# Nonlinear Boundary Value Problems
# PhD Thesis

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# Chapter 0

# Introduction

In 1900, the same year as Fredholm developed, in *Sur une nouvelle méthode pour la résolution du problème de Dirichlet*, his theory of integral equations, and two years before Lebesgue's dissertation, *Intégrale, longueur, aire*, Hilbert poses 23 problems at the Second International Congress of Mathematicians in Paris as a challenge for the 20th century. The problems include the continuum hypothesis, the mathematical treatment of the axioms of physics, Goldbach's conjecture, the transcendence of powers of algebraic numbers, the Riemann hypothesis and many more. Some of the problems were solved during the 20th century, and each time one of the problems was solved it was a major event for mathematics.

The 20th problem was called 'The general problem of boundary values'. In his speech, Hilbert describes it in the following terms:

> An important problem closely connected with the foregoing is the question concerning the existence of solutions of partial differential equations when the values on the boundary of the region are prescribed. This problem is solved in the main by the keen methods of H. A. Schwarz, C. Neumann, and Poincaré for the differential equation of the potential. These methods, however, seem to be generally not capable of direct extension to the case where along the boundary there are prescribed either the differential coefficients or any relations between these and the values of the function. Nor can they be extended immediately to the case where the inquiry is not for potential surfaces but, say, for surfaces of least area, or surfaces of constant positive gaussian curvature, which are to pass through a prescribed twisted curve or to stretch over a given ring surface. It is my conviction that it will be possible to prove these existence theorems by means of a general principle whose nature is indicated by Dirichlet's principle. This general principle will then perhaps enable us to approach the question: *Has not every regular variation problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied* (say that the functions concerned in these boundary conditions are continuous and have in sections one or more derivatives), and provided also if need be that *the notion of a solution shall be suitably extended?*

Many of the most outstanding mathematicians of the twentieth century (among which there were people of the rank of Hilbert himself for the first part of the question and Schwartz and Sobolev for the second) devoted their efforts to study this problem and related ones. Their works established a well-developed theory of boundary value problems for *linear* differential equations, and gave rise to disciplines with the modern relevance of convex analysis, monotone operators theory, distribution theory, critical point theory, Sobolev spaces, etc.

However, most phenomena in our world seem to display an intrinsically *nonlinear* behavior. Thus, it became a priority to understand, as well, nonlinear problems. In many cases, problems arising in biology, mechanics,... may be seen as nonlinear perturbations of linear ones.

In this memory we mainly deal with second order, elliptic, semilinear boundary value problems, or periodic problems associated with nonlinear ordinary differential equations. All these can be represented in the abstract form

$$\mathcal{L}u = \mathcal{N}u\,, \tag{$\natural$}$$

where $\mathcal{L} : X \to Y$ and $\mathcal{N} : Y \to Y$ are suitable operators between Banach spaces $X, Y$, and $X \subset Y$ compactly. When $\mathcal{L}$ is inversible, ($\natural$) can be rewritten as a fixed point equation:

$$u = [\mathcal{L}^{-1}\mathcal{N}]u\,.$$

In case $\mathcal{L}^{-1} : Y \to X$ and $\mathcal{N} : Y \to Y$ are continuous and carry bounded sets into bounded sets, $\mathcal{L}^{-1}\mathcal{N} : Y \to Y$ is completely continuous. Thus, the 1930 Schauder's Fixed Point Theorem ([76]), which extended to completely continuous operators on infinite dimensional Banach spaces the well-known Brouwer's fixed point theorem, was a landmark in the treatment of such problems. Schauder's paper was closely followed by the introduction, in 1934, of the Leray-Schauder topological degree [50]. In this remarkable paper already appeared some facts -such as the existence of continua of solutions for some semilinear equations depending on a one-dimensional parameter- which were going to be systematically employed by mathematicians in the last quarter of the century.

The World War II broke out, and Schauder's and Leray-Schauder's papers had little impact outside the scope of nonlinear perturbations of *invertible* operators. In this background, the sixties saw a tremendous development of the the theory of *operators of monotone type*, ([9], [51]), which could treat some problems outside the scope of the Leray-Schauder theory, but could hardly be applied to solve problems where the equation is a nonlinear perturbation of a linear operator with nontrivial, sign-changing kernel, or even a nonmonotonous perturbation of a linear operator with a constant-sign kernel. Thus, the 1970 paper by Landesman and Lazer [46] was a mayor step in the treatment of such problems. In this work, the authors considered the boundary value problem

$$\begin{array}{c}\Delta u + \lambda_k u + g(u) = h(x) \quad \text{in } D \\ u|_{\partial D} = 0\,,\end{array} \tag{1}$$

where $D$ is a bounded domain in $\mathbb{R}^N$, $h \in L^2(D)$, $\lambda_k$ is a simple eigenvalue of $-\Delta$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that the limits $g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$ exist and are finite, and

$$g(-\infty) < g(\xi) < g(+\infty) \quad \forall \xi \in \mathbb{R}\,.$$

It was shown that, if $\varphi_k$ is an eigenfunction corresponding to $\lambda_k$, $D^- := \{x \in D : \varphi_k(x) < 0\}$, and $D^+ := \{x \in D : \varphi_k(x) > 0\}$, then the condition

$$g(-\infty)\int_{D^+}\varphi_k dt + g(+\infty)\int_{D^-}\varphi_k dt < \int_D h\varphi_k dt < g(+\infty)\int_{D^+}\varphi_k dt + g(-\infty)\int_{D^-}\varphi_k dt$$

is both necessary and sufficient for the existence of a weak solution of (1).

In the mesure that this paper was able to overcome the monotonicity requirements, it had a tremendous impact, and the just born Landesman-Lazer condition was rapidly adopted for similar situations in other problems at resonance (see, for instance, [58], for a quick survey). It is fair to say, however, that results of Landesman-Lazer type had already appeared in [35] and [48], shortly before the publication of [46]. Moreover, [48] presented an additional difficulty, since it dealt with perturbations of linear operators having a bidimensional kernel. Both papers were looking for almost periodic and periodic solutions of ordinary differential equations and they did not find the same *resonance* among experts as the Landesman-Lazer monograph.

In a separate context, the forced pendulum equation

$$-u'' + A\sin u = h(t) \tag{2}$$

had been thoroughly studied since 1922, when Hamel published, in the special issue of the Matematische Annalen dedicated to Hilbert's sixtieth birthday anniversary, the first general existence results for $2\pi$-periodic solutions of equation (2) when $h(t) = b\sin t$. Using the *direct method of the calculus of variations*, Hamel was able to show the existence of, at least, one solution of this problem.

Almost 60 years later, and when the interest in the pendulum equation had decayed, Fučik reopened the problem and wrote, in 1969:

*Description of the set $P$ of $h$ for which the equation $u'' + \sin u = h(t)$ has a $T$-periodic solution seems to remain a terra incognita.*

In this task, Landesman-Lazer's conditions did not help much; unfortunately, the limits $\lim_{x\to\pm\infty} \sin(x)$ do not exist. It motivated works such as [21], [25], [86], [62] which reintroduced, in the early eighties, the use of variational methods in the study of the periodic solutions of the forced pendulum equation. Since then, this equation has become *a paradigm for nonlinear analysis and dynamical systems* [57], and been the object of a extensive research by many mathematicians, see Section 3.1. We refer to [56] for a more complete survey on this equation.

Let now $c > 0$ be given. The equation

$$-u'' - cu' + A\sin u = h(t) \tag{3}$$

models the swing of a planar pendulum rod immersed in a medium with constant friction coefficient $c$, under the action of a time-dependent external force $h = h(t)$. When $h$ is $T$-periodic and, say, continuous, questions such as the existence, or the geometric multiplicity (that is, multiplicity up to constant multiples of $2\pi$), or the stability of periodic solutions of (3) appear. Even when many partial answers to these problems have been given, they are far from closed, and open questions remain. In Chapter 3 we give a step in the second direction, and we deal with the problem of the number of geometrically different, periodic solutions of (3) depending on $h$. The problem had already been studied, among others, by Mawhin and Willem [62], and later on, by Ortega [65] in the conservative case $c = 0$, and Katriel [42] when the function $A\sin(\cdot)$ is replaced by a $2\pi$-periodic, $C^2$ function $g$ verifying certain additional conditions concerning its Fourier coefficients - conditions which do not hold for $g(u) = A\sin u$ -. Along this chapter we generalize their results and further establish, for conservative pendulum-type equations, exact multiplicity results.

More recent is the history of the analogous Dirichlet boundary value problem

$$\begin{aligned} -u'' - u + A\sin(u) &= h(t) \\ u(0) = u(\pi) &= 0 \end{aligned} \tag{4}$$

and its PDE generalizations. It has been detailed in [41] that the differential equation in (4) models the swing of a pendulum clock as it is excited by the external force $-h$. Of course, Landesman-Lazer conditions do not apply here either, and the use of the stationary phase principle was already suggested in Dancer's paper [25] in order to obtain existence and multiplicity results through the Lyapunov-Schmidt decomposition of this problem. We refer to Sections 1.1 and 2.1 for a more detailed overview of its history. In some sense, this problem is more difficult to deal with than (2), since, for instance, the periodicity of the action functional is lost, but, in other sense, it is easier, since it is precisely this fact which allows the use of asymptotics (which are, at the end, based in the so-called Riemann-Lebesgue Lemma) to obtain solvability or multiplicity results, and we employ this idea along the first and second chapters of this memory.

More precisely, the first chapter is devoted to the study of non self-adjoint, Dirichlet boundary value problems of the type

$$\begin{aligned} -u'' - \alpha u' - \lambda_1(\alpha)u + g(u) &= h(t), \quad t \in [0,\pi] \\ u(0) = u(\pi) &= 0\,, \end{aligned} \tag{5}$$

where $\alpha \in \mathbb{R}$ is given, $\lambda_1(\alpha) := 1 + \frac{\alpha^2}{4}$ is the first eigenvalue of the linear problem

$$\begin{aligned} -u''(t) - \alpha u'(t) &= \lambda u(t), \quad t \in [0,\pi] \\ u(0) = u(\pi) &= 0\,, \end{aligned}$$

$g \in C(\mathbb{R}/T\mathbb{Z})$ is continuous and periodic, and $h \in L^1[0,\pi]$. In case $h$ is a continuous function and $\alpha = 0$, this problem had already been studied by authors such as Dancer [25], Ward [85], Schaaf and Schmitt [73], Arcoya and Cañada [6], Cañada [11], or Cañada and Roca [15], [16]. We extend their results for the more general framework described above. Our main tools here are the Lyapunov-Schmidt decomposition of the problem together with the Riemann-Lebesgue Lemma as developed in [85]. In particular, we show the solvability hyperplane $\mathcal{R}$ of the associated linear problem to be included in (the interior of) the solvability set of (5). As it is well-known since Ortega's first counterexample [64], the analogous thing does not occur for the $T-$periodic problem associated to (3). We further establish some multiplicity and asymptotic results.

The second chapter is devoted to the study of solvability and multiplicity results for elliptic self-adjoint problems of the type

$$\begin{cases} -\Delta u - \lambda_1 u + g(u) = h(x) = \tilde{h}(x) + \bar{h}\varphi(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{6}$$

where $\Omega$ is a bounded, regular domain in $\mathbb{R}^N$, $\lambda_1$ is the first eigenvalue of $-\Delta$ when acting on $H_0^1(\Omega)$, $g : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, and $h : \Omega \to \mathbb{R}$ is Lipschitz. This problem, which can be seen as the natural extension to PDE of (4), had already been considered by authors such as Amann, Ambrosetti and Mancini [5], Solimini [78], Costa, Jeggle, Schaaf and Schmitt, [23], or Schaaf and Schmitt, [74], [75] among others. Under some geometric assumptions on $\Omega$ which hold, in particular, if $\Omega$ is convex or Steiner-symmetric, we use asymptotic techniques to obtain solvability, nondegeneracy and multiplicity results. These are particulary fruitful in case $N = 3$; for $N \geq 4$ the asymptotics only provide generic results, and open problems remain. We also display examples showing qualitative differences appearing in the multiplicity problem (with respect to the cases of low dimensions $N = 1, 2, 3$) when the dimension $N$ is big; precisely, $N \geq 5$.

In recent times, a lot of attention has been given, both in the ordinary and the partial differential cases, to extend spectral, bifurcation or existence results from semilinear equations of second order to perturbations of nonlinear operators such as the $p$-Laplacian

$$u \mapsto \Delta_p u = div(|\nabla u|^{p-2}\nabla u)$$

or some suitable generalization (see, for instance, [29], [53], [54], [30], [31]). In Chapter 4 we extend to the vector $p$-Laplacian case a former result first proved by Hartman [39] and later improved by Knobloch [44] on the existence of Dirichlet or periodic solutions of nonlinear perturbations of the ordinary Laplacian which verify a Nagumo condition. Our main tools here are a suitable extension of the so-called Hartman-Nagumo inequality, -which is going to provide a priori bounds-, together with a continuation theorem proved in [60].

A more detailed overview on the history of the previously mentioned problems, as well as the main contributions of this doctoral thesis and related open questions, may be found at the beginning of each one of the four chapters in which it is divided.

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# Contents

# Chapter 1

# Dirichlet problems for resonant, pendulum-like equations: the first eigenvalue

## 1.1 Introduction

Second order, ordinary differential equations of the type

$$-u'' - \alpha u' + f(t,u) = 0$$

and associated boundary value problems, have been extensively studied in the literature since Newton's times, not only because of its intrinsic mathematical importance, but also because of the huge variety of phenomena in nature they may be used to model.

A particular case is given by nonlinearities $f$ of the form $f(t,u) = r(u) - h(t)$. They give rise to equations of the type

$$-u'' - \alpha u' + r(u) = h(t)$$

In this chapter we deal with nonlinear, resonant boundary value problems having the form

$$\begin{gathered} -u'' - \alpha u' - \lambda_1(\alpha)u + g(u) = h(t) = \tilde{h}(t) + \bar{h}\psi(t), \quad t \in [0,\pi] \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.1}$$

where the following set of hypothesis is assumed

**[H$_1$]** $\alpha$ is a given real number, $\lambda_1(\alpha) = 1 + \alpha^2/4$ is the first eigenvalue of the linear problem

$$\begin{gathered} -u''(t) - \alpha u'(t) = \lambda u(t), \quad t \in [0,\pi] \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.2}$$

$g \in \tilde{C}(\mathbb{R}/T\mathbb{Z})$ is a continuous and $T$-periodic function with zero mean, and $h \in L^1[0,\pi]$ is an integrable function.

