Proof of Lemma 6. To prove this lemma we will use Moser's ideas, presented in [56] in a more general framework. Let us break the proof in several steps.

Step 1. Let $\bar{B}^+ = \mathbb{T} \times [a_0, b] \subset \bar{B}$. It results

$$\mu(\bar{B}^+) = \int_{\bar{B}^+} \det \bar{g}_1'(\theta, r) d\theta dr.$$

Let $\bar{g}_1(\theta, r) = (\theta_1, r_1)$. The 1-form $r_1 d\theta_1 - r d\theta$ is exact symplectic on $\bar{A}$ so its integral over any closed path in $\bar{A}$ must vanish, in particular we have

$$\int_{\mathbb{T}\times\{a_0\}} r d\theta = \int_{\bar{g}_1(\mathbb{T}\times\{a_0\})} r d\theta.$$

Moreover, because $\bar{g}_1$ is the identity over $\mathbb{T} \times \{b\}$ we have

$$\int_{\mathbb{T}\times\{b\}} r d\theta = \int_{\bar{g}_1(\mathbb{T}\times\{b\})} r d\theta$$

so that

$$\int_{\partial\bar{B}^+} r d\theta = \int_{\partial\bar{B}^+} r_1 d\theta_1.$$

Notice that $\theta_1 = \theta_1(\theta, r)$ and $r_1 = r_1(\theta, r)$ so that $d\theta_1 = \frac{\partial\theta_1}{\partial\theta} d\theta + \frac{\partial\theta_1}{\partial r} dr$. Finally, by Green's formula,

$$\int_{\bar{B}^+} d\theta dr = \int_{\bar{B}^+} \det \bar{g}_1'(\theta, r) d\theta dr$$

that implies our claim.

Step 2. Define $\Omega(\theta, r) = 1 - \det \bar{g}_1'(\theta, r)$. Then there exist two C^1 functions $\alpha(\theta, r)$ and $\beta(\theta, r)$ 1-periodic in θ that vanish on $\partial\bar{B}$ and such that $\Omega = \frac{\partial\beta}{\partial r} - \frac{\partial\alpha}{\partial\theta}$.
Consider the two functions

$$\alpha(\theta, r) = -\int_0^\theta [\Omega(\Theta, r) - \tilde{\Omega}(r)] d\Theta \quad \text{and} \quad \beta(\theta, r) = \int_{a_0}^r \tilde{\Omega}(\rho) d\rho.$$

with $\tilde{\Omega}(r) = \int_0^1 \Omega(\theta, r) d\theta$. First of all they are of class C^1 because $\bar{g}_1$ is of class C^2.
Notice that from Step 1 we have that

$$\int_{a_0}^b \int_0^1 \Omega(\theta, r) d\theta dr = 0, \tag{2.7}$$

and, remembering that $\bar{g}_1$ is the identity on a neighbourhood of $\partial\bar{B}$, it results

$$\Omega(\theta, b) = 0 \tag{2.8}$$

Moreover, an exact symplectic map is also area preserving so the determinant of the Jacobian is 1. It follows that

$$\Omega(\theta, r) = 0 \text{ on } \mathbb{T} \times [a_0, a]. \tag{2.9}$$

By computation we get $\Omega = \frac{\partial \beta}{\partial r} - \frac{\partial \alpha}{\partial \theta}$ on $\bar{B}^+$. The fact that α and β are 1-periodic with respect to θ is trivial for β, while comes from (2.7) for α. Moreover, remembering (2.8) we have

$$\alpha(\theta, b) = -\int_0^\theta [\Omega(\Theta, b) - \tilde{\Omega}(b)] d\Theta = 0$$

and by (2.7)

$$\beta(\theta, b) = \int_{a_0}^b \int_0^1 \Omega(\theta, r) d\theta = 0$$

and, by (2.9), α and β vanish on $\mathbb{T} \times [a_0, a]$. We can do the same on $\mathbb{T} \times [-b_0, -a_0]$ and find $\alpha(\theta, r)$ and $\beta(\theta, r)$ with the same property. Finally we can extend these functions to all $\bar{B}$ setting $\alpha(\theta, r) = 0$ and $\beta(\theta, r) = 0$ on A_0 and the properties on $\mathbb{T} \times [a_0, a]$ guarantee the regularity.

Step 3. Consider the function, for $t \in [0, 1]$, $\Omega_t(\theta, r) = (1 - t) + t \det g_1'(\theta, r)$ and define the vector field

$$X_1(t, \theta, r) = \frac{1}{\Omega_t(\theta, r)} \alpha(\theta, r), \quad X_2(t, \theta, r) = -\frac{1}{\Omega_t(\theta, r)} \beta(\theta, r)$$

and the associated differential equation

$$\dot{\theta} = X_1(t, \theta, r), \quad \dot{r} = X_2(t, \theta, r)$$

with solution $\bar{\phi}_t = (\Theta_t, R_t)$ passing through (θ, r) at time $t = 0$. The solution is unique because X_1 and X_2 are of class C^1 (this justifies the hypothesis of f being C^2). We claim that

$$\Omega_t(\Theta_t, R_t) \det(\frac{\partial(\Theta_t, R_t)}{\partial(\theta, r)}) = 1, \quad t \in [0, 1].$$

Remember that a map isotopic to the identity is also orientation preserving, while the converse is false in the cylinder (as a counterexample take the map $(\theta, r) \mapsto (-\theta, -r)$). Hence we have that $\det \bar{g}_1' > 0$ so that the vector field is well defined. Notice that if $(\theta, r) \in \bar{B}$ the solution does not leave $\bar{B}$ because the boundary circles of $\bar{B}$ are continua of fixed points: it implies that $\bar{\phi}_t(\bar{B}) = \bar{B}$. Using Liouville formula for the linearized equation we have, for every $t \in [0,1]$

$$\begin{aligned}
&\Omega_t(\Theta_t, R_t)\det(\frac{\partial(\Theta_t, R_t)}{\partial(\theta, r)}) \\
&= \Omega_t(\Theta_t, R_t)\exp\{\int_0^t tr(\frac{\partial X_1}{\partial \theta} + \frac{\partial X_2}{\partial r})(s, \Theta_s, R_s)ds\} \\
&= \Omega_t(\Theta_t, R_t)\exp\{\int_0^t (-\frac{1}{\Omega_t^2}\alpha\frac{\partial \Omega_t}{\partial \theta} + \frac{1}{\Omega_t^2}\beta\frac{\partial \Omega_t}{\partial r} - \frac{1}{\Omega_t}\frac{\partial \Omega_t}{\partial t})ds\}
\end{aligned}$$

where in the last equality we used the properties of Ω and the fact that $\partial\Omega_t/\partial t = -\Omega$.
So, we have to prove that

$$\Omega_t(\Theta_t, R_t)\exp\{\int_0^t (-\frac{1}{\Omega_t^2}\alpha\frac{\partial \Omega_t}{\partial \theta} + \frac{1}{\Omega_t^2}\beta\frac{\partial \Omega_t}{\partial r} - \frac{1}{\Omega_t}\frac{\partial \Omega_t}{\partial t})ds\} = 1.$$

Passing to the logarithm and differentiating with respect to t this equality is true if and only if

$$-\frac{1}{\Omega_t^2}\alpha\frac{\partial \Omega_t}{\partial \theta} + \frac{1}{\Omega_t^2}\beta\frac{\partial \Omega_t}{\partial r} - \frac{1}{\Omega_t}\frac{\partial \Omega_t}{\partial t} = -\frac{1}{\Omega_t}[\frac{\partial \Omega_t}{\partial \theta}\dot{\Theta}_t + \frac{\partial \Omega_t}{\partial r}\dot{R}_t + \frac{\partial \Omega_t}{\partial t}]$$

that is the case remembering the definition of Θ_t and R_t.

Step 4. The function $\bar{g}_2 = \bar{g}_1 \circ \bar{\phi}_1$ satisfies the lemma.
Indeed, by the previous step

$$\det \bar{g}_2' = \det(\bar{g}_1' \circ \bar{\phi}_1)\det \bar{\phi}_1' = \Omega_1(\Theta_1, R_1)\det(\frac{\partial(\Theta_1, R_1)}{\partial(\theta, r)}) = 1$$

that means that is area preserving. Moreover, by the definition of the vector field (X_1, X_2) we have $\bar{\phi}_1|_{\partial\bar{B}} = Id$ and $\bar{\phi}_1|_{\bar{A}_0} = Id$ that imply $\bar{g}_2|_{\partial\bar{B}} = Id$ and $\bar{g}_2|_{\bar{A}_0} = \bar{f}$. □

With the same notation let us conclude with the proof of Lemma 7. This part does not involve differential forms so we report the version by Franks.

Proof of Lemma 7. Let $g_2 : B \to B$ be the lift (fixed by the boundary twist condition) of $\bar{g}_2$ that extends $f : A \to B$. Now consider

$$M_0 := \sup_{x \in B} d(g_2(x), x)$$

where d is the distance in $\mathbb{R}^2$ and fix $M > M_0$. So we have that M is greater than the distance that a point in B could be moved by g_2. Now consider the strip $A^+ = \mathbb{R} \times [a_0, a_0 + \epsilon] \subset A$ such that by the boundary twist condition and continuity we have that for all $x \in A^+$

$$P(g_2(x)) - P(x) > \epsilon \tag{2.10}$$

where $P(x_1, x_2) = x_1$ is the projection on the first component.
Define $h : B \to B$ by

$$h(\theta, r) = (\theta + M\rho(r), r)$$

where $\rho(r)$ is smooth, monotone such that $\rho(r) = 0$ for $r < a_0$ and $\rho(r) = 1$ for $r > a_0 + \epsilon$. Notice that h is area preserving, it is the identity for $r < a_0$, it is a translation by M if $r > a_0 + \epsilon$, if $x \in A^+$ then $h(x) \in A^+$ and $P(h(x)) > P(x)$.
Finally consider

$$g_3 = g_2 \circ h.$$

For $x \in A^+$ we have, using (2.10)

$$P(g_3(x)) - P(x) = P(g_2(h(x))) - P(x) > \epsilon + P(h(x)) - P(x) > \epsilon > 0$$

which means that we do not have fixed points in A^+. Moreover, if we take $x = (\theta, r) \in \mathbb{R} \times [a_0 + \epsilon, b]$ then $g_3(x) = g_2(h(x)) = g_2(\theta + M, r)$ and by definition of M that means that we do not have fixed points in $\mathbb{R} \times [a_0 + \epsilon, b]$ and the boundary twist condition is satisfied on $\mathbb{R} \times \{b\}$.
To conclude we consider $A^- = \mathbb{R} \times [-a_0 - \epsilon, -a_0]$ and define analogously $\tilde{h}(\theta, r) = (\theta - M\rho(r), r)$ with similar properties of h. Defining $g = g_3 \circ \tilde{h}$ we get also the complete boundary twist condition. □

Let us conclude with a remark on the stability of such fixed points. Remember that a fixed point p of a one-to-one continuous map $f : U \subset \mathbb{R}^N \to \mathbb{R}^N$ is said to be stable in the sense of Liapunov if for every neighbourhood V of p there exists another neighbourhood $W \subset V$ such that, for each $n > 0$, $f^n(W)$ is well defined and $f^n(W) \subset V$. We have:

Corollary 1. *If f is analytic, at least one of the fixed points coming from theorem 2 is unstable.*

Proof. For the special case of dimension two, there exists a relation between the stability of a fixed point and its fixed point index. In fact it was proved in [62] that if a continuous one-to-one map f which is also orientation and area preserving has a stable fixed point p then either $f = Id$ in some neighbourhood of p or there exists a sequence of Jordan curves $\{\Gamma_n\}$ converging to p such that, for each n,

$$\Gamma_n \cap Fix(f) = \emptyset, \quad i(f, \hat{\Gamma}_n) = 1$$

where $\hat{\Gamma}_n$ is the bounded component of $\mathbb{R}^2 \setminus \Gamma_n$.
The set of fixed points can be described by the equation

$$(\Theta(\theta, r) - \theta)^2 + (R(\theta, r) - r)^2 = 0$$

This is an analytic subset of the plane, indeed is the set of the zeros of an analytic function. The local structure of these sets is described in [32]. There are two alternative possibilities: they can be isolated points or points with a finite number of branches emanating from them. Suppose that we are in the case of a non isolated fixed point and that there exists a sequence of Jordan curves converging to the point and not crossing the set. Every one of such curves has index 1 so it must contain a fixed point and this is in contradiction with the the local structure of the set of fixed points as new fixed points appear.

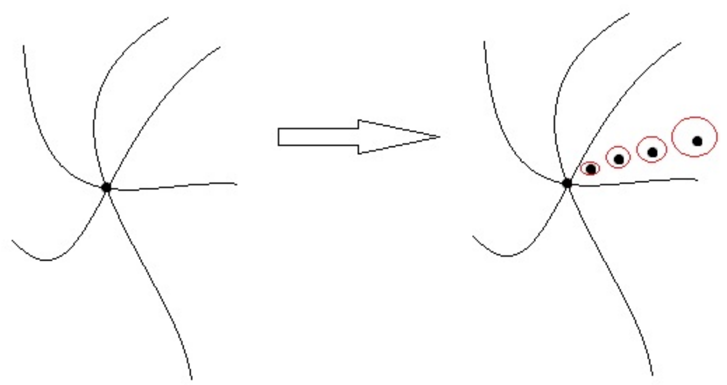


Figure 2.1: The contradiction in the non isolated case

In the isolated case by a standard compactness argument we can have only finitely many fixed points. The Poincaré-Hopf index formula for manifolds with boundary [22, p.447 Theorem 3.1], shows that

$$\lambda(f) = \sum_{j=1}^{n} i(f, p_j)$$

where $\lambda(f)$ denotes the Lefschetz number and $p_1, \ldots, p_n$ are the fixed points. Notice that here f is understood as the isotopic extension to the larger strip B so that $f : B \to B$ and $\lambda(f)$ is well defined. Remembering that f is isotopic to the inclusion, we have that $\lambda(f) = \lambda(Id)$. But from [22, p.233 Proposition 4.5] we have $\lambda(Id) = \chi(B)$ where $\chi(B)$ is the Euler characteristic of the strip B that is null. So

$$0 = \sum_{j=1}^{n} i(f, p_j)$$

and at least one fixed point does not have positive index and so it is unstable. □

So far we have not considered the twist condition and in fact it is not necessary for the proof of Franks version. Anyway, if we suppose that also the twist property is satisfied, something more can be said. We are considering the map $f : \Omega \to \mathbb{R}$ where $\Omega = \{(\theta, r) \in \mathbb{R}^2 : a < r < \psi(\theta)\}$, a is a fixed constant and $\psi : \mathbb{R} \to]a, +\infty]$ is a 1-periodic, lower semi-continuous function. We remember that the twist condition reads as

$$\frac{\partial \Theta}{\partial r} > 0$$

on Ω. Note that the twist condition and the boundary twist condition used in Franks Theorem or in the Poincaré-Birkhoff theorem are independent. Indeed, consider the annulus $A = \mathbb{T} \times [-1, 1]$: the map

$$\begin{cases} \theta_1 = \theta + e^r \\ r_1 = r \end{cases}$$

satisfies the twist condition but not the boundary twist one; the map

$$\begin{cases} \theta_1 = \theta + r^2 + \frac{3}{2} r \\ r_1 = r \end{cases}$$

satisfies the boundary twist condition but not the twist one. Moreover the two conditions coexist in the map

$$\begin{cases} \theta_1 = \theta + r \\ r_1 = r. \end{cases}$$

Using the twist condition, a first result that can be proved is the following

Theorem 3 ([63]). *Assume that f is exact symplectic and satisfies the twist condition. Fix an integer N and assume that for each $\theta \in \mathbb{R}$ there exists $r_\theta \in]a, \psi(\theta)[$ with*

$$\Theta(\theta, a) < \theta + N < \Theta(\theta, r_\theta). \tag{2.11}$$

Then the system

$$\begin{cases} \Theta(\theta, r) = \theta + N \\ R(\theta, r) = r, \end{cases} \tag{2.12}$$

with $\theta \in [0, 1[, (\theta, r) \in \Omega$, has at least two solutions.

Notice that the solutions of system (2.12) can be seen as fixed point of the map defined on the cylinder. The integer N stands for the number of revolutions given by the point around the cylinder. Some interesting corollaries of this theorem are given:

Corollary 2. *If in theorem 3 we require also that $f(\theta, r)$ is analytic then the set of the solutions of system (2.12) is either finite or the graph of an analytic 1-periodic function.*

Proof. By (2.11) and the twist condition, we get that for each θ the equation

$$\Theta(\theta, r) = \theta + N \tag{2.13}$$

has a unique solution $r := \phi(\theta)$. By the uniqueness we have that ϕ is 1-periodic. Moreover, because of the twist condition we can apply, for a fixed θ, the analytic version of the implicit function theorem and get an open neighbourhood U_θ and an analytic function $\tilde{\phi} : U_\theta \to \mathbb{R}$ such that $\Theta(\theta, \tilde{\phi}(\theta)) = \theta + N$ on U_θ. But, by uniqueness, $\phi(\theta) = \tilde{\phi}(\theta)$ on U_θ. Repeating the argument for each θ, we get that ϕ is also analytic. So, $f(\theta, \phi(\theta))$ is the graph of an analytic function in the cylinder: let us call it $\phi_1(\theta)$. This comes from the analyticity and the periodicity of f and the fact that $\phi(\theta)$ satisfies equation (2.13). So, by Lemma 3, ϕ and ϕ_1 must intersect in at least two points of the cylinder that are the solutions of system (2.12) when lifted. Moreover, from the theory of analytic functions, we know that either the set $\{\theta \in [0, 1] : \phi(\theta) = \phi_1(\theta)\}$ is finite or $\phi(\theta) = \phi_1(\theta)$ for every θ.

□

Corollary 3. *Suppose that in theorem 3, condition (2.11) is satisfied for some $N \in \mathbb{Z}$. Let $(\hat{\theta}, \hat{r})$ be an isolated solution of system (2.12) and define the map $T(\theta, r) = (\theta + 1, r)$. Then $i(T^{-N} f, (\hat{\theta}, \hat{r}))$ is either -1 or 0 or 1.*

Proof. First of all notice that $i(T^{-N} f, (\hat{\theta}, \hat{r}))$ is well defined because $(\hat{\theta}, \hat{r})$ is an isolated fixed point of $T^{-N} f$. To compute the index remember that

$$i(T^{-N} f, (\hat{\theta}, \hat{r})) = \deg(T^{-N} f - Id, B_\delta(\hat{\theta}, \hat{r}))$$

where deg indicates the Brouwer degree and δ could be chosen small enough by the excision property. So we will deal with the degree of the map

$$\begin{aligned}(T^{-N}f - Id)(\theta, r) &= (\Theta(\theta, r) - N - \theta, R(\theta, r) - r)\\ &:= (\tilde{\theta}(\theta, r), \tilde{R}(\theta, r)) := \tilde{f}(\theta, r)\end{aligned}$$

and to compute it we will use a technique by Krasnosel'skii [33] that allows to reduce the dimension.
By the hypothesis, the point $(\hat{\theta}, \hat{r})$ is an isolated zero of $\tilde{\Theta}(\theta, r)$ and by the twist condition $\frac{\partial\tilde{\Theta}}{\partial r} = \frac{\partial\Theta}{\partial r} > 0$. So we can apply the implicit function theorem to the equation $\tilde{\Theta}(\theta, r) = 0$ and find a C^1 function $\phi(\theta)$ defined on a neighbourhood of $\hat{\theta}$ such that $\tilde{\Theta}(\theta, \phi(\theta)) = 0$ and $\phi(\hat{\theta}) = \hat{r}$. Hence it is well defined the function

$$\Phi(\theta) = \tilde{R}(\theta, \phi(\theta))$$

which has $\hat{\theta}$ as an isolated zero. Now consider the homotopy

$$H((\theta, r), \lambda) = \begin{cases}\lambda\tilde{\Theta}(\theta, r) + (1-\lambda)(r - \phi(\theta))\\ \lambda\tilde{R}(\theta, r) + (1-\lambda)\Phi(\theta).\end{cases}$$

We claim that it is admissible i.e. $(\hat{\theta}, \hat{r})$ is an isolated zero for every λ. Indeed consider the system

$$\begin{cases}\lambda\tilde{\Theta}(\theta, r) + (1-\lambda)(r - \phi(\theta)) = 0\\ \lambda\tilde{R}(\theta, r) + (1-\lambda)\Phi(\theta) = 0.\end{cases} \tag{2.14}$$

Because of the twist condition, if we define $\mathcal{F}(\theta, r, \lambda) = \lambda\tilde{\Theta}(\theta, r) + (1-\lambda)(r - \phi(\theta))$, we have

$$\frac{\partial\mathcal{F}}{\partial r}(\hat{\theta}, \hat{r}, \lambda) = \lambda\frac{\partial\tilde{\Theta}}{\partial r}(\hat{\theta}, \hat{r}) + (1-\lambda) > 0$$

and so we can apply the implicit function theorem to solve the first equation in a neighbourhood of $\hat{\theta}$ and by the uniqueness the only solution is $r = \phi(\theta)$. Substituting it in the second equation we get, because of the definition of $\Phi(\theta)$,

$$\lambda\tilde{R}(\theta, \phi(\theta)) + (1-\lambda)\Phi(\theta) = 0 \Rightarrow \lambda\Phi(\theta) + (1-\lambda)\Phi(\theta) = 0 \Rightarrow \Phi(\theta) = 0$$

that, remember, has $\hat{\theta}$ as an isolated solution. So $(\hat{\theta}, \hat{r})$ is an isolated solution of system (2.14) and we can choose δ small enough such that $(\hat{\theta}, \hat{r})$ is the only solution in $B_\delta(\hat{\theta}, \hat{r})$. So we are led to the computation of the degree of the map

$$(\theta, r) \longmapsto (r - \phi(\theta), \Phi(\theta))$$

that, if $\Phi'(\hat{\theta}) \neq 0$, can be easily computed by linearization.
However it could happen that $\Phi'(\hat{\theta}) = 0$. So consider the other homotopy

$$H((\theta,r)\lambda) = \begin{cases} \lambda\tilde{\Theta}(\hat{\theta},r) + (1-\lambda)(r-\phi(\theta)) \\ \Phi(\theta) \end{cases}$$

where $\hat{\theta}$ is fixed. To prove that it is admissible, consider the system

$$\begin{cases} \lambda\tilde{\Theta}(\hat{\theta},r) + (1-\lambda)(r-\phi(\theta)) = 0 \\ \Phi(\theta) = 0. \end{cases}$$

By the definition of $\Phi(\theta)$ we have that $\hat{\theta}$ is an isolated solution of the second equation that, substituted in the first one, gives $\lambda\tilde{\Theta}(\hat{\theta},r)+(1-\lambda)(r-\hat{r}) = 0$. We have that $\hat{r}$ is a solution and is also the only one, because by the twist condition we have

$$\frac{\partial}{\partial r}[\lambda\tilde{\Theta}(\hat{\theta},r) + (1-\lambda)(r-\hat{r})] > 0.$$

So $(\hat{\theta},\hat{r})$ is an isolated solution of the system, the homotopy is admissible and we can compute the degree of the function

$$W(\theta,r) = (\tilde{\Theta}(\hat{\theta},r), \Phi(\theta)).$$

To use the factorization property of the degree consider the function $L(x,y) = (y,x)$. We have

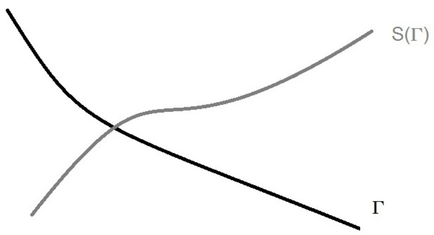
$$\deg(L \circ W, B_\delta(\hat{\theta},\hat{r})) = \deg(L, B_\delta(0,0)) \deg(W, B_\delta(\hat{\theta},\hat{r})) = -\deg(W, B_\delta(\hat{\theta},\hat{r})).$$

Now, by the factorization property

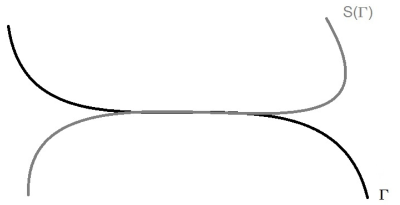
$$\begin{aligned} \deg(W, B_\delta(\hat{\theta},\hat{r})) &= -\deg(\tilde{\Theta}, I_{\hat{r}}) \deg(\Phi, I_{\hat{\theta}}) \\ &= -\,sign\{\frac{\partial F}{\partial r}(\hat{\theta},\hat{r})\} \deg(\Phi, I_{\hat{\theta}}) = -\deg(\Phi, I_{\hat{\theta}}). \end{aligned}$$

The function Φ is defined in dimension 1 so its degree can be either 0 or 1 or -1. Finally $i(T^{-N}f,(\hat{\theta},\hat{r}))$ can be either 0 or 1 or -1. □

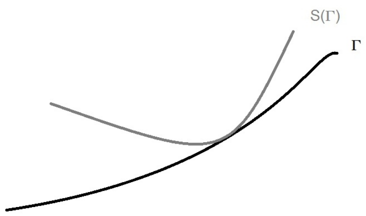
Remark 1. *An intuitive idea of when these cases could occur is given by figure 1.*



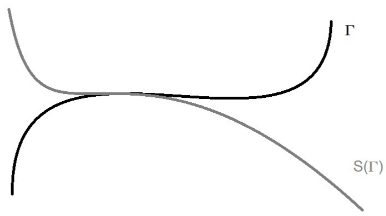
(a) $i = +1$, $\Phi'(\hat{\theta}) \neq 0$



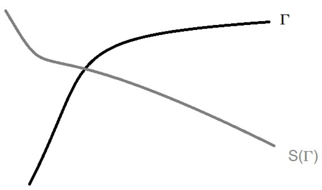
(b) $i = +1$, $\Phi'(\hat{\theta}) = 0$



(c) $i = 0$, $\Phi'(\hat{\theta}) = 0$



(d) $i = -1$, $\Phi'(\hat{\theta}) = 0$



(e) $i = -1$, $\Phi'(\hat{\theta}) \neq 0$

Figure 2.2: Possibilities of intersections

2.2 Invariant curves and KAM theory

In the previous section we saw that the classical Poincaré-Birkhoff theorem can be applied in the case we have a map of the cylinder that leaves a strip invariant. Anyway we said that this condition is quite difficult to verify in the applications. This section is devoted to the study of the invariant curves that guarantee, between other properties, the existence of these invariant region.