Here, $h$ is usually called *the forcing term*, or *the external force* of the equation. We decompose it as $h = \tilde{h} + \bar{h}\psi$, being

$$\psi(t) = \frac{1}{\sqrt{\int_0^\pi \left[e^{\frac{\alpha}{2}s}\sin s\right]^2 ds}}\, e^{\frac{\alpha}{2}t}\sin t, \qquad 0 \le t \le \pi,$$

and

$$\bar{h} \in \mathbb{R}, \qquad \tilde{h} \in L^1[0,\pi], \quad \int_0^\pi \tilde{h}(s)\psi(s)ds = 0.$$

One important reason to do this is that, in case $g$ is trivial, the resulting linear problem (1.1) is solvable if and only if $\bar{h} = 0$. We will call $\psi^\perp$ this hyperplane in $L^1[0,\pi]$ :

$$\psi^\perp := \left\{ \tilde{h} \in L^1[0,\pi] : \int_0^\pi \tilde{h}(s)\psi(s)ds = 0 \right\} = \left\{ h \in L^1[0,\pi] : (1.1) \text{ with } g \equiv 0 \text{ is solvable } \right\} .$$

The question of the solvability of nonlinear problems of the type (1.1) had already been considered by different authors, starting with the pioneering work of Dancer ([25]). Here, the boundary value problem (1.1) was first explored in detail, in the more restricted framework of $h$ being continuous, $\alpha = 0$, $g(u) = \Lambda \sin u$. In this setting, it was shown ([25], Theorem 4, pp. 182) that, for any given $\tilde{h}$, there exists $\epsilon_0 = \epsilon_0(\tilde{h}) > 0$ such that problem (2.1) has solution for any $|\bar{h}| \leq \epsilon_0$. Further, the problem was also seen to have infinitely many solutions for $\bar{h} = 0$.

Many subsequent efforts were devoted to extend the Theorem above to general periodic nonlinearities $g$. We briefly describe some of them in a chronological order. Still assuming the continuity of the forcing term $h$, Ward [85] extended Dancer's results for arbitrary oscillating functions $g$, showing, in the non-friction case ($\alpha = 0$), that if $\bar{h} = 0$, problem (1.1) has at least one solution. His result was improved two years later by Schaaf and Schmitt [73], who used methods from global bifurcation theory to show that Ward's problem has in fact infinitely many positive and infinitely many negative solutions.

A related problem was to show *nondegeneracy*. It follows from the lower and upper solutions method, the boundedness of $g$, and Ward's results ([85]), that, for any given $\tilde{h} \in C[0,\pi]$, there exists a nonempty, closed and bounded interval $\mathcal{I}_{\tilde{h}} \ni 0$ of real numbers such that (1.1) is solvable if and only if $\bar{h} \in \mathcal{I}_{\tilde{h}}$. In case this interval contains a neighborhood of zero, the equation is said to be *nondegenerate*. Thus, in Dancer's work already appeared the nondegeneracy of (1.1) when $\alpha = 0$ and $g(u) = \Lambda \sin(u)$. It was extended for general periodic nonlinearities (still assuming the continuity of $h$ and the absence of friction) by Cañada and Roca in [15], using a Lyapunov-Schmidt reduction of this problem and a suitable generalization of the Riemann-Lebesgue lemma developed in [85].

In the first part of this chapter, we further generalize these results for problem (1.1) in the broader framework established in $[\mathbf{H}_1]$. First of all, we have an analogous non-degeneracy result.

**Theorem 1.1.1.** *Assume $g \not\equiv 0$. For any given $\tilde{h} \in \psi^\perp$ there exist real numbers $-\varepsilon_-(\tilde{h}) < 0 < \varepsilon_+(\tilde{h})$ such that problem (1.1) is solvable if and only if $-\varepsilon_-(\tilde{h}) \leq \bar{h} \leq \varepsilon_+(\tilde{h})$.*

*Further, if $\mathcal{H} \subset \psi^\perp$ is an equi-integrable subset, that is,*

$$\text{there exists } h_0 \in L^1[0,\pi] \text{ such that } |h| \leq h_0 \ \forall h \in \mathcal{H},$$

*then, there exists a positive constant $\epsilon > 0$ such that*

$$\varepsilon_-(\tilde{h}),\ \varepsilon_+(\tilde{h}) \geq \epsilon \ \forall \tilde{h} \in \mathcal{H}.$$

We also have a multiplicity result, showing that the number of solutions of (1.1) diverges to infinity whenever $|\bar{h}|$ is small enough. This divergence can be seen to be uniform with respect to $\tilde{h}$ belonging to equi-integrable subsets of $\psi^\perp$:

**Theorem 1.1.2.** *Let $\mathcal{H} \subset \psi^\perp$ be an equi-integrable subset. Then, for each $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that problem (1.1) has at least $m$ different solutions for any $\tilde{h} \in \mathcal{H},\ |\bar{h}| \leq \varepsilon_m$.*

Finally, it is possible to show that the length of the solvability interval $[-\varepsilon_-(\tilde{h}), \varepsilon_+(\tilde{h})]$ tends to 0 as the damping $\alpha$ becomes large. This convergence is uniform with respect to $\tilde{h}$.

**Theorem 1.1.3.** $\lim_{|\alpha| \to \infty} \{\varepsilon_+ + \varepsilon_-\} = 0$ *uniformly in $\psi^\perp$*

Thus, we generalize the existing results in two different directions. Firstly, a damping $\alpha$ is taken into consideration. And, secondly, we deal with forcing terms which are no longer continuous or bounded. This latter fact is, with much, which introduces the main new difficulties in our problem, and a delicate computations are needed to tackle it. As in [25, 85, 15], our approach is based in the Lyapunov-Schmidt decomposition of equation (1.1).

In the last two sections of this chapter we develop a multi-dimensional generalization of the Riemann-Lebesgue Lemma, which we use to explore the behavior at infinity of $\varepsilon_\pm$. Our arguments there may be extended to a much wider variety of problems -say, elliptic problems associated to PDE of the type of these considered in Chapter II, or resonant problems in higher eigenvalues such as

$$-u'' - \alpha u' - \lambda_k(\alpha)u + g(u) = h(t), \quad t \in [0,\pi]$$
$$u(0) = u(\pi) = 0.$$

where $\lambda_k(\alpha) = k^2 + \alpha^2/4$ is the $k^{th}$ eigenvalue of (1.2). However, *to decide whether a similar theorem to (1.1.1) holds for this problem seems to remain an open problem*, even though some partial answers have been given (see [38], [83]).

## 1.2 Preliminary results. The alternative system

Let us fix $\tilde{h} \in \psi^{\perp}$ and define

$$\mathcal{I}_{\tilde{h}} := \left\{ \bar{h} \in \mathbb{R} : \text{ problem (1.1) is solvable} \right\}. \tag{1.3}$$

We consider the linear differential operator

$$\mathcal{L} : W_0^{2,1}[0,\pi] \to L^1[0,\pi], \qquad \mathcal{L}u = -u'' - \alpha u' - \lambda_1(\alpha)u, \qquad \forall u \in W_0^{2,1}[0,\pi],$$

and the Nemytskii operator associated with $-g$

$$\mathcal{N} : W_0^{2,1}[0,\pi] \to L^1[0,\pi], \qquad \mathcal{N}u(t) = -g(u(t)), \qquad \forall u \in W_0^{2,1}[0,\pi], \ \forall t \in [0,\pi],$$

so that (1.1) is equivalent to the operator equation

$$\mathcal{L}u = \mathcal{N}u + h \tag{1.4}$$

It is well known that $\mathcal{L}$ is a linear Fredholm operator of zero index. Furthermore,

$$\ker \mathcal{L} = \langle \varphi \rangle, \qquad \operatorname{im} \mathcal{L} = \psi^{\perp}$$

where

$$\varphi(t) = \frac{1}{\sqrt{\int_0^\pi \left[e^{-\frac{\alpha}{2}s} \sin s\right]^2 ds}} e^{-\frac{\alpha}{2}t} \sin t, \qquad 0 \le t \le \pi \tag{1.5}$$

is a normalized generator of the eigenspace associated with the first eigenvalue $\lambda = \lambda_1(\alpha)$ of the linear problem (1.2). Observe that

$$W_0^{2,1}[0,\pi] = \varphi^{\perp} \oplus \langle \varphi \rangle,$$

being

$$\varphi^{\perp} := \left\{ \tilde{u} \in W_0^{2,1}[0,\pi] : \int_0^\pi \tilde{u}(s)\varphi(s)ds = 0 \right\}.$$

This splitting is also well adapted to our problem; just note that $\mathcal{L} : \varphi^{\perp} \to \psi^{\perp}$ is a topological isomorphism, and we will use it to rewrite any element $u \in W_0^{2,1}[0,\pi]$ as $u = \tilde{u} + \bar{u}\varphi$, where $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in \varphi^{\perp}$. We call $\mathcal{K} : \psi^{\perp} \to \varphi^{\perp}$ the inverse of this isomorphism and define

$$Q : L^1[0,\pi] \to L^1[0,\pi], \qquad h \mapsto \left( \int_0^\pi h(s)\psi(s)ds \right) \psi$$

In this way, equation (1.4) becomes equivalent to the so-called *Lyapunov-Schmidt* system

$$\tilde{u} = \mathcal{K}(I - Q)\mathcal{N}(\bar{u}\varphi + \tilde{u}) + \mathcal{K}\tilde{h} \tag{1.6}$$

$$\bar{h} = \int_0^\pi g(\bar{u}\varphi(s) + \tilde{u}(s))\psi(s)\, ds \tag{1.7}$$

Let us firstly study the so-called *auxiliary equation* (1.6). Let us call $\mathcal{T}$ the set of its solutions

$$\mathcal{T} := \left\{ (\bar{u}, \tilde{u}) \in \mathbb{R} \times \varphi^{\perp} : \tilde{u} = \mathcal{K}(I - Q)\mathcal{N}(\bar{u}\varphi + \tilde{u}) + \mathcal{K}\tilde{h} \right\} \ . \tag{1.8}$$

Something can be said about the set $\mathcal{T}$. Observe, to start, that being $\mathcal{N}$ bounded, and $\mathcal{K}$ compact, the Schauder fixed point theorem implies the existence, for any $\bar{u} \in \mathbb{R}$, of $\tilde{u} \in \varphi^{\perp}$ such that $(\bar{u}, \tilde{u}) \in \mathcal{T}$:

$$pr_{\mathbb{R}}\mathcal{T} = \mathbb{R}.$$

Another important fact, $\mathcal{T}$ is contained in a cylinder:

$$\begin{gathered} \text{There exist } M > 0 \text{ and an integrable function } m \in L^1[0, \pi], \text{ such that} \\ |\tilde{u}|,\ |\tilde{u}'| \ \le M, \qquad\qquad |\tilde{u}''| \ \le \ m \ \text{ a.e. in } [0, \pi], \\ \text{for any } (\bar{u}, \tilde{u}) \in \mathcal{T}. \end{gathered} \tag{1.9}$$

But then, the Riemann-Lebesgue lemma ([85]) together with (1.6) imply indeed

$$\lim_{\bar{u} \to \infty} \|\tilde{u} - \mathcal{K}\tilde{h}\|_{C^1[0,\pi]} = 0 \tag{1.10}$$

uniformly for $(\bar{u}, \tilde{u}) \in \mathcal{T}$.

Finally, $\mathcal{T}$ is locally compact, as it can be easily checked from its definition (1.8).

Other well-known properties of $\mathcal{T}$, related with the existence of 'large' connected subsets, will be described later. In our next step, we are going to start the proof of theorem 1.1.1 by showing the set $\mathcal{I}_{\tilde{h}}$ (defined in (1.3)) to be an interval.

Using the well-known change of variables $v = u - \mathcal{K}\tilde{h}$, problem (1.1) becomes

$$\begin{gathered} -v'' - \alpha v' - \lambda_1(\alpha)v + g(v + \mathcal{K}\tilde{h}) = \bar{h}\psi(t), \quad t \in [0, \pi] \\ v(0) = v(\pi) = 0. \end{gathered} \tag{1.11}$$

Thus, the lower and upper solutions method in a particular version which does not require any ordering of the lower and upper solutions (see [5, 12]), shows that $\mathcal{I}_{\tilde{h}}$ is an interval. Moreover, if it contains both positive and negative values, it must be closed. To check this, we will just show that in case $m := \inf \mathcal{I}_{\tilde{h}} < 0$, then $m \in \mathcal{I}_{\tilde{h}}$. It is indeed a consequence of Ward's version of the Riemann-Lebesgue lemma ([85]). Let $\{\varepsilon_n\}_n$ be any sequence in $\mathcal{I}_{\tilde{h}}$ with $\{\varepsilon_n\} \to m$ and let $\{(\bar{u}_n, \tilde{u}_n)\} \subset \mathbb{R} \times \varphi^{\perp}$ be such that

$$\tilde{u}_n = \mathcal{K}(I - Q)\mathcal{N}(\bar{u}_n\varphi + \tilde{u}_n) + \mathcal{K}(\tilde{h}) \tag{1.12}$$

and

$$\varepsilon_n = \int_0^{\pi} g(\bar{u}_n\varphi(s) + \tilde{u}_n(s))\psi(s)ds \tag{1.13}$$

We showed in (1.9) that the sequence $\{\tilde{u}_n\}$ must be bounded in $C^1[0, \pi]$. It implies the sequence $\{\bar{u}_n\}$ has, at least, a bounded subsequence, since the contrary would mean $\{|\bar{u}_n|\} \to \infty$ and the Riemann-Lebesgue lemma [85] would imply

$$m = \lim_{n \to \infty} \{\varepsilon_n\} = \lim_{n \to \infty} \int_0^{\pi} g(\bar{u}_n\varphi(s) + \tilde{u}_n(s))\psi(s)\ ds = 0.$$

Thus, we may find subsequences $\{\tilde{u}_{n_r}\}_r$ of $\{u_n\}$ and $\{\bar{u}_{n_r}\}_r$ of $\{\bar{u}_n\}$, together with elements $\tilde{u} \in C[0, \pi]$, $\bar{u} \in \mathbb{R}$ such that $\{\tilde{u}_{n_r}\} \to \tilde{u}$ in $C[0, \pi]$, $\{\bar{u}_{n_r}\} \to \bar{u}$. Taking limits as $r \to \infty$ in (1.12) and (1.13) we deduce:

$$\tilde{u} = \mathcal{K}(I - Q)\mathcal{N}(\bar{u}\varphi + \tilde{u})$$

and

$$m = \int_0^{\pi} g(\bar{u}\varphi(s) + \tilde{u}(s))\psi(s)\ ds, \tag{1.14}$$

that is, $m \in \mathcal{I}_{\tilde{h}}$.

The idea to show that $\mathcal{I}_{\tilde{h}}$ contains both positive and negative values is roughly that, for $|\bar{u}|$ big enough, $(\bar{u}, \tilde{u}) \in \mathcal{T}$,

$$\int_0^\pi g(\bar{u}\varphi + \tilde{u})\psi \, ds \approx \int_0^\pi g(\bar{u}\varphi + \mathcal{K}\tilde{h})\psi ds$$

as a consequence of (1.10). But this latter integral can be shown to oscillate around zero as $\bar{u} \to \pm\infty$. This is something where the fact that we are dealing with integrals on an interval plays an important role; even when it remains valid for dimensions 2,3, it is no longer true for $N \geq 4$. We will study this in detail in Chapter 2.