With the notation used so far, let $f : \Sigma \to \mathscr{C}$ be an embedding of the cylinder. An *invariant curve* Γ is a non-contractible simple Jordan curve contained in Σ such that $f(\Gamma) = \Gamma$. Notice that there can exist curves Γ such that $f(\Gamma) = \Gamma$ that are contractible: we are excluding this kind of curves. Invariant curves are very important for the dynamics of the homeomorphism f as they give bounds to the orbits. More precisely suppose that there exists an invariant curve Γ for the homemorphism f. First of all it comes from the definition that every orbit starting on Γ stays on Γ so that

$$\sup_{n\in\mathbb{Z}} |r_n| < \infty.$$

Moreover, the curve Γ divides the cylinder in two connected components, the upper and the lower one. Being f an homeomorphism, we have that the image of a connected component is another connected component so that there cannot exists orbits (θ_n, r_n) such that

$$\liminf_{n\to-\infty} r_n = -\infty \quad \text{and} \quad \limsup_{n\to+\infty} r_n = +\infty.$$

or

$$\limsup_{n\to-\infty} r_n = \infty \quad \text{and} \quad \liminf_{n\to+\infty} r_n = -\infty.$$

It means that there cannot exist orbits connecting the top and the bottom of the cylinder. Another interesting case occurs when we have two invariant curves Γ_1 and Γ_2 that do not intersect. Let Σ be the bounded region delimited by such curves. Once again, from the fact that f is an homeomorphism we have that $f(\Sigma) = \Sigma$. Notice that we have to use the fact that F sends connected components to connected components and preserves the boundaries. So, if the curves are graphs of two functions $\mu_1(\theta)$ and $\mu_2(\theta)$ such that $\mu_1 < \mu_2$, the strip Σ satisfies the condition on the invariance required by the classical Poincaré-Birkhoff theorem. Notice that from the fact that $f(\Sigma) = \Sigma$ we can also deduce that every orbit starting in Σ is bounded in the sense that if $(\theta, r) \in \Sigma$ then

$$\sup_{n\in\mathbb{Z}} |r_n| < \infty.$$

We have just seen the importance of invariant curves in the study of the dynamics of an homeomorphism of the cylinder, so we are going to present a theory that investigates conditions guaranteeing the existence of such curves: the so-called KAM theory.
The name KAM comes from the initial of Kolmogorov, Arnol'd and Moser who initiated the theory. The pioneer was Kolmogorov who in the International Congress of Mathematicians, held in Amsterdam in 1954 [31], gave a sketch of the proof that a small analytic perturbation of an integrable analytic Hamiltonian systems preserve most of the invariant tori. In [31], Kolmogorov refers to the paper [30] for a proof of this result. He gave the sketch of how to dominate small divisors through the Newton's method of convergence. This is the point in which a diophantine condition appears. We quote that also Siegel [70] dealt with small divisors and diophantine conditions in the context of holomorphic functions. Arnol'd [3], on the line of Kolmogorov, gave a complete proof of Kolmogorov result. It was Moser [54] that interpreted the theory in the background of discrete dynamics giving an analogous result in the case of applications with only a finite number of derivatives. Moser was the reviewer of the paper [31], and seemed not to be convinced at all by Kolmogorv argument as he wrote

> [. . .] the convergence discussion does not seem convincing to the reviewer. [. . .]

In fact, he proved the result combining Newton method with a different result by Nash [58].

As we are interested in maps of the cylinder we will consider for the presentation only Moser framework. Beside Moser's fundamental theorem [54], known as invariant curve theorem, we will present some consequences and generalizations taken from [24, 25, 60, 61].
We are looking for invariant curves of maps f defined from a closed cylinder $\mathcal{A} = \mathbb{T} \times [a, b]$ to $\mathscr{C}$. Let us start with some example. A first example of a map having invariant curves is the following

$$\begin{cases} \theta_1 = \theta + \omega \\ r_1 = r \end{cases} \tag{2.15}$$

with $\omega \in \mathbb{R}$. It is obvious that every curve Γ of the form $r = const$ is invariant under f. Moreover, the map f restricted to Γ is conjugated to a rotation $\mathcal{R}_\omega$ of angle ω, in the sense there exists an homeomorphism ψ such that the following diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ f\ } & \Gamma \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi} \\ \mathbb{T} & \xrightarrow[\mathcal{R}_\omega]{} & \mathbb{T} \end{array}$$

commutes. In this case we have that $\psi(\theta, r) = \theta$. Note that we can assign to every invariant curve a *rotation number*, defined through the diagram. Note that the map f restricted to Γ is still isotopic to the identity so that preserves the orientation on Γ. It allows to say that the rotation number is well defined. In this case every invariant curve has rotation number ω.
Another example is given by the twist map

$$\begin{cases} \theta_1 = \theta + \omega + \alpha(r) \\ r_1 = r \end{cases} \tag{2.16}$$

with $\alpha' > 0$. Also in this case we have that the curves of the form $r = const$ are invariant and the map restricted to them is conjugated to the rotation $\mathcal{R}_{\omega+\alpha(r)}$. Notice that in this case we have that for every $\beta \in [\omega+\alpha(a), \omega+\alpha(b)]$ there exists an invariant curve with rotation number α. Moser invariant curve theorem deals with the following problem: does some invariant curve of the twist map survive if a small perturbation of the map is considered? If the answer is positive, how many of them survive?. We will see that the answer depends on the arithmetic properties of the rotation number β. We remember that a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is said of *constant type* if the quantity γ defined by

$$\gamma = \inf\{q^2|\alpha - p/q| : p, q \in \mathbb{Z}, q \geq 1\}$$

is strictly positive. The number γ is called the *Markoff constant* of α. Heuristically, the numbers of constant type are the ones that are badly approximated by rational numbers. One can see that there are infinitely many of these numbers in every compact interval of $\mathbb{R}$. For more details and comments, see [24].
We first notice that if we want that some invariant curve is preserved, we need some hypothesis more. Take as an example the map

$$\begin{cases} \theta_1 = \theta + \omega + \alpha(r) \\ r_1 = r + \epsilon \end{cases} \tag{2.17}$$

that, for ϵ small can be considered as a small perturbation of the twist map. It is clear that every orbit leaves the cylinder $\mathcal{A}$ in a finite number of iterations so that there cannot be invariant curves. A way of excluding such maps is to impose that the map f satisfies the intersection property. In fact map (2.17)

does not satisfies this property as, for example, the curve $r = (a+b)/2$ is mapped on the curve $r = (a+b)/2 + \epsilon$ and clearly does not intersect its image. Maps (2.15) and (2.16) satisfy the intersection property as they are exact symplectic.
To state Moser invariant curve theorem it is important to point out the smoothness. So we will suppose that $\alpha \in C^4[a,b]$ with $\alpha'(r) > 0$ for every $r \in [a,b]$, and that the map f has the form

$$\begin{cases} \theta_1 = \theta + \omega + \alpha(r) + R_1(\theta, r) \\ r_1 = r + R_2(\theta, r) \end{cases} \tag{2.18}$$

with $R_1, R_2 \in C^4(\mathcal{A})$. Moreover we will suppose that f has the intersection property.

Theorem 4. *There exists $\epsilon > 0$, depending on $b - a$ and α such that the map f has invariant curves if*

$$||R_1||_{C^4(\mathcal{A})} + ||R_2||_{C^4(\mathcal{A})} < \epsilon.$$

Remark 2. *The theorem was proved by Moser in [54] assuming that f was of class* 333*. An analytic version can be find in [71] while this version comes from the work of Herman [24, 25].*

Remark 3. *It is known that one can find infinitely many numbers of constant type α belonging to $[\omega+\alpha(a), \omega+\alpha(b)]$. For each of them, it is possible to find an invariant curve Γ such that f restricted to Γ is conjugated to a rotation $\mathcal{R}_\alpha$. These are some of the curves that survive to the perturbation.*

Notice that the role of the twist α is fundamental. Consider the following map defined from the cylinder $\mathcal{A} = \mathbb{T} \times [1,2]$ to $\mathscr{C}$

$$\begin{cases} \theta_1 = \theta + \pi \\ r_1 = r + \epsilon \sin 4\pi\theta. \end{cases} \tag{2.19}$$

One can prove that it is exact symplectic and that all orbits with $\theta_0 \in [0,1) \setminus \{0, 1/4, 1/2, 3/4\}$ escapes from the cylinder $\mathcal{A}$ so that there cannot be invariant curves for ϵ small.
In several applications the map f can be seen as a perturbation of the following Small Twist map

$$\begin{cases} \theta_1 = \theta + \omega + \delta\alpha(r) \\ r_1 = r \end{cases} \tag{2.20}$$

where δ is a small positive parameter. Notice that in this case one cannot apply directly the previous theorem but a modification, called Small twist theorem holds. Suppose that f can be written as

$$\begin{cases} \theta_1 = \theta + \omega + \delta[\alpha(r) + R_1(\theta, r)] \\ r_1 = r + \delta R_2(\theta, r) \end{cases}$$

where $\alpha \in C^4[a, b]$, and $R_1, R_2 \in C^4(\mathcal{A})$. The number $\omega \in \mathbb{R}$ is arbitrary and $\delta \in (0, 1]$ is a parameter. Suppose that the function α satisfies

$$c_0^{-1} \leq \alpha'(r) \leq c_0 \quad \forall r \in [a, b], \quad ||\alpha||_{C^4[a,b]} \leq c_0$$

for some constant $c_0 > 1$. Moreover, we suppose that f satisfies the intersection property. In this framework, it comes from the work of Herman [24, 25] on Moser theorem that

Theorem 5. *There exists $\epsilon > 0$, depending on only on c_0, such that the map f has invariant curves if*

$$||R_1||_{C^4(\mathcal{A})} + ||R_2||_{C^4(\mathcal{A})} \leq \epsilon$$

Remark 4. *The proof does not come directly from the previous theorem because ϵ is independent on δ. Moreover, once again we can get many invariant curves. See [60, Appendix] for more details.*

2.3 Aubry-Mather Theory

The KAM theory provided sufficient conditions for finding invariant curves of area preserving maps of the cylinder. We saw that the presence of invariant curves is a very strong characteristic of a map and in fact we needed to impose relatively strong hypothesis, namely regularity and being a small perturbation of an integrable case. The latter condition can be a great obstacle when trying to apply in concrete cases coming from differential equations. If we try to relax the regularity or we consider large perturbations of the integrable map then the existence of invariant curves is no longer guaranteed. The Aubry-Mather theory uses variational techniques to have the requirements on the differentiability very moderate and to drop the condition on proximity of integrability. All these generalizations have a cost: in fact one can find invariant sets but it is not possible to guarantee that they are curves as in KAM theory. Actually the case of a Cantor set will be typical. This theory has the origin in the works of Aubry-LeDaeron [4] on solid state physics and Mather [46] on twist mappings. A good presentation of the theory can be find in [5], in which other applications are given, and in [48] in which the case of composition of twist maps is considered. During the presentation we will refer mainly to these two works.

The Aubry-Mather theory deals with diffeomorphism of the cylinder belonging to the following class:

Definition 1. *Let $\mathcal{P}^\infty = \bigcup_{\beta>0} \mathcal{P}_\beta$, where $\mathcal{P}_\beta$ is the class of maps $f \in \mathcal{E}^1(\mathscr{C})$ that*

1. *are isotopic to the identity*
2. *are exact symplectic*
3. *are β-twist*
4. *preserve the ends of the infinite cylinder, in the sense that $R(\theta, r) \to \pm\infty$ as $r \to \pm\infty$ uniformly in θ*
5. *twist each end infinitely, in the sense that $\Theta(\theta, r) - \theta \to \pm\infty$ as $r \to \pm\infty$ uniformly in θ.*

Consider a map $f \in \mathcal{P}^\infty$. Being f exact symplectic and twist one can consider its generating function h that, by the infinite amount of twist at infinity is a C^2 function from $\mathbb{R}^2$ to $\mathbb{R}$. By a *configuration* we will mean a sequence of real numbers $(\theta_i)_{i\in\mathbb{Z}}$. We can extend the definition of h to a finite segment $(\theta_j, \ldots, \theta_k)$ of a configuration by setting

$$h(\theta_j, \ldots, \theta_k) = \sum_{i=j}^{k-1} h(\theta_i, \theta_{i+1}).$$

A segment $(\theta_j, \ldots, \theta_k)$ is minimal with respect to h if

$$h(\theta_j, \ldots, \theta_k) \le h(\theta_j^*, \ldots, \theta_k^*)$$

for all segments $(\theta_j^*, \ldots, \theta_k^*)$ such that $\theta_j^* = \theta_j$ and $\theta_k^* = \theta_k$. We say that a configuration $(\bar\theta_i)$ is minimal if any finite segment is minimal. The importance of minimal configurations comes from the fact that, as h is differentiable, the configuration is *stationary* in the sense that

$$\partial_1 h(\bar\theta_i, \bar\theta_{i+1}) + \partial_2 h(\bar\theta_{i-1}, \bar\theta_i) = 0 \quad \text{for every } i. \tag{2.21}$$

This property allows to construct a complete orbit $(\bar\theta_i, \bar r_i)$ of f defining

$$\bar r_i = -\partial_1 h(\bar\theta_i, \bar\theta_{i+1}) = \partial_2 h(\bar\theta_{i-1}, \bar\theta_i)$$

that we call *minimal orbit*. So this emphasizes the importance of minimal configurations and all this section is dedicated to the study of them. In [47, §3], Mather proved that the generating function of a map in $\mathcal{P}^\infty$ satisfies the following properties

(H1) $h(\theta+1,\theta_1+1) = h(\theta,\theta_1)$,

(H2) $\lim_{|\xi|\to\infty} h(\theta,\theta+\xi) = \infty$ uniformly in θ,

(H3) $h(\gamma,\theta_1) + h(\theta,\gamma_1) - h(\theta,\theta_1) - h(\gamma,\gamma_1) > 0$ if $\theta < \gamma$ and $\theta_1 < \gamma_1$

(H4) if the segments $(\bar\theta,\theta,\theta_1)$ and $(\bar\xi,\theta,\xi_1)$ are minimal and distinct, then $(\bar\theta-\bar\xi)(\theta_1-\xi_1) < 0$.

Remark 5. *If h is C^2 and such that $\partial_{12}h \le -\delta < 0$, then (H2)-(H4) follow. To obtain (H2) one integrates in the triangle with vertices $(\theta,\theta),(\theta,\theta+\xi),(\theta+\xi,\theta+\xi)$, to obtain (H3) on the quadrangle $(\theta,\theta_1),(\gamma,\theta_1),(\theta,\gamma_1),(\gamma,\gamma_1)$. Property (H4) follows from the monotonicity of $\theta_1 \mapsto \partial_1 h(\theta,\theta_1)$ and $\theta \mapsto \partial_2 h(\theta,\theta_1)$.*

These conditions themselves do not imply the differentiability of h but Bangert proved that they imply the existence of minimal configurations. Anyway if h comes from a map f in $\mathcal{P}^\infty$ it is differentiable and so we can pass from a minimal configuration of h to a minimal obit of f. Throughout the text we will deal with generating functions that are only continuous and so something more will be needed to generate a stationary configuration. The results that we are going to quote are valid under the sole hypothesis that h is continuous and satisfies hypothesis (H1)-(H4).

Given two configurations $\Theta = (\theta_i)$ and $\Gamma = (\gamma_i)$ we say that $\Theta < \Gamma$ if $\theta_i < \gamma_i$ for every i. Two configurations Θ and Γ are comparable if either $\Theta = \Gamma$ or $\Theta > \Gamma$ or $\Theta < \Gamma$. From now on we will refer to a configuration $\Theta = (\theta_i)$. We can define for $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ its translate $T_{p,q}\Theta$ by

$$(T_{p,q}\theta)_i = \theta_{i+q} - p.$$

If $\Theta = T_{p,q}\Theta$ then it is said $(p,q)-$*periodic*. Note that the translates of a minimal configuration are minimal, and we have that any two translate of a minimal configuration are comparable. From this last property it follows that if Θ is minimal then either Θ is constant or is monotone.

We can define the rotation number of Θ as

$$\lim_{i\to\infty} \frac{\theta_i}{i} \tag{2.22}$$

whenever the limit exists and we call it $\rho(\Theta)$. If the configuration is minimal the limit (2.22) exists so that the rotation number $\rho(\Theta)$ is well defined. Moreover, for every $i \in \mathbb{Z}$ we have that

$$|\theta_i - \theta_0 - i\rho(\Theta)| < 1$$

and if Θ is $(p, q)-$periodic then $\rho(\Theta) = p/q$. Conversely we have the main theorem of the Aubry-Mather theory

Theorem 6. *For every $\omega \in \mathbb{R}$ there exists a minimal orbit $\Theta = (\bar{\theta}_i)$ of h such that*

$$\rho(\Theta) = \omega.$$

We will call $\mathcal{M}_\omega$ the set of minimal configuration with rotation number ω. The structure of $\mathcal{M}_\omega$ depends on the arithmetic properties of ω. If it is irrational then the crucial fact is that the set $\mathcal{M}_\omega$ is completely ordered with respect to the order relation defined before. Moreover, the projection $p : \mathscr{C} \to \mathbb{T}$ maps $\mathcal{M}_\omega$ homeomorphically onto $p(\mathcal{M}_\omega)$. We have just the following two alternatives:

(a) $p(\mathcal{M}_\omega) = \mathbb{T}$,

(b) $p_0(\mathcal{M}_\omega)$ is a Cantor set.

If ω is rational, say $\omega = p/q$ then $\mathcal{M}_\omega$ is the disjoint union of three stets $\mathcal{M}_\omega^{per}$, $\mathcal{M}_\omega^{+}$, $\mathcal{M}_\omega^{-}$. To describe this sets we start with $\mathcal{M}_\omega^{per}$, the set of periodic configurations in $\mathcal{M}_\omega$. We have that it is non-empty, closed, totally ordered and every configuration has minimal period (p, q). Being the set totally ordered, it makes sense to say that two configurations of $\mathcal{M}_\omega^{per}$ are *neighboring* if there does not exist an element of $\mathcal{M}_\omega^{per}$ between them. So, letting $\theta^- < \theta^+$ be two neighboring elements of $\mathcal{M}_\omega^{per}$ we can define

$$\mathcal{M}_\omega^{+}(\theta^-, \theta^+) = \{\theta \in \mathcal{M}_\omega : \lim_{i\to-\infty} |\theta_i - \theta_i^-| = 0 \text{ and } \lim_{i\to+\infty} |\theta_i - \theta_i^+| = 0\}$$

and analogously

$$\mathcal{M}_\omega^{-}(\theta^-, \theta^+) = \{\theta \in \mathcal{M}_\omega : \lim_{i\to-\infty} |\theta_i - \theta_i^+| = 0 \text{ and } \lim_{i\to+\infty} |\theta_i - \theta_i^-| = 0\}.$$

We denote by $\mathcal{M}_\omega^{-}$ the union of the sets $\mathcal{M}_\omega^{-}(\theta^-, \theta^+)$ extended over all pairs of neighboring elements (θ^-, θ^+). Analogously one can define the set $\mathcal{M}_\omega^{+}$. One can prove that the existence of two neighboring elements $\theta^- < \theta^+$ is sufficient to say that the sets $\mathcal{M}_\omega^{+}(\theta^-, \theta^+)$ and $\mathcal{M}_\omega^{-}(\theta^-, \theta^+)$ are non-empty. This is the basic theory of minimal configurations, for more details and proof we refer to Bangert [5].

We saw that if we have differentiability of the function h then we can construct from a minimal configuration $(\bar{\theta}_i)$, a minimal orbit $(\bar{\theta}_i, \bar{r}_i)$ of the corresponding diffeomorphism f. The set of minimal orbits is indicated as $M(f)$ and called *Aubry-Mather set*. Every minimal orbit inherits the rotation number of the corresponding minimal configuration so that we can define $M_\omega(f)$ as the subset of $\mathscr{C}$ of all the minimal orbits with rotation number ω. Aubry-Mather sets have a particular structure given by the following proposition proved in [48]

Proposition 1. *If $f \in \mathcal{P}^\infty$ is β-twist, then $M_\omega(f)$ is a compact subset of $\mathscr{C}$. Let π denote the projection of $\mathscr{C}$ on its first factor $\mathbb{R}/\mathbb{Z}$. Then $\pi|_{M_\omega(f)}$ is injective. Consequently, $M_\omega(f)$ is the graph of a suitable function $u : \pi(M_\omega(f)) \to \mathbb{R}$. This function is Lipschitz, more precisely for every $\theta, \theta_1 \in \mathbb{R}$,*

$$|u(\theta_1) - u(\theta)| \leq \cot\beta |\theta_1 - \theta|.$$

Remark 6. *We began this section introducing the Aubry-Mather theory as a generalization of the KAM theory. Hence a question arise: which is the relation between the invariant curves coming from KAM theory and the Aubry-Mather set with irrational rotation number? Note that the theorem before shows that the set $M_\omega(f)$ can be seen as the graph of a Lipschitz function. In general the domain of this function is not the whole $\mathbb{R}$, actually we saw that it can be a Cantor set and then $M_\omega(f)$ in general is not an invariant curve. The answer is given by Mather [48]: if Γ is an invariant curve then $\Gamma \subset M(f)$.*

We stress once more that to pass from a minimal configuration of h to a minimal orbit of f we needed the differentiability of h that is not guaranteed by the sole hypothesis (H1)-(H4). As we will be dealing with non-differentiable generating functions, this problem cannot be eluded. In [47, §4], Mather proved that a diffeomorphism $f \in \mathcal{P}^\infty$ also satisfies the following two properties:

(H5) There exists a positive continuous function ρ on $\mathbb{R}^2$ such that

$$h(\gamma, \theta_1) + h(\theta, \gamma_1) - h(\theta, \theta_1) - h(\gamma, \gamma_1) \geq \int_\theta^\gamma \int_{\theta_1}^{\gamma_1} \rho$$

if $\theta < \gamma$ and $\theta_1 < \gamma_1$, (H6α) there exists $\alpha > 0$ such that

$$\theta \to \alpha\theta^2/2 - h(\theta, \theta_1) \text{ is convex for every } \theta_1$$
$$\theta_1 \to \alpha\theta_1^2/2 - h(\theta, \theta_1) \text{ is convex for every } \theta.$$

Condition (H6) is understood when condition (H6θ) is satisfied for some θ. Conditions (H1)-(H6) are not independent, indeed Mather proved in [47, §4] that (H5) and (H6) implies (H3) and (H4). It means that if we consider hypothesis (H1), (H2), (H5), (H6) then all the results we have cited on minimal configurations still hold. The advantage of considering stronger assumptions is that, even if differentiability cannot be guaranteed, it can be proved [47, §4] that if $(\bar{\theta}_j, \ldots, \bar{\theta}_k)$ is minimal for h then the partial derivatives $\partial_1 h(\bar{\theta}_i, \bar{\theta}_{i+1})$ and $\partial_2 h(\bar{\theta}_{i-1}, \bar{\theta}_i)$ both exist and satisfy

$$\partial_1 h(\bar{\theta}_i, \bar{\theta}_{i+1}) + \partial_2 h(\bar{\theta}_{i-1}, \bar{\theta}_i) = 0 \quad \text{for every } i. \tag{2.23}$$

So as before we can construct minimal orbits and define the Aubry-Mather sets with the same properties in the case h is only continuous and satisfies properties (H1), (H2), (H5), (H6). The results we were stating are in the framework of Mather work [48]. Anyway in the original paper [46] the results appear in a different form. The following lemma gives the relation between the two formalisms.

Lemma 8. *For every $(\bar{\theta}_i, \bar{r}_i) \in M_\omega(f)$ there exist two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ satisfying, for every $t \in \mathbb{R}$*

$$\phi(t+1) = \phi(t) + 1, \quad \eta(t+1) = \eta(t)$$
$$f(\phi(t), \eta(t)) = (\phi(t+\omega), \eta(t+\omega)) \tag{2.24}$$

where ϕ is monotone (strictly if $\omega \notin \mathbb{Q}$) and η is of bounded variation. Moreover, $(\bar{\theta}_i, \bar{r}_i) \in graph(\phi(t), \eta(t))$.

Proof. Inspired by [57], let us consider, for every ω, the set

$$\Sigma = \{t \in \mathbb{R} : \ t = j\omega - k \text{ for some } (j,k) \in \mathbb{Z}^2\}. \tag{2.25}$$

We have to distinguish whether ω is rational or not.
− If ω is irrational, Σ is a dense additive subgroup of $\mathbb{R}$ and every pair (j,k) gives rise to a different number. We proceed by steps.
STEP 1: definition of ϕ on Σ. If $t \in \Sigma$ we define

$$\phi(t) = \bar{\theta}_j - k. \tag{2.26}$$

We claim that the function $\phi : \Sigma \to \mathbb{R}$ is strictly increasing: we have to prove that

$$j\omega - k < j'\omega - k' \Rightarrow \bar{\theta}_j - k < \bar{\theta}_{j'} - k'$$

that is, calling $r = j' - j$ and $s = k' - k$,

$$0 < r\omega - s \Rightarrow \bar{\theta}_j < \bar{\theta}_{j+r} - s.$$

The case $r = 0$ is obvious, so suppose $r \neq 0$. Suppose by contradiction that for some $j \in \mathbb{Z}$

$$\bar{\theta}_j \geq \bar{\theta}_{j+r} - s \tag{2.27}$$

we have, from the comparison property of the translated, that either

$$\bar{\theta}_i > \bar{\theta}_{i+r} - s \quad \text{for every } i.$$

or

$$\bar{\theta}_i = \bar{\theta}_{i+r} - s \quad \text{for every } i.$$

In the second case the orbit would be periodic and this is not compatible with an irrational rotation number. So, from (2.27) we can prove by induction that for every $n \in \mathbb{N}$

$$\bar{\theta}_j > \bar{\theta}_{j+nr} - ns.$$

Now suppose that $r > 0$. Taking the limit for $n \to \infty$ after having divided by nr we get

$$0 \geq \omega - \frac{s}{r}.$$

that leads to a contradiction as we multiply by r. Notice that we can repeat the same argument and get the same contradiction for $r < 0$.
Moreover, ϕ satisfies the periodicity property

$$\phi(t+1) = \phi(t) + 1 \quad \text{for each } t \in \Sigma.$$

STEP 2: extension of ϕ outside Σ. Given $\tau \in \mathbb{R} - \Sigma$, the limits

$$\phi(\tau\pm) = \lim_{t \to \tau^{\pm}, t \in \Sigma} \phi(t)$$

exist and $\phi(\tau-) \leq \phi(\tau+)$. To extend ϕ to a monotone function on the whole real line it is sufficient to impose $\phi(\tau) \in [\phi(\tau-), \phi(\tau+)]$ and we choose $\phi(\tau) = \phi(\tau-)$. In this way $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing and satisfies

$$\phi(t+1) = \phi(t) + 1 \quad \text{for each } t \in \mathbb{R}.$$

STEP 3: definition of η on Σ. Define, for $t \in \Sigma$

$$\eta(t) = \partial_2 h(\phi(t - \omega), \phi(t)) \tag{2.28}$$

where h is the variational principle associated to F. We claim that for every $t, s \in \Sigma$

$$|\eta(s) - \eta(t)| \leq \alpha(\phi(s) - \phi(t)) \tag{2.29}$$

where α comes from (H6α). Supposing $t < s$ we have from the monotonicity

$$\phi(t - \omega) < \phi(s - \omega), \quad \phi(t) < \phi(s), \quad \phi(t + \omega) < \phi(s + \omega).$$

Inspired by [48, Proposition 2.6], we notice that if in (H5) we set $\gamma = \phi(s-\omega)$, $\theta = \phi(t - \omega)$, $\theta_1 = \phi(t) - \epsilon$, $\gamma_1 = \phi(t)$ with $\epsilon > 0$, divide by ϵ and let $\epsilon \to 0$ we get

$$\partial_2 h(\phi(s - \omega), \phi(t)-) \leq \partial_2 h(\phi(t - \omega), \phi(t)).$$

remembering that the partial derivatives exist on the orbits. Moreover, from (H6α) and remembering that the one side partial derivatives of a convex function exist and are non decreasing, we have

$$\partial_2 h(\phi(s - \omega), \phi(s)) \leq \partial_2 h(\phi(s - \omega), \phi(t)-) + \alpha(\phi(s) - \phi(t)).$$

Combining these two inequalities we have

$$\eta(s) \leq \eta(t) + \alpha(\phi(s) - \phi(t)).$$

Using using (2.23) we can see that also $\eta(t) = -\partial_1 h(\phi(t), \phi(t+\omega))$ so we can get analogously

$$\eta(t) \leq \eta(s) + \alpha(\phi(s) - \phi(t))$$

and conclude.
STEP 4: extension of η *outside* Σ. If $\tau \notin \Sigma$ we define

$$\eta(\tau) = \lim_{t \uparrow \tau, t \in \Sigma} \eta(t) \tag{2.30}$$

This is a correct definition. Indeed, from (2.29) we have that

$$|\eta(t_{n+k}) - \eta(t_n)| \leq \alpha|\phi(t_{n+k}) - \phi(t_n)|,$$

and, being $\phi(t_n)$ a Cauchy sequence, we have that $\eta(t_n)$ converges and the limit (2.30) exists. In principle the limit could depend on the sequence. This is not the case, indeed in case that $\eta(t_n^1) \to l_1$ and $\eta(t_n^2) \to l_2$ we can construct a new increasing sequence (τ_n) having (t_n^1) and (t_n^2) as sub-sequences. So also $\eta(\tau_n)$ has to converge to a limit that is the same as l_1 and l_2. So the definition (2.30) makes sense. With this definition we have that estimate (2.29) holds for every $t, s \in \mathbb{R}$. Since ϕ is monotone and hence of bounded variation, we have that η is of bounded variation.

Now, from the periodicity property of h and ϕ we get that $\eta(t+1) = \eta(t)$.
STEP 5: property (2.24) holds. Let us assume first that $t \in \Sigma$. Then $t = j\omega - k$ and

$$\phi(t) = \bar{\theta}_j - k, \quad \phi(t+\omega) = \bar{\theta}_{j+1} - k.$$

Moreover,

$$\eta(t) = \partial_2 h(\phi(t-\omega), \phi(t)) = \partial_2 h(\bar{\theta}_{j-1}, \bar{\theta}_j) = \bar{r}_j$$

and similarly $\eta(t+\omega) = \bar{r}_{j+1}$. Since $(\bar{\theta}_j - k, \bar{r}_j)$ is an orbit of F we conclude that

$$F(\phi(t), \eta(t)) = (\phi(t+\omega), \eta(t+\omega)).$$

Let us assume now that $t \in \mathbb{R} \setminus \Sigma$. So we select a sequence (t_n) converging to t with $t_n \in \Sigma$ and $t_n < t$. Then we can pass to the limit in the identity

$$F(\phi(t_n), \eta(t_n)) = (\phi(t_n+\omega), \eta(t_n+\omega)).$$

The irrational case is done.
− The case $\omega = \frac{p}{q}$ rational is simpler. We can suppose that p and q are relative primes and that the corresponding sequence $(\bar{\theta}_i)$ is periodic (in the sense that $\bar{\theta}_{i+q} = \bar{\theta}_i + p$). First of all notice that in this case, the subgroup Σ defined in (2.25) is discrete, precisely,

$$\Sigma = \{\frac{d}{q} : \ d \in \mathbb{Z}\}.$$

The representation $t = j\omega - k$ is not unique, indeed $t = j\frac{p}{q} - k = j'\frac{p}{q} - k'$ whenever $k' - k = Np$ and $j' - j = Nq$ for some $N \in \mathbb{N}$. Anyway the periodicity of $(\bar{\theta}_i)$ implies that

$$j\frac{p}{q} - k = j'\frac{p}{q} - k' \Rightarrow \bar{\theta}_j - k = \bar{\theta}_{j'} - k'.$$

So we can define ϕ on Σ as in (2.26). As before one can prove that $\phi : \Sigma \to \mathbb{R}$ is increasing (non strictly). We extend it to a monotone function on the whole $\mathbb{R}$ as a piecewise constant function that is continuous from the left and taking only the values $\bar{\theta}_j - k$.

Finally, as before, one can prove that $\phi(t+1) = \phi(t) + 1$. Moreover, the fact that ϕ takes only values at points of a minimal orbit, we can define directly for $t \in \mathbb{R}$

$$\eta(t) = \partial_2 h(\phi(t-\omega), \phi(t)).$$

This function is of bounded variation and condition (2.24) is satisfied as well. To prove this we just have to repeat the same arguments as in the irrational case. Note that this time it is not necessary to pass to the limit. □

In many applications one has to consider a map F that is not twist but can be written as a finite composition of twist maps. So let $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^\infty$ for $i = 1, \ldots, N$. One can easily verify, remembering the preliminaries discussions we made, that F satisfies all the properties of the class $\mathcal{P}^\infty$ apart from the twist condition. If one wants to repeat all the machinery that holds for one single function, the loss of the twist condition is a great fault as we cannot define a generating function, the main tool of the setting. Anyway, every component f_i belongs to $\mathcal{P}^\infty$ and so possesses a generating function h_i satisfying properties (H1), (H2), (H5), (H6). Mather proved that this is sufficient to define an analogous to the generating function for the composition F. This is based on the notion of conjunction, introduced in [47, §5]. Let h_1 and h_2 be two real-valued continuous functions defined on $\mathbb{R}^2$ that satisfy (H2). We set

$$h_1 * h_2(\theta, \theta_1) = \min_{\xi}(h_1(\theta, \xi) + h_2(\xi, \theta_1)).$$

This operation is called *conjunction* and $h_1 * h_2$ is defined and continuous. So, given $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^\infty$ and the corresponding generating functions $h_1, \ldots, h_2$, it is well defined the function

$$h = h_1 * \cdots * h_N.$$

Notice that h is no longer a generating function but it inherits enough properties in order to define the Aubry-Mather sets $M_\omega(F)$. The function h is called *variational principle* of F. The basic property of conjunction is that if h_1 and h_2 are continuous real valued functions that satisfy properties (H1), (H2), (H5), (H6), then so is $h_1 * h_2$. It means that the variation principle h acts as a generating function and we can repeat all the machinery we developed for the case of a single twist map f. Summing up we have the theorem

Theorem 7 ([48]). *Let $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^\infty$ for $i = 1, \ldots, N$. Then for every $\omega \in \mathbb{R}$ there exists an orbit $(\bar{\theta}_i, \bar{r}_i)$ of F such that any two translates of $(\bar{\theta}_i)$ are comparable and the sequence $(\bar{\theta}_i)$ is increasing. Moreover,*

$$\lim_{i \to \infty} \frac{\bar{\theta}_i}{i} = \omega$$

and ω is called rotation number.

Corollary 4. *In the hypothesis of the theorem there exist two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ satisfying the same properties as in Lemma 8.*

Hypothesis 5 in the definition of the class $\mathcal{P}^\infty$ gives some problems in the applications as it is not always satisfied. So we will present a modified version of theorem 7 in which we use the class $\mathcal{P}^{\rho_+,\rho_-}$ defined ad follows

Definition 2. *Let $\mathcal{P}^{\rho_+,\rho_-}$ be the class of maps $f \in \mathcal{E}^2(\mathscr{C})$ that satisfy properties* 1., 2., 4. *of the class $\mathcal{P}^\infty$ and*

3'. are twist

5'. are such that $\Theta(\theta, r) - \theta \to \rho_\pm$ as $r \to \pm\infty$ uniformly in θ,

6. there exists M such that $|R(\theta, r) - r| \leq M$ for every $(\theta, r) \in \mathbb{T} \times \mathbb{R}$

Having relaxed the hypothesis of an infinite twist at infinite has as a counterpart the need of adding some hypothesis on the diffeomorphism F. In particular the presence of invariant curves will be crucial. Hence we give the following notation: let Γ_k be a sequence of invariant curves. We say that $\Gamma_k \uparrow +\infty$ uniformly if there exists a sequence $r_k \to +\infty$ as $k \to +\infty$ such that $\Gamma_k \subset \mathbb{T} \times (r_k, +\infty)$. The reader can easily guess the meaning of $\Gamma_k \downarrow -\infty$ uniformly.

We can prove

Theorem 8. *Consider a finite family $\{f_i\}_{i=1,\dots,N}$ where $f_i \in \mathcal{P}^{\rho_+,\rho_-}$. Let $F = f_1 \circ \dots \circ f_N$. Suppose that F possesses a sequence (Γ_k) of invariant curves such that $\Gamma_k \uparrow +\infty$ uniformly as $k \to +\infty$ and $\Gamma_k \downarrow -\infty$ uniformly as $k \to -\infty$ Then, for every $\omega \in (N\rho_-, N\rho_+)$ there exists an orbit $(\bar\theta_i, \bar r_i)$ of F such that*

$$\lim_{i\to\infty} \frac{\bar\theta_i}{i} = \omega.$$

Moreover there exist two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ satisfying the same properties as in Lemma 8.

We will get the proof of the theorem passing through several lemmas. First we need to recall some basic facts on the wave equation.

Lemma 9. *Consider $f \in C^1(\mathbb{R}^2)$, $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Suppose that $f(t_0, t_1) = f(t_0 + 1, t_1 + 1)$ and that ϕ and ψ are 1-periodic. Then, for $k \in \mathbb{R}$, the following problem*

$$\begin{cases} u_{t_0 t_1} = f(t_0, t_1) \\ u(t_0, t_0 + k) = \phi(t_0) \\ (u_{t_1} - u_{t_0})(t_0, t_0 + k) = \psi(t_0) \end{cases} \tag{2.31}$$

has a unique solution $u \in C^2(\mathbb{R}^2)$ such that $u(t_0, t_1) = u(t_0 + 1, t_1 + 1)$.

Remark 7. *The uniqueness has to be understood in the following sense: for every (t_0^*, t_1^*), consider the characteristic triangle Δ defined by the lines $r_1 : t_0 = t_0^*$, $r_2 : t_1 = t_1^*$, $r_3 : t_1 = t_0 + k$ and let J be the segment of r_3 defined by r_1 and r_2. See figure 2.3. If $f_1 = f_2$ on Δ, $\phi_1 = \phi_2$ on J and $\psi_1 = \psi_2$ on J then we have that $u_1(t_*, x_*) = u_2(t_*, x_*)$, where u_1 and u_2 are the corresponding solutions.*

Proof. Perform the change of variables $t_0 = x - t, t_1 = x + t$. We get

$$\begin{cases} u_{tt} - u_{xx} = \tilde{f}(t, x) \\ u(\frac{k}{2}, x) = \tilde{\phi}(x) \\ u_t(\frac{k}{2}, x) = \tilde{\psi}(x) \end{cases} \tag{2.32}$$

where $\tilde{f}(t, x) = -4f(x - t, x + t)$, $\tilde{\phi}(x) = \phi(x - \frac{k}{2})$ and $\tilde{\psi}(x) = \psi(x - \frac{k}{2})$. Moreover $\tilde{f}(t, x + 1) = \tilde{f}(t, x)$, $\tilde{\phi}(x + 1) = \tilde{\phi}(x)$ and $\tilde{\psi}(x + 1) = \tilde{\psi}(x)$. In the new variables the characteristic triangle $\tilde{\Delta}$ turns to be defined, for every (t_*, x_*) by the lines $\tilde{r}_1 : x - x_* = t - t_*$, $\tilde{r}_2 : x - x_* = t_* - t$, $\tilde{r}_3 : t = \frac{k}{2}$. Let $\tilde{J}$ be the segment on $\tilde{r}_3$ defined by $\tilde{r}_1$ and $\tilde{r}_2$. See figure 2.3.

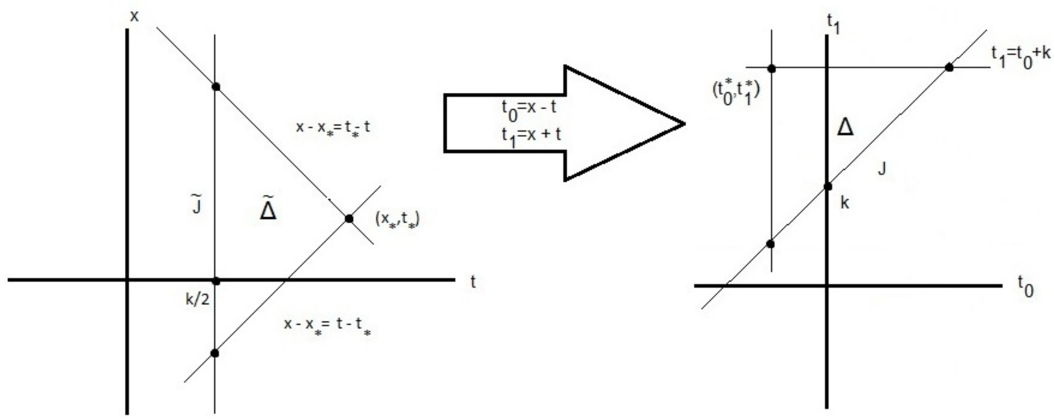


Figure 2.3: The characteristic triangle

The classical theory of the wave equation guarantee the existence of a unique solution $\tilde{u} \in C^2(\mathbb{R}^2)$ such that $\tilde{u}(t, x + 1) = \tilde{u}(t, x)$. Here the uniqueness in understood in a similar way as in the previous remark. Undoing the change of variables we get the thesis. □

Now we are ready to start.

Lemma 10. *Consider* $f \in \mathcal{P}^{\rho_+,\rho_{,-}}$. *Fix an interval* $[a, b]$. *Then, there exists* $\tilde{f} \in \mathcal{P}^{\infty}$ *such that* $f = \tilde{f}$ *on* $\mathbb{T} \times [a, b]$.

Proof. It is convenient to work with the generating function $h(\theta, \theta_1)$. In this case it is a C^3 function defined on the set $\tilde{\Sigma} = \{\rho_- < \theta_1 - \theta < \rho_+\} \subset \mathbb{R}^2$ such that $h(\theta + 1, \theta_1 + 1) = h(\theta, \theta_1)$ and satisfies the Legendre condition $\partial_{12}h < 0$. It generates f in the sense that the map f is defined implicitly by the equations

$$\begin{cases} \partial_1 h(\theta, \theta_1) = -r \\ \partial_2 h(\theta, \theta_1) = r_1. \end{cases} \tag{2.33}$$

Notice that the strip $\mathbb{T} \times [a, b]$ of the cylinder corresponds to the set $\tilde{\Sigma}_2 = \{\alpha(\theta) \leq \theta_1 - \theta \leq \beta(\theta)\} \subset \tilde{\Sigma}$ where α and β are implicitly defined by

$$-\partial_1 h(\theta, \theta + \alpha(\theta)) = a$$
$$-\partial_1 h(\theta, \theta + \beta(\theta)) = b.$$

The functions α and β are C^2, 1-periodic and the Legendre condition implies that $\alpha(\theta) < \beta(\theta)$. Moreover, we have that $\alpha(\theta) \downarrow \rho_-$ as $a \to -\infty$ and $\beta(\theta) \uparrow \rho_+$ as $b \to +\infty$. Now take two larger strips $\tilde{\Sigma}_1 = \{\tilde{a} \leq \theta_1 - \theta \leq \tilde{b}\}$ and $\tilde{\Sigma}_\epsilon = \{\tilde{a} + \epsilon < \theta_1 - \theta < \tilde{b} - \epsilon\}$ such that $\tilde{\Sigma}_2 \subset \tilde{\Sigma}_\epsilon \subset \tilde{\Sigma}_1 \subset \tilde{\Sigma}$ (cfr figure). Notice that, by compactness, there exists $\delta > 0$ such that $\partial_{12}h < -\delta$ on $\tilde{\Sigma}_1$. Now, fix $\varepsilon > 0$ small and extend $\partial_{12}h$ out of $\{\rho_- + \varepsilon \leq \theta_1 - \theta_0 \leq \rho_+ - \varepsilon\}$ as a C^1 bounded function (it is not important how you do it). So we can suppose that there exists a constant $M_1 > 0$ such that

$$\sup_{(\theta_0, \theta_1) \in \mathbb{R}^2} |\partial_{12}h| \leq M_1. \tag{2.34}$$

Consider χ a C^∞ cut-off function of $\mathbb{R}^2$ such that

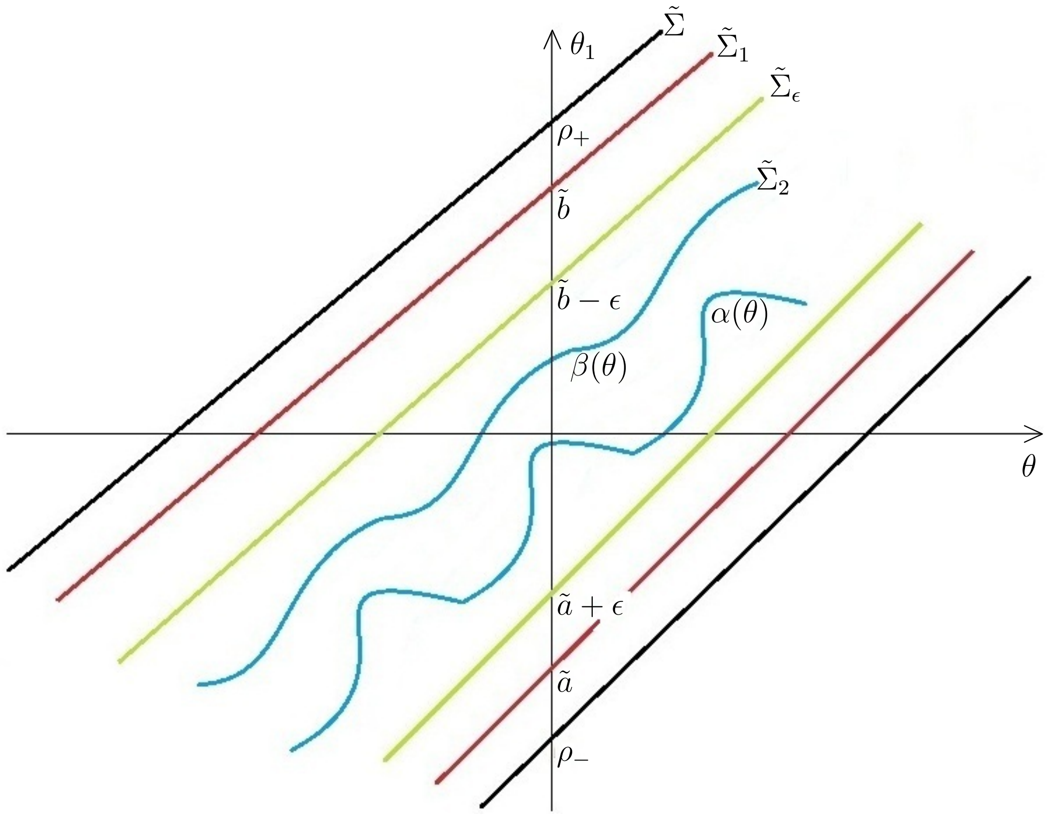
$$\begin{cases} \chi = 1 \text{ on } \tilde{\Sigma}_\epsilon \\ \chi = 0 \text{ on } \{\theta_1 - \theta > \tilde{b}\}. \end{cases}$$

Moreover we can suppose that $\chi = \chi(\theta_1 - \theta)$, $0 \leq \chi \leq 1$ and $\chi > 0$ on $\{\tilde{b} - \epsilon < \theta_1 - \theta < \tilde{b}\}$. Define the new function

$$\Delta = \chi \partial_{12}h + (\chi - 1)\delta.$$

We notice that $\Delta \in C^1$, $\Delta(\theta_1 + 1, \theta + 1) = \Delta(\theta_1, \theta)$ and

$$\begin{cases} \Delta = \partial_{12}h \text{ on } \tilde{\Sigma}_\epsilon \\ \Delta = -\delta \text{ on } \{\theta_1 - \theta > \tilde{b}\} \end{cases}$$



Consider the following Cauchy problem for the wave equation

$$\begin{cases} \partial_{12}u = \Delta(\theta, \theta_1) \\ u(\theta, \theta+\tilde{a}) = h(\theta, \theta+\tilde{a}) \\ (\partial_2 u - \partial_1 u)(\theta, \theta+\tilde{a}) = (\partial_2 h - \partial_1 h)(\theta, \theta+\tilde{a}). \end{cases}$$

Applying lemma 9, the solution h^+ is defined on the set $\{\theta_1 - \theta > \tilde{a}+\epsilon\}$, is such that $h^+ \in C^2$, $h^+(\theta_1+1, \theta+1) = h^+(\theta_1, \theta)$, $\partial_{12}h^+ = \Delta$ and $h^+ = h$ on $\tilde{\Sigma}_\epsilon$. Now perform an analogous argument to modify $\partial_{12}h$ also in the zone $\{\theta_1 - \theta < \tilde{a}\}$ finding h^-. Finally glue h^+ and h^- through the common part $\tilde{\Sigma}_\epsilon$ to get a function $\tilde{h}$. Notice that $\partial_{12}\tilde{h} \leq -\delta$ on $\mathbb{R}^2$. The function $\tilde{h}$ generates via (2.33) a diffeomorphism $\tilde{f}(\theta, r) = (\theta_1, r_1)$ such that the relation

$$\frac{\partial\theta_1}{\partial r} = -\frac{1}{\partial_{12}\tilde{h}}$$

holds. So the diffeomorphism $\tilde{f}$ is β-twist with $\beta = 1/\max\{-\partial_{12}\tilde{h}\}$ and satisfies property 5'. Moreover, as $h = \tilde{h}$ on $\tilde{\Sigma}_\epsilon$, the diffeomorphism $\tilde{f}$ coincides with f on $\mathbb{T} \times [a, b]$. $\square$

It is not hard to guess that we are going to use this lemma to modify the diffeomorphism F through its components f_i. So, it is worth introducing some notation. Given $f \in \mathcal{P}^{\rho_-,\rho_+}$ and an interval $[a, b]$ then the modified diffeomorphism $\tilde{f}$ with support $[a, b]$ is the diffeomorphism coming from lemma 10. Given $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^{\rho_-,\rho_+}$, we will call $\tilde{F}$ with support $[a, b]$ the diffeomorphism given by $\tilde{F} = \tilde{f}_1 \circ \cdots \circ \tilde{f}_N$ where every $\tilde{f}_i$ is supported in $[a, b]$. Moreover, notice that, if $f_i \in \mathcal{P}^\infty$ then trivially $\tilde{f}_i \equiv f_i$. Finally, F has coordinates $(\Theta(\theta, r), R(\theta, r))$ while f_i has coordinates $(\Theta^{(i)}(\theta, r), R^{(i)}(\theta, r))$ and the corresponding modifications have coordinates $(\tilde{\Theta}(\theta, r), \tilde{R}(\theta, r))$ and $(\tilde{\Theta}^{(i)}(\theta, r), \tilde{R}^{(i)}(\theta, r))$.