## 1.3 Study of some oscillating integrals

We may rewrite the *bifurcation equation* (1.7) as follows:

$$\mathcal{I}_{\tilde{h}} = \left\{ \int_0^\pi g(\bar{u}\varphi(s) + \tilde{u}(s))\psi(s) \, ds : (\tilde{u}, \bar{u}) \in \mathcal{T} \right\}. \tag{1.15}$$

On the other hand, well-known results, based upon the continuity property of the Leray-Schauder topological degree (see, for instance, [25]), show the existence of a continuum (i.e., a closed, connected set) $\mathcal{S} \subset \mathbb{R} \times \varphi^\perp$ of solutions

$$\tilde{u} = \mathcal{K}(I - Q)\mathcal{N}(\bar{u}\varphi + \tilde{u}) + \mathcal{K}\tilde{h} \qquad\qquad \forall (\bar{u}, \tilde{u}) \in \mathcal{S}$$

with projection on $\mathbb{R}$ covering the whole real line

$$pr_{\mathbb{R}}(\mathcal{S}) = \mathbb{R}.$$

Of course, from (1.15) we know

$$\int_0^\pi g(\bar{u}\varphi(s) + \tilde{u}(s))\psi(s) \, ds \in \mathcal{I}_{\tilde{h}} \quad \forall (\bar{u}, \tilde{u}) \in \mathcal{S}. \tag{1.16}$$

Relation (1.16) will be later used at big values of $|\bar{u}|$ to deduce that $\mathcal{I}_{\tilde{h}}$ contains both positive and negative values. And, in order to study the asymptotic behaviour of functions defined by integrals of this type, it is firstly convenient to understand the shape of $\varphi$:

**Lemma 1.3.1.** *There is an (unique) real number $\theta \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$ such that:*

1. $\varphi'(t) > 0, \ \forall t \in \left[0, \frac{\pi}{2} - \theta\right[, \ \varphi'(t) < 0, \ \forall t \in \left]\frac{\pi}{2} - \theta, \pi\right].$
2. $\varphi'(\frac{\pi}{2} - \theta) = 0, \ \varphi''(\frac{\pi}{2} - \theta) < 0.$

*As a consequence, the maximum value of $\varphi$ on $[0, \pi]$ is attained at $\frac{\pi}{2} - \theta$.*

*Proof.* We recall the explicit expression of $\varphi$, given in (1.5). If, for simplicity, we call

$$C := \sqrt{\int_0^\pi \left[e^{-\frac{\alpha}{2}s} \sin s\right]^2 ds} \ ,$$

we have

$$\varphi(t) = \frac{1}{C} e^{-\frac{\alpha}{2}t} \sin t,$$

$$\varphi'(t) = \frac{1}{C} e^{-\frac{\alpha}{2}t} \left(-\frac{\alpha}{2} \sin t + \cos t\right) = \frac{A}{C} e^{-\frac{\alpha}{2}t} \left(\frac{1}{A} \cos t - \frac{\alpha/2}{A} \sin t\right),$$

being

$$A = \sqrt{1 + \frac{\alpha^2}{4}} \ . \tag{1.17}$$

Since

$$\left(\frac{1}{A}\right)^2 + \left(\frac{\alpha/2}{A}\right)^2 = 1,$$

there is an unique point $\theta \in ]-\pi/2, \pi/2[$ satisfying

$$\cos\theta = \frac{1}{A}\ , \qquad\qquad \sin\theta = \frac{\alpha/2}{A}\ .$$

Then,

$$\varphi'(t) = \frac{A}{C} e^{\frac{-\alpha}{2}t}\cos(t+\theta), \qquad\qquad \varphi''(t) = -A^2 e^{\frac{-\alpha}{2}t}\sin(t+2\theta),$$

and the lemma follows. □

Let $G$ be the primitive with zero mean of $g$. We choose $b_-, b_+ \in \mathbb{R}$ such that

$$0 \le b_-, b_+ < T, \qquad G(b_-) = \min_{\mathbb{R}} G, \qquad G(b_+) = \max_{\mathbb{R}} G.$$

Being $\mathcal{S}$ connected, for each $n \in \mathbb{N}$ there exist $(\bar{u}_n, \tilde{u}_n),\ (\bar{v}_n, \tilde{v}_n) \in \mathcal{S} \subset \mathcal{T}$ with

$$\bar{u}_n\varphi\left(\frac{\pi}{2} - \theta\right) + \tilde{u}_n\left(\frac{\pi}{2} - \theta\right) = b_- + nT. \tag{1.18}$$

$$\bar{v}_n\varphi\left(\frac{\pi}{2} - \theta\right) + \tilde{v}_n\left(\frac{\pi}{2} - \theta\right) = b_+ + nT; \tag{1.19}$$

We plan to prove that, for sufficiently large $n$,

$$\int_0^\pi g(\bar{u}_n\varphi + \tilde{u}_n)\psi\ ds < 0 \tag{1.20}$$

and

$$\int_0^\pi g(\bar{v}_n\varphi + \tilde{v}_n)\psi\ ds > 0. \tag{1.21}$$

uniformly with respect to $\tilde{h}$ as it varies on equi-integrable subsets of $\varphi^\perp$. This motivates Theorem 1.3.2 below.

**Theorem 1.3.2.** *Let $\Omega \subset W_0^{2,1}([0,\pi])$ be bounded in the $C^1[0,\pi]$ topology. Assume, further, that*

$$\left\{\Omega'' : \Omega \in \Omega\right\} \quad \textit{is equi-integrable.}$$

*Then, there exists $n_* \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \ge n_*$, there exists a positive constant $K_n > 0$ verifying*

$$\int_0^\pi g(r\varphi + \Omega)\psi dt \ge K_n \quad \forall (r,\Omega) \in \mathbb{R}\times\Omega \textit{ with } r\varphi\left(\frac{\pi}{2} - \theta\right) + \Omega\left(\frac{\pi}{2} - \theta\right) = b_+ + nT, \tag{1.22}$$

$$\int_0^\pi g(r\varphi + \Omega)\psi dt \le -K_n \quad \forall (r,\Omega) \in \mathbb{R}\times\Omega \textit{ with } r\varphi\left(\frac{\pi}{2} - \theta\right) + \Omega\left(\frac{\pi}{2} - \theta\right) = b_- + nT. \tag{1.23}$$

All the remaining of this Section is consecrated to the proof of this theorem. However, let us firstly see how to deduce Theorems 1.1.1 and 1.1.2 from this.

*Proof of Theorem 1.1.1.* Let $\mathcal{H} \subset \psi^\perp$ be equi-integrable. Then, the set

$$\Omega := \bigcup_{\tilde{h}\in\mathcal{H},\ \bar{v}\in\mathbb{R}} \left\{\tilde{v} \in \varphi^\perp : \tilde{v} = \mathcal{K}(I-Q)\mathcal{N}(\bar{v}+\tilde{v}) + \mathcal{K}\tilde{h}\right\} \subset W_0^{2,1}[0,\pi]$$

is bounded in the $C^1[0,\pi]$-topology, while

$$\left\{\tilde v'' : \tilde v \in \Omega\right\}$$

is equi-integrable. Thus, $n_* \in \mathbb{N}$ may be chosen such that for any $n \in \mathbb{N}$ with $n \geq n_*$, there exists a positive constant $K_n > 0$ verifying (1.22) and (1.23).

Then, in view of (1.19), (1.18) and (1.3), it implies, for the particular case of $n = n_*$,

$$\mathcal{I}_{\tilde h} \supset [-K_{n_*}, K_{n_*}] \qquad\qquad \forall \tilde h \in \mathcal{H},$$

showing Theorem 1.1.1. □

*Proof of Theorem 1.1.2.* Let $\mathcal{H} \subset \psi^\perp$ be equi-integrable and choose $n_* \in \mathbb{N}$ as in the proof of Theorem 1.1.1 above.

It follows from our auxiliary equation (1.6) that the limit in (1.10) is indeed uniform with respect to $\tilde h$ belonging to bounded subsets of $\psi^\perp$. Thus, a natural number $n^* \geq n_*$ may be found such that given any $n \geq n^*$ and

$$\left\{(\bar u_n, \tilde u_n)\right\}_{n\geq n^*}, \qquad\qquad \left\{(\bar v_n, \tilde v_n)\right\}_{n\geq n^*},$$

sequences of solutions of (1.6) for some $\tilde h \in \mathcal{H}$, verifying (1.18) and (1.19) for each $n \geq n_*$, we have

$$\bar u_n\varphi(t) + \tilde u_n(t) < \bar v_n\varphi(t) + \tilde v_n(t) < \bar u_{n+1}\varphi(t) + \tilde u_{n+1}(t) \quad \forall t \in ]0,\pi[ \text{ in case } b_- < b_+ ; \tag{1.24}$$

$$\bar v_n\varphi(t) + \tilde v_n(t) < \bar u_n\varphi(t) + \tilde u_n(t) < \bar v_{n+1}\varphi(t) + \tilde v_{n+1}(t) \quad \forall t \in ]0,\pi[ \text{ in case } b_+ < b_- . \tag{1.25}$$

We observe that we further have

$$\bar u_n\varphi'(0) + \tilde u_n'(0) < \bar v_n\varphi'(0) + \tilde v_n'(0), \qquad \text{in case } b_- < b_+ ;$$

$$\bar u_n\varphi'(0) + \tilde u_n'(0) < \bar v_{n+1}\varphi'(0) + \tilde v_{n+1}'(0), \qquad \text{in case } b_+ < b_- ,$$

the nonstrict inequalities being a consequence of (1.24) and (1.25), while equalities cannot occur.

Thus, the lower and upper solutions method gives us, for any $\tilde h \in \mathcal{H}$ and

$$|\tilde h| < \min_{n^*\leq n\leq n^*+m} \{K_n\},$$

the existence of at least $m$ different solutions of (1.1).

□

*Proof of Theorem 1.3.2.* Of course, (1.22) and (1.23) are analogous, and we may restrict ourselves to prove, for instance, (1.22). In order to achieve that we may well concentrate in the case of $\{v_n\}$ and limit ourselves to show

$$\lim_{n\to\infty} \bar v_n \int_0^\pi g(\bar v_n\varphi + \tilde v_n)\psi \, ds = +\infty\,, \tag{1.26}$$

uniformly with respect to $\tilde h$ belonging to equi-integrable subsets of $\psi^\perp$. Thus, we choose any sequence $\{(\bar\Omega_n, \tilde\Omega_n)\}$ in $\mathbb{R} \times \varphi^\perp$, (in fact, our proof will only need a sequence $\{(\bar\Omega_n, \tilde\Omega_n)\}$ in $\mathbb{R} \times W_0^{2,1}[0,\pi]$), from which we assume:

$$\exists M > 0,\ \exists m \in L^1[0,\pi] \text{ such that } |\Omega_n| \leq M,\ |\Omega_n'| \leq M,\ |\Omega_n''| \leq m \quad \forall n \in \mathbb{N}, \tag{1.27}$$

$$\bar\Omega_n\varphi\left(\frac{\pi}{2} - \theta\right) + \tilde\Omega_n\left(\frac{\pi}{2} - \theta\right) = b_+ + nT \quad \forall n \in \mathbb{N}, \tag{1.28}$$

and we are going to show that

$$\lim_{n\to\infty} \bar\Omega_n \int_0^\pi g(\bar\Omega_n\varphi + \tilde\Omega_n)\psi \, ds = +\infty. \tag{1.29}$$

Note that, from (1.27) and (1.28) we deduce that $\bar\Omega_n \to +\infty$ as $n \to +\infty$. Thus, it is not restrictive to assume $\bar\Omega_n > 0$ $\forall n \in \mathbb{N}$. Let us define

$$p_n := \varphi + \frac{1}{\bar\Omega_n}\tilde\Omega_n \tag{1.30}$$

As $\{\bar\Omega_n\} \to +\infty$ and $\varphi'(t) \neq 0$ if $t \neq \frac{\pi}{2} - \theta$, it seems reasonable we will be able to control by below $|p_n'|$ by $\frac{1}{2}|\varphi'|$ whenever $n$ is big and we are not too close to $\frac{\pi}{2} - \theta$. This is shown next:

**Lemma 1.3.3.** *There exist $D > 0$ and $n_1 \in \mathbb{N}$ such that*

$$\left|p_n'(t) - \varphi'(t)\right| \leq \frac{1}{2}\left|\varphi'(t)\right|, \ \forall t \in [0,\pi] \setminus \left[\frac{\pi}{2} - \theta - \frac{D}{\bar{\Omega}_n}, \frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right], \ \forall \ n \geq n_1. \tag{1.31}$$

*Proof.* Pick $0 < \epsilon < -\varphi''(\frac{\pi}{2} - \theta)$ and $D > 0$ satisfying

$$D > \frac{2M}{-\varphi''(\frac{\pi}{2} - \theta) - \epsilon} \tag{1.32}$$

where $M$ is the constant given in (1.27). Moreover, since

$$\varphi''\left(\frac{\pi}{2} - \theta\right) = \lim_{t \to \frac{\pi}{2} - \theta} \frac{\varphi'(t)}{t - (\frac{\pi}{2} - \theta)}$$

we may choose $\delta > 0$ such that $[\frac{\pi}{2} - \theta - \delta, \frac{\pi}{2} - \theta + \delta] \subset [0,\pi]$, and

$$\begin{aligned} &\varphi'(t) \geq \left(\varphi''(\tfrac{\pi}{2} - \theta) + \epsilon\right)\left(t - (\tfrac{\pi}{2} - \theta)\right) \ \forall t \in \left[\tfrac{\pi}{2} - \theta - \delta, \tfrac{\pi}{2} - \theta\right], \\ &\varphi'(t) \leq \left(\varphi''(\tfrac{\pi}{2} - \theta) + \epsilon\right)\left(t - (\tfrac{\pi}{2} - \theta)\right), \ \forall t \in \left[\tfrac{\pi}{2} - \theta, \tfrac{\pi}{2} - \theta + \delta\right]. \end{aligned} \tag{1.33}$$

Finally, select $n_1 \in \mathbb{N}$ such that

$$\left|\varphi'(t)\right| \geq \frac{2M}{\bar{\Omega}_n} \ \forall t \in [0,\pi] \setminus \left[\frac{\pi}{2} - \theta - \delta, \frac{\pi}{2} - \theta + \delta\right], \quad \frac{D}{\bar{\Omega}_n} \leq \delta \qquad \forall n \geq n_1. \tag{1.34}$$

Then, if $n \geq n_1$, and $t \in \left[0, \frac{\pi}{2} - \theta - \delta\right]$, we have from (1.34)

$$p_n'(t) \geq \varphi'(t) - M/\bar{\Omega}_n = \frac{1}{2}\varphi'(t) + \left(\varphi'(t)/2 - M/\bar{\Omega}_n\right) \geq \frac{1}{2}\varphi'(t) + \left(1/2\frac{2M}{\bar{\Omega}_n} - M/\bar{\Omega}_n\right) = \frac{1}{2}\varphi'(t)$$