Lemma 11. *Consider $f \in \mathcal{P}^{\rho_-,\rho_+}$. There exists $K > 0$ such that for every modified $\tilde{f}$ with support $[a, b]$*

$$|\tilde{R}(\theta, r) - r| \leq K \quad \textit{for every } (\theta, r) \in \mathbb{T} \times \mathbb{R}$$

uniformly in $[a, b]$.

Proof. We have to prove that, given a modification with support $[a, b]$, we have the estimate with the constant K independent on $[a, b]$. Consider the generating function $\tilde{h}$ of $\tilde{f}$. We have to estimate the quantity

$$|\partial_2 \tilde{h}(\theta, \theta_1) + \partial_1 \tilde{h}(\theta, \theta_1)|.$$

Notice that, with the notation of the previous lemma, in $[\tilde{b} - \epsilon, \tilde{a} + \epsilon]$ we have $h \equiv \tilde{h}$ so the estimate comes directly from property 6. in the definition of the class $f \in \mathcal{P}^{\rho_-,\rho_+}$. If $\theta_1 - \theta > \tilde{b}$ or $\theta_1 - \theta < \tilde{a}$ then $\tilde{R}(\theta, r) = r$ and $K = 0$. So we only have to study the cases $\tilde{b} - \epsilon \leq \theta_1 - \theta \leq \tilde{b}$ and $\tilde{a} \leq \theta_1 - \theta \leq \tilde{a} + \epsilon$. Let us study the first, being the second similar. We need d'Alambert formula, valid for a function $V \in C^2(\mathbb{R}^2)$:

$$V(\theta, \theta_1) = -\int_{\theta+\delta}^{\theta_1} d\eta \int_{\theta}^{\eta-\delta} \partial_{12} V(\xi, \eta) d\xi + V(\theta, \theta + \delta) + \int_{\theta+\delta}^{\theta_1} \partial_2 V(\eta - \delta, \eta) d\eta$$

where $\delta \in \mathbb{R}$. Applying it to $\tilde{h}$ and choosing $\delta = \tilde{b} - \epsilon$ we get

$$\tilde{h}(\theta, \theta_1) = -\int_{\theta+\tilde{b}-\epsilon}^{\theta_1} d\eta \int_{\theta}^{\eta-\tilde{b}+\epsilon} \Delta(\xi, \eta) d\xi + h(\theta, \theta + \tilde{b} - \epsilon) + \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \partial_2 h(\eta - \tilde{b} + \epsilon, \eta) d\eta$$

Let us compute the partial derivatives. The fundamental theorem of calculus gives

$$\partial_1 \tilde{h}(\theta, \theta_1) = \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \Delta(\theta, \eta) d\eta + \partial_1 h(\theta, \theta + \tilde{b} - \epsilon).$$

Remembering the definition of Δ we have, integrating by parts

$$\int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \Delta(\theta, \eta) d\eta = \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \chi(\eta - \theta) \partial_{12} h(\theta, \eta) d\eta + \delta \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \{\chi(\eta - \theta) - 1\} d\eta =$$

$$\chi(\theta_1 - \theta) \partial_1 h(\theta, \theta_1) - \partial_1 h(\theta, \theta + \tilde{b} - \epsilon) - \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \chi'(\eta - \theta) \partial_1 h(\theta, \eta) d\eta$$

$$+ \delta \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \{\chi(\eta - \theta) - 1\} d\eta$$

where we used the fact that $\chi(\tilde{b} - \epsilon) = 1$. So

$$\begin{aligned} \partial_1 \tilde{h}(\theta, \theta_1) =& \chi(\theta_1 - \theta) \partial_1 h(\theta, \theta_1) + \delta \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \{\chi(\eta - \theta) - 1\} d\eta \\ &- \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \chi'(\eta - \theta) \partial_1 h(\theta, \eta) d\eta. \end{aligned}$$

Similarly,

$$\begin{aligned} \partial_2 \tilde{h}(\theta, \theta_1) =& \chi(\theta_1 - \theta) \partial_2 h(\theta, \theta_1) - \delta \int_{\theta}^{\theta_1 - \tilde{b} + \epsilon} \{\chi(\theta_1 - \xi) - 1\} d\xi \\ &- \int_{\theta}^{\theta_1 - \tilde{b} + \epsilon} \chi'(\theta_1 - \xi) \partial_2 h(\xi, \theta_1) d\xi. \end{aligned}$$

Now we can concentrate on the quantity

$$|\partial_2 \tilde{h}(\theta, \theta_1) + \partial_1 \tilde{h}(\theta, \theta_1)|.$$

To estimate it we first notice that

$$|\chi(\theta_1 - \theta) \partial_2 h(\theta, \theta_1) + \chi(\theta_1 - \theta) \partial_1 h(\theta, \theta_1)| = |\chi(\theta_1 - \theta)| |\partial_2 h(\theta, \theta_1) + \partial_1 h(\theta, \theta_1)| \leq M$$

using property 6 in the definition of the class $\mathcal{P}^{\rho_+, \rho_-}$. Moreover, with the change of variable $\theta_1 - \xi = \eta - \theta$ we get

$$\delta \int_{\theta+\tilde{b}-\epsilon}^{\theta_1} \{\chi(\eta - \theta) - 1\} d\eta = \delta \int_{\theta}^{\theta_1 - \tilde{b} + \epsilon} \{\chi(\theta_1 - \xi) - 1\} d\xi = 0.$$

So we just have to estimate the quantity

$$|\int_{\theta}^{\theta_1-\tilde b+\epsilon} \chi'(\theta_1-\xi)\partial_2 h(\xi,\theta_1)d\xi + \int_{\theta+\tilde b-\epsilon}^{\theta_1} \chi'(\eta-\theta)\partial_1 h(\theta,\eta)d\eta|$$

that, after the change of variable $\eta = \xi + \tilde b - \epsilon$ in the first integral and having noticed that $|\chi'|$ is bounded, reduces to an estimate of

$$\begin{aligned}&\int_{\theta+\tilde b-\epsilon}^{\theta_1} |\partial_2 h(\eta-\tilde b+\epsilon,\theta_1)+\partial_1 h(\theta,\eta)|d\eta\\ &\le |\theta_1-\theta-\tilde b+\epsilon| \max_{\theta+\tilde b-\epsilon\le\eta\le\theta_1} |\partial_2 h(\eta-\tilde b+\epsilon,\theta_1)+\partial_1 h(\theta,\eta)|.\end{aligned}$$

Remembering that we are working in the region $\tilde b - \epsilon \le \theta_1 - \theta \le \tilde b$,

$$|\theta_1-\theta-\tilde b+\epsilon| \le \epsilon. \tag{2.35}$$

Now, by the Legendre condition, the function

$$\Psi(\eta) = \partial_2 h(\eta-\tilde b+\epsilon,\theta_1)+\partial_1 h(\theta,\eta)$$

is monotone, so $\max_{\theta+\tilde b-\epsilon\le\eta\le\theta_1}|\Psi(\eta)|$ is either $|\Psi(\theta_1)|$ or $|\Psi(\theta+\tilde b-\epsilon)|$. Suppose we are in the first case, being the other similar. We have

$$\begin{aligned}|\Psi(\theta_1)| &\le |\partial_2 h(\theta_1-\tilde b+\epsilon,\theta_1)-\partial_2 h(\theta,\theta_1)| + |\partial_1 h(\theta,\theta_1)+\partial_1 h(\theta,\theta_1)|\\ &\le |\partial_{12} h(c,\theta_1)||\theta_1-\theta-\tilde b+\epsilon| + M\end{aligned}$$

for some $c \in [\theta, \theta_1 - \tilde b + \epsilon]$. Now we can conclude using (2.35) and (2.34). □

Lemma 12. *Let $F(\theta, r)$ be a diffeomorphism of $\mathbb{T}\times\mathbb{R}$. Assume that $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^{\rho_+,\rho_-}$ for $i = 1,\dots,N$. Then, for every $\omega \in (N\rho_-, N\rho_+)$ there exists three non negative constant r_*, A and B such that*

$$\begin{cases}\Theta(\theta,r)-\theta \ge \omega+\eta & \text{for } r > r_*\\ \tilde\Theta(\theta,r)-\theta \ge \omega+\eta & \text{for } r > r_*\\ \Theta(\theta,r)-\theta \le \omega-\eta & \text{for } r < -r_*\\ \tilde\Theta(\theta,r)-\theta \le \omega-\eta & \text{for } r < -r_*\end{cases}$$

where $\tilde F$ has support $[-r_* - A^*, r_* + B^*]$ with $A^* > A$ and $B_* > B$.

Proof. For simplicity of notation, let us prove it for $N = 2$. The proof goes by induction. If $N = 1$, then $\omega \in (\rho_-, \rho_+)$ and then by property $5'$ in the definition of the class $\mathcal{P}^{\rho_+,\rho_-}$ there exist $r_* > 0$ and $\eta > 0$ such that

$$\begin{cases}\Theta(\theta,r)-\theta \ge \omega+\eta & \text{for } r > r_*\\ \Theta(\theta,r)-\theta \le \omega-\eta & \text{for } r < -r_*\end{cases}$$

Every modified $\tilde{F}$ outside $[-r_*, r_*]$ is twist, so

$$\frac{\partial(\tilde{\Theta}(\theta, r) - \theta)}{\partial r} > 0$$

and, remembering that $F(\theta, \pm r_*) = \tilde{F}(\theta, \pm r_*)$ for every θ, one can verify that also

$$\begin{cases} \tilde{\Theta}(\theta, r) - \theta \geq \omega + \eta & \text{for } r > r_* \\ \tilde{\Theta}(\theta, r) - \theta \leq \omega - \eta & \text{for } r < -r_*. \end{cases}$$

Now suppose that $F = f_1 \circ f_2$ so that we fix $\omega \in (2\rho_-, 2\rho_+)$. From the case $N = 1$ there exist ρ_* and η such that, for $i = 1, 2$,

$$\begin{cases} \Theta^{(i)}(\theta, r) - \theta \geq \frac{\omega+\eta}{2} & \text{for } r > \rho_* \\ \tilde{\Theta}^{(i)}(\theta, r) - \theta \geq \frac{\omega+\eta}{2} & \text{for } r > \rho_* \\ \Theta^{(i)}(\theta, r) - \theta \leq \frac{\omega-\eta}{2} & \text{for } r < -\rho_* \\ \tilde{\Theta}^{(i)}(\theta, r) - \theta \leq \frac{\omega-\eta}{2} & \text{for } r < -\rho_*. \end{cases} \tag{2.36}$$

Moreover, as f_2 preserves the end, there exists $r_* > \rho_*$ such that $R^{(2)}(\theta, r) > \rho_*$ for $r > r_*$. So, for $r > r_*$

$$\Theta(\theta, r) - \theta = \Theta^{(1)}(\Theta^{(2)}(\theta, r), R^{(2)}(\theta, r)) - \Theta^{(2)}(\theta, r) + \Theta^{(2)}(\theta, r) - \theta \geq \omega + \eta$$

Analogously we can suppose that

$$\Theta(\theta, r) - \theta \leq \omega - \eta \quad \text{for } r < -r_*.$$

Now take the modified $\tilde{f}_i$ with support bigger than $[-r_* - K, r_* + K]$ where K is the constant coming from lemma 11. Let us estimate the quantity

$$\tilde{\Theta}(\theta, r) - \theta = \tilde{\Theta}^{(1)}(\tilde{\Theta}^{(2)}(\theta, r), \tilde{R}^{(2)}(\theta, r)) - \tilde{\Theta}^{(2)}(\theta, r) + \tilde{\Theta}^{(2)}(\theta, r) - \theta$$

for $r > r_*$. It comes from (2.36) that $\tilde{\Theta}^{(2)}(\theta, r) - \theta \geq \frac{\omega+\eta}{2}$. It remains to prove that

$$\tilde{\Theta}^{(1)}(\tilde{\Theta}^{(2)}(\theta, r), \tilde{R}^{(2)}(\theta, r)) - \tilde{\Theta}^{(2)}(\theta, r) \geq \frac{\omega + \eta}{2}.$$

If $r_* < r \leq r_* + K$ then $\tilde{R}^{(2)}(\theta, r) = R^{(2)}(\theta, r) > \rho_*$ and we get the estimation through (2.36). If $r > r_* + K$ then, by the definition of K, we have $\tilde{R}^{(2)}(\theta, r) > r_* > \rho_*$ and we conclude as before. In an analogous way we have the others estimates. □

Lemma 13. *Let $F(\theta, r)$ be a diffeomorphism of $\mathbb{T} \times \mathbb{R}$ that possesses an invariant curve Γ. Assume that $F = f_1 \circ \cdots \circ f_N$ with $f_i \in \mathcal{P}^{\rho_+,\rho_-}$ for $i = 1, \ldots, N$. Then, for every $\omega \in (N\rho_-, N\rho_+)$ there exist three non negative constants r_*, A and B, such that the following holds. Let (θ_n, r_n) be an orbit of F or of a modified $\tilde{F}$ with support $[-r_* - A^*, r_* + B^*]$ with $A^* > A$ and $B_* > B$. Suppose that for every $\eta > 0$ we have*

$$\liminf_{n\to\infty} \frac{\theta_n}{n} < \omega + \eta \quad \text{and} \quad \limsup_{n\to\infty} \frac{\theta_n}{n} > \omega - \eta$$

then there exists $\bar{n} \in \mathbb{Z}$ such that

$$(\theta_{\bar{n}}, r_{\bar{n}}) \in \mathbb{T} \times (-r_*, r_*).$$

Proof. Let r_*, A and B the constant coming from lemma 12. We can suppose that r_* is large enough to have $\Gamma \subset \mathbb{T} \times (-r_*, r_*)$. The invariant curve divides the cylinder in two components, the upper and the lower and both are F-invariant (resp. $\tilde{F}$-invariant). Notice that to prove it we must use the fact that F (resp. $\tilde{F}$) preserves the ends. It means that there cannot exist orbits that jump from the top to the bottom of the cylinder. So, if an orbit (θ_n, r_n) of F or $\tilde{F}$ is such that $r_n > r_*$ for every n or $r_n < -r_*$ for every n then

$$\liminf_{n\to\infty} \frac{\theta_n}{n} \geq \omega + \eta \quad \text{or} \quad \limsup_{n\to\infty} \frac{\theta_n}{n} \leq \omega - \eta$$

respectively, in contradiction with the hypothesis. □

Now we are ready for the

Proof of theorem 8. Fix $\omega \in (N\rho_-, N\rho_+)$, consider the constants r_*, A and B coming from lemma 13. By hypothesis, we can find two invariant curves Γ_+ and Γ_- contained, respectively in $r > r_*$ or $r < r_*$. Let Σ be the compact region defined by such curves. Let $F^{(j)} = f_1 \circ \cdots \circ f_j$ for $j = 1, \ldots, N$. The sets $F^{(j)}(\Sigma)$ are compacts and so one can find a region $\tilde{\Sigma}$, defined by two invariant curves such that

$$\Sigma \cup F^{(1)}(\Sigma) \cup F^{(2)}(\Sigma) \cup \cdots \cup F^{(N)}(\Sigma) \subset int\tilde{\Sigma}.$$

Analogously, we can find and a region $\Sigma_1 = \mathbb{T} \times [-r_* - A^*, r_* + B^*]$ with $A_* > A$ and $B_* > B$ such that

$$\tilde{\Sigma} \cup F^{(1)}(\tilde{\Sigma}) \cup F^{(2)}(\tilde{\Sigma}) \cup \cdots \cup F^{(N)}(\tilde{\Sigma}) \subset int\Sigma_1.$$

Now modify every f_i outside the strip Σ_1 applying lemma 10 and find the corresponding $\tilde{f}_i$. So we get $\tilde{F} = \tilde{f}_1 \circ \cdots \circ \tilde{f}_n$. The diffeomorphisms $\tilde{F}$ satisfies the hypothesis of theorem 7 so we get an orbit $(\bar{\theta}_n, \bar{r}_n)$ of $\tilde{F}$ with rotation number ω. By lemma 13 there exists $\bar{n}$ such that $(\bar{\theta}_{\bar{n}}, \bar{r}_{\bar{n}}) \in \Sigma$. Notice that Γ_+ and Γ_- are also invariant curves for $\tilde{F}$ and so by the invariance on Σ we have that $(\bar{\theta}_n, \bar{r}_n) \in \tilde{\Sigma}$ for every n. But in $\tilde{\Sigma}$ we have $F = \tilde{F}$ so that $(\bar{\theta}_n, \bar{r}_n)$ is also an orbit of F. Remembering corollary 4 we get the thesis. $\square$

Chapter 3

Applications to Hamiltonian systems

In this chapter we are going to give an example of application of the theorems given in the previous chapter. We consider particular systems of ordinary differential equations (ODEs) called planar Hamiltonian systems. These are systems of ODEs that can be written in the form

$$\begin{cases} q'(t) = \frac{\partial H}{\partial p}(t; q(t), p(t)) \\ p'(t) = -\frac{\partial H}{\partial q}(t; q(t), p(t)) \end{cases} \tag{3.1}$$

where $H(t; q, p)$ is a continuous real valued function differentiable in the variables q and p. It is called Hamiltonian function. Throughout the text we will refer to system (3.1) with the notation

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(t; q, p) \\ \dot{p} = -\frac{\partial H}{\partial q}(t; q, p) \end{cases} \tag{3.2}$$

Generally, the variables (q, p) are referred as position-momentum coordinates. A relevant example of Hamiltonian system comes from the so called Euler-Lagrangian equation. Consider a function $L = L(t; q, \dot{q})$ and the corresponding Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{3.3}$$

If

$$\frac{\partial^2 L}{\partial \dot{q}^2} > 0 \tag{3.4}$$

we can consider the following change of variable

$$\begin{cases} q = q \\ p = \frac{\partial L}{\partial \dot{q}} \end{cases} \tag{3.5}$$

and define a new Hamiltonian through the formula

$$H(t;q,p) = \dot{q}p - L(t;q,\dot{q}) \tag{3.6}$$

where, by (3.4), $\dot{q} = \dot{q}(t;q,p)$.

3.1 The Poincaré map of Hamiltonian systems

The link between Hamiltonian systems and the maps on the cylinder that we were discussing is given by a map describing the dynamics of the solutions. As we want a map defined on a cylinder, some periodicity assumptions have to be imposed: from now on we will be dealing with an Hamiltonian of the form

$$H(t;q,p) = \mathcal{H}(t;q,p) + f(t)q \tag{3.7}$$

for a continuous function f. We will impose the following periodicity assumption

$$\mathcal{H}(t;q+1,p) = \mathcal{H}(t;q,p). \tag{3.8}$$

Consider the Hamiltonian system coming from an Hamiltonian H of the form (3.7)-(3.8). Given the initial conditions

$$\begin{cases} q(0) = \theta \\ p(0) = r \end{cases} \tag{3.9}$$

we suppose that there exists a unique solution denoted as $(q(t;\theta,r), p(t;\theta,r))$ and defined for every $t \in \mathbb{R}$. So, for every $\bar{t} \in \mathbb{R}$, it is well defined the *time-$\bar{t}$ map* $\Pi_{\bar{t}} : \mathbb{R}^2 \to \mathbb{R}^2$ with components

$$\begin{cases} \Theta(\theta,r) = q(\bar{t};\theta,r) \\ R(\theta,r) = p(\bar{t};\theta,r). \end{cases} \tag{3.10}$$

It follows from (3.8) and the uniqueness that the solutions are such that

$$q(t;\theta+1,r) = q(t;\theta,r) + 1 \quad \text{and} \quad p(t;\theta+1,r) = p(t;\theta,r).$$

Hence, the components of $\Pi_{\bar{t}}$ satisfy

$$\Theta(\theta+1,r) = \Theta(\theta,r) + 1 \quad \text{and} \quad R(\theta+1,r) = R(\theta,r)$$

so that the time-$\bar{t}$ map is defined on the cylinder $\mathscr{C}$. Moreover, from the theorem of differentiability with respect to the initial conditions, we have that if all the partial derivatives of $H(t;q,p)$ with respect to the variables (q,p) of order less than or equal to $k+1$ are continuous in the variables (t,q,p) then $\Pi \in C^k(\mathbb{R}^2)$. By uniqueness and the definition for all t, $\Pi_{\bar{t}}$ is a diffeomorphism of $\mathbb{R}^2$ and the function $K : [0,1] \times \mathbb{R}^2$ given by

$$K(\lambda;\theta,r) = (q(\lambda\bar{t};\theta,r), p(\lambda\bar{t};\theta,r))$$

gives the isotopy to the identity. Note that we have to use the generalized periodicity of the solution to prove that for every λ, the function $K(\lambda;\cdot,\cdot)$ is defined on the cylinder. So we have that $\Pi_{\bar{t}} \in \mathcal{E}^k(\mathscr{C})$. It is well known that the time map of Hamiltonian systems is symplectic, but our assumptions allow to say more:

Proposition 2. *Consider $\bar{t} \in \mathbb{R}$. The time-$\bar{t}$ map of an Hamiltonian system satisfying (3.7) and (3.8) is exact symplectic if*

$$\int_0^{\bar{t}} f(t)dt = 0. \tag{3.11}$$

Proof. Let $H = H(t;q,p)$ be the Hamiltonian and $(q = q(t;\theta,r), p = p(t;\theta,r))$ be a solution of the corresponding Hamiltonian system (3.2). We have to find a C^1 function $V = V(\theta,r)$, 1-periodic in θ, such that

$$dV = p(\bar{t};\theta,r)dq(\bar{t};\theta,r) - rd\theta$$

Consider the function

$$V(\theta,r) = \int_0^{\bar{t}} [p(\frac{\partial H}{\partial p}) - H]dt.$$

First of all, remembering (3.7), it follows from the generalized periodicity of the solution and (3.8) that

$$V(\theta+1,r) = \int_0^{\bar{t}} [p(\frac{\partial H}{\partial p}) - \mathcal{H} - f(t)q]dt - \int_0^{\bar{t}} f(t)dt.$$

Using (3.11) we have that $V(\theta+1,r) = V(\theta,r)$ so that V is good candidate. Now let us compute the differential dV. We have

$$\begin{aligned} V_\theta &= \int_0^{\bar{t}} [p\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial p}\right) - \frac{\partial H}{\partial q}\frac{\partial q}{\partial\theta}]dt \\ &= \int_0^{\bar{t}} [p\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial p}\right) + \dot{p}\frac{\partial q}{\partial\theta}]dt \end{aligned} \tag{3.12}$$

using the second equation in (3.2). Now, integrating by parts and using the first equation in (3.2) we get

$$\int_0^{\bar{t}} \dot{p}\frac{\partial q}{\partial \theta}dt = [p\frac{\partial q}{\partial \theta}]_0^{\bar{t}} - \int_0^{\bar{t}} p\frac{\partial \dot{q}}{\partial \theta}dt = [p\frac{\partial q}{\partial \theta}]_0^{\bar{t}} - \int_0^{\bar{t}} p\frac{\partial}{\partial \theta}\left(\frac{\partial H}{\partial p}\right)dt$$

that, substituted in (3.12) gives

$$V_\theta = p(\bar{t})\frac{\partial q}{\partial \theta}(\bar{t}) - p(0)\frac{\partial q}{\partial \theta}(0).$$

Analogously we can get

$$V_r = p(\bar{t})\frac{\partial q}{\partial r}(\bar{t}) - p(0)\frac{\partial q}{\partial r}(0).$$

Hence $dV = p(\bar{t})[\frac{\partial q}{\partial \theta}(\bar{t}) + \frac{\partial q}{\partial r}(\bar{t})] - p(0)[\frac{\partial q}{\partial \theta}(0) + \frac{\partial q}{\partial r}(0)]$, from which we have the thesis. □

Remark 8. *Notice that if we were dealing with a Lagrangian system with Lagrangian L of the form*

$$\begin{aligned} L(t;q,p) &= \mathcal{L}(t;q,\dot{q}) + f(t)q \\ \mathcal{L}(t;q+1,\dot{q}) &= \mathcal{L}(t;q,\dot{q}), \end{aligned} \tag{3.13}$$

then the corresponding time-$\bar{t}$ map would have been exact symplectic as well. It would be sufficient considering

$$V(\theta,r) = \int_0^{\bar{t}} L(t;q,\dot{q})dt$$

with the notation used in the proposition.

Remark 9. *Notice that we are able to give an explicit formula for the primitive V. If condition (3.11) fails then the result is no longer guaranteed. Take as an example the Hamiltonian system*

$$\begin{cases} \dot{q} = p \\ \dot{p} = f(t). \end{cases} \tag{3.14}$$

We are clearly in the framework given by conditions (3.7) and (3.8). Moreover the time-$\bar{t}$ map has the explicit form

$$\begin{cases} \Theta(\theta,r) = \theta + r\bar{t} + \int_0^{\bar{t}} F(t)dt \\ R(\theta,r) = r + F(\bar{t}) \end{cases} \tag{3.15}$$

where $F(t) = \int_0^t f(s)ds$. *So we can apply condition (1.7) to say that the time-*$\bar{t}$ *map is exact symplectic if and only if*

$$\int_0^1 F(\bar{t})d\theta = 0$$

that is if and only if condition (3.11) holds.

We saw that the time-$\bar{t}$ map fits quite well in our theoretical setting of maps in the cylinder. Now we are going to describe how we can use it to describe the dynamics of the solutions of the Hamiltonian system.
To this aim add the hypothesis

$$\mathcal{H}(t+T;q,p) = \mathcal{H}(t;q,p), \quad f(t+T) = f(t). \tag{3.16}$$

for some $T > 0$. Consider the corresponding time-T map and call it $\Pi(\theta, r) = (\Theta(\theta, r), R(\theta, r))$. Due to its importance it is called Poincaré map. Suppose that $(\bar{\theta}, \bar{r})$ is a fixed point of Π. Consider the corresponding solution $(q(t;\bar{\theta},\bar{r}), p(t;\bar{\theta},\bar{r}))$. The uniqueness of the solution and the periodicity condition (3.16) implies that

$$(q(t+T;\bar{\theta},\bar{r}), p(t+T;\bar{\theta},\bar{r})) = (q(t;\bar{\theta},\bar{r}), p(t;\bar{\theta},\bar{r})) \quad \text{for every } t$$

and the solution is said periodic. The Poincaré map also gives informations in the case we are able to prove that for every (θ, r),

$$\sup_{n\in\mathbb{Z}} |r_n| < \infty$$

where $(\theta_n, r_n) = \Pi^n(\theta, r)$. If we suppose that

$$|\frac{\partial H}{\partial q}(t;q,p)| < C \tag{3.17}$$

for some positive constant C then

$$\sup_{t\in\mathbb{R}} |p(t;\theta,r)| < \infty. \tag{3.18}$$

To see this, notice that every t we can find n such that $nT \leq t < (n+1)T$. The mean value theorem implies

$$|p(t) - p(nT)| \leq |\dot{p}||t - nT| \leq CT.$$

Condition (3.18) follows from the fact that CT is independent on t and that $p(nT) = r_n$ so that the sequence $\{p(nT)\}$ is uniformly bounded. Analogously one can prove that the condition

$$\lim_{n\to\infty} |r_n| = \infty$$

for some $(\bar\theta, \bar r)$, is equivalent to

$$\lim_{t\to\infty} |p(t;\bar\theta,\bar r)| = \infty$$

Notice that if we drop condition (3.17) we can only ensure that

$$\limsup_{t\to\infty} |p(t;\bar\theta,\bar r)| = \infty.$$

Finally suppose that there exist two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ such that for every $\xi \in \mathbb{R}$

$$\phi(\xi+1) = \phi(\xi)+1, \quad \eta(\xi+1) = \eta(\xi) \tag{3.19}$$

and

$$\Pi(\phi(\xi), \eta(\xi)) = (\phi(\xi+\omega), \eta(\xi+\omega)). \tag{3.20}$$

for some $\omega \in \mathbb{R}$. Moreover we suppose that ϕ is monotone and η is of bounded variation. Let $X_\xi(t) = (q_\xi(t), p_\xi(t))$ be the solution of the Hamiltonian system with initial condition $(\phi(\xi), \eta(\xi))$. Notice that from (3.19) and uniqueness we have that

$$X_{\xi+1}(t) = X_\xi(t) + (1,0)$$

and from (3.20) and the definition of Π,

$$X_\xi(t+T) = X_{\xi+\omega}(t).$$

To understand in a better way these solutions, we define, inspired by [59],

$$\Phi_\xi(\theta_1,\theta_2) = X_{\theta_2 - \frac{\omega}{T}\theta_1 + \xi}(\theta_1).$$

It satisfies

$$\Phi_\xi(\theta_1+T,\theta_2) = \Phi_\xi(\theta_1,\theta_2), \quad \Phi_\xi(\theta_1,\theta_2+1) = \Phi_\xi(\theta_1,\theta_2) + (1,0)$$

and this says that the function Φ_ξ is doubly periodic once it takes values on the phase space $\mathbb{T}\times\mathbb{R}$. The solution is recovered by the formula

$$X_\xi(t) = \Phi_\xi(t, \frac{\omega}{T}t)$$

when Φ_ξ is continuous as a function of the three variables (ξ,θ_1,θ_2). This kind of solutions are called *quasi-periodic*. Again we are assuming that it takes values on $\mathbb{T}\times\mathbb{R}$. In the discontinuous case the solution will not be quasi-periodic in the classical sense but the bounded variation of the initial conditions implies that quasi-periodicity in the sense of Mather will appear.

When the number ω is rational, say $\omega = \frac{a}{b}$ with a and b relatively prime, then

$$X_\xi(t + bT) = X_\xi(t) + (a, 0)$$

and the solution is periodic with period bT. Once more we are assuming that X_ξ takes values on $\mathbb{T} \times \mathbb{R}$. Classically these solutions are called subharmonic solutions of the second kind. When looked in the covering space and $b = 1$ these solutions are called running solutions. In the general case the solution is said generalized periodic. Finally, consider the limit

$$\lim_{t\to\infty} \frac{q_\xi(t)}{t}.$$

We have that, for $nT \leq t \leq (n+1)T$

$$\frac{q_\xi(t)}{t} = \frac{q_\xi(t) - q_\xi(nT)}{t} + \frac{q_\xi(nT)}{nT}\frac{nT}{t}$$

where the quantity $q_\xi(t) - q_\xi(nT)$ is bounded. So we can compute

$$\lim_{t\to\infty} \frac{q_\xi(t)}{t} = \lim_{n\to\infty} \frac{q_\xi(nT)}{nT} = \lim_{n\to\infty} \frac{q_{\xi+n\omega}(0)}{nT} = \lim_{n\to\infty} [\frac{q_{\xi+\{n\omega\}}(0)}{nT} + \frac{[n\omega]}{nT}] = \frac{\omega}{T}$$

where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$. So, ω/T can be considered as a rotation number of the solution of the Hamiltonian system.

Summing up, we have the following correspondence between an Hamiltonian system in our conditions and its Poincaré map Π:

$$\begin{array}{rcl} \text{Fixed point of } \Pi & \leftrightsquigarrow & \text{Periodic solution,} \\ \text{Minimal orbit with irrational rotation number of } \Pi & \leftrightsquigarrow & \text{Quasi-periodic solution in a generalized sense,} \\ \text{Minimal orbit with rational rotation number of } \Pi & \leftrightsquigarrow & \text{Generalized periodic solution.} \end{array} \tag{3.21}$$

Moreover, if the Hamiltonian H satisfies

$$|\frac{\partial H}{\partial q}| < C$$

then

$$\begin{array}{rcl} \text{Bounded orbit of } \Pi & \leftrightsquigarrow & \text{Bounded solution,} \\ \text{Unbounded orbit of } \Pi & \leftrightsquigarrow & \text{Unbounded solution.} \end{array} \tag{3.22}$$

3.2 An example: the forced relativistic pendulum

We are going to show how to use all the machinery we developed in the first chapter in a concrete case. The equation of the so-called forced relativistic pendulum is

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + a\sin x = f(t), \tag{3.23}$$

where $a > 0$ is a parameter. Moreover we will suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic and such that

$$\int_0^T f(t)dt = 0 \tag{3.24}$$

The motivation and the physical meaning of this equation have been discussed in the introduction. To have a first insight into the problem, consider the case $f = 0$, i.e. the autonomous equation

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + a\sin x = 0 \tag{3.25}$$

with $a \leq \frac{\pi^2}{T^2}$, that can be treated with a phase portrait analysis. Let us consider the case $T = 2\pi$, so that $a \leq 1/4$.
First of all it is easily seen that the points $(k\pi, 0)$, $k \in \mathbb{Z}$ are constant solutions in the phase space $(x, \dot{x})$. This analysis is quite simple because the energy

$$E(x, \dot{x}) = \frac{1}{\sqrt{1-\dot{x}^2}} - a\cos 2\pi x + a \tag{3.26}$$

is a first integral and we suddenly reach the conditions

$$E \geq 1 \quad \text{and} \quad -1 < \dot{x} < 1. \tag{3.27}$$

Remembering that $a \leq 1/4$ we get the phase portrait in figure 3.2 where we have the constant solution $(0, 0)$ for $E = 1$, periodic orbits for $1 < E < 1+2a$, the heteroclinic orbits for $E = 1 + 2a$ and the unbounded solutions for $E > 1 + 2a$. Moreover, from the first integral (3.26) we can see the velocity as a function of the time and energy and $\dot{x} \to \pm 1$ as $E \to +\infty$ depending on the sign of $\dot{x}(0)$.
Now we turn to the study of the period of the periodic orbits, in particular, we will study the number of 2π-periodic orbits.

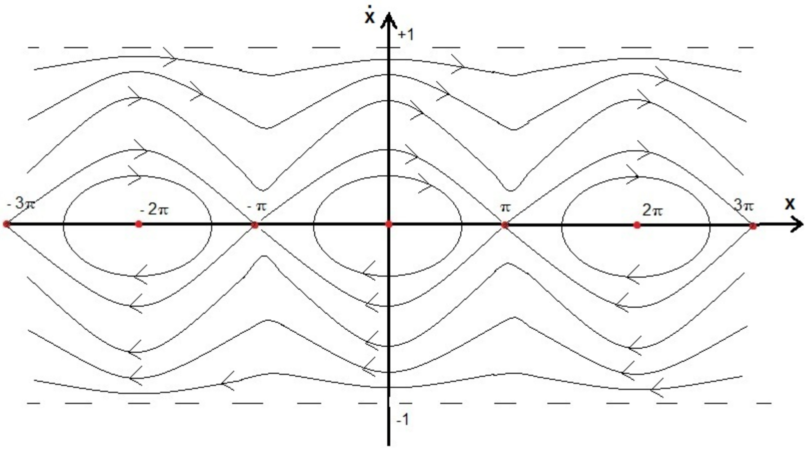


Figure 3.1: Phase portrait

Proposition 3. *The only 2π-periodic solutions of equation (3.25) are the constant ones $(k, 0)$ with $k \in \mathbb{Z}$.*

Proof. We will prove that the period of every orbit (except for the constants one) is strictly greater than 2π. To do so, by the symmetries of the phase portrait, it is enough to prove that for every non-constant periodic orbit, $\dot{x}(\pi, x_0, 0) > 0$ with $x_0 < 0$.
Let us write the solution $x(t, E)$ such that $x(0, E) = \arccos(1 - E/a)$ and $\dot{x}(0, E) = 0$ and compute $\frac{\partial \dot{x}}{\partial E}(\pi, E)$. Remembering (3.27) and $a \leq 1/4$ we have

$$\frac{\partial \dot{x}}{\partial E} > 0 \quad \text{for} \quad E > 1.$$

Notice that the point $(0, 0)$ is a strict minimum of $E(x_0, \dot{x}_0)$ an so

$$\dot{x}(\pi, E(0,0)) < \dot{x}(\pi, E(x_0, \dot{x}_0)) \quad \forall (x_0, \dot{x}_0) \neq (0, 0).$$

Now, remembering that E is constant on the solutions, we have that for every initial condition $(x_0, \dot{x}_0)$ such that $1 < E(x_0, \dot{x}_0) < 1 + \frac{a}{\pi}$ there exists $\hat{x} < 0$ such that $E(x_0, \dot{x}_0) = E(\hat{x}, 0)$ and $\dot{x}(\pi, E(\hat{x}, 0)) > \dot{x}(\pi, E(0, 0)) = 0$.

□

Looking for running solutions can do the following. By the phase portrait analysis we got that for $E > 1 + 2a$ the solution is unbounded and the orbit in the phase plane is the graph of a function. In this case we will show

Proposition 4. *Fix $|\omega| < 1$, $\omega \neq 0$. Then there exists exactly one value of the energy $E > 1 + 2a$ such that*

$$\lim_{t \to \infty} \frac{x(t, E)}{t} = \omega.$$

Proof. Remembering the energy (3.26) we can define a function $T_\omega(E)$ such that $x(T_\omega(E)) = T\omega$ for some $T > 0$ (i.e. $T_\omega(E)$ is the time needed by a solution starting from 0 at $t = 0$ to reach $T\omega$), namely

$$T_\omega(E) = \int_0^{T\omega} \frac{dx}{\sqrt{1 - \frac{1}{(E + a\cos x - a)^2}}}.$$

Notice that it is continuous, monotone decreasing in E and

$$\lim_{E \to 1+2a} T_\omega(E) = +\infty, \qquad \lim_{E \to +\infty} T_\omega(E) = T\omega.$$

Using the properties just mentioned and the fact that by hypothesis $|\omega| < 1$ one has that there exists exactly one value of the energy $E > 1 + 2a$ such that $T_\omega(E) = T$. We have just found a solution such that

$$x(t + T, E) = x(t, E) + T\omega. \tag{3.28}$$

Now let us compute the limit. First of all notice that, for $nT \leq t < (n+1)T$,

$$\frac{x(t, E)}{t} = \frac{x(t, E) - x(nT, E)}{t} + \frac{x(nT, E)}{nT}\frac{nT}{t}.$$

The quantity $x(t, E) - x(nT, E)$ is bounded so the limit exists and is equal to

$$\lim_{n \to \infty} \frac{x(nT, E)}{nT}$$

Using (3.28) on gets

$$\lim_{n \to \infty} \frac{x(nT, E)}{nT} = \lim_{n \to \infty} \frac{x(0, E) + nT\omega}{nT} = \omega.$$

□

The study of the non autonomous systems is more complicated, and we will need all the abstract tools we were discussing in the previous chapter. Our methods allow to consider a more general equation, say

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}}\right) + g(x) = f(t), \tag{3.29}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a differentiable 2π-periodic function such that

$$\int_0^{2\pi} g(s)ds = 0. \tag{3.30}$$

At the moment we do not give more information on the differentiability of the function g as we will point it out throughout the discussion. To apply directly the abstract results, avoiding tedious changes of variable ad adjustments, we will suppose that g is 1-periodic and satisfying

$$\int_0^1 g(s)ds = 0. \tag{3.31}$$

Equation (3.29) can be seen as the Euler-Lagrange equation coming from the Lagrangian

$$L(x, \dot{x}, t) = -\sqrt{1-\dot{x}^2} - G(x) + f(t)x$$

where $G(x) = \int_0^x g(s)ds$ is 1-periodic by (3.31). On the other hand, if we perform the change of variables given by the Legendre Transform

$$\begin{cases} q = x \\ p = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1-\dot{x}^2}}, \end{cases} \tag{3.32}$$

we get the Hamiltonian

$$H(q, p, t) = \sqrt{p^2+1} + G(q) - f(t)q$$

and the new Hamiltonian system

$$\begin{cases} \dot{q} = \frac{p}{\sqrt{1+p^2}} \\ \dot{p} = -g(q) + f(t). \end{cases} \tag{3.33}$$

Notice that the vector field $(H_q, -H_p)$ is bounded so that there exists a unique solution defined for every $t \in \mathbb{R}$. Hence we can define the Poincaré map $\Pi(q_0, p_0) = (\Theta(q_0, p_0), R(q_0, p_0))$, where the coordinates (q_0, p_0) are to be understood as initial conditions. The Hamiltonian is of the form (3.7)-(3.8) so, if $g \in C^k$ then $\Pi \in \mathcal{E}^k(\mathscr{C})$. Moreover, by (3.24) we can apply proposition 2 and say that it is exact symplectic. Moreover, by the boundedness of the vector field and the T-periodicity of f, we can describe the dynamics of the solution through Π, having in mind (3.21)-(3.22).
We are going to study the twist condition. We can prove that

Proposition 5. *If $||g'||_\infty < \frac{\pi^2}{T^2}$, then the Poincaré map Π is β-twist.*

Proof. We have to prove that $\frac{\partial q}{\partial p_0}(T, q_0, p_0) > 0$.
If we call

$$x(t) = \frac{\partial q}{\partial p_0}(t, q_0, p_0) \quad y(t) = \frac{\partial p}{\partial p_0}(t, q_0, p_0)$$

we know from the elementary theory of ODEs that the vector $(x(t), y(t))$ satisfies the variational equation

$$\begin{cases} \dot{x} = \frac{1}{(1+p^2(t,q_0,p_0))^{3/2}} y \\ \dot{y} = -g'(q(t,q_0,p_0))x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$$

that is equivalent to the problem

$$\begin{cases} \frac{d}{dt}(\dot{x}(1+p^2(t,q_0,p_0))^{3/2}) + g'(q(t,q_0,p_0))x = 0 \\ x(0) = 0 \\ \dot{x}(0) = (\frac{1}{p_0^2+1})^{3/2}. \end{cases} \tag{3.34}$$

Now consider the equation

$$\ddot{z} + \frac{\pi^2}{T^2} z = 0 \tag{3.35}$$

and first suppose that $||g'||_\infty < \pi^2/T^2$. In this case we have that

$$(1 + p(t)^2)^{3/2} \geq 1 \quad \text{and} \quad g'(q) < \pi^2/T^2$$

then (3.35) is a strict Sturm majorant of (3.34). So the Sturm theory and the fact that the function $z(t) = \sin(t\frac{\pi}{T})$ is a solution of (3.35), prove that there exists $\beta > 0$ such that $x(T) \geq \beta > 0$ and the thesis will follow.

□

Remark 10. *Let us study the case* $||g'||_\infty = \frac{\pi^2}{T^2}$ *in the simpler contest of equation (3.23). So we have to suppose* $a = \pi^2/T^2$. *If* $f(t) \not\equiv 0$ *then* $q = 2k\pi$ *from being a solution. This means that there exists an open subset of positive measure of* $[0, T]$ *on which* $q \neq 2k\pi$ *and so*

$$\int_0^T \frac{\pi^2}{T^2} \cos q(t) dt < \int_0^T \frac{\pi^2}{T^2} dt.$$

In this case we can use a generalization of the classical Sturm separation theorem. It can be achieved adapting the classical proof (cf. [23]) to our framework. Consider the argumentum θ_1 *and* θ_2 *respectively of (3.34) and (3.35) coming from the Prufer change of variables; then we can conclude that* $\theta_1(T) > \theta_2(T)$. *Remembering that in this framework we have that* $x(\tilde{t}) = 0 \Leftrightarrow \theta(\tilde{t}) = k\pi$ *for some* $k \in \mathbb{Z}$ *and we are rotating in the clockwise sense, we can conclude using the same argumentation of the previous case translated into the phase-space* $(x, p\dot{x})$.

Remark 11. *The condition* $a \leq \frac{\pi^2}{T^2}$ *is optimal. Indeed suppose* $a > \frac{\pi^2}{T^2}$ *and consider the autonomous system*

$$\begin{cases} \dot{q} = \frac{p}{\sqrt{1+p^2}} \\ \dot{p} = -a\sin(q). \end{cases}$$

Notice that $(p = 0, q = 0)$ *is an obvious solution. As before consider the variational equation*

$$\frac{d}{dt}(\dot{x}(1 + p^2(t, q_0, p_0))^{3/2}) + a\cos(q(t, q_0, p_0))x = 0.$$

Notice that evaluated in the above solution it is nothing but

$$\ddot{x} + ax = 0.$$

Using Sturm comparison with $\ddot{y} + \frac{\pi^2}{T^2}y = 0$ *we can conclude analogously as before that* $x(T) < 0$*: it means that we do not have the twist condition. Finally note that in the case* $a = \frac{\pi^2}{T^2}$ *we have* $x(T) = 0$ *and again the twist condition fails.*

Boundedness and unboundedness of the momentum

As we saw in the introduction, equation (3.23) can be seen as a relativistic counterpart of the classical Newtonian pendulum

$$\ddot{x} + a\sin x = f(t). \tag{3.36}$$

This equation has been analysed from many points of view. In particular Levi [38] and You [76] proved that all the solutions of (3.36) have bounded velocity $\dot{x}(t)$ whenever (3.24) holds. The relativistic framework implies that $|\dot{x}(t)| < 1$ and so the boundedness of the velocity is automatic. However we will prove that the results by Levi and You have a relativistic parallel when the velocity is replaced by the momentum

$$p(t) = \frac{\dot{x}(t)}{\sqrt{1 - \dot{x}(t)^2}}.$$

The main result of this section says that if $f(t)$ satisfies (3.24) then all solutions of (3.23) satisfy

$$\sup_{t\in\mathbb{R}} |p(t)| < \infty. \tag{3.37}$$

Moreover we will prove that condition (3.24) is essential for this conclusion. To achieve these result we will consider the Poincaré map Π of system (3.33),

in particular the boundedness result will come from Moser invariant curve theorem.
So remember that, roughly speaking, Moser's theorem gives the existence of invariant curves a class of sufficiently regular maps of the cylinder whose lift has the form

$$\begin{cases}\theta_1 = \theta + \omega + \delta[\alpha(r) + R_1(\theta, r)] \\ r_1 = r + \delta R_2(\theta, r)\end{cases} \tag{3.38}$$

supposing that the reminders R_1 and R_2 were small in some C^k norm. Here δ plays the role of a small parameter.
First of all some regularity is needed, so we will suppose

$$g \in C^7(\mathbb{R}).$$

The coordinates (q, p) are not the best ones to have the Poincaré map written in form (3.38), so perform the following symplectic change of variables

$$\begin{cases} q = Q \\ p = P + G(q) + F(t)\end{cases}$$

where $F(t)$ is a primitive of f. Note that that $F(t)$ is T-periodic and C^1. We get the system

$$\begin{cases}\dot{Q} = \frac{P+G(Q)+F(t)}{\sqrt{1+(P+G(Q)+F(t))^2}} \\ \dot{P} = g(Q)(1 - \frac{P+G(Q)+F(t)}{\sqrt{1+(P+G(Q)+F(t))^2}})\end{cases} \tag{3.39}$$

Now we can introduce the small parameter $\delta > 0$ through the following change of scale

$$Q = u, \quad P = \frac{1}{\delta v} \quad v \in [1/2, 7/2]. \tag{3.40}$$

It is important to note that the strip $\mathbb{R} \times [1/2, 7/2]$ corresponds in the original variables to the time dependent region

$$A_\delta = \{(q, p) \in \mathbb{R}^2 : \frac{2}{7\delta} + G(q) + F(t) \leq p \leq \frac{2}{\delta} + G(q) + F(t)\}$$

and so from the boundedness of F and G

$$p \to \infty \text{ as } \delta \to 0 \text{ uniformly in } v. \tag{3.41}$$

System (3.39) transforms into

$$\begin{cases}\dot{u} = \frac{1+\delta v[G(u)+F(t)]}{\sqrt{\delta^2 v^2+(1+\delta v[G(u)+F(t)])^2}} \\ \dot{v} = -\delta v^2 g(u)[1 - \frac{1+\delta v[G(u)+F(t)]}{\sqrt{\delta^2 v^2+(1+\delta v[G(u)+F(t)])^2}}].\end{cases} \tag{3.42}$$

The change of variables (3.40) is not symplectic, but the Poincaré map of systems (3.42) is still conjugated to Π.
Note that if $\delta = 0$ system (3.42) transforms into

$$\begin{cases} \dot{u} = 1 \\ \dot{v} = 0 \end{cases}$$

and taking any initial condition $(u_0, v_0) \in \mathbb{R} \times (1/2, 7/2)$ we have that the solution is well-defined for $t \in [0, T]$. So, by continuous dependence, there exists $\Delta > 0$ such that the if $\delta \in [0, \Delta]$ the solution is still well-defined for $t \in [0, T]$. The coordinates (u, v) are the good ones to have the Poincaré map written in form (3.38). To have a rough idea of why this is true, one can see trough a formal computation that system (3.42) has the following expansion for small δ

$$\begin{cases} \dot{u} = 1 - \frac{1}{2}\delta^2 v^2 + O(\delta^3) \\ \dot{v} = O(\delta^3). \end{cases}$$

Notice the fundamental fact that up to second order F and G do not play any role. Now one can obtain the Poincaré map integrating and evaluating at $t = T$.
We are going to make this argument rigorous and the key is the theory of differentiability with respect to the parameters. So, inspired by [61], let us recall some general facts. Consider a differential equation depending on a parameter

$$\frac{dz}{dt} = \Psi(t, z, \delta) \tag{3.43}$$

where $\Psi : [0, T] \times \mathcal{D} \times [0, \Delta] \to \mathbb{R}^n$ is of class $C^{0,\nu+2,\nu+2}$, $\nu \geq 1$ and $\mathcal{D}$ is an open connected subset of $\mathbb{R}^n$ and $\Delta > 0$. The general theory of differential equations says that the solution $z(t, z_0, \delta)$ is of class $C^{0,\nu+2,\nu+2}$ in its three arguments. The following lemma will be crucial for our purpose, and generalizes the result [61, Proposition 6.4].