Also, if $n \geq n_1$, and $t \in \left[\frac{\pi}{2} - \theta - \delta, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n\right]$, we have from (1.32)

$$\begin{aligned} p_n'(t) &\geq \frac{1}{2}\varphi'(t) + \left(\frac{1}{2}\varphi'(t) - M/\bar{\Omega}_n\right) \geq \\ &\geq \frac{1}{2}\varphi'(t) + \left(\frac{1}{2}\left[\left(\varphi''\left(\frac{\pi}{2} - \theta\right) + \epsilon\right)\left(t - \left(\frac{\pi}{2} - \theta\right)\right)\right] - M/\bar{\Omega}_n\right) \geq \\ &\geq \frac{1}{2}\varphi'(t) + \left(\frac{1}{2}\left[\left(\varphi''(\frac{\pi}{2} - \theta) + \epsilon\right)\left(-D/\bar{\Omega}_n\right)\right] - M/\bar{\Omega}_n\right) \geq \\ &\geq \frac{1}{2}\varphi'(t) - \left(\frac{1}{2}\left(\varphi''\left(\frac{\pi}{2} - \theta\right) + \epsilon\right)\frac{1}{\bar{\Omega}_n}\frac{2M}{-\varphi''(\frac{\pi}{2} - \theta) - \epsilon}\right) - M/\bar{\Omega}_n = \frac{1}{2}\varphi'(t) \end{aligned}$$

Therefore, if $n \geq n_1$, we obtain

$$p_n'(t) \geq \frac{1}{2}\varphi'(t), \forall \ t \in \left[0, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n\right] \tag{1.35}$$

$$p_n'(t) \leq \frac{3}{2}\varphi'(t), \forall \ t \in \left[0, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n\right] \tag{1.36}$$

To see this, it is sufficient to show

$$\varphi'(t) \geq 2M/\bar{\Omega}_n, \ \forall \ t \in \left[0, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n\right].$$

However, it is shown in (1.34) that previous inequality holds in $\left[0, \frac{\pi}{2} - \theta - \delta\right]$. Moreover, from (1.33), we obtain that if $t \in [\frac{\pi}{2} - \theta - \delta, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n]$,

$$\varphi'(t) \geq \left(\varphi''\left(\frac{\pi}{2} - \theta\right) + \epsilon\right)\left(t - \left(\frac{\pi}{2} - \theta\right)\right) \geq 2M/D\left(\frac{\pi}{2} - \theta - t\right) \geq 2M/\bar{\Omega}_n.$$

Now, (1.35) and (1.36) prove (1.31) in $[0, \frac{\pi}{2} - \theta - D/\bar{\Omega}_n]$. And an analogous reasoning establishes (1.31) on $[\frac{\pi}{2} - \theta + D/\bar{\Omega}_n, \pi]$. ☐

Recall from (1.28) that the sequence $p_n$ was chosen in such a way that $p_n\left(\frac{\pi}{2} - \theta\right) = b_+ + nT$. In these points $g$ vanishes, making the sequence

$$\left[\frac{\pi}{2} - \theta - \frac{D}{\bar{\Omega}_n}, \frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right] \to \mathbb{R}, \qquad s \mapsto g(\bar{\Omega}_n p_n(s))\psi(s),$$

(and thus, also its mean) converge to 0. We write the detailed proof below.

**Lemma 1.3.4.**

$$\lim_{n\to\infty} \bar{\Omega}_n \int_{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}}^{\frac{\pi}{2}-\theta+\frac{D}{\bar{\Omega}_n}} g(\bar{\Omega}_n p_n(t))\psi(t)\ dt = 0$$

*Proof.* By using the substitution $t = \frac{\pi}{2} - \theta + s/\bar{\Omega}_n$, previous limit becomes

$$\lim_{n\to\infty} \int_{-D}^{D} g\left(\bar{\Omega}_n\varphi\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) + \tilde{\Omega}_n\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right)\right)\psi\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds = 0$$

However, $\left\{t \mapsto g\left(\bar{\Omega}_n\varphi(\frac{\pi}{2} - \theta + t/\bar{\Omega}_n) + \tilde{\Omega}(\frac{\pi}{2} - \theta + t/\bar{\Omega}_n)\right)\right\} \to 0$ uniformly on $[0, \pi]$. To check this, let $\epsilon \in \mathbb{R}^+$ be given and $\delta > 0$ such that $|g(v)| \leq \epsilon,\ \forall\ v \in [b_+ - \delta, b_+ + \delta]$. Since $g$ is $T$-periodic, we have $|g(v)| \leq \epsilon,\ \forall\ v \in [b_+ + nT - \delta, b_+ + nT + \delta],\ \forall\ n \in \mathbb{N}$. Also, from the equality $\varphi'(\frac{\pi}{2} - \theta) = 0$, it follows that for sufficiently large $n$,

$$\left|\bar{\Omega}_n\varphi\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) - \bar{\Omega}_n\varphi\left(\frac{\pi}{2} - \theta\right)\right| \leq \delta/2 \qquad \forall\ s \in [-D, D].$$

Moreover, from (1.9), we deduce that if $n$ is sufficiently large, then

$$\left|\tilde{\Omega}_n\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) - \tilde{\Omega}_n\left(\frac{\pi}{2} - \theta\right)\right| \leq \delta/2 \qquad \forall\ s \in [-D, D].$$

Thus, previous relations and (1.19) imply that for sufficiently large $n$,

$$\left|g\left(\bar{\Omega}_n\varphi\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) + \tilde{\Omega}_n\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right)\right)\right| \leq \epsilon,\ \forall\ s \in [-D, D]$$

☐

In our next result we show we may substitute the sequence of integrals

$$\left\{\bar{\Omega}_n \int_0^{\pi} g(\bar{\Omega}_n p_n(t))\psi(t)\ dt\right\}$$

by another more appropriate sequence which points out the oscillations of the function $G$.

**Lemma 1.3.5.** *Let $D$ be chosen as in Lemma 1.3.3. Then*

$$\lim_{n\to\infty} \left\{\bar{\Omega}_n \int_0^{\pi} g(\bar{\Omega}_n p_n(t))\psi(t)\ dt - \mathcal{J}_n^+ - \mathcal{J}_n^-\right\} = 0,$$

*where*

$$\mathcal{J}_n^+ = \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} \left(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\right) \frac{d}{dt}\left(\frac{\psi(t)}{p_n'(t)}\right)\, dt,$$
$$\mathcal{J}_n^- = \int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi} \left(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\right) \frac{d}{dt}\left(\frac{\psi(t)}{p_n'(t)}\right)\, dt$$

*Proof.* From Lemma 1.3.4 we know

$$\lim_{n\to\infty}\Bigg[\bar\Omega_n \int_0^{\pi} g(\bar\Omega_n p_n(t))\psi(t)\, dt - \bar\Omega_n \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} g(\bar\Omega_n p_n(t))\psi(t)\, dt - \\ - \bar\Omega_n \int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi} g(\bar\Omega_n p_n(t))\psi(t)\, dt\Bigg] = 0.$$

On the other hand, Lemma 1.3.3 gives us that, for $n \geq n_1$, the function

$$t \mapsto \frac{\psi(t)}{p_n'(t)}$$

is absolutely continuous both on $\left[0, \frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right]$ and on $\left[\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}, \pi\right]$. Therefore, for $n$ sufficiently large, it follows from integration by parts that

$$\bar\Omega_n \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} g\big(\bar\Omega_n p_n(t)\big)\psi(t) dt = \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} g\big(\bar\Omega_n p_n(t)\big)\bar\Omega_n p_n'(t)\left(\frac{\psi(t)}{p_n'(t)}\right) dt =$$
$$= G\left(\bar\Omega_n p_n\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)\right)\frac{\psi\big(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\big)}{p_n'\big(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\big)} - \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} G\big(\bar\Omega_n p_n(t)\big)\frac{d}{dt}\left(\frac{\psi(t)}{p_n'(t)}\right) dt =$$
$$= \left[G\left(\bar\Omega_n p_n\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)\right) - \max_{\mathbb{R}} G\right]\frac{\psi\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)}{p_n'\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)} + \\ + \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}} \left(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\right)\frac{d}{dt}\left(\frac{\psi(t)}{p_n'(t)}\right) dt$$

and, analogously,

$$\bar\Omega_n \int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi} g\left(\bar\Omega_n p_n(t)\right)\psi(t) dt = \left(-G\left(\bar\Omega_n p_n\left(\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}\right)\right) + \max_{\mathbb{R}} G\right)\frac{\psi(\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n})}{p_n'(\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n})} + \\ + \int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi} \left(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\right)\frac{d}{dt}\left(\frac{\psi(t)}{p_n'(t)}\right) dt$$

Therefore, to finish the proof of this lemma it is sufficient to show that

$$\lim_{n\to\infty}\left[\frac{\max_{\mathbb{R}} G - G\left(\bar\Omega_n p_n\left(\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}\right)\right)}{p_n'\left(\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}\right)}\right] = 0 \tag{1.37}$$

$$\lim_{n\to\infty}\left[\frac{-\max_{\mathbb{R}} G + G\left(\bar\Omega_n p_n\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)\right)}{p_n'\left(\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}\right)}\right] = 0 \tag{1.38}$$

and we will restrict ourselves to prove (1.37). Of course, (1.38) would we proved analogously.

From Lemma 1.3.3, we know

$$0 \le \frac{1}{-p_n'(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n})} \le \frac{2}{-\varphi'(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n})} \; .$$

Since

$$\lim_{n\to\infty} \frac{\varphi'(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n})}{D/\bar{\Omega}_n} = \varphi''(\frac{\pi}{2} - \theta) < 0,$$

we need only to prove

$$\lim_{n\to\infty} \; \bar{\Omega}_n \left[\max_{\mathbb{R}} G - G\left(\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right)\right)\right] = 0.$$

However,

$$\begin{aligned}\bar{\Omega}_n \left[\max_{\mathbb{R}} G - G\left(\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right)\right)\right] &= \bar{\Omega}_n \int_{\bar{\Omega}_n p_n\left(\frac{\pi}{2}-\theta+\frac{D}{\bar{\Omega}_n}\right)}^{\bar{\Omega}_n p_n\left(\frac{\pi}{2}-\theta\right)} g(s)ds \le \\ &\le \bar{\Omega}_n \left|\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right) - \bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta\right)\right| \max_{J_n} \; |g|\end{aligned}$$

where $J_n = \left[\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right), \bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta\right)\right]$. Moreover,

$$\begin{aligned}\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right) - \bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta\right) &= \bar{\Omega}_n \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}-\theta+\frac{D}{\bar{\Omega}_n}} p_n'(s) \; ds = \\ &= \int_0^D \left(\varphi'\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) + \frac{1}{\bar{\Omega}_n} \tilde{\Omega}_n' \left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right)\right) ds \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

as a consequence of (1.27). Then, from (1.19), $\max_{J_n} |g| \to 0$ as $n \to \infty$. So, the proof will be finished if we show the sequence of real numbers

$$\left\{\bar{\Omega}_n \left[\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right) - \bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta\right)\right]\right\}_n$$

to be bounded. However,

$$\begin{aligned}\bar{\Omega}_n \left[\bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta + \frac{D}{\bar{\Omega}_n}\right) - \bar{\Omega}_n p_n \left(\frac{\pi}{2} - \theta\right)\right] &= \bar{\Omega}_n \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}-\theta+\frac{D}{\bar{\Omega}_n}} \bar{\Omega}_n p_n'(s)ds = \\ &= \bar{\Omega}_n \int_0^D \varphi'\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds + \int_0^D \tilde{\Omega}_n' \left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds = \\ &= \bar{\Omega}_n \int_0^D \left(\varphi'\left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) - \varphi'\left(\frac{\pi}{2} - \theta\right)\right) ds + \int_0^D \tilde{\Omega}_n' \left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds = \\ &= \int_0^D \left(\bar{\Omega}_n \int_0^{s/\bar{\Omega}_n} \varphi''\left(\frac{\pi}{2} - \theta + r\right) dr\right) ds + \int_0^D \tilde{\Omega}_n' \left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds = \\ &= \int_0^D \left(\int_0^s \varphi''\left(\frac{\pi}{2} - \theta + t/\bar{\Omega}_n\right) \; dt\right) \; ds + \int_0^D \tilde{\Omega}_n' \left(\frac{\pi}{2} - \theta + s/\bar{\Omega}_n\right) ds\end{aligned}$$

whose boundedness derives from (1.27). □

*Proof of* (1.29). It follows from Lemma 1.3.5 above that what we want to prove is nothing but

$$\lim_{n\to\infty} \{\mathcal{J}_n^+ + \mathcal{J}_n^-\} = +\infty \tag{1.39}$$

Indeed, what happens here is that both the terms in the sum above diverge:

$$\lim_{n\to\infty} \mathcal{J}_n^- = \lim_{n\to\infty} \mathcal{J}_n^+ = +\infty$$

To see this, we consider the functions $\psi_1, \psi_2 : [0,\pi] \to \mathbb{R}$ given by

$$\psi_1(t) = e^{-\alpha t/2}\sin(t+\theta), \qquad \psi_2(t) = e^{-\alpha t/2}\cos(t+\theta),$$

where $\theta$ is the real constant defined in Lemma 1.3.1. Then, since

$$\psi\left(\frac{\pi}{2}-\theta\right) > 0, \qquad \psi_1\left(\frac{\pi}{2}-\theta\right) > 0, \qquad \psi_2\left(\frac{\pi}{2}-\theta\right) = 0, \qquad \psi_2'\left(\frac{\pi}{2}-\theta\right) > 0,$$

there exist (unique) real constants $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta_1 > 0$, and a function $R \in C^\infty[0,\pi]$ with $R(\frac{\pi}{2}-\theta) = R'(\frac{\pi}{2}-\theta) = 0$, such that

$$\psi = \beta_1\psi_1 + \beta_2\psi_2 + R \tag{1.40}$$

Let us divide the proof of (1.39) into two steps.