Lemma 14. *Let K be a compact set of $\mathcal{D}$ such that for every $z_0 \in K$ and $\delta \in [0, \Delta]$ the solution is well defined in $[0, T]$. Then, for every $(t, z, \delta) \in [0, T] \times K \times [0, \Delta]$ the following expansion holds*

$$z(t, z_0, \delta) = z(t, z_0, 0) + \delta \frac{\partial z}{\partial \delta}(t, z_0, 0) + \frac{\delta^2}{2} \frac{\partial^2 z}{\partial \delta^2}(t, z_0, 0) + \frac{\delta^2}{2} R(t, z_0, \delta)$$

where

$$||R(t, \cdot, \delta)||_{C^\nu(K)} \to 0 \quad \text{as } \delta \to 0$$

uniformly in $t \in [0, T]$.

Proof. For a function $\phi \in C^{0,\nu+2,\nu+2}([0,T] \times K \times [0,\Delta])$, the Taylor formula with remainder in integral form gives

$$\phi(t,z_0,\delta) = \phi(t,z_0,0) + \frac{\partial\phi}{\partial\delta}(t,z_0,0)\delta + \frac{\delta^2}{2}\frac{\partial^2\phi}{\partial\delta^2}(t,z_0,0) + R_2(t,z_0,\delta)$$

where

$$R_2(t,z_0,\delta) = \frac{1}{2}\int_0^\delta \frac{\partial^3\phi}{\partial\delta^3}(t,z_0,\xi)(\delta-\xi)^2 d\xi.$$

Integrating by parts one gets

$$R_2(t,z_0,\delta) = \frac{1}{2}\{2\int_0^\delta \frac{\partial^2\phi}{\partial\delta^2}(t,z_0,\xi)(\delta-\xi)d\xi - \frac{\partial^2\phi}{\partial\delta^2}(t,z_0,0)\delta^2\}$$

and through the change of variable $\xi = \delta s$ we get

$$R_2(t,z_0,\delta) = \delta^2\int_0^1 (1-s)[\frac{\partial^2\phi}{\partial\delta^2}(t,z_0,\delta s) - \frac{\partial^2\phi}{\partial\delta^2}(t,z_0,0)]ds.$$

from which it is easy to conclude using the regularity of the solution. □

Note that, by means of this lemma we have a semi-explicit formula for the solution of (3.43). This is very useful to compute its Poincaré map. So, let us apply the previous lemma to system (3.42). First of all, calling $Z = (u,v)$, system (3.42) can be written in the form

$$\dot{Z} = \Psi(t;Z,\delta).$$

The initial condition will be denoted by $Z(0) = z_0 = (u_0,v_0)$ and the corresponding solution by $z(t;z_0,\delta) = (u(t;u_0,v_0,\delta), v(t;u_0,v_0,\delta))$. We will suppose, by periodicity, that $z_0 \in [0,1] \times [1,3]$. From (3.2) we have that

$$z(t;u_0,v_0,0) = (u_0+t, v_0). \tag{3.44}$$

To compute the first derivative with respect to the parameter let us call $X(t;z_0,\delta) = \frac{\partial z}{\partial\delta}(t;z_0,\delta)$. We need $X(t;z_0,0)$ that solves the Cauchy problem

$$\begin{cases} \dot{X} = A(t)X + a(t) \\ X(0) = 0. \end{cases}$$

where

$$A(t) = \frac{\partial\Psi}{\partial Z}(t;z(t;z_0,0),0), \quad a(t) = \frac{\partial\Psi}{\partial\delta}(t;z(t;z_0,0),0).$$

A simple computation gives

$$\frac{\partial\Psi}{\partial Z}(t;Z,0) = 0 \quad \frac{\partial\Psi}{\partial\delta}(t;Z,0) = 0 \tag{3.45}$$

so that

$$X(t; u_0, v_0, 0) = 0. \tag{3.46}$$

Now let us compute the second derivative. Let us call $Y(t; z_0, \delta) = \frac{\partial^2 z}{\partial \delta^2}(t; z_0, \delta)$ with components $(\xi(t; z_0, \delta), \eta(t; z_0, \delta))$. We need $Y(t; z_0, 0)$ that solves the Cauchy problem

$$\begin{cases} \dot{Y} = A(t)Y + b(t) \\ Y(0) = 0 \end{cases}$$

where

$$\begin{aligned} b(t) = \frac{\partial^2 \Psi}{\partial \delta^2}(t; z(t; z_0, 0), 0) &+ 2\frac{\partial^2 \Psi}{\partial Z \partial \delta}(t; z(t; z_0, 0), 0) X(t; z_0, 0) \\ &+ \frac{\partial^2 \Psi}{\partial Z^2}(t; z(t; z_0, 0), 0)[X(t; z_0, 0), X(t; z_0, 0)] \end{aligned}$$

and $\frac{\partial^2 \Psi}{\partial Z^2}(t; z(t; z_0, 0), 0)$ is interpreted as a bilinear form from $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{R}^2$. A simple computation gives

$$\frac{\partial^2 \Psi}{\partial \delta^2}(t; z(t; z_0, 0), 0) = (-v_0^2, 0).$$

From (3.45) and (3.46) we get the system

$$\begin{cases} \dot{\xi} = -v_0^2, \quad \xi(0) = 0 \\ \dot{\eta} = 0, \quad \eta(0) = 0 \end{cases}$$

leading to

$$Y(t; u_0, v_0, 0) = (-v_0^2 t, 0). \tag{3.47}$$

Next we apply lemma 14 using (3.44), (3.46) and (3.47). We have that

$$Z(t; u_0, v_0, \delta) = (u_0 + t, v_0) + \frac{\delta^2}{2}(-v_0^2 t, 0) + \frac{\delta^2}{2} R(t; u_0, v_0, \delta),$$

where the remainder R satisfies the estimate

$$||R(t, \cdot, \delta)||_{C^5([0,1]\times[1,3])} \to 0 \quad \text{as } \delta \to 0$$

uniformly in $t \in [0, T]$. Finally, evaluating at $t = T$ we get the following expression for the Poincaré map

$$\begin{cases} u_1 = u_0 + T - \frac{\delta^2}{2} T v_0^2 + \frac{\delta^2}{2} R_1(u_0, v_0, \delta) \\ v_1 = v_0 + \frac{\delta^2}{2} R_2(u_0, v_0, \delta) \end{cases} \tag{3.48}$$

and

$$||R_1(\cdot, \cdot, \delta)||_{C^5(\mathbb{R}/\mathbb{Z}\times[1,3])} + ||R_2(\cdot, \cdot, \delta)||_{C^5(\mathbb{R}/\mathbb{Z}\times[1,3])} \to 0 \quad \text{as } \delta \to 0. \tag{3.49}$$

Now it is easy to see that the Poincaré map expressed in the form (3.48)-(3.49) satisfies all the hypothesis of theorem 5. In this case, $\theta = u_0$, $r = v_0$, $\omega = T$, $\alpha(v_0) = -\frac{T}{2}v_0^2$ and δ is small enough. Concerning the intersection property, remember that the Poincaré map associated to system (3.33) is exact symplectic and so has the intersection property. Finally we can say that also map (3.48) has the intersection property because this property is preserved by conjugacy. So, an application of theorem 5 proves the existence of invariant curves for the map (3.48) as $\delta \to 0$. In view of (3.41), we can find a sequence of invariant curves Γ_n approaching uniformly the top of the cylinder. Analogously one can prove the existence of a sequence of invariant curves approaching the bottom of the cylinder. It means that we can put every initial condition between two invariant curve and prove (3.37).

In the proof of the boundedness of the momentum, hypothesis (3.24) was essential. More prescisely, if, for example, we suppose that

$$\bar{f} = \frac{1}{T}\int_0^T f(s)ds > 0,$$

then we can prove that there exists R sufficiently large such that if $|p_0| \geq R$ then the corresponding orbit of the Poincaré map Π is unbounded. This will lead to solutions with unbounded momentum.
To prove this result, a less subtle expansion of Π, coming directly from system (3.33), will be sufficient. So, integrate (3.33) and get, for $t \in [0, T]$

$$\begin{cases} q(t; q_0, p_0) = q_0 + t + \tilde{\varepsilon}(t, q_0, p_0) \\ p(t; q_0, p_0) = p_0 + \int_0^t g(q(s; q_0, p_0))ds + \int_0^t f(s)ds \end{cases} \tag{3.50}$$

where

$$\tilde{\varepsilon}(t, q_0, p_0) = \int_0^t \left\{ \frac{p(s; q_0, p_0)}{\sqrt{1 + p^2(s; q_0, p_0)}} - 1 \right\} ds$$

As $p(t; q_0, p_0) \to \infty$ as $p_0 \to \infty$ uniformly in q_0 and $t \in [0, T]$, we have $\tilde{\varepsilon} \to 0$ as $p_0 \to \infty$, uniformly in q_0 and $t \in [0, T]$.
Adding and subtracting $\int_0^t g(q_0 + s)ds = G(q_0 + t) - G(q_0)$ in the second equation of (3.50) we get

$$p(t; q_0, p_0) = p_0 + G(q_0 + t) - G(q_0) + \int_0^t f(s)ds + \varepsilon(t, q_0, p_0)$$

where

$$\varepsilon(t, q_0, p_0) = \int_0^t \{g(q_0 + s + \tilde{\varepsilon}(s, q_0, p_0)) - g(q_0 + s)\}ds.$$

The mean value theorem implies that $\varepsilon \to 0$ as $p_0 \to \infty$ uniformly in q_0 and $t \in [0, T]$. Evaluating in $t = T$ we get the following expansion of Π:

$$\begin{cases} q_1 = q_0 + T + \tilde{\varepsilon}(T, q_0, p_0) \\ p_1 = p_0 + G(q_0 + T) - G(q_0) + T\bar{f} + \varepsilon(T, q_0, p_0) \end{cases}$$

where ε and $\tilde{\varepsilon}$ tends to zero uniformly in q_0 as p_0 tends to $+\infty$.

Now, inspired by [2], consider the function

$$V(q, p) = p - G(q).$$

and notice that

$$V(\Pi(q, p)) = V(q, p) + \Gamma(q, p)$$

where

$$\Gamma(q, p) = -G(q + T + \tilde{\varepsilon}(T, q, p)) + G(q + T) + \varepsilon(T, q, p) + T\bar{f}.$$

Now, using the fact that G is bounded, one can find V_* such that if $V(q_0, p_0) \geq V_*$ then p_0 is sufficiently large in order to have $\Gamma(q_0, p_0) > \frac{T\bar{f}}{2}$. For such a p_0 we have

$$V(\Pi(q_0, p_0)) > V(q_0, p_0) + \frac{T\bar{f}}{2} > V^*.$$

So, by induction we can prove that

$$V(q_n, p_n) > V(q_0, p_0) + n\frac{T\bar{f}}{2}, \quad n \geq 1.$$

Finally we have that

$$\lim_{n\to\infty} V(q_n, p_n) = +\infty$$

and remembering the definition of V and the boundedness of G we get that $p_n \to +\infty$.

Periodic and running solutions

Once we have proved that all the solutions have bounded momentum, let us look for some particular kind of solutions. In this chapter we are going to study the existence of running solutions of equation (3.29). For this part it will be sufficient to assume that

$$g \in C^2(\mathbb{R}).$$

We remember that a running solution is a solution such that

$$x(t+T) = x(t) + N \quad \text{for every } t \in \mathbb{R}$$

where $N \in \mathbb{Z}$. First of all notice that physical intuition suggests that it should not be possible to have such solutions for every N and T, because of the bound given by the speed of light. This is a necessary condition that holds for a larger class of equations, namely:

Proposition 6. *Let $x(t)$ be a running solution of*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) = F(t,x) \tag{3.51}$$

where $F(t,x)$ is continuous and T-periodic in t.
Then

$$\left|\frac{N}{T}\right| < 1. \tag{3.52}$$

Proof. By Lagrange theorem we get

$$|N| = |x(t+T) - x(t)| = |\dot{x}(c)T|$$

for some $c \in (t, t+T)$. But the domain of equation (3.51) is $|\dot{x}(t)| < 1$ for all t, so

$$|N| < T.$$

□

In this section we will see why the relativistic condition (3.52) is also sufficient to have, for every $N \in \mathbb{Z}$ satisfying (3.52) at least two running solutions. The proof will be an application of theorem 2 considering to the Poincaré map Π.

First of all, let us perform the change of variables

$$y(t) = x(t) - \frac{N}{T}t. \tag{3.53}$$

Notice that in this way $y(t+T) = y(t)$ and running solutions of (3.29) correspond to classical T-periodic solution of

$$\frac{d}{dt}\left(\frac{\dot{y}+\frac{N}{T}}{\sqrt{1-(\dot{y}+\frac{2N\pi}{T})^2}}\right) + g(y+\frac{N}{T}t) = f(t). \tag{3.54}$$

We will find T-periodic solutions of equation (3.54) as fixed points of the Poincaré map.
Equation (3.54) can be seen as the Euler-Lagrange equation coming from the Lagrangian

$$L(y, \dot{y}, t) = -\sqrt{1 - (\dot{y} + \frac{2N\pi}{T})^2} - G(y + \frac{N}{T}t) + f(t)y.$$

from which we can get the Hamiltonian system

$$\begin{cases} \dot{q} = \frac{p}{\sqrt{1+p^2}} - \frac{N}{T} \\ \dot{p} = -g(q + \frac{N}{T}t) + f(t). \end{cases} \tag{3.55}$$

It is easily seen that the corresponding Poincaré map Π belongs to $\mathcal{E}^2(\mathscr{C})$ and, by proposition 2 is exact symplectic. So to apply theorem 2 we just need the following

Lemma 15. *If* $|\frac{N}{T}| < 1$ *then there exists* $\tilde{p} > 0$ *and* $\epsilon > 0$ *such that*

$$\Theta(q_0, -\tilde{p}) - q_0 < -\epsilon \quad \textit{and} \quad \Theta(q_0, \tilde{p}) - q_0 > \epsilon.$$

for every $q_0 \in \mathbb{T}$.

Proof. Let us prove the first inequality, being the second similar. Let us call $K = \frac{N}{T}$. By hypothesis we have $|K| < 1$. Consider the function, coming from system (3.55),

$$A(p) = \frac{p}{\sqrt{p^2 + 1}}.$$

We have that $A(p)$ is an odd increasing function such that $A(0) = 0$ and $\lim_{p \to \pm\infty} A(p) = \pm 1$. Since $|K| < 1$, by continuity, we can find $\hat{p} > 0$ such that

$$\begin{cases} A(p) > K & \text{for } p > \hat{p} \\ A(p) < K & \text{for } p < -\hat{p}. \end{cases}$$

Now, integrating the second equation of (3.55) we get, for $t \in [0, T]$

$$p(t) = p_0 - \int_0^t g(q(s) + K)ds + \int_0^t f(s)ds \le p_0 + tC$$

for some constant C coming from the boundedness and periodicity of f and g. So we can find $\tilde{p} > 0$ large enough so that if $p_0 < -\tilde{p}$ then $p(t) < -\hat{p}$ for $t \in [0, T]$. It means that

$$\dot{q}(t) = \frac{p(t)}{\sqrt{1 + p^2(t)}} - \frac{N}{T} < 0 \quad t \in [0, T]$$

that is $q(t)$ is decreasing if $t \in [0, T]$ so,

$$\Theta(q_0, -\tilde{p}) = q(T, q_0, -\tilde{p}) < q(0, q_0, -\tilde{p}) = q_0.$$

A standard compactness argument concludes the proof. □

Now it is straightforward the application of theorem 2 choosing the strip $A = \mathbb{R} \times [-\tilde{p}, \tilde{p}]$ and the fact that solutions of system (3.42) are globally defined implies that we can find a larger strip B such that $\Pi(A) \subset intB$. It means that we can find two fixed points of Π that corresponds to two running solutions of (3.29). The case $N = 0$ obviously satisfies condition (3.52) and we recover two periodic solutions.

Remark 12. *If we suppose that g is analytic, then the right-hand side of (3.55) is analytic in (q, p). By analytic dependence on initial conditions, also the Poincaré map is analytic. Notice that we do not need the analyticity of f [37, p.44]. So, using corollary 1 we get the instability of one solution.*

Remark 13. *Similar results on the classical pendulum have been obtained by Franks in [20, Proposition 5.1]. He proved the existence of fixed points for the Poincaré map using his version of the Poincaré-Birkhoff theorem and affirmed that they should have positive or negative index. This result needs some clarification. In fact there is another possibility: there could be only a continuum of fixed points and the fixed point index could not be defined. Consider the equation of the classical pendulum: the existence or not of forcing terms f of null mean value such that the periodic solutions are represented only by a continuum in still an open problem. Anyway, as a related example consider the equation*

$$\ddot{y} + a\sin(y + \frac{2\pi}{T}t) = 0$$

where the potential depends on time. Its T-periodic solutions correspond, via the change of variables $x = y + \frac{2\pi}{T}t$ to solutions $x(t)$ of

$$\ddot{x} + a\sin x = 0$$

such that $x(t+T) = x(t) + 2\pi$. These solutions forms the graph of a function in the phase space, so it is impossible to define the index.

If we suppose that

$$||g'|| < \frac{\pi^2}{T^2}$$

then the Poincaré map is twist and we can get more informations on the solutions. To state it remember that a running solution is said to be *isolated* if there exists $\delta > 0$ such that every solution $(q(t), p(t))$ satisfying

$$0 < |q(0) - \hat{q}(0)| + |p(0) - \hat{p}(0)| < \delta$$

is not T-periodic with winding number N. Moreover, to state these results, we adapt a definition given in [64] saying that a planar first order system in the variables (q, p) is *degenerate* if there exists a curve $(q_s(0), p_s(0))$ such that the application $s \mapsto q_s(0)$ is defined from $\mathbb{R}$ onto $\mathbb{R}$, satisfies $q_{s+1}(t) = q_s(t)+1$ and $p_{s+1}(t) = p_s(t)$, is bijective in $[0, 1)$ and continuous and for every $s \in [0, 1)$ the point $(q_s(0), p_s(0))$ is the initial condition of a running solution. We have

Theorem 9. *If g is analytic and $||g'|| < \pi^2/T^2$ either the number of isolated running solutions is finite or we are in the degenerate case and every degenerate solution is unstable. Moreover, in the first situation, the index of such solution is either -1 or 0 or 1.*

Proof. By the twist condition we can apply the results of the previous chapter. In particular theorem 3 runs with $\Omega = \{(q, p) \in \mathbb{R}^2 : -\tilde{p} < p < \tilde{p}\}$ where $\tilde{p}$ comes from Lemma 15. Indeed, if we take $r_\theta = \tilde{p} - \epsilon$ with ϵ sufficiently small, condition (2.11) holds with $N = 0$ by continuous dependence, and the Poincaré map is exact symplectic. This is another way to find two periodic solutions. Notice that it is a weaker result because we need the restriction on the period T.
Anyway the Poincaré map is analytic, so, by corollary 2, we have that fixed points either are isolated or form the graph of an analytic 1-periodic function. Moreover by corollary 3 we have the informations on the degree.
The translation of these results from the Poincaré map to the differential equation gives informations on the periodic solutions of system (3.55) and, by the change of variables (3.53) we get analogous results on the running solutions of system (3.29). □

Remark 14. *As in remark 10, let us study the case $||g'||_\infty = \frac{\pi^2}{T^2}$ in the simpler contest of equation (3.23). As we are considering running solutions, we have to consider system (3.55) where $g(s) = \sin s$. Reasoning in a similar way as in remark 10 we have that the same results as in theorem 9 hold if we suppose that $f(t) \not\equiv \sin(\frac{N}{T}t)$.*

Quasi periodic solutions

To find generalized quasi-periodic solutions, remembering the discussion of section 3.1, it is sufficient to find two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ (the first monotone and the second of bounded variation) such that for every $\xi \in \mathbb{R}$

$$\phi(\xi + 1) = \phi(\xi) + 1, \quad \eta(\xi + 1) = \eta(\xi) \tag{3.56}$$

and

$$\Pi(\phi(\xi), \eta(\xi)) = (\phi(\xi + \omega), \eta(\xi + \omega)). \tag{3.57}$$

for some $\omega \in \mathbb{R}$. We will be able to prove the following

Theorem 10. *Suppose that $g \in C^7(\mathbb{R})$. For every $\omega \in (-T, T)$, there exists a family of generalized quasi-periodic solutions of (3.33), $X_\xi(t) = (q_\xi(t), p_\xi(t))$, with $\xi \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{q_\xi(t)}{t} = \frac{\omega}{T}.$$

Remark 15. *If ω is rational then we have generalized periodic solutions and if ω is an integer then we recover one of the two running solutions of the previous paragraph.*

To prove theorem 10, consider the change of variable

$$\begin{cases} Q = q \\ P = p - F(t) \end{cases}$$

where $F(t) = \int_0^t f(s)ds$. System (3.33) transforms into

$$\begin{cases} \dot{Q} = \frac{P+F(t)}{\sqrt{1+(P+F(t))^2}} \\ \dot{P} = g(Q) \end{cases} \tag{3.58}$$

with Hamiltonian

$$H(t; Q, P) = \sqrt{1 + (P + F(t))^2} - G(Q). \tag{3.59}$$

Notice that from condition (3.24), the function F is still T-periodic, so that the Poincaré map Π relative to system (3.58) is still a good tool to study the properties of the solutions of system (3.33).
Remembering theorem 8, we will have completed the proof, as soon as we will have proved that the Poincaré map of system (3.58) is a finite composition of maps belonging to the class $\mathcal{P}^{-L,L}$ for some $L > 0$. Notice that from the results concerning the boundedness of the momentum we have already proved the existence of invariant curves for the Poincaré map of system (3.33). It is not difficult to transfer these invariant curves to the Poincaré map of system

3.58, so that we have the sequence required by the theorem.
Consider a partition of the interval $[0, T]$ in N sub intervals of equal length

$$L = \frac{T}{N} < \frac{\pi}{\sqrt{||g'||_\infty}} \tag{3.60}$$

and consider the map $\Pi_{L,\tau}(Q_0, P_0) = (Q(\tau+L; \tau, Q_0, P_0), P(\tau+L; \tau, Q_0, P_0)) = (Q_1, P_1)$ where $(Q(t; \tau, Q_0, P_0), P(t; \tau, Q_0, P_0))$ is the solution of (3.58) with initial condition (Q_0, P_0) at time τ. The Poincaré map Π of the system can be written as composition of such maps, precisely we have that

$$\Pi = \Pi_{T,0} = \Pi_{L,(N-1)L} \circ \cdots \circ \Pi_{L,L} \circ \Pi_{L,0}.$$

So let us study such maps. Notice that by the periodicity of (3.58) it can be seen as a map defined on the cylinder $\mathbb{T} \times \mathbb{R}$. Moreover we have that $\Pi_{\tau,L} \in \mathcal{E}^7(\mathscr{C})$. For every τ, the map is also exact symplectic. Indeed, the Hamiltonian (3.59) is of the form (3.7)-(3.8) with $f \equiv 0$, so that condition (3.11) is trivially verified and proposition 2 applies. This is not the only property satisfied by the map. In fact we have

Proposition 7. *For every $\tau \in [0, T]$, we have $\Pi_{\tau,L} \in \mathcal{P}^{-L,L}$*

Proof. The map $\Pi_{L,\tau}$ is exact symplectic and by a similar argument as in proposition 5 condition (3.60) implies that for every $\tau \in [0, T]$, the map $\Pi_{L,\tau}$ is twist. From equation (3.58) we have

$$\begin{cases} Q(t; \tau, Q_0, P_0) = Q_0 + \int_\tau^t \frac{P(s;\tau,Q_0,P_0)+F(s)}{\sqrt{1+(P(s;\tau,Q_0,P_0)+F(s))^2}} ds \\ P(t; \tau, Q_0, P_0) = P_0 + \int_\tau^t g(Q(s; \tau, Q_0, P_0))ds. \end{cases}$$

Evaluating the second equation in $t = \tau + L$, the boundedness of g gives that $\Pi_{\tau,L}$ preserves the end of the infinite cylinder. Moreover, evaluating the first equation in $t = \tau + L$ and using the second we easily get

$$\lim_{P_0 \to \pm\infty} (Q_1 - Q_0) = \pm L$$

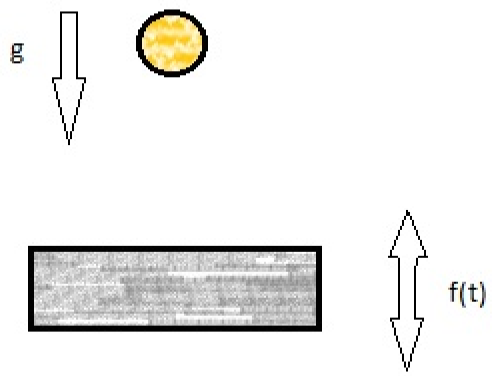
uniformly in Q_0. Finally, property 6. is a trivial consequence of the boundedness of g. □

So, summing up we have that the Poincaré map of system (3.58) can be written as a composition of maps in $\mathcal{P}^{-L,L}$ and we can conclude applying theorem 8 to the Poincaré map Π of system (3.58) and undoing the change of variable.