*Step 1:* Here, we plan to establish the equality

$$\lim_{n\to\infty}\left[\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\Omega_n}} \left(\max_{\mathbb{R}} G - G(\bar\Omega_n p_n(t))\right)\frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right)dt + \right.$$
$$\left. + \int_{\frac{\pi}{2}-\theta+\frac{D}{\Omega_n}}^{\pi} \left(\max_{\mathbb{R}} G - G(\bar\Omega_n p_n(t))\right)\frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right)dt\right] = 0.$$

To show this, recall (1.27) and Lemma 1.3.3 to write, for $n$ is sufficiently large and $t \in \left[0, \frac{\pi}{2}-\theta-\frac{D}{\Omega_n}\right] \cup \left[\frac{\pi}{2}-\theta+\frac{D}{\Omega_n}, \pi\right]$,

$$\left|\frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right)\right| \le \left|\frac{R'(t)}{p_n'(t)}\right| + \left|\frac{R(t)}{p_n'(t)^2}\right| |p_n''(t)| \le 2\left|\frac{R'(t)}{\varphi'(t)}\right| + 4\left|\frac{R(t)}{\varphi'(t)^2}\right| \Big(|\varphi''(t)| + m(t)\Big) \tag{1.41}$$

Moreover, both functions

$$t \mapsto \frac{R'(t)}{\varphi'(t)}, \qquad t \mapsto \frac{R(t)}{\varphi'(t)^2},$$

are continuous on $[0,\pi]$. Thus, we may apply the Lebesgue Convergence Theorem to obtain that the sequence of $L^1[0,\pi]$-functions given by

$$t \to \begin{cases} \frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right) & \text{if } t \in \left[0, \frac{\pi}{2}-\theta-\frac{D}{\Omega_n}\right] \cup \left[\frac{\pi}{2}-\theta+\frac{D}{\Omega_n}, \pi\right], \\ 0 & \text{if } t \in \left[\frac{\pi}{2}-\theta-\frac{D}{\Omega_n}, \frac{\pi}{2}-\theta+\frac{D}{\Omega_n}\right] \end{cases}$$

converges in $L^1[0,\pi]$ to

$$t \mapsto \frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right),$$

and we conclude that

$$\lim_{n\to\infty}\Bigg[\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right)dt+ \\ +\int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{R(t)}{p_n'(t)}\right)dt- \\ -\int_0^{\pi}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right)dt\Bigg]=0.$$

Thus, to finish this step, it remains to show that

$$\lim_{n\to\infty}\int_0^{\pi}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right)dt=0. \tag{1.42}$$

However,

$$\int_0^{\pi}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right)dt= \\ =\Big(\max_{\mathbb{R}} G\Big)\left(\frac{R(\pi)}{\varphi'(\pi)}-\frac{R(0)}{\varphi'(0)}\right)-\int_0^{\pi} G(\bar\Omega_n p_n(t))\frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right)\,dt.$$

At this point, we apply the Riemann-Lebesgue lemma ([85]), which says that

$$\lim_{n\to\infty}\int_0^{\pi} G\big(\bar\Omega_n p_n(t)\big)\frac{d}{dt}\left(\frac{R(t)}{\varphi'(t)}\right)dt=0,$$

while it follows from (1.40) that

$$\frac{R(\pi)}{\varphi'(\pi)}=\frac{R(0)}{\varphi'(0)},$$

showing (1.42).

*Step 2.* We claim that

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar\Omega_n}}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{\beta_1\psi_1(t)+\beta_2\psi_2(t)}{p_n'(t)}\right)dt= \\ =\lim_{n\to\infty}\int_{\frac{\pi}{2}-\theta+\frac{D}{\bar\Omega_n}}^{\pi}\Big(\max_{\mathbb{R}} G - G\big(\bar\Omega_n p_n(t)\big)\Big)\frac{d}{dt}\left(\frac{\beta_1\psi_1(t)+\beta_2\psi_2(t)}{p_n'(t)}\right)dt=+\infty$$

To prove this, we may concentrate, for example, in the first limit, the second one being analogous. The following equality is straightforward:

$$\frac{d}{dt}\left(\frac{\beta_1\psi_1(t)+\beta_2\psi_2(t)}{p_n'(t)}\right)=e^{-\alpha t}\frac{\mathcal{A}_n(t)}{p_n'(t)^2}-\frac{1}{\bar\Omega_n}e^{-\alpha t}\frac{\mathcal{B}_n(t)}{p_n'(t)^2} \tag{1.43}$$

where

$$\mathcal{A}_n(t)=\beta_1 A+\frac{\beta_1}{\bar\Omega_n}e^{\alpha t/2}\cos(t+\theta)\tilde\Omega_n'(t)-\frac{\beta_2}{\bar\Omega_n}e^{\alpha t/2}\sin(t+\theta)\tilde\Omega_n'(t);$$
$$\mathcal{B}_n(t)=\Big(\beta_1\sin(t+\theta)+\beta_2\cos(t+\theta)\Big)\frac{d}{dt}\Big(e^{\alpha t/2}\tilde\Omega_n'(t)\Big),$$

being $A=\sqrt{1+\frac{\alpha^2}{4}}$ as defined in (1.17). Moreover, from (1.27) we deduce

1. $\{\mathcal{A}_n\}\to\beta_1 A$, uniformly on $[0,\pi]$.

2. There exists a function $B \in L^1[0,\pi]$ such that $|\mathcal{B}_n| \leq B$ for $n \in \mathbb{N}$ sufficiently large.

At this point we consider the sequences $\{r_n\}_n$ and $\{s_n\}_n$ of real numbers defined below:

$$r_n := \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}} \left(\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)\right) e^{-\alpha t} \frac{\mathcal{A}_n(t)}{p_n'(t)^2} dt;$$
$$s_n := \frac{1}{\bar{\Omega}_n} \int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}} \left(\max_{\mathbb{R}} G - G(\bar{\Omega}_n p_n(t))\right) e^{-\alpha t} \frac{\mathcal{B}_n(t)}{p_n'(t)^2} dt,$$

so that, for each $n \in \mathbb{N}$, from (1.43) we may write

$$\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}} \left(\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)\right) \frac{d}{dt}\left(\frac{\beta_1\psi_1(t)+\beta_2\psi_2(t)}{p_n'(t)}\right) dt = r_n - s_n/\bar{\Omega}_n.$$

To finish, we will show that

$$\lim r_n = +\infty; \tag{1.44}$$
$$\{s_n\} \text{ is bounded.} \tag{1.45}$$

In orden to check (1.44), choose $n$ large enough so that

$$A_n(t) \geq \beta_1 \frac{A}{2} \qquad\qquad \forall t \in [0,\pi],$$

and fix any $\rho \in \left]0, \frac{\pi}{2}-\theta\right[$ with

$$\frac{\pi}{2} - \theta - \frac{D}{\bar{\Omega}_n} \geq \rho. \tag{1.46}$$

Then, from Lemma 1.3.3, we have

$$\begin{aligned}
r_n/\beta_1 \geq \frac{A}{2}\frac{4}{9}\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}} &\left(\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)\right) e^{-\alpha t}\frac{1}{\varphi'(t)^2} dt \geq \\
&\geq \frac{2A}{9}\int_0^{\rho} \left(\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)\right) e^{-\alpha t}\frac{1}{\varphi'(t)^2} dt = \\
&= \frac{2A}{9}\int_0^{\rho} \left(\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)\right) \frac{1}{A^2\cos^2(t+\theta)}\ dt = \\
&= \frac{2}{9A}\left(\max_{\mathbb{R}} G\right)\int_0^{\rho} \frac{1}{\cos^2(t+\theta)}\ dt - \frac{2}{9A}\int_0^{\rho} G\big(\bar{\Omega}_n p_n(t)\big)\frac{1}{\cos^2(t+\theta)}\ dt
\end{aligned}$$

and letting $n$ tend to infinity, the Riemann-Lebesgue lemma ([85]) gives

$$\liminf_{n\to\infty}\ r_n \geq \frac{2}{9A}\max_{\mathbb{R}} G \int_0^{\rho}\frac{1}{\cos^2(t+\theta)} dt.$$

Since $\rho \in \left]0, \frac{\pi}{2}-\theta\right[$ was chosen with the only restriction of verifying (1.46) for $n$ big enough, we may let it tend to $\frac{\pi}{2}-\theta$ to obtain (1.44).

Finally, to show (1.45), we observe the following consequence of Lemma 1.3.3:

$$|s_n| \leq 4\int_0^{\frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}} \frac{\max_{\mathbb{R}} G - G\big(\bar{\Omega}_n p_n(t)\big)}{\bar{\Omega}_n\varphi'(t)^2} e^{-\alpha t}\mathcal{B}(t)\ dt = 4\int_0^{\frac{\pi}{2}-\theta} \gamma_n(t)\ dt$$

where $\gamma_n : [0, \frac{\pi}{2}-\theta] \to \mathbb{R}$, is defined by

$$\gamma_n(t) = \begin{cases} \frac{\max_{\mathbb{R}} G - G(\bar{\Omega}_n p_n(t))}{\bar{\Omega}_n\varphi'(t)^2}, & \text{if } 0 \leq t \leq \frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n}, \\ 0, \ \text{if } \frac{\pi}{2}-\theta-\frac{D}{\bar{\Omega}_n} \leq t \leq \frac{\pi}{2}-\theta \end{cases}$$

Since $\gamma_n$ converges pointwise to zero, (1.45) will be proved if the sequence $\gamma_n$ is uniformly bounded on $L^1[0,\pi]$. But, for $n$ sufficiently large and $t \in \left[0, \frac{\pi}{2} - \theta - \frac{D}{\bar{\Omega}_n}\right]$, we have (see 1.19):

$$0 \le \gamma_n(t) = \frac{G\left(\bar{\Omega}_n p_n\left(\frac{\pi}{2}-\theta\right)\right) - G\left(\bar{\Omega}_n p_n(t)\right)}{\bar{\Omega}_n \varphi'(t)^2} \le \left(\max_{\mathbb{R}} g\right) \frac{|p_n(\frac{\pi}{2}-\theta) - p_n(t)|}{\varphi'(t)^2} \le$$

$$\le \left(\max_{\mathbb{R}} g\right) \frac{\varphi\left(\frac{\pi}{2}-\theta\right) - \varphi(t)}{\varphi'(t)^2} + \frac{1}{\bar{\Omega}_n}\left(\max_{\mathbb{R}} g\right) \frac{|\tilde{\Omega}_n(\frac{\pi}{2}-\theta) - \tilde{\Omega}_n(t)|}{\varphi'(t)^2}$$

However, it follows from Lemma 1.3.1 the existence of a constant $k > 0$ satisfying

$$\left(t - \left(\frac{\pi}{2} - \theta\right)\right)^2 \le k\varphi'(t)^2, \ \forall\, t \in [0,\pi].$$

Consequently, for $n$ sufficiently large and $t \in \left[0, \frac{\pi}{2} - \theta - \frac{D}{\bar{\Omega}_n}\right]$,

$$0 \le \gamma_n(t) \le k\left(\max_{\mathbb{R}} g\right) \frac{|\varphi(t) - \varphi\left(\frac{\pi}{2}-\theta\right)|}{|t - \left(\frac{\pi}{2}-\theta\right)|^2} + \frac{k}{D}\left(\max_{\mathbb{R}} g\right)\left(\max_{[0,\pi]} \tilde{\Omega}'_n\right),$$

whose boundedness is a consequence of (1.27). This ends the proof. □

Theorem 1.3.2 follows. As seen before, it implies Theorems 1.1.1 and 1.1.2. □

At this point, we turn ourselves to the proof of Theorem 1.1.3.

*Proof of Theorem 1.1.3.* Since up to now, the damping coefficient $\alpha$ was always fixed, it was not necessary to introduce it in the notation of the functions $\varphi,\ \psi,\ \varepsilon_\pm, etc.$ However, here we will make $\alpha$ vary and we will adapt our notation by adding $\alpha$ as a superscript; in this way we will write $\varphi^\alpha,\ \psi^\alpha,\ \varepsilon^\alpha_\pm$, etc. From our bifurcation equation (1.7) we deduce:

$$|\bar{h}| \le \|g\|_\infty \|\psi^\alpha\|_{L^1[0,\pi]} \qquad \forall \bar{h} \in \mathcal{I}^\alpha_{\tilde{h}},\ \ \forall \tilde{h} \in \left(\varphi^\alpha\right)^\perp,$$

and, consequently,

$$0 < \epsilon^\alpha_\pm \le \|g\|_\infty \|\psi^\alpha\|_{L^1[0,\pi]}\,.$$

However, the quantity $\|\psi^\alpha\|_{L^1[0,\pi]}$ is easily computable, and we have

$$\|\psi^\alpha\|_{L^1[0,\pi]} = \frac{\int_0^\pi e^{\frac{\alpha}{2}t}\sin t dt}{\sqrt{\int_0^\pi \left[e^{\frac{\alpha}{2}}t \sin t\right]^2 dt}} = \frac{\left(1+\exp(\alpha\pi/2)\right)/\left(1+\alpha^2/4\right)}{\sqrt{\left(\exp\left(\alpha\pi\right)-1\right)4/\left(8\alpha+2\alpha^3\right)}} \underset{|\alpha|\to\infty}{\longrightarrow} 0,$$

which finishes our proof. □

## 1.4 The continuity of $\varepsilon_\pm$

In this section, we want to show the continuity of the functionals $\varepsilon_\pm$. This problem was already studied, in a broader, PDE setting, by Dancer [24], Ortega, [63]. We briefly recall here their argument, adapted to our framework. In order to show lower semicontinuity of, say, $\varepsilon_+$ (upper semicontinuity is straightforward), take some point $\tilde{h}_* \in \varphi^\perp$. Corresponding solutions $u_-$ and $u_+$ of problem (1.1) with $\bar{h} = 0,\ \varepsilon_+(\tilde{h})$ respectively, may be found. Moreover, after some further work which was basically carried out in [5], it is possible to choose $u_-$ and $u_+$ with $u_-(t) < u_+(t)\ \forall t \in ]0,\pi[,\ u'_-(0) < u'_+(0),\ u'_-(\pi) > u'_+(\pi)$. We then consider the set $\mathcal{G}$ of $C^1[0,\pi]$ functions $u$ with $u'_-(0) < u'(0) < u'_+(0),\ u_-(t) < u(t) < u_+(t)\ \forall t \in ]0,\pi[,\ u'_-(\pi) > u'(\pi) > u'_+(\pi)$, which is an open and bounded subset of $C^1[0,\pi]$. Given any $0 < a < \varepsilon_+(\tilde{h})$, problem (1.1) with $\bar{h} = a$ may be equivalently reformulated as: find the fixed points of a suitable completely continuous operator $P$ on $C^1[0,\pi]$. Being $u_-$ a lower solution and $u_+$ an upper solution, Hopf's Lemma shows that there are not solutions in $\partial\mathcal{G}$, and the Leray-Schauder degree of $I-P$ in $\mathcal{G}$ is shown to be 1. Both things keep true under small perturbations of $P$, so that, for $\tilde{h}$ near $\tilde{h}_*$ and $\bar{h}=a$, we have still solutions of (1.1) in $\mathcal{G}$ and $\varepsilon_+(\tilde{h})\geq a$.

When it uses Hopf's Lemma, this argument needs the nonlinearity $g$ to be *Lipschitz*, at least in a neighborhood of 0. In our setting, $g$ is merely assumed to be continuous. We devote this section to overcome this difficulty by replacing the function $\psi$, which vanishes at 0 and 1, by a strictly positive function.

**Lemma 1.4.1.** *Let $E$ be a metric space, $e_-,\ e_+ : E\to\mathbb{R}^+$ positive functions defined on $E$. We consider the set*

$$\mathcal{E} := \Big\{(x,t)\in E\times\mathbb{R} : -e_-(x)\leq t\leq e_+(x)\Big\}$$

*Then,*

1. *$\mathcal{E}$ is closed in $E\times\mathbb{R}$ if and only if $e_-$ and $e_+$ are upper semicontinuous.*
2. *$\mathcal{E}\subset\overline{int\,(\mathcal{E})}$ if and only if $e_-$ and $e_+$ are lower semicontinuous.*

The proof is straightforward (see, for instance, [27]). However, it gives rise to the following interesting consequence:

**Corollary 1.4.2.** *Let $X$ be a Banach space, $H\subset X$ a closed hyperplane and $u,v\in X\backslash H$. Assume that $e^u_\pm,\ e^v_\pm : H\to\mathbb{R}^+$ are functions verifying*

$$\Big\{h+tu : h\in H, -e^u_-(h)\leq t\leq e^u_+(h)\Big\} = \Big\{h+tv : h\in H, -e^v_-(h)\leq t\leq e^v_+(h)\Big\}$$

*Then, $e^u_\pm$ is lower (respectively, upper) semi-continuous if and only if $e^v_\pm$ has the same property.*

**Theorem 1.4.3.** *The functionals $\varepsilon_\pm : \psi^\perp\to\mathbb{R}^+$, as defined in Theorem 1.1.1, are continuous.*

*Proof.* In view of (1.4.1), the upper continuity of both $e_-$ and $e_+$ is granted as soon we check the set

$$\mathcal{R} := \{(\tilde{h},\bar{h})\in\psi^\perp\times\mathbb{R} : -\varepsilon_-(\tilde{h})\leq\bar{h}\leq\varepsilon_+(\tilde{h})\}$$

to be closed in $\psi^\perp\times\mathbb{R}$. This argument is not new; was already used, for instance, in [15]. Given $\{(\tilde{h}_n,\bar{h}_n)\}\to(\tilde{h}_*,\bar{h}_*)$ a sequence in $\mathcal{R}$, either we have $\bar{h}_*=0$ (and then, $(\tilde{h}_*,\bar{h}_*)\in\mathcal{R}$), or $\bar{h}_*\neq 0$, and we pick a corresponding sequence $\big\{(\tilde{u}_n,\bar{u}_n)\big\}\subset\varphi^\perp\times\mathbb{R}$ with

$$\tilde{u}_n = \mathcal{K}(I-Q)\mathcal{N}(\bar{u}_n\varphi+\tilde{u}_n)+\mathcal{K}\tilde{h}_n \tag{1.47}$$

$$\bar{h}_n = \int_0^\pi g(\bar{u}_n\varphi(s)+\tilde{u}_n(s))\psi(s)\ ds \tag{1.48}$$

It follows from (1.47) that the sequence $\{\tilde{u}_n\}$ is bounded in $\varphi^\perp$, in particular, it is bounded in $C^1[0,\pi]$. Thus, $\{\bar{u}_n\}$ should also be bounded in $\mathbb{R}$, since, otherwise, the Riemann-Lebesgue lemma together with (1.48) would imply $\bar{h}_*=0$. Consequently, convergent subsequences $\{\tilde{u}_{r_n}\}\to\tilde{u}_*$ in $C[0,\pi]$ and $\{\bar{u}_{r_n}\}\to\bar{u}_*$ may be found. Passing to the limit along these subsequences in (1.47) and (1.48) we deduce:

$$\tilde{u}_* = \mathcal{K}(I-Q)\mathcal{N}(\bar{u}_*\varphi+\tilde{u}_*)+\mathcal{K}\tilde{h}_*$$

$$\bar{h}_* = \int_0^\pi g(\bar{u}_*\varphi(s)+\tilde{u}_*(s))\psi(s)\ ds$$

and then, $(\tilde{h}_*,\bar{h}_*)\in\mathcal{R}$.

In orden to show the lower semicontinuity of $\varepsilon_\pm$, we are going to use Corollary 1.4.2. We write our solvability set in an alternative form:

$$\Big\{\tilde{h}+t\varphi : \tilde{h}\in\psi^\perp,\ -\varepsilon_-(\tilde{h})\leq t\leq\varepsilon_+(\tilde{h})\Big\} = \Big\{\tilde{h}+t1 : \tilde{h}\in\psi^\perp,\ -\epsilon_-(\tilde{h})\leq t\leq\epsilon_+(\tilde{h})\Big\}$$

for suitable functions $\epsilon_\pm : \psi^\perp\to\mathbb{R}^+_0$. This may be done thanks to the upper and lower solutions method ([5]), and Theorem 1.1.1 above, which guarantees in particular that problem (1.1) is always solvable if $\bar{h}=0$. It will be easier to show the lower continuity of $\epsilon_\pm$ instead of $\varepsilon_\pm$. Of course, we may (and we will) restrict ourselves to study $\epsilon_+$, the case of $\epsilon_-$ being analogous. We choose any function $\tilde{h}_0\in\psi^\perp$.

1. In case $\epsilon_+(\tilde{h}_0) = 0$, $\epsilon_+$ being nonnegative, it is lower semicontinuous at $\tilde{h}_0$.
2. In case $\epsilon_+(\tilde{h}_0) > 0$, choose $0 < a < \epsilon_+(\tilde{h}_0)$ and take a solution $u_+$ of problem (1.1) with $h(t) = h_0(t) + \epsilon_+(\tilde{h}_0)1$. We call $v_+ := u_+ - \mathcal{K}\tilde{h}_0$, which is a solution of the alternative problem

$$\begin{array}{c} -v'' - \alpha v' - \lambda_1(\alpha)v + g(v + (\mathcal{K}\tilde{h}_0)(t)) = \gamma 1, \quad t \in [0,\pi] \\ u(0) = u(\pi) = 0 \end{array} \tag{1.49}$$

for $\gamma = \epsilon_+(\tilde{h}_0)$, and thus, a strictly upper solution of (1.49) for $\gamma = a$. On the other hand, using the Riemann-Lebesgue lemma and letting $\bar{u} \to -\infty$ in (1.6, 1.7), a $C^2[0,\pi]$ function $v_-$ with

$$\begin{array}{c} v_-(0) = v_-(\pi) = 0 \ ; \quad v_-'(0) < v_+'(0) \ ; \quad v_-'(\pi) > v_+'(\pi); \\ -v_-''(t) - \alpha v_-'(t) - \beta_1(\alpha)v_-(t) + g(v_-(t) + (\mathcal{K}\tilde{h}_0)(t)) \leq a/2 < a \quad \forall t \in [0,\pi]; \\ v_-(t) < v_+(t) \quad \forall t \in ]0,\pi[, \end{array}$$

may be found. In this way, the set

$$\mathcal{O} := \Big\{ w \in C^1[0,\pi] : v_-'(0) < w'(0) < v_+'(0), \quad v_-'(\pi) > v_+'(\pi), \quad v_-(t) < v_+(t) \quad \forall t \in ]0,\pi[ \Big\}$$

is open in $C^1[0,\pi]$, and problem (1.49) with $\gamma = a$ has not solutions in $\partial\mathcal{O}$, since $v_-$ and $v_+$ are, respectively, *strictly* lower and upper solutions. The Theorem is now a consequence of known arguments, based upon the continuity of the Leray-Schauder topological degree (see, for instance, [63], pp. 38, 39).

□

## 1.5 Generic asymptotic behavior of the solvability set

In the second part of this chapter we plan to study the asymptotic behavior along lines of the functionals

$$\varepsilon_\pm : \psi^\perp \to \mathbb{R}^+$$

defined in Theorem 1.1.1. These functionals delimite the solvability set of problem (1.1).

Roughly, our results can be abridged by saying that, generically, $\varepsilon_\pm$ converge to zero along lines. Further, this convergence is uniform in bounded sets. However, there exists at the same time a dense set of directions where this fails to happen. A basic ingredient of our proofs is a multidimensional generalization of the Riemann-Lebesgue lemma which is developed in Lemma 1.5.1.

For the corresponding periodic problem, an asymptotic result on the behavior of the functionals $\varepsilon_\pm$ was developed in [43]. In this paper, it was shown how these functionals converge to zero generically along lines; that is, there exists an open, dense set of directions for which this happens. Examples were also given to show that exceptional directions, where the functional do not converge to zero, also exist. In the proofs, the fact that the principal eigenfunction of the periodic problem is constant played a key role, and the problem was solved using the more classical, one-dimensional version of the Riemman-Lebesgue Lemma (see, for instance, [85]).

We recall some of the notation that will be extensively used in what follows. A function $h \in L^1([a,b])$ will be called a *step function* if there exists a partition $a = t_0 < t_1 < ... < t_{m-1} < t_m = b$ of $[a,b]$ such that $h|_{]t_{i-1},t_i[}$ is constant for all $i : 1, ...m$. In case all these constants are not 0, we will say that $h$ is a *non-vanishing step function.* A function $u \in W^{2,1}[a,b]$ will be called a *parabolic spline* if $u''$ is a step function and the set $\{u \in W^{2,1}([0,\pi]) : u \text{ is a parabolic spline}\}$ will be denoted as $\mathcal{P}$. For given $\delta > 0$, if there exists a partition $a = t_0 < t_1 < ... < t_{m-1} < t_m = b$ of the interval $[a,b]$, and $1 \leq i_0 \leq m$, $\mu, C \in \mathbb{R}$ with $C \neq 0$ such that

$$t_{i_0} - t_{i_0 - 1} < \delta \tag{1.50}$$

$$u(t) = \mu\varphi(t) + C \ \forall t \in [t_{i_0-1}, t_{i_0}] \tag{1.51}$$

and

$$u''_{|]t_{i-1},t_i[} \text{ is constant for every } 1 \le i \le m,\ i \ne i_0, \tag{1.52}$$

$u$ will be called a *$\delta$-singular function*, and the set of all $W^{2,1}([0,\pi])$ $\delta$-singular functions will be denoted as $\mathcal{S}_\delta$. Finally, for every measurable set $I \subset [0,\pi]$, we will denote by meas $I$ its one-dimensional Lebesgue measure.

Let $\mathcal{B} \subset \psi^\perp$ be a bounded subset, let $\tilde{h} \in \psi^\perp$ be given. For any $\tilde{b} \in \mathcal{B}$ and $\bar{h}, \lambda \in \mathbb{R}$, the Lyapunov-Schmidt system (1.6), (1.7) associated to the forcing term $h = \tilde{b} + \lambda\tilde{h} + \bar{h}$ becomes:

$$\tilde{u} = \mathcal{K}\tilde{b} + \lambda\mathcal{K}\tilde{h} + \mathcal{K}(I-Q)\mathcal{N}(\bar{u}\varphi + \tilde{u}) \tag{1.53}$$

$$\bar{h} = \int_0^\pi g\big(\bar{u}\varphi(t) + \tilde{u}(t)\big)\psi(t)dt, \tag{1.54}$$

the operators $\mathcal{K}$, $\mathcal{N}$, and $Q$ being defined as in Section 1.2. Thus,

$$\varepsilon_\pm(\tilde{b} + \lambda\tilde{h}) = \max\left\{\pm\int_0^\pi g\big(\bar{u}\varphi + \tilde{u}\big)\psi\ dt : \begin{array}{c}\bar{u} \in \mathbb{R},\ \tilde{u} \in \varphi^\perp \\ \tilde{u} = \mathcal{K}\tilde{b} + \lambda\mathcal{K}\tilde{h} + \mathcal{K}(I-Q)\mathcal{N}(\bar{u}\varphi + \tilde{u})\end{array}\right\} =$$

$$= \max\left\{\pm\int_0^\pi g\Big(\mathcal{K}\tilde{b} + \lambda\mathcal{K}\tilde{h} + \bar{u}\varphi + \mathcal{K}(I-Q)\mathcal{N}\big(\bar{u}\varphi + \lambda\mathcal{K}\tilde{h} + \tilde{u}\big)\Big)\psi\ dt : \begin{array}{cc}\bar{u} \in \mathbb{R}, & \tilde{u} \in \psi^\perp \\ (1.53) & \text{holds.}\end{array}\right\}. \tag{1.55}$$

Of course, both functionals $\varepsilon_\pm$ may be studied in an analogous way, so that we will concentrate ourselves with $\varepsilon_+$. For any $\lambda \in \mathbb{R}$, $\tilde{b} \in \mathcal{B}$, there exists an element $(\tilde{u}_{\lambda,\tilde{b}}, \bar{u}_{\lambda,\tilde{b}}) \in \varphi^\perp \times \mathbb{R}$ such that

$$\tilde{u}_{\lambda,\tilde{b}} = \mathcal{K}\tilde{b} + \lambda\mathcal{K}\tilde{h} + \mathcal{K}(I-Q)\mathcal{N}(\bar{u}_{\lambda,\tilde{b}}\varphi + \tilde{u}_{\lambda,\tilde{b}})\ ; \tag{1.56}$$

$$\varepsilon_+(\tilde{b} + \lambda\tilde{h}) = \int_0^\pi g\big(\bar{u}_{\lambda,\tilde{b}}\varphi + \lambda\mathcal{K}\tilde{h} + \mathcal{K}(I-Q)\mathcal{N}(\bar{u}_{\lambda,\tilde{b}}\varphi + \tilde{u}_{\lambda,\tilde{b}})\big)\psi dt\ . \tag{1.57}$$

Since $\mathcal{N}$ is bounded, it is possible to find a constant $M > 0$, not depending on $\lambda \in \mathbb{R}$ or $\tilde{b} \in \mathcal{B}$, such that

$$\left\|\mathcal{K}(I-Q)\mathcal{N}(\bar{u}_{\lambda,\tilde{b}}\varphi + \tilde{u}_{\lambda,\tilde{b}})(t)\right\|_{C^1[0,\pi]} \le M \tag{1.58}$$

All this motivates us to the study of following multidimensional generalization of the Riemann-Lebesgue lemma:

**Lemma 1.5.1.** *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, bounded, and have a bounded primitive, and let $u_1, \dots, u_N \in C^1[0,\pi]$ be given functions satisfying the following property:*

**[P]** *If $\rho_1, \dots, \rho_N$ are real numbers such that*

$$\text{meas}\left\{t \in [0,\pi] : \sum_{i=1}^N \rho_i u_i'(t) = 0\right\} > 0\,,$$

*then $\rho_1 = \cdots = \rho_N = 0$.*

*(We say that $u_1, \dots, u_N$ are linearly independent on sets of positive mesure).*
*Let $B \subset C^1[0,\pi]$ be such that*

$$\{b' : b \in B\}$$

*is bounded in $C[0,\pi]$. Then, for any given function $r \in L^1[0,\pi]$, we have*

$$\lim_{\|\rho\|\to\infty} \int_0^\pi g\left(\sum_{i=1}^N \rho_i u_i(t) + b(t)\right) r(t)\, dt = 0 \tag{1.59}$$

*uniformly with respect to $b \in B$.*

*Proof.* Let $r \in L^1[0,\pi]$ be any integrable function and let $\{\rho^n\}_n \subset \mathbb{R}^N$, $\{b^n\}_n \subset B$ be given sequences with $\|\rho^n\| \to \infty$. The sequence

$$\mu^n := \rho^n/\|\rho^n\|$$

being bounded we have, at least for a subsequence, $\mu^n \to \mu$ for some $\mu \in \mathbb{R}^N$ with $\mu_1^2 + \cdots + \mu_N^2 = 1$. We write

$$u := (u_1, \ldots, u_N),$$

so that, by hypothesis, $\text{meas}(Z) = 0$, where

$$Z = \{t \in [0,\pi] : \langle \mu, u'(t) \rangle = 0\}.$$

This implies that the linear span of the set

$$S = \{\langle \mu, u' \rangle \chi_I : \ I \text{ is any compact subinterval of } [0,\pi], I \cap Z = \emptyset\} \tag{1.60}$$

is a dense set in $L^1[0,\pi]$. To see this, let us define

$$S_1 = \{\chi_I : \ I \text{ is any compact subinterval of } [0,\pi], I \cap Z = \emptyset\} \tag{1.61}$$

Then, for any open subset $A \subset [0,\pi]$ (in particular, for any open subinterval of $[0,\pi]$), $\text{meas}(A \setminus Z) = \text{meas}(A)$. Since $A \setminus Z$ is also open, there exists an at most countable collection $\{I_i, \ i \in \mathbb{N}\}$ of pairwise disjoint open intervals such that $A \setminus Z = \cup_{i \in \mathbb{N}} I_i$ and $\text{meas}(A \setminus Z) = \sum_{i \in \mathbb{N}} \text{meas}(I_i)$. Consequently, the linear span of the set $S_1$ is a dense set in the set of step functions and therefore in $L^1[0,\pi]$.