Chapter 4

Applications to impact problems

In this chapter we are going to see an application of some of the abstract theorem to model of bouncing ball. This model describes the motion of a free falling ball with mass 1 under the gravity force bouncing on a moving plate.



4.1 Statement of the problem

Consider the model of a free falling ball with mass 1 under the gravity force g and let $z(t)$ be its vertical position. The plate is supposed to move according to a $C^4(\mathbb{R})$ 1-periodic function $f(t) \leq z(t)$. We can consider a system of reference joined with the plate performing the change of variable

$x(t) = z(t) - f(t)$. At an instant τ of impact, the change of velocity is assumed to be elastic. So we will consider the problem

$$\begin{cases} \ddot{x} = -(g + \ddot{f}(t)) \\ x(t) \geq 0 \\ x(\tau) = 0 \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-) \end{cases} \tag{4.1}$$

As in [34], by a solution we understand a function $y \in C(\mathbb{R})$ and a sequence (t_n^*) of impact times such that

1. $\inf_n(t_{n+1}^* - t_n^*) > 0$
2. $y(t_n^*) = 0$ for every n and $y(t) > 0$ for $t \in (t_n^*, t_{n+1}^*)$
3. the function y is of class C^2 on every interval $[t_n^*, t_{n+1}^*]$ and satisfies the linear differential equation on this interval.
4. $\dot{y}(t_n^+) = -\dot{y}(t_n^-)$

Moreover, the solution is called bounded if it also satisfies

5. $\sup_n(t_{n+1} - t_n) < \infty$.

Notice that in such a case we have

$$\sup_{t\in\mathbb{R}} |y(t)| + \operatorname{ess\,sup}_{t\in\mathbb{R}} |\dot{y}(t)| < \infty.$$

The problem can be formulated in a discrete form. We can solve the initial value problem

$$\begin{cases} \ddot{x} = -(g + \ddot{f}(t)) \\ x(t_{n-1}) = 0, \quad \dot{x}(t_{n-1}) = w_{n-1} \end{cases} \tag{4.2}$$

and impose the conditions

$$x(t_n) = 0, \quad \dot{x}(t_n) = -w_n$$

to obtain

$$t_n = t_{n-1} + \frac{2}{g}w_{n-1} - \frac{2}{g}f[t_n, t_{n-1}] + \frac{2}{g}\dot{f}(t_{n-1})$$

and

$$-w_n = w_{n-1} - g(t_n - t_{n-1}) - \dot{f}(t_n) + \dot{f}(t_{n-1}).$$

where

$$f[t_n, t_{n-1}] = \frac{f(t_n) - f(t_{n-1})}{t_n - t_{n-1}}.$$

Substituting the first in the second we get the formulas

$$\begin{cases} t_n = t_{n-1} + \frac{2}{g}w_{n-1} - \frac{2}{g}f[t_n, t_{n-1}] + \frac{2}{g}\dot{f}(t_{n-1}) \\ w_n = w_{n-1} - 2f[t_n, t_{n-1}] + \dot{f}(t_n) + \dot{f}(t_{n-1}). \end{cases} \tag{4.3}$$

Inspired by such formulas we can consider the following map $S(t_0, w_0) = (t_1, w_1)$ defined by

$$\begin{cases} t_1 = t_0 + \frac{2}{g}w_0 - \frac{2}{g}f[t_1, t_0] + \frac{2}{g}\dot{f}(t_0) \\ w_1 = w_0 - 2f[t_1, t_0] + \dot{f}(t_1) + \dot{f}(t_0). \end{cases} \tag{4.4}$$

Notice that this is an implicit definition but we have

Lemma 16. *There exists $\bar{w} > 0$, depending on $||\dot{f}||_\infty$, such that if $w_0 > \bar{w}$ and for $t_0 \in \mathbb{R}$ the map $S(t_0, w_0) = (t_1, w_1)$ is well defined and C^3.*

Proof. First of all notice that, since f is C^4 and periodic

$$t_1 - t_0 = \frac{2}{g}w_0 + O(1)$$

so that if $w_0 \to \infty$ then $t_1 - t_0 \to \infty$. Now, considering the function

$$F(t_0, t_1, w_0) = t_1 - t_0 - \frac{2}{g}w_0 + \frac{2}{g}f[t_1, t_0] - \frac{2}{g}\dot{f}(t_0)$$

we have

$$\partial_{t_1}F(t_0, t_1, w_0) = 1 + \frac{2}{g}\frac{\dot{f}(t_1)(t_1 - t_0) - f(t_1) + f(t_0)}{(t_1 - t_0)^2}$$

that is strictly positive for $t_1 - t_0 \to \infty$. So, taking w_0 sufficiently big, we have that for every t_0 we have a unique $t_1 = t_1(t_0, w_0) > t_0 + 1$ that solves the first equation in (4.4). Moreover applying the implicit function theorem we have, by uniqueness, that $t_1(t_0, w_0)$ is a C^4 function. Substituting in the second we have the thesis. □

It has been showed in [34] that a good strategy to face this problem is to take a sequence (t_n^*) of impact time such that $\inf_n(t_{n+1}^* - t_n^*)$ were sufficiently big in order to have a positive solution of the corresponding Dirichlet problem

$$\begin{cases} \ddot{x} = -(g + \ddot{f}(t)) \\ x(t_{n+1}^*) = x(t_n^*) = 0. \end{cases} \tag{4.5}$$

Then we have to glue such solutions in a way that the elastic bounce condition holds. To this aim we have to pass to the discrete version of the problem, given by the map $S(t_0, w_0) \mapsto (t_1, w_1)$ coming from lemma 16. This map is not exact symplectic but $S(t_0, E_0) \mapsto (t_1, E_1)$ where $E_0 := \frac{1}{2}w_0^2$ is exact symplectic. The coordinates (t_n, E_n) are conjugate, so the map can be expressed in terms of a generating function $h(t_0, t_1)$ such that

$$\begin{cases} \partial_1 h(t_0, t_1) = -E_0 \\ \partial_2 h(t_0, t_1) = E_1. \end{cases} \tag{4.6}$$

and that can be explicitly computed giving

$$\begin{aligned} h(t_0, t_1) = & \frac{g^2}{24}(t_1 - t_0)^3 + \frac{g}{2}(f(t_1) + f(t_0))(t_1 - t_0) - \frac{(f(t_1) - f(t_0))^2}{2(t_1 - t_0)} \\ & - g \int_{t_0}^{t_1} f(t)dt + \frac{1}{2}\int_{t_0}^{t_1} \dot{f}^2(t)dt. \end{aligned} \tag{4.7}$$

So the good sequence (t_n^*) giving the elastic bounce condition turns to be one such that

$$\partial_{t_1} h(t_{n-1}^*, t_n^*) + \partial_{t_0} h(t_n^*, t_{n+1}^*) = 0. \tag{4.8}$$

See [34] for more details and note the relation with the stationary condition (2.23) of the Aubry-Mather theory. Moreover, we can introduce an order relation between two different bouncing solution $x_1(t)$ and $x_2(t)$, saying that $x_1(t) \prec x_2(t)$ if and only if, called $(\tau_i^1)_i$ and $(\tau_i^2)_i$, the corresponding sequences of impact times, we have $\tau_i^1 \leq \tau_i^2$ for every i.

4.2 Existence of Aubry-Mather bouncing solutions

Motivated by condition (4.8), We are going apply the Aubry-Mather theory in order to find special orbits. Remember that the theory will apply if the following hypothesis on the generating function hold:

(H1) $h \in C^2(\mathbb{R}^2)$,

(H2) $h(t_0 + 1, t_1 + 1) = h(t_0, t_1)$ for all $(t_0, t_1) \in \mathbb{R}^2$,

(H3) $\partial_{t_0 t_1} h \leq \epsilon < 0$ for all $(t_1, t_0) \in \mathbb{R}^2$.

We will refer to them as hypothesis (H). In this case we can apply theorem 6 to have monotone increasing configurations $t = (t_n^*)$ satisfying condition (4.8) and characterized by a rotation number defined as

$$\alpha(t) = \lim_{n \to \infty} \frac{t_n^*}{n}.$$

To these configurations correspond orbits (t_n^*, E_n^*) for the diffeomorphism that are contained in a compact invariant set called Aubry-Mather set. In our case, the generating function (4.7) does not satisfies the whole hypothesis (H), but we have

Proposition 8. *The generating function (4.7) is in $C^3(\mathbb{R}^2)$, satisfies (H2) and $\partial_{t_0 t_1} h \le \epsilon < 0$ for $t_1 - t_0$ sufficiently large.*

Proof. First of all notice that (H2) follows directly by the periodicity properties of f. The regularity comes from the fact that

$$\frac{(f(t_1) - f(t_0))^2}{(t_1 - t_0)} = [\int_0^1 \dot{f}(\lambda t_1 + (1-\lambda)t_0) d\lambda]^2 (t_1 - t_0).$$

Finally, a direct calculus of the second derivative of h gives

$$\partial_{t_1 t_0} h = -\frac{g^2}{4}(t_1 - t_0) + O(1) \quad \text{as } t_1 - t_0 \to \infty$$

from which we conclude. □

The fact that our generating function satisfies almost all the hypothesis that we need suggests the following strategy: to look for a modification $\tilde{h}$ of the generating function h, that satisfies properties (H1),(H2),(H3) and that coincide with h for $t_1 - t_0$ sufficiently large. An idea of how to do this is presented in [49].

Proposition 9. *Consider a function $h : \mathbb{R}^2 \to \mathbb{R}$, $h \in C^3(\mathbb{R}^2)$, satisfying property (H2) and such that $\partial_{t_0 t_1} h \le \epsilon < 0$ for $t_1 - t_0$ sufficiently large. Suppose that $\partial_{t_0 t_1 t_1} h$ and $\partial_{t_0 t_0 t_1} h$ are bounded. Then there exists a function $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ that satisfies property (H1),(H2),(H3) and such that it coincides with h for $t_1 - t_0$ sufficiently large.*

Proof. Let us start considering, for any constant $A > 0$, $C \in \mathbb{R}$, $x_0 \in \mathbb{R}$, the function

$$\phi(x) = \frac{A + C(x - x_0)}{(x - x_0)^2}$$

and notice that $\phi(x)$ is bounded from below and

$$\inf_{x < x_0} \phi(x) \le 0.$$

Let us call $d(t_0, t_1) = \partial^2_{t_1 t_0} h(t_0, t_1)$. By hypothesis we have that there exists $k > 0$ sufficiently large and $\epsilon < 0$ such that, if $t_1 - t_0 \ge k$ then

$$d(t_0, t_1) \le \epsilon < 0. \tag{4.9}$$

Now choose $\tilde{\epsilon}$ such that $\epsilon < \tilde{\epsilon} < 0$. In the definition of ϕ let

$$A = \tilde{\epsilon} - \epsilon, \quad C = \frac{1}{2}||\partial_{t_1} d - \partial_{t_0} d||_\infty, \quad x_0 = k.$$

Notice that A is positive due to the choice of $\tilde{\epsilon}$ and that C is finite due to the assumption on $\partial_{t_0 t_1 t_1} h$ and $\partial_{t_0 t_0 t_1} h$. So we can define

$$I := \inf_{x<k} \phi(x).$$

As we said we have $-\infty < I \leq 0$ so that we can fix H such that $H < I$. So let

$$\begin{aligned} D(t_0, t_1) =& d(\frac{t_0 + t_1 - k}{2}, \frac{t_0 + t_1 + k}{2}) + \\ & \frac{1}{2}(\partial_{t_1} d(t_0, t_1) - \partial_{t_0} d(t_0, t_1))(t_1 - t_0 - k) + H(t_1 - t_0 - k)^2 \end{aligned} \tag{4.10}$$

and define

$$\tilde{d}(t_0, t_1) = \begin{cases} d(t_0, t_1) & \text{if } t_1 - t_0 \geq k \\ D(t_0, t_1) & \text{if } t_1 - t_0 < k \end{cases} \tag{4.11}$$

It is easily seen that $\tilde{d}(t_0 + 1, t_1 + 1) = \tilde{d}(t_0, t_1)$. Moreover we claim that $\tilde{d} \in C^1(\mathbb{R}^2)$. Indeed clearly, it comes from the regularity of h that every piece of the definition is C^1. Moreover it is also immediate that

$$d(t_0, t_0 + k) = D(t_0, t_0 + k),$$

and a long but straight computation of the partial derivatives gives the requested regularity. Now let us study how to satisfy property (H3). We claim that,

$$\tilde{d}(t_0, t_1) \leq \tilde{\epsilon} < 0 \tag{4.12}$$

where $\tilde{\epsilon}$ comes from the definition of H. Indeed, it is clear by hypothesis for $t_1 - t_0 \geq k$. For $t_1 - t_0 < k$, noticing that $\frac{t_0+t_1+k}{2} - \frac{t_0+t_1-k}{2} = k$ and remembering the definition of H and the definition of I as an infimum we have:

$$\begin{aligned} & D(t_0, t_1) \leq \epsilon + C|t_1 - t_0 - k| + I(t_1 - t_0 - k)^2 \leq \\ & \epsilon - C(t_1 - t_0 - k) + \frac{\tilde{\epsilon} - \epsilon + C(t_1 - t_0 - k)}{(t_1 - t_0 - k)^2}(t_1 - t_0 - k)^2 = \tilde{\epsilon} < 0 \end{aligned} \tag{4.13}$$

Now consider the Cauchy problem

$$\begin{cases} u_{t_0 t_1} = \tilde{d}(t_0, t_1) \\ u(t_0, t_0 + k) = h(t_0, t_0 + k) =: \tilde{\phi}(t_0) \in C^2 \\ (u_{t_1} - u_{t_0})(t_0, t_0 + k) = (h_{t_1} - h_{t_0})(t_0, t_0 + k) =: \tilde{\psi}(t_0) \in C^1. \end{cases} \tag{4.14}$$

We can apply lemma 9 to have a solution $\tilde{h} \in C^2(\mathbb{R}^2)$ that is unique on every characteristic triangle, such that $\tilde{h}(t_0, t_1) = \tilde{h}(t_0 + 1, t_1 + 1)$ and that, by construction, satisfies (H3). Finally, since $\tilde{d} = d$ if $t_1 > t_0 + k$, we have by uniqueness that $\tilde{h} = h$ if $t_1 > t_0 + k$, so that $\tilde{h}$ satisfies the thesis.

□

Now we are ready to apply Aubry-Mather theory. First of all notice that Proposition 8 guarantees most of the hypothesis required by Proposition 9. To verify the boundedness of $\partial_{t_0 t_1 t_1} h$ and $\partial_{t_0 t_0 t_1} h$ consider, first of all, in (4.7), the term

$$\frac{(f(t_1) - f(t_0))^2}{2(t_1 - t_0)}.$$

A direct computation gives the boundedness of the third derivatives for $t_1 - t_0$ large. If $t_1 - t_0$ is small we just have to remember that

$$\frac{(f(t_1) - f(t_0))^2}{(t_1 - t_0)} = [\int_0^1 \dot{f}(\lambda t_1 + (1 - \lambda)t_0) d\lambda]^2 (t_1 - t_0)$$

and perform a direct computation. The boundedness of the third derivatives is trivial for the other terms remembering the regularity and the periodicity of f. So we can apply Proposition 9 to (4.7) to have a generating function $\tilde{h}(t_0, t_1)$ to which we can apply theorem 6. Using the terminology of Mather we can find for every $\alpha \in \mathbb{R}$ a minimal configuration $t = (t_n^*)_{n \in \mathbb{Z}}$ with rotation number $\alpha(t) = \alpha$. Moreover, for this configuration we have that

$$\partial_2 \tilde{h}(t_{n-1}^*, t_n^*) + \partial_1 \tilde{h}(t_n^*, t_{n+1}^*) = 0 \tag{4.15}$$

and

$$|t_n^* - t_0^* - n\alpha(t)| < 1. \tag{4.16}$$

Furthermore $\tilde{h}$ generates a diffeomorphism $\tilde{S}$ in the sense that

$$\begin{cases} \partial_1 \tilde{h}(t_0, t_1) = -E_0 \\ \partial_2 \tilde{h}(t_0, t_1) = E_1 \end{cases} \Leftrightarrow \tilde{S}(t_0, E_0) = (t_1, E_1), \tag{4.17}$$

so, letting $E_n^* = \partial_2 \tilde{h}(t_{n-1}^*, t_n^*) = -\partial_1 \tilde{h}(t_n^*, t_{n+1}^*)$, we have that (t_n^*, E_n^*) is a complete orbit of $\tilde{S}$: we call it minimal orbit. Yet, from (4.16) we have

$$t_0^* + n\alpha(t) - 1 < t_n^* < t_0^* + n\alpha(t) + 1$$

and

$$t_0^* + n\alpha(t) + \alpha(t) - 1 < t_{n+1}^* < t_0^* + n\alpha(t) + \alpha(t) + 1$$

from which we have

$$\alpha(t) - 2 < t^*_{n+1} - t^*_n < \alpha(t) + 2 \quad \text{for every } n. \tag{4.18}$$

It means that there exists α_* such that if $\alpha > \alpha_*$ we have that for every n

$$t^*_{n+1} - t^*_n > k. \tag{4.19}$$

where $k > 0$ is a large positive constant such that $h(t_0, t_1) = \tilde{h}(t_0, t_1)$ if $t_1 - t_0 > k$. Finally we claim that the orbit (t^*_n, E^*_n) of $\tilde{S}$ is actually an orbit of S. Indeed we have, remembering (4.19), that

$$S(t^*_n, t^*_{n+1}) = \begin{cases} -\partial_1 h(t^*_n, t^*_{n+1}) \\ \partial_2 h(t^*_n, t^*_{n+1}) \end{cases} = \begin{cases} -\partial_1 \tilde{h}(t^*_n, t^*_{n+1}) \\ \partial_2 \tilde{h}(t^*_n, t^*_{n+1}) \end{cases} = (E^*_n, E^*_{n+1}). \tag{4.20}$$

So, coming back to the physical problem of the bouncing ball we have

Corollary 5. *There exists α_* such that for every $\alpha > \alpha_*$ there exists a bouncing solution such that*

$$\lim_{n\to\infty} \frac{t^*_n}{n} = \alpha.$$

Moreover,

- *If $\alpha = p/q$ is rational then*
  - *there exists a p-periodic bouncing solutions of (4.1) with q bounces in a period,*
  - *the periodic bouncing solutions with the same rotation number α are totally ordered with respect to $\prec$,*
  - *if there exist two different periodic solutions $x^1(t)$ and $x^2(t)$, $x^1(t) \prec x^2(t)$ with the same rotation number α such that there is not another periodic solution $x^*(t)$ with the same rotation number such that $x^1(t) \prec x^*(t) \prec x^2(t)$ then there exist two different solutions $x^+(t)$ and $x^-(t)$ with rotation number α such that the corresponding sequences of impact times satisfy*

$$\lim_{i\to-\infty} |t^+_i - t^1_i| = 0 \quad \textit{and} \quad \lim_{i\to\infty} |t^+_i - t^2_i| = 0$$

and

$$\lim_{i\to-\infty} |t^-_i - t^2_i| = 0 \quad \textit{and} \quad \lim_{i\to\infty} |t^-_i - t^1_i| = 0$$

- *If α is irrational then*
  - *and the bouncing solutions with the same rotation number α are totally ordered with respect to $\prec$,*
  - *the sequence of impact times t_i^* of the solution $x^*(t)$ with rotation number α is such that the set $\{t_i^* + \mathbb{Z}, i \in \mathbb{Z}\}$ is either dense in $\mathbb{R}/\mathbb{Z}$ or a Cantor set in $\mathbb{R}/\mathbb{Z}$.*

Proof. Condition (4.15) is the one that guaranties the condition of elastic bouncing. So we have that to every minimal orbit of S with rotation number $\alpha > \alpha_*$ corresponds a bouncing solution of problem (4.1) such that

$$\lim_{n\to\infty} \frac{t_n^*}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (t_{k+1}^* - t_k^*) = \alpha$$

where t_n^* represent the time of the n-th bounce on the racket. The thesis follows directly from the general Aubry-Mather theory, remembering that in the case of a rational rotation number the periodic minimal orbits (t_n^*, E_n^*) for S are such that

$$\begin{cases} t_{n+q}^* - t_n^* = p \\ E_{n+q}^* = E_n^* \end{cases} \tag{4.21}$$

□

Remark 16. *Notice that we can interpret the rotation number of a bouncing solution as an average of the distance between two consecutive impact times. The solutions with irrational rotation number can be seen as quasi-periodic solutions.*

Remark 17. *Notice that if we suppose that $||\dot{f}||_{C^4}$ is sufficiently small then, we can apply theorem 4 to the map S written in the form (4.4). We have the existence of a sequence of invariant curves approaching the top of the cylinder. Notice that we can recover the intersection property from the fact that the conjugated map in the variables (t, E) is exact symplectic. The existence of invariant curves implies that the velocity is always bounded. In corollary 5 we have proved that we can have bounded solutions without smallness assumptions on $\dot{f}$. We will see in the next section that, however, the existence of unbounded orbits is also allowed in the general case.*

4.3 Unbounded orbits

In this section we will consider the change of variable

$$\Psi : \quad t = t, \quad w = v - \dot{f}(t)$$

applied to the map (4.4). This a diffeomorphism of the cylinder $\mathbb{T} \times \mathbb{R}$ and represents a shift from the mobile system of reference to the inertial one. The new coordinate v is nothing but the inertial velocity. The conjugate map $P = \Psi^{-1} \circ S \circ \Psi$ turns to be

$$P : \begin{cases} t_1 = t_0 + \frac{2}{g} v_0 - \frac{2}{g} f[t_1, t_0] \\ v_1 = v_0 + 2\dot{f}(t_1) - 2f[t_1, t_0] \end{cases} \tag{4.22}$$

that, remembering lemma 16 is well defined for $v_0 > \bar{v}$ for a certain sufficiently big $\bar{v}$. This is the formulation considered by Pustil'nikov in [67].

Notice that if $(t_n^*, v_n^*)_{n \in \mathbb{Z}}$ is a complete orbit satisfying

$$f(t_n^*) = f(t_0^*) \quad \text{for every } n \in \mathbb{Z} \tag{4.23}$$

then $f[t_n^*, t_{n-1}^*] = 0$ for every $n \in \mathbb{Z}$ and $(t_n^*, v_n^*)_{n \in \mathbb{Z}}$ becomes a complete orbit for the generalized standard map

$$GS : \begin{cases} t_1 = t_0 + \frac{2}{g} v_0 \\ v_1 = v_0 + 2\dot{f}(t_1) \end{cases} \tag{4.24}$$

Clearly the converse is also true, if $(t_n^*, v_n^*)_{n \in \mathbb{Z}}$ is a complete orbit of GS with $v_n > \bar{v}$ for every n and satisfying condition (4.23) then it is also an orbit for P. This fact will be crucial in the following. We start constructing unbounded orbits for GS.

Proposition 10. *Let $t_0^* < t_1^*$ be real numbers and let $(t_n^*, v_n^*)_{n \in \mathbb{Z}}$ be the orbit of (4.24) with initial conditions $t_0 = t_0^*, v_0 = v_0^* = g(t_1^* - t_0^*)/2$. Suppose that there exist $N, W, V \in \mathbb{N} \setminus \{0\}$ such that*

1. $N(t_1^* - t_0^*) + \frac{4}{g} \sum_{k=1}^{N-1} (N-k) \dot{f}(t_k^*) = W$,
2. $\frac{4}{g} \sum_{k=0}^{N-1} \dot{f}(t_k^*) = V$,

Then

$$t_{n+N}^* = t_n^* + \sigma_n, \quad \sigma_n \in \mathbb{N}$$
$$v_{n+N}^* = v_n^* + \frac{g}{2} V.$$

Moreover, there exists $T > 0$ such that if $t_1^* - t_0^* > T$ then $v_n^* > \bar{v}$ for every $n \geq 0$.