Now, let $\chi_I$ be a given element of $S_1$. Write $w = \langle \mu, u' \rangle$ and $m = \inf_I \ |w|$ $(m > 0)$. Finally, fix $\epsilon > 0$. Choose a partition of $I = [a,b]$, $a = a_0 < a_1 < \cdots < a_{m-1} < a_m = b$ such that if $t, y \in J_i = [a_{i-1}, a_i]$, $1 \le i \le m$, then $|w(t) - w(y)| \le \epsilon$. Then, for any $t \in I$, there is some $i$, $1 \le i \le m$, such that $t \in J_i$ and

$$\left| \chi_I(t) - \sum_{i=1}^m \frac{w\chi_{J_i}(t)}{w(a_i)} \right| = \left| \frac{w(a_i) - w(t)}{w(a_i)} \right| \le \epsilon/m,$$

so that

$$\left\| \chi_I - \sum_{i=1}^m \frac{w\chi_{J_i}}{w(a_i)} \right\|_1 \le \epsilon\pi/m\,.$$

We deduce from this all that the linear span of $S$ is dense in $S_1$ and therefore in $L^1[0,\pi]$.

On the other hand, let us denote $l^\infty$ the Banach space of bounded sequences of real numbers endowed with the uniform norm, $l^0$ the closed subspace of converging to 0 sequences, and $\mathcal{T} : L^1[0,\pi] \to l^\infty$, $s \mapsto \mathcal{T}s = \{(\mathcal{T}s)^n\}_n$ the linear operator defined by

$$(\mathcal{T}s)^n = \int_0^\pi g(\langle \rho^n, u(t) \rangle + b^n(t))s(t)\, ds, \ \forall\ s \in L^1[0,\pi], \ \forall\ n \in \mathbb{N},$$

which is trivially continuous $\big(\|\mathcal{T}s\|_\infty \le \|g\|_\infty \|s\|_{L^1[0,\pi]}\big)$. Since our objective is to prove the inclusion $T\big(L^1[0,\pi]\big) \subset l^0$ and $T$ is continuous, to prove the lemma it is sufficient to demonstrate that $T(S) \subset l^0$, i.e.,

$$\lim_{n\to\infty} \int_I g\big(\langle \rho^n, u(t) \rangle + b^n(t)\big) \langle \mu, u'(t) \rangle \, dt = 0, \tag{1.62}$$

for any compact subinterval $I$ of $[0,\pi]$ such that $I \bigcap Z = \emptyset$. But, if $v^n, v : I \to \mathbb{R}$ are defined as

$$v^n(t) = \langle \mu^n, u(t) \rangle + b^n(t)/\|\rho^n\|,$$
$$v(t) = \langle \mu, u(t) \rangle, \ \forall\ t \in [0,\pi],$$

we trivially have

$$\lim_{n\to\infty} \int_I g\big(\|\rho^n\| v^n(t)\big) \big(v'(t) - (v^n)'(t)\big)\, dt = 0 \tag{1.63}$$

and

$$\lim_{n\to\infty} \int_I g\big(\|\rho^n\| v^n(t))(v^n)'(t)\big)\, dt = \lim_{n\to\infty} \frac{G(\|\rho^n\| v^n(\max I)) - G(\|\rho^n\| v^n(\min I))}{\|\rho^n\|} = 0 \tag{1.64}$$

where $G$ is any primitive of the function $g$. Now, (1.63) plus (1.64) imply (1.62). $\square$

We observe that inside the set of continuous and bounded functions having a bounded primitive are the periodic functions with zero mean. In our next result we use Lemma 1.5.1 in this particular case to obtain that, for generic $\tilde{h}$, $\lim_{|\lambda|\to\infty} \varepsilon_{\pm}(\lambda\tilde{h}) = 0$.

**Corollary 1.5.2.** *Let $\tilde{h} \in \psi^{\perp}$ be a given function and suppose that the functions $K\tilde{h}$ and $\varphi$ satisfy the following relationship:*

**[P1]** *If $\rho_1, \rho_2$ are real numbers such that*

$$\text{meas}\{t \in [0,\pi] : \rho_1 (K\tilde{h})'(t) + \rho_2 \varphi'(t) = 0\} > 0,$$

*then $\rho_1 = \rho_2 = 0$.*

*Let $B \subset \psi^{\perp}$ be any bounded subset. Then*

$$\lim_{|\lambda|\to\infty} \varepsilon_{+}(\lambda\tilde{h} + b) = \lim_{|\lambda|\to\infty} \varepsilon_{-}(\lambda\tilde{h} + b) = 0, \tag{1.65}$$

*uniformly with respect to $b \in B$.*

*Proof.* Immediate from (1.57), (1.58) and Lemma 1.5.1 above. □

The following equivalent version of previous corollary will be very useful for our purposes.

**Corollary 1.5.3.** *Let $\tilde{h} \in \psi^{\perp}$ be a given function and suppose that, for every $\rho \in \mathbb{R}$,*

$$\text{meas}\{t \in [0,\pi] : \ (K\tilde{h})'(t) = \rho\varphi'(t)\} = 0. \tag{1.66}$$

*Let $B \subset \psi^{\perp}$ be any bounded subset. Then*

$$\lim_{|\lambda|\to\infty} \varepsilon_{+}(\lambda\tilde{h} + b) = \lim_{|\lambda|\to\infty} \varepsilon_{-}(\lambda\tilde{h} + b) = 0, \tag{1.67}$$

*uniformly with respect to $b \in B$.*

However, the set of functions $\tilde{h} \in \psi^{\perp}$ not verifying (1.66) for some $\rho \in \mathbb{R}$, can be seen to be residual in $\psi^{\perp}$. This will lead us to one of our main results in this chapter:

**Theorem 1.5.4.** *There exists a subset $F \subset \psi^{\perp}$, of first Baire category in this space, such that for any $\tilde{h} \in \psi^{\perp} \setminus F$, and each given bounded subset $B \subset \psi^{\perp}$, one has*

$$\lim_{|\lambda|\to\infty} \varepsilon_{+}(\lambda\tilde{h} + b) = \lim_{|\lambda|\to\infty} \varepsilon_{-}(\lambda\tilde{h} + b) = 0 \tag{1.68}$$

*uniformly with respect to $b \in B$.*

*Proof.* Let

$$F = \Big\{\tilde{h} \in \psi^{\perp} : \ \exists\, \rho \in \mathbb{R} \text{ with } \text{meas}\big(\{t \in [0,\pi] : (\mathcal{K}\tilde{h})'(t) = \rho\varphi'(t)\}\big) > 0\Big\}$$

Then $F = \bigcup_{n\in\mathbb{N}} F_n$, where

$$F_n = \Big\{\tilde{h} \in \psi^{\perp} : \ \exists\, \rho \in \mathbb{R} \text{ with } \text{meas}\big(\{t \in [0,\pi] : (\mathcal{K}\tilde{h})'(t) = \rho\varphi'(t)\}\big) \geq 1/n\Big\}$$

Let us prove that each subset $F_n$ is closed and has an empty interior. Given $n \in \mathbb{N}$, since $\mathcal{K} : \psi^{\perp} \to \varphi^{\perp}$ is a topological isomorphism, $F_n$ is closed in $\psi^{\perp}$ if and only if $G_n := \mathcal{K}(F_n)$ is a closed subset of $\varphi^{\perp}$. However,

$$G_n = \{u \in \varphi^{\perp} : \exists \rho \in \mathbb{R} \text{ with } \ \text{meas}\{t \in [0,\pi] : u'(t) = \rho\varphi'(t)\} \geq 1/n\}$$

Let $\{u_m\}_m \subset G_n$ be a sequence such that $\{u_m\} \to u$ in $\varphi^{\perp}$. For any $m \in \mathbb{N}$, we can find $\rho_m \in \mathbb{R}$ such that

$$\text{meas}(\{t \in [0,\pi] : u_m'(t) = \rho_m\varphi'(t)\}) \geq 1/n$$

Since

$$\text{meas}(\{t \in [0, \pi] : \varphi'(t) = 0\}) = 0,$$

the sequence $\{\rho_m\}$ must be bounded and, after possibly passing to a subsequence, we can suppose, without loss of generality, that $\{\rho_m\} \to \rho$. Moreover, if we define

$$M_m = \{t \in [0, \pi] : u'_m(t) = \rho_m \varphi'(t)\}$$

then $\text{meas}(M_m) \geq 1/n$, $\forall\ m \in \mathbb{N}$ and $\text{meas}\left(\bigcap_{m=1}^{\infty} [\bigcup_{s=m}^{\infty} M_s]\right) \geq 1/n$. Finally, let us observe that if $t \in \bigcap_{m=1}^{\infty} [\bigcup_{s=m}^{\infty} M_s]$, then $u'(t) = \rho\varphi'(t)$, so that

$$\text{meas}\{t \in [0, \pi] : u'(t) = \rho\varphi'(t)\} \geq 1/n$$

and, consequently, $u \in G_n$.

Next, we are going to show that $F$ (and therefore each $F_n$) has an empty interior. To do that, let $\hat{\varphi}$ be the only solution of the linear problem

$$\hat{\varphi}'' = \varphi$$
$$\hat{\varphi}(0) = \hat{\varphi}(\pi) = 0$$

Then, $\hat{\varphi} \in C[0, \pi]$, $\hat{\varphi}(t) < 0\ \forall t \in ]0, \pi[$ by the maximum principle, and, for any $u \in W_0^{2,1}[0, \pi]$,

$$\int_0^{\pi} u\varphi = -\int_0^{\pi} u'\hat{\varphi}' = \int_0^{\pi} u''\hat{\varphi}\ .$$

As a consequence, the mapping

$$\Phi : \varphi^{\perp} \to \hat{\varphi}^{\perp}\,, \qquad\qquad u \mapsto u'',$$

is a topological isomorphism, where

$$\hat{\varphi}^{\perp} = \left\{ h \in L^1[0, \pi] : \int_0^{\pi} h(t)\hat{\varphi}(t)\, dt = 0 \right\}\ .$$

We deduce from this all that $F$ has an empty interior in $\psi^{\perp}$ if and only if $\Phi(\mathcal{K}(F))$ has an empty interior in $\hat{\varphi}^{\perp}$. This last result will follow from points *2.* and *3.* of our next lemma. □

**Lemma 1.5.5.** *Let us denote by $\mathcal{A}$ the subset of $L^1[0, \pi]$ given by all the step functions and by $\mathcal{B}$ the subset of $L^1[0, \pi]$ given by all the non-vanishing step functions. Then,*

1. *$\mathcal{A} \cap \hat{\varphi}^{\perp}$ is dense in $\hat{\varphi}^{\perp}$;*
2. *$\mathcal{B} \cap \hat{\varphi}^{\perp}$ is dense in $\hat{\varphi}^{\perp}$;*
3. *$\mathcal{B} \cap \Phi(\mathcal{K}(F)) = \emptyset$.*

*Proof.* 1. Take any $h \in \hat{\varphi}^{\perp}$ and $\epsilon > 0$. Then, there exists $s \in \mathcal{A}$ such that

$$\|h - s\|_1 < \min\left\{ \epsilon/2\pi,\ \frac{\|\hat{\varphi}\|_1}{\|\hat{\varphi}\|_{\infty}} \right\}.$$

Now,

$$\tilde{s} = s + \frac{1}{\|\hat{\varphi}\|_1} \int_0^{\pi} s\hat{\varphi}$$

is again a step function which belongs to $\hat{\varphi}^{\perp}$ and verifies $\|h - \tilde{s}\|_1 < \epsilon$.

2. Let us show that $\mathcal{B} \cap \hat{\varphi}^{\perp}$ is dense in $\mathcal{A} \cap \hat{\varphi}^{\perp}$. Thus, take a function $u \in \mathcal{A} \cap \hat{\varphi}^{\perp}$. Given $a, b \in \mathbb{R}$, define $u_{a,b} = u + a\chi_{[0,\pi/2]} + b\chi_{[\pi/2,\pi]}$. The condition for $u_{a,b}$ to belong to $\hat{\varphi}^{\perp}$ is

$$a\int_0^{\pi/2} \hat{\varphi} + b\int_{\pi/2}^{\pi} \hat{\varphi} = 0$$

Since $\int_0^{\pi/2} \hat{\varphi} < 0$ and $\int_{\pi/2}^{\pi} \hat{\varphi} < 0$ $\left(\text{recall that } \hat{\varphi}(t) < 0\ \forall t \in ]0, \pi[\right)$, we may choose $a$ and $b$ both different from zero but with small absolute value such that $u_{a,b} \in \mathcal{B} \cap \hat{\varphi}^{\perp}$.

3. Assume, instead, that an element $s \in \mathcal{B} \cap \Phi(\mathcal{K}(F))$ may be found. Then, there exist $\tilde{h} \in F$, $u \in W^{2,1}([0,\pi])$, such that $\mathcal{K}(\tilde{h}) = u$, $\Phi(u) = u'' = s$. Since $\tilde{h} \in F$, $\rho \in \mathbb{R}$ can be chosen such that $\text{meas}(\{t \in [0,\pi] : u'(t) = \rho\varphi'(t)\}) > 0$. Choose some nontrivial compact interval $I \subset [0,\pi]$ satisfying $s|_I \equiv c \neq 0$ and such that $\text{meas}(\{t \in I : u'(t) = \rho\varphi'(t)\}) > 0$. This implies that

$$\text{meas}(\{t \in I : c = u''(t) = \rho\varphi''(t)\}) > 0,$$

a contradiction.

□

## 1.6 Many 'exceptional' functions coexisting together

Now that we know that, generically, the functionals $\varepsilon_\pm$ converge to zero along lines, we are inevitably confronted with the following questions:

1. Are there any functions $\tilde{h} \in \psi^\perp$ such that $\liminf_{|\lambda|\to\infty} a_\pm(\lambda\tilde{h}) > 0$?

2. If 'yes', how big is this set of such 'exceptional' functions?

In orden to respond to these questions, we need to approximate accurately the term $\tilde{u}$ appearing in the Lyapunov-Schmidt system (1.6), (1.7). This is the aim of next proposition.