Proof. Notice that from (4.24) we obtain the following expression for the n-th iterate:

$$v_n = v_0 + 2 \sum_{k=1}^{n} \dot{f}(t_k) \tag{4.25}$$

$$t_n = t_0 + \frac{2}{g} n v_0 + \frac{4}{g} \sum_{k=1}^{n-1} (n-k) \dot{f}(t_k). \tag{4.26}$$

We claim that for every $j \in \mathbb{N}$, there exists $\sigma_j \in \mathbb{N}$ such that

$$t^*_{N+j} = t^*_j + \sigma_j. \tag{4.27}$$

Let us prove it by induction on j. The fact that $v^*_0 = g(t^*_1 - t^*_0)/2$ and the hypothesis, together with (4.26) give the first step for $j = 0$ with $\sigma_0 = W$. Notice that by periodicity we have also $\dot{f}(t^*_N) = \dot{f}(t^*_0)$.
Now suppose that $t^*_{N+i} = t^*_i + \sigma_i$ for every $i < j$. Using (4.24) we have

$$\begin{aligned} t^*_{N+j} &= t^*_{N+j-1} + \frac{2}{g} v^*_{N+j-1} = t^*_{j-1} + \sigma_{j-1} + \frac{2}{g}[v^*_{j-1} + 2 \sum_{k=0}^{N-1} \dot{f}(t^*_{k+j})] = \\ &(t^*_{j-1} + \frac{2}{g} v^*_{j-1}) + \sigma_{j-1} + \frac{4}{g} \sum_{k=0}^{N-1} \dot{f}(t^*_{k+j}) = t^*_j + \sigma_{j-1} + \frac{4}{g} \sum_{k=0}^{N-1} \dot{f}(t^*_{k+j}). \end{aligned} \tag{4.28}$$

We just have to prove that the last term is an integer. Notice that for every k, there exist $d \in \mathbb{N}$ and $r \in \{0, \ldots, N-1\}$ such that $k + j = Nd + r$. Moreover, the fact that $k \in \{0, \ldots, N-1\}$ implies that $N(d-1) + r < j$. This allows to use the inductive hypothesis several times and get

$$t^*_{k+j} = t^*_{Nd+r} = t^*_{N+N(d-1)+r} = t^*_{N(d-1)+r} + \sigma_{N(d-1)+r} = \cdots = t^*_r + \sigma,$$

where $\sigma \in \mathbb{N}$. Moreover, from the definition, we have that r takes all the values in $\{0, \ldots, N-1\}$ as k goes from 0 to $N-1$. Finally we have

$$\frac{4}{g} \sum_{k=0}^{N-1} \dot{f}(t^*_{k+j}) = \frac{4}{g} \sum_{r=0}^{N-1} \dot{f}(t^*_r) = V$$

and we conclude by hypothesis.
This allows us to write, from (4.25),

$$v^*_{N+n} = v^*_n + 2 \sum_{k=n+1}^{n+N} \dot{f}(t^*_k) = v^*_n + 2 \sum_{k=0}^{N-1} \dot{f}(t^*_k) = v^*_n + \frac{g}{2} V.$$

Finally, once more from (4.25) we have the last assertion remembering that $v^*_0 = g(t^*_1 - t^*_0)/2$ and $\dot{f}$ is bounded. □

Remark 18. *This result has a well-known geometrical interpretation. The map GS satisfies*

$$GS(t_0 + 1, v_0) = GS(t_0, v_0) + (1, 0)$$

$$GS(t_0, v_0 + \frac{g}{2}) = GS(t_0, v_0) + (1, \frac{g}{2}).$$

It means that GS induces a map on the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\frac{g}{2}\mathbb{Z}$ and the orbit $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$ becomes an N-cycle on this torus.

We shall use this proposition to find unbounded orbit for the original map P.

Corollary 6. *Suppose that the hypothesis of proposition 10 hold. Moreover, let $t_1^* - t_0^* > T$ and*

3. $f(t_0^*) = f(t_1^*) = \cdots = f(t_{N-1}^*)$,

then $(t_n^*, v_n^*)_{n\geq 0}$ is an unbounded orbit for P.

Proof. The generalized periodicity of the sequence (t_n) implies that the condition (4.23) holds and we can repeat the discussion of the beginning of this section. □

Remark 19. *For $N = 1$ all the condition are satisfied if*

$$\frac{4}{g}\dot{f}(t_0^*) \in \mathbb{N} \setminus \{0\}. \tag{4.29}$$

In this case we can select t_1^* such that $t_1^* - t_0^*$ is a sufficiently large integer. The condition (4.29) was essentially found by Pustyl'nikov in [67]. Next we show that our result, for $N \geq 2$ gives new information.

Remark 20. *Consider the function*

$$f(t) = \frac{g}{16\pi}\sin(4\pi t)\chi(t)$$

where $\chi(t)$ is a 1-periodic C^∞ bump function such that

$$\chi(t) = \begin{cases} 1 & \textit{if } 0 \leq t \leq \frac{2}{3} \\ < 1 & \textit{if } \frac{2}{3} < t < \frac{3}{4} \\ 1 & \textit{if } \frac{3}{4} \leq t < 1 \end{cases} \tag{4.30}$$

and $|\chi'| < 1$. It easily seen that $f(t)$ satisfies hypothesis $1), 2)$ and $3)$ of the preceding proposition 10 and corollary 6 $N = 2$, $t_0^* = 0$ and $t_1^* = \frac{1}{2} + k$ where $k \in \mathbb{Z}$ have to be chosen large enough to have v_0^* sufficiently big. In this case we have $W = 2k + 2$ and $V = 2$. Moreover, we have that f has minimal period 1 and $\dot{f}(t) < \frac{g}{2}$ so the function does not satisfy Pustil'nikov's result, as he needed a point t_0 such that $\dot{f}(t_0) = \frac{k_0 g}{2}$ with k_0 a positive integer.

Remark 21. *The existence of an unbounded orbit breaks the invariant curves, so, in the case of the corollary we have that the Mather sets previously found are not invariant curves.*

By now we have an unbounded solution. We shall prove the existence of a continuum of unbounded solutions. The key role is played by the following theorem. It is a discrete version of a classical theorem of differential equations and for completeness we report the sketch of the proof.

Theorem 11. *Consider the difference equation*

$$x_{n+1} = Ax_n + R_n(x_n)$$

where A is a $m \times m$ matrix such that k eigenvalues have modulus less than 1 and the remaining $m-k$ have modulus greater than 1. Suppose that $R_n : U \subset \mathbb{R}^m \to \mathbb{R}^m$ is continuous for every n and $R_n(0) = 0$. Moreover suppose that for every $\epsilon > 0$ there exist $\delta > 0$ and $M > 0$ such that for every $n > M$ and $u, v \in B_\delta$

$$|R_n(u) - R_n(v)| \leq \epsilon |u - v|$$

with B_δ representing the ball centred in 0 with radius δ. Then for every n_0 sufficiently large there exists a k-dimensional topological manifold $S = S_{n_0}$ passing through zero such that if $x_{n_0} \in S$ then $x_n \to 0$ as $n > n_0$ tends to $+\infty$.

Proof. The proof is inspired by [15, Theorem 4.1 pp 330]. First of all there exists a change of variables such that our system becomes equivalent to

$$y_{n+1} = By_n + g_n(y_n) \tag{4.31}$$

where

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \tag{4.32}$$

and B_1 is a $k \times k$ matrix with all eigenvalues with modulus less than one and B_2 is a $m-k \times m-k$ matrix with all eigenvalues with modulus greater than one. Now define

$$U_1(n) = \begin{pmatrix} B_1^n & 0 \\ 0 & 0 \end{pmatrix} \tag{4.33}$$

and

$$U_2(n) = \begin{pmatrix} 0 & 0 \\ 0 & B_2^n \end{pmatrix}. \tag{4.34}$$

Notice that the $U(n) = U_1(n) + U_2(n)$ is the matrix solution of the linear system $y_{n+1} = By_n$. Moreover, there exist $K > 0$, $\alpha_1 < 1$ and $\alpha_2 < 1$ such that

$$|U_1(n)| \leq K\alpha_1^n, \tag{4.35}$$

$$|U_2(-n)| \leq K\alpha_2^n \tag{4.36}$$

where $|\cdot|$ is the associated matrix norm. Fix $\alpha_1 < \alpha < 1$ and choose ϵ such that $\epsilon K(\frac{\alpha_2}{1-\alpha_2\alpha} + \frac{1}{\alpha-\alpha_1}) < \frac{1}{2}$. Using the hypothesis and passing to the variable y we can find M_ϵ and δ_ϵ such that

$$|g_n(u) - g_n(v)| \leq \epsilon|u - v| \tag{4.37}$$

for $n > M_\epsilon$ and $u, v \in B_{\delta_\epsilon}$. Let $n_0 > M_\epsilon$ and $a \in \mathbb{R}^m$ such that $2K|a| < \delta_\epsilon$. For each $l \geq 0$ consider the sequence $\{\theta_n^{(l)}(a)\}_{n \geq n_0}$ defined by induction on l by

$$\begin{cases} \theta_n^{(0)}(a) = 0 \\ \theta_n^{(l+1)}(a) = U_1(n-n_0)a + \sum_{s=n_0}^{n-1} U_1(n-s-1)g_s(\theta_s^{(l)}(a)) \\ \qquad\qquad - \sum_{s=n}^{\infty} U_2(n-s-1)g_s(\theta_s^{(l)}(a)) \end{cases} \tag{4.38}$$

Let us prove by induction on l that this sequence is well-defined and for $n \geq n_0$

$$\begin{aligned} |\theta_n^{(l+1)}(a) - \theta_n^{(l)}(a)| \leq \frac{K|a|\alpha^{n-n_0}}{2^l} \quad \text{and} \\ |\theta_n^{(l)}(a)| < \delta_\epsilon \end{aligned} \tag{4.39}$$

The first step comes readily from estimate (4.35) and the choice of α. Notice that the hypothesis $|\theta_n^{(l)}(a)| < \delta_\epsilon$ permits to say that $|\theta_n^{(l+1)}(a)|$ is well defined because the last sum is dominated by a convergent geometric series. Moreover, by inductive hypothesis we have that for $h = 0, \dots, l$,

$$\begin{aligned} |\theta_n^{(h+1)}(a)| &\leq |\theta_n^{(h+1)}(a) - \theta_n^{(h)}(a)| + |\theta_n^{(h)}(a) - \theta_n^{(h-1)}(a)| + \cdots + |\theta_n^{(1)}(a)| \\ &\leq \sum_{i=0}^{h} \frac{K|a|\alpha^{n-n_0}}{2^i} \leq \sum_{i=0}^{+\infty} K|a|\frac{1}{2^i} = 2K|a| < \delta_\epsilon \end{aligned} \tag{4.40}$$

remembering the choice of a. So also $|\theta_n^{(l+2)}|$ is well defined and we have

$$\begin{aligned} |\theta_n^{(l+2)}(a) - \theta_n^{(l+1)}(a)| \leq &\sum_{s=n_0}^{n-1} K\alpha_1^{n-s}|g_s(\theta_s^{(l+1)}(a)) - g_s(\theta_s^{(l)}(a))| \\ &- \sum_{s=n}^{\infty} K\alpha_2^{s-n}|g_s(\theta_s^{(l+1)}(a)) - g_s(\theta_s^{(l)}(a))|. \end{aligned} \tag{4.41}$$

So, by (4.40) we can use (4.37) and the inductive hypothesis to get

$$|\theta_n^{(l+2)}(a) - \theta_n^{(l+1)}(a)| \leq \frac{K|a|}{2^l}(\epsilon K(\frac{\alpha^{n-n_0} - \alpha_1^{n-n_0}}{\alpha - \alpha_1} + \frac{\alpha_2\alpha^{n-n_0}}{1 - \alpha_2\alpha}))$$

from which, remembering that $\alpha - \alpha_1 > 0$

$$|\theta_n^{(l+2)}(a) - \theta_n^{(l+1)}(a)| \leq \frac{K|a|\alpha^{n-n_0}}{2^l}(\epsilon K(\frac{1}{\alpha - \alpha_1} + \frac{\alpha_2}{1-\alpha_2\alpha}))$$

that allows us to conclude remembering the choice of ϵ.
From estimate (4.39) we have that for $n \geq n_0$ the sequence $\{\theta_n^{(l)}(a)\}_l$ tends uniformly to a limit $\theta_n(a)$ such that

$$|\theta_n(a)| < \delta_\epsilon \alpha^{n-n_0}. \tag{4.42}$$

Moreover, by the Weierstrass Test the function $a \mapsto \theta_n(a)$ is continuous for $n \geq n_0$ and $|a| < \frac{\delta_\epsilon}{2K}$. Now we want to pass to the limit in (4.38). Notice that in order to pass the limit into the last sum we have to use the dominated convergent theorem noticing that for every s

$$U_2(n-s-1)g_s(\theta_s^{(l)}(a)) \to U_2(n-s-1)g_s(\theta_s(a)) \quad \text{uniformly}$$

and

$$|U_2(n-s-1)g_s(\theta_s^{(l)}(a))| \leq C\alpha^n.$$

So we are lead to the equation

$$\theta_n(a) = U_1(n-n_0)a + \sum_{s=n_0}^{n-1} U_1(n-s-1)g_s(\theta_s(a)) - \sum_{s=n}^{\infty} U_2(n-s-1)g_s(\theta_s(a)) \tag{4.43}$$

Now it is easily seen that the just defined sequence $\{\theta_n(a)\}_{n\geq n_0}$ satisfies the difference equation (4.31) and by (4.42) tends to 0 as $n \to \infty$. Consider, for $|a|$ sufficiently small

$$\theta_{n_0}(a) = a - \sum_{s=n_0}^{\infty} U_2(n-s-1)g_s(\theta_s(a)).$$

Notice that, if we consider the decomposition of the ambient space $\mathbb{R}^m = \mathbb{R}^k \oplus \mathbb{R}^{m-k}$ and suppose $a \in \mathbb{R}^k$, then we can see $\theta_{n_0}(a)$ in the form

$$\theta_{n_0}(a) = (a, \phi_{n_0}(a))$$

where $\phi_{n_0}(a) = -\sum_{s=n_0}^{\infty} U_2(n-s-1)g_s(\theta_s(a))$ is continuous in a. So, by continuity, $graph\phi_{n_0}$ defines a k-dimensional manifold in $\mathbb{R}^{m-k}$. Coming back to the x variables we have the thesis. □

In order to apply Theorem 11 let us prove the following technical lemma

Lemma 17. *The equation*

$$t_1 = t + \frac{2}{g}v - \frac{2}{g}f[t, t_1]$$

has a unique solution $t_1 = T(t, v) \geq t + 1$ *for large* v. *Moreover* T *is of class* C^4 *on* $\{(t, v) : v > \bar{v}\}$ *and*

$$\begin{aligned} T &= t + \frac{2}{g}v + O(\frac{1}{v}) \\ \frac{\partial T}{\partial t} &= 1 + O(\frac{1}{v}), \qquad \frac{\partial T}{\partial v} = \frac{2}{g} + O(\frac{1}{v}) \end{aligned} \tag{4.44}$$

as $v \to +\infty$.

Proof. The first part follows on the lines of lemma 16, so let us prove the validity of the asymptotic expansions. Let $\Delta = \Delta(t, v) = T - t$ so that, from the definition of T, we have

$$\Delta = \frac{2}{g}v - \frac{2}{g}\frac{f(T) - f(t)}{\Delta}$$

that is equivalent to the quadratic equation

$$\Delta^2 - \frac{2}{g}\Delta v + \frac{2}{g}(f(T) - f(t)) = 0$$

with roots

$$\Delta_{\pm} = \frac{1}{g}v \pm \sqrt{\frac{1}{g^2}v^2 - \frac{2}{g}(f(T) - f(t))}.$$

Then $\Delta_+ \to +\infty$ and $\Delta_- \to 0$ as $v \to +\infty$. Since we know that $\Delta \geq 1$, then it must coincide with the positive branch for large v. So

$$\Delta = \frac{1}{g}v + \sqrt{\frac{1}{g^2}v^2 - \frac{2}{g}(f(T) - f(t))} = \frac{2}{g}v + O(\frac{1}{v})$$

that is the expansion for T.

We can obtain the expansions for the partial derivatives differentiating the formula

$$T = t + \frac{2}{g}v - \frac{2}{g}\frac{f(T) - f(t)}{T - t}$$

and remembering the previous expansion of T. □

Remember that we are considering the map

$$\begin{cases} t_{n+1} = t_n + \frac{2}{g}v_n - \frac{2}{g}f[t_{n+1}, t_n] \\ v_{n+1} = v_n + 2\dot{f}(t_{n+1}) - 2f[t_{n+1}, t_n] \end{cases} \tag{4.45}$$

and we proved the existence of an unbounded orbit $(t_n^*, v_n^*)_{n\geq 0}$ such that $t_n^* \equiv t_{n+N}^* \mod 1$. Moreover, from condition 3. of corollary 6 we have

$$f[t_n^*, t_{n+1}^*] = 0 \quad \text{for every } n.$$

Also

$$t_{n+1}^* - t_n^* = \frac{2}{g}v_n^* \to +\infty \quad \text{as } n \to +\infty. \tag{4.46}$$

Now we are ready to prove the following

Proposition 11. *Consider the matrix*

$$A_n = \begin{pmatrix} 1 & \frac{2}{g} \\ 2\ddot{f}(t_n^*) & 1 + \frac{4}{g}\ddot{f}(t_n^*) \end{pmatrix}$$

If to the hypothesis of Corollary 6 we add the following

4. $|Trace(A_0 A_1 \ldots A_{N-1})| > 2$

then there exists a one dimensional continuum of initial data leading to unbounded solutions.

Proof. Performing the change of variables

$$\begin{cases} \tau_n = t_n - t_n^* \\ \nu_n = v_n - v_n^* \end{cases} \tag{4.47}$$

we have that system (4.45) transforms into

$$\begin{cases} \tau_{n+1} = \tau_n + \frac{2}{g}\nu_n - \frac{2}{g}\lambda_n(\tau_n, \nu_n) \\ \nu_{n+1} = \nu_n + 2\phi_n(\tau_n, \nu_n) - 2\dot{f}(t_{n+1}^*) - 2\lambda_n(\tau_n, \nu_n) \end{cases} \tag{4.48}$$

where

$$\lambda_n(\tau, \nu) = f[\tau + t_n^*, T(\tau + t_n^*, \nu + v_n^*)]$$

and

$$\phi_n(\tau, \nu) = \dot{f}(T(\tau + t_n^*, \nu + v_n^*)).$$

Rewriting (4.48) as a perturbation of the linear map induced by the matrix A_n, we get the following map F defined implicitly by

$$\begin{cases} \tau_{n+1} = \tau_n + \frac{2}{g}\nu_n - \frac{2}{g}\lambda_n(\tau_n, \nu_n) \\ \nu_{n+1} = \nu_n + 2\ddot{f}(t_{n+1}^*)(\tau_n + \frac{2}{g}\nu_n) + 2r_n(\tau_n, \nu_n) - 2\lambda_n(\tau_n, \nu_n) \end{cases} \tag{4.49}$$

where

$$r_n(\tau,\nu) = \phi_n(\tau,\nu) - \dot{f}(t^*_{n+1}) - \ddot{f}(t^*_{n+1})(\tau + \frac{2}{g}\nu).$$

The function λ_n and r_n are well defined and of class C^3 in a common neighbourhood U of the origin. Moreover, as $T(t^*_n, v^*_n) = t^*_{n+1}$, we have that $\lambda_n(0,0) = r_n(0,0) = 0$. From the asymptotic estimates of lemma 17 we deduce that

$$T(\tau + t^*_n, \nu + v^*_n) - t^*_{n+1} - \tau - \frac{2}{g}\nu = O(\frac{1}{v^*_n}) \quad \text{as } n \to +\infty \tag{4.50}$$

uniformly in U. In consequence, remembering (4.46),

$$T(\tau + t^*_n, \nu + v^*_n) - t^*_n \to +\infty \quad \text{as } n \to +\infty$$

uniformly in U. From this it is easy to verify that

$$||\nabla\lambda_n||_{L^\infty(U)} \to 0 \quad \text{as } n \to +\infty.$$

Now a simple application of the mean value theorem gives, for every ϵ, the existence of $M > 0$ such that

$$|\lambda_n(x) - \lambda_n(y)| < \epsilon|x-y| \tag{4.51}$$

for every $x, y \in U$ and $n > M$. To estimate ∇r_n we have to proceed with more care. We will consider $\frac{\partial r_n}{\partial\tau}$, being the other case similar. We have

$$\begin{aligned}\frac{\partial r_n}{\partial\tau} =& \ddot{f}(T(\tau + t^*_n, \nu + v^*_n))\frac{\partial T}{\partial\tau}(\tau + t^*_n, \nu + v^*_n) - \ddot{f}(t^*_{n+1}) \\ =& \ \ddot{f}(T(\tau + t^*_n, \nu + v^*_n) - \ddot{f}(t^*_{n+1}) + O(\frac{1}{v^*_n}) \quad \text{as } n \to +\infty\end{aligned} \tag{4.52}$$

where we have used the estimate of lemma 17. Now, the mean value theorem imply that

$$\begin{aligned}|\ddot{f}(T(\tau + t^*_n, \nu + v^*_n) - \ddot{f}(t^*_{n+1})| &\le ||\dddot{f}||_\infty |T - t^*_{n+1}| \\ &= ||\dddot{f}||_\infty |\tau + \frac{2}{g}\nu| + O(\frac{1}{v^*_n})\end{aligned} \tag{4.53}$$

where we have used (4.50). A similar estimates holds for $\frac{\partial r_n}{\partial\nu}$, so, for every ϵ we can find a neighbourhood U of the origin such that

$$||\nabla r_n||_{L^\infty(U)} < \epsilon \quad \text{as } n \to +\infty.$$

As before, using the mean value theorem we can find for every ϵ, the existence of $M > 0$ such that

$$|r_n(x) - r_n(y)| < \epsilon|x - y| \tag{4.54}$$

for every $x, y \in U$ and $n > M$. This last estimate, together with (4.51), is sufficient to prove that the condition imposed in thoerem 11 on the remainder are satisfied in our case.

So, calling $x_n = (\tau_n, \nu_n)$ and $\Omega_n(x_n) = (-\frac{2}{g}\lambda_n(x_n), 2r_n(x_n) - 2\lambda_n(x_n))$ we have

$$x_{n+1} = A_{n+1}x_n + \Omega_n(x_n)$$

Consider now the map F^N, we have

$$x_{n+N} = Ax_n + R_n(x_n, \ldots, x_{n+N-1})$$

where remembering (4.27)

$$A = A_N \circ A_{N-1} \ldots A_1 = A_0 \circ A_{N-1} \ldots A_1$$

In order to apply theorem 11 we have to prove that the matrix A is hyperbolic so let us compute its trace. Remembering that the trace operator applied to a finite product of matrices is commutative if and only if the corresponding permutation is diedral [1], we have

$$Trace(A) = Trace(A_0 \circ A_1 \ldots A_{N-1}).$$

Remembering that a 2×2 matrix with determinant 1 is hyperbolic with one eigenvalue in modulus greater than 1 and the other smaller if $|Trace(A)| > 2$, by hypothesis, A is hyperbolic. Moreover, the remaining R_n is a finite composition of λ_i, r_i and A_i, $i = 0, \ldots, N-1$ and, recalling what we said before about the Lipschitz constants we have, by the chain rule, that for n sufficiently big, the Lipschitz constant of R_n can be made as small as we want in a sufficiently small ball. So we can apply theorem 11 and get the thesis coming back to the variables (t_n, v_n). □

Remark 22. *Consider the function*

$$f(t) = [1 + \frac{g}{16\pi}\sin(4\pi t)]\Theta(t)$$

where $\Theta(t)$ is a C^∞ function with minimal period 1 *such that*

$$\begin{cases} \Theta(0) = \Theta(\frac{1}{2}) = 1 \\ \dot{\Theta}(0) = \dot{\Theta}(\frac{1}{2}) = 0 \\ \ddot{\Theta}(0) = -\frac{1}{4}, \quad \ddot{\Theta}(\frac{1}{2}) = -\frac{1}{3} \\ |\dot{\Theta}| < 1. \end{cases} \tag{4.55}$$

It satisfies hypothesis $1), 2), 3)$ *and* $4)$ *of the propositions 10 and 11 and corollary 6 with* $N = 2$ *choosing* $t_0^* = 0$ *and* $t_1^* = \frac{1}{2} + k$*. In this case we have* $W = 2k + 2$ *and* $V = 2$ *. Notice that furthermore the points* t_0^* *and* t_1^* *do not satisfy Pustil'nikov's hypothesis* $\ddot{f}(t^*) > 0$ *or* $\ddot{f}(t^*) < -g$.

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