**Proposition 1.6.1.** *Let $\mathcal{B} \subset C^1[0,\pi]$ be a bounded set. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that, for any $w \in \mathcal{S}_\delta$,*

$$\limsup_{\|(\lambda,\mu)\|\to\infty} \|\mathcal{K}(I-Q)\mathcal{N}(\lambda\varphi + \mu w + b)\|_\infty < \epsilon, \tag{1.69}$$

*uniformly with respect to $b \in \mathcal{B}$.*

*Proof.* Assume the stated result is false. Then, there exists $\epsilon_0 > 0$ and, for any $\delta > 0$, $w_\delta \in W^{2,1}[0,\pi]$ such that $w_\delta \in \mathcal{S}_\delta$, and

$$\limsup_{\|(\lambda,\mu)\|\to\infty} \left( \sup_{b\in\mathcal{B}} \left\| \mathcal{K}(I-Q)\mathcal{N}(\lambda\varphi + \mu w_\delta + b) \right\|_\infty \right) \geq \epsilon_0, \qquad \forall \delta > 0. \tag{1.70}$$

Let $\{h_n : n \in \mathbb{N}\}$ be a dense and countable subset of $L^1[0,\pi]$. For any $n \in \mathbb{N}$, let us choose $\delta_n > 0$ such that

$$\int_A |h_n(t)|dt < \frac{1}{n(1+\|g\|_\infty)} \qquad \forall A \subset [0,\pi] \text{ measurable with } \text{meas}(A) < \delta_n. \tag{1.71}$$

Using (1.70) and Lemma 1.5.1, for any $n \in \mathbb{N}$ there exist $w_n \in \mathcal{S}_{\delta_n}$, $0 \leq p_n < q_n \leq \pi$, $(\lambda_n, \mu_n) \in \mathbb{R}^2$ and $b_n \in \mathcal{B}$ with

$$q_n - p_n < \delta_n; \tag{1.72}$$

$$\left| \int_0^{p_n} g\big(\lambda_n\varphi(t) + \mu_n w_{\delta_n}(t) + b_n(t)\big) h_r(t)\, dt \right| < 1/n \qquad \forall r : 1 \dots n; \tag{1.73}$$

$$\left| \int_{q_n}^{\pi} g\big(\lambda_n\varphi(t) + \mu_n w_{\delta_n}(t) + b_n(t)\big) h_r(t)\, dt \right| < 1/n \qquad \forall r : 1 \dots n; \tag{1.74}$$

$$\left\| \mathcal{K}(I-Q)\mathcal{N}(\lambda_n\varphi + \mu_n w_{\delta_n} + b_n) \right\|_\infty \geq \epsilon_0/2 \tag{1.75}$$

We deduce from (1.71), (1.72), (1.73) and (1.74) that

$$\left| \int_0^{\pi} g\big(\lambda_n\varphi(t) + \mu_n w_{\delta_n}(t) + b_n(t)\big) h_r(t)\, dt \right| < 3/n \qquad \forall r, n \in \mathbb{N} \text{ with } r \leq n,$$

and then, since $\{h_n : n \in \mathbb{N}\}$ is dense in $L^1[0,\pi]$ and $g$ is bounded,

$$\lim_{n\to\infty} \int_0^\pi g\big(\lambda_n \varphi(t) + \mu_n w_{\delta_n}(t) + b_n(t)\big) h(t)\, dt = 0 \qquad \forall h \in L^1[0,\pi].$$

In particular,

$$\lim_{n\to\infty} \int_0^\pi g\big(\lambda_n \varphi(t) + \mu_n w_{\delta_n}(t) + b_n(t)\big) h(t)\, dt = 0 \qquad \forall h \in L^\infty[0,\pi],$$

that is, the $L^1[0,\pi]$ sequence $\{\mathcal{N}(\lambda_n\varphi+\mu_n w_{\delta_n}+b_n)\}_n$ converges weakly to zero. Since the operator $\mathcal{K}(I-Q) : L^1[0,\pi] \to C[0,\pi]$ is compact, it is weakly-strong continuous, and

$$\lim_{n\to\infty} \Big\{\mathcal{K}(I-Q)\mathcal{N}(\lambda_n\varphi + \mu_n w_{\delta_n} + b_n)\Big\} \to 0 \qquad \text{uniformly on } [0,\pi],$$

contradicting (1.75).

□

**Theorem 1.6.2.** *Let $g \not\equiv 0$ be given. Then, there exists $\delta > 0$ such that for every $\tilde{h} \in \psi^\perp$ with $\mathcal{K}(\tilde{h}) \in \mathcal{S}_\delta$, $\varepsilon_\pm(\lambda\tilde{h}) \underset{\lambda\to+\infty}{\not\to} 0$. In particular,*

$$\Big\{\tilde{h} \in \psi^\perp : \varepsilon_\pm(\lambda\tilde{h}) \underset{\lambda\to+\infty}{\not\to} 0\Big\}$$

*is $\|\cdot\|_1$-dense in $\psi^\perp$.*

*Proof.* Since $g \not\equiv 0$ has mean value, it is possible to find $\epsilon, \rho > 0$ and $\gamma_-,\ \gamma_+ \in \mathbb{R}$ such that $g(u) < -\rho \quad \forall u \in [\gamma_- - \epsilon, \gamma_- + \epsilon]$ and $g(u) > \rho \quad \forall u \in [\gamma_+ - \epsilon, \gamma_+ + \epsilon]$.

Observe that

$$\mathcal{B} := \big\{\mathcal{K}[I-Q](a) : a \in C[0,\pi],\ \|a\|_\infty \le \|g\|_\infty\big\}$$

is a bounded subset of $C^1[0,\pi]$. Thus, using Proposition 1.6.1 above, we may find $\delta > 0$ such that (1.69) holds for every $w \in \mathcal{S}_\delta$.

Let $\tilde{h}_0 \in \psi^\perp$ such that $\mathcal{K}\tilde{h}_0 = \tilde{u}_0 \in \mathcal{S}$ be given. Find a partition $0 = t_0 < t_1 < \ldots < t_m = \pi$ of the interval $[0,\pi]$ and $1 \le i_0 \le m$, $\mu, C \in \mathbb{R}$ with $C \neq 0$, such that (1.50), (1.51), (1.52) with $u = \tilde{u}_0$ are satisfied. Define, for each $n \in \mathbb{N}$, $\lambda_n := \frac{nT+\gamma_+}{C}$. We claim that

$$\liminf_{n\to+\infty} \epsilon_+(\lambda_n\tilde{h}_0) \ge \rho \int_{x_{i_0-1}}^{x_{i_0}} \psi(t)dt\ .$$

In orden to check this, take some $n \in \mathbb{N}$. The Schauder fixed point theorem provides the existence of some $\tilde{u}_n \in \varphi^\perp \subset C[0,\pi]$ such that

$$\tilde{u}_n = \mathcal{K}(I-Q)\mathcal{N}(-\mu\lambda_n\varphi + \lambda_n\tilde{u}_0 + \tilde{u}_n)\,.$$

Thus, $\tilde{u}_n$ belongs indeed to $\mathcal{B}$. By the choice of $\delta$, and since $\|(-\mu\lambda_n, \lambda_n)\| \ge \lambda_n \to \infty$, for $n$ big enough we have the inequality

$$\|u_n\|_\infty < \epsilon, \tag{1.76}$$

so that, remembering (1.55),

$$\epsilon_+(\lambda_n\tilde{h}) \ge \int_0^\pi g\big(-\mu\lambda_n\varphi + \lambda_n\tilde{u}_0 + \tilde{u}_n\big)\, dt = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} g(-\mu\lambda_n\varphi + \lambda_n\tilde{u}_0 + \tilde{u}_n)\psi dt$$

Observe now that, for each $i \neq i_0$,

$$\lim_{n\to\infty} \int_{t_{i-1}}^{t_i} g\big(-\mu\lambda_n\varphi + \lambda_n\tilde{u}_0 + \tilde{u}_n\big)\psi\, dt = 0, \tag{1.77}$$

as a consequence of Lemma 1.5.1 if $\tilde{u}_0'$ is not constantly 0 on $[x_{i_0-1}, x_{i_0}]$ and as a consequence of the classical Riemann-Lebesgue Lemma (see [85]) otherwise. On the other hand, for each $n \in \mathbb{N}$,

$$\int_{t_{i_0-1}}^{t_{i_0}} g\big(-\mu\lambda_n\varphi + \lambda_n\tilde{u}_0 + \tilde{u}_n\big)\psi\, dt = \int_{t_{i_0-1}}^{t_{i_0}} g\big(\lambda_n C + \tilde{u}_n\big)\psi dt$$

To end the proof, observe that, thanks to (1.76) and the choice of $\epsilon$,

$$g\big(\lambda_n C + \tilde{u}_n(t)\big)\psi(t) = g\big(\gamma_+ + nT + \tilde{u}_n(t)\big)\psi(t) = g\big(\gamma_+ + \tilde{u}_n(t)\big)\psi(t) \geq \rho\psi(t) \qquad \forall t \in [0,\pi],$$

so that,

$$\liminf_{n\to\infty} \varepsilon_+(\lambda_n\tilde{h}_0) \geq \liminf \int_{t_{i_0-1}}^{t_{i_0}} g\big(\lambda_n C + \tilde{u}_n(t)\big)\psi(t)\, dt \geq \rho \int_{x_{i_0-1}}^{x_{i_0}} \psi(t)\, dt.$$

Of course, an analogous reasoning would give

$$\liminf_{n\to\infty} \varepsilon_-(\mu_n\tilde{h}_0) \geq \rho \int_{x_{i_0-1}}^{x_{i_0}} \psi(t)\, dt.$$

for the sequence $\mu_n := \frac{\gamma_- + nT}{C}$. This proves the theorem. □

The final part of this chapter is devoted to check the claimed density of

$$\{\tilde{h} \in \psi^\perp : \mathcal{K}(\tilde{h}) \in \mathcal{S}_\delta\}$$

in $\psi^\perp$ for any $\delta > 0$.

*Proof.* Since $\mathcal{K} : \psi^\perp \to \varphi^\perp$ is a topological isomorphism, we can equivalently prove the density of $\mathcal{S}_\delta \cap \varphi^\perp$ in $\varphi^\perp$. Now, recall that $\mathcal{P} \cap \varphi^\perp$, and hence, $\mathcal{P} \cap \varphi^\perp\backslash\{0\}$ are dense in $\varphi^\perp$ (this was proven in Lemma 1.5.5, 1). With this in mind we are finished if we prove that $(\mathcal{P} \cap \varphi^\perp)\backslash\{0\} \subset \overline{\mathcal{S}_\delta \cap \varphi^\perp}$. Take, therefore, $u \in \mathcal{P} \cap \varphi^\perp$, $u \not\equiv 0$. It will be shown that there exists $\epsilon > 0$ and a continuous curve $U : [0,\epsilon[\to W_0^{2,1}[0,\pi]$ such that $U(0) = u$; $U(t) \in \mathcal{S}_\delta \cap \varphi^\perp$ $\forall t \in ]0,\epsilon[$.

Being $u \in \varphi^\perp = \big\{w \in W_0^{2,1}[0,\pi] : \int_0^\pi w(s)\varphi(s)ds = 0\big\}$ we know that $u$ must achieve both positive and negative values in $]0,\pi[$. Let $\rho_0 > 0$ be the greater positive number $\rho$ such that

$$\mathcal{J}_\rho := \{s \in ]0,\pi[ : u(s) = \rho(\varphi(s) + 1)\} \neq \emptyset \tag{1.78}$$

and choose some point $t_0 \in \mathcal{J}_{\rho_0}$.

Being $u \in \mathcal{P}$, it should be possible to find a partition $0 = r_0 < r_1 < r_2 < \ldots < r_{p-1} < r_p = t_0 = s_p < s_{p-1} < \ldots < s_2 < s_1 < t_0 = \pi$ of the interval $[0,\pi]$ such that $p \geq 3$ and $u''_{|]r_{i-1},r_i[}$, $u''_{|]s_i,s_{i-1}[}$ are constant for each $i : 1...p$. Define $\epsilon := \min\{r_p - r_{p-1}, s_{p-1} - s_p, \delta/3\}$. We will explicitly describe only $U(\xi)_{|[0,s_0]}$ for any $\xi \in [0,\epsilon[$; $U(\xi)_{|[s_0,\pi]}$ would be constructed similarly.

Consider the linear mapping $T : \mathbb{R}^3 \to W^{2,1}[r_0, r_2]$ defined by

$$T(y_0'; m_1, m_2) := \text{The solution } y \text{ of the linear IVP } \begin{cases} y(0) = 0; \quad y'(0) = y_0'; \\ y''(t) = m_1;\ r_0 < t < r_1; \\ y''(t) = m_2;\ r_1 < t < r_2. \end{cases} \tag{1.79}$$

Next, define $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$(y_0'; m_1, m_2) \mapsto \left(\big[T(y_0'; m_1, m_2)\big](r_2), \big[T(y_0'; m_1, m_2)\big]'(r_2), \int_0^{r_2} \big[T(y_0'; m_1, m_2)\big](t)\varphi(t)dt\right)$$

which is easily seen to have a trivial kernel, being, therefore, a linear isomorphism.

Finally, for each $\xi \in [0,\epsilon[$, $U(\xi)_{|[0,s_0]}$ will be built as follows:

- If $t \in [r_p - \xi, r_p]$, define $[U(\xi)](t) := \rho_0(\varphi(t) + 1)$.

- Next, extend $U(\xi)$ to $[r_2, r_p]$ in the only way that keeps $U(\xi)$ being a $W^{2,1}$ function which, in addition, verifies $[U(\xi)]'' = u''$ in $]r_2, r_p - \xi[$.
- To finish, extend $U(\xi)$ to $[0, s_0]$ by setting, for any $t \in [0, r_2[$,

$$[U(\xi)](t) := T\left[\Psi^{-1}\left(\big[U(\xi)\big](r_2), \big[U(\xi)\big]'(r_2), \int_0^{s_0} u(s)\varphi(s)ds - \int_{r_2}^{s_0} \big[U(\xi)\big](s)\varphi(s)ds\right)\right](t).$$

For any $\xi \in [0, \epsilon[$, $U(\xi)$ is built similarly on $[s_0, \pi]$. Eventually, it is clear that, as assured, $U : [0, \epsilon[ \to W_0^{2,1}[0, \pi]$ is a continuous mapping verifying $U(0) = u$, $U(\xi) \in \varphi^\perp \cap \mathcal{S}$ $\forall \xi \in ]0, \epsilon[$. The result is now proven. □

# Chapter 2

# Periodic perturbations of linear, resonant, elliptic operators in bounded domains

## 2.1 Introduction

In this chapter we are concerned with self-adjoint, elliptic boundary value problems of the type

$$\begin{cases} -\Delta u - \lambda_1 u + g(u) = h(x) = \tilde{h}(x) + \bar{h}\varphi(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

where the following hypothesis are made:

**[$\mathbf{H}_2$]**

1. $\Omega$ is a bounded, smooth domain in $\mathbb{R}^N$ for some $N \geq 2$,
2. $\lambda_1$ is the **first eigenvalue** associated to the operator $-\Delta$ when acting on $H_0^1(\Omega)$; our problem is **resonant**. We call $\varphi$ an associated normalized eigenfunction

$$-\Delta\varphi = \lambda_1\varphi\,, \qquad\qquad \max_{\Omega} \varphi = 1\,.$$

3. $g : \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz, periodic, and to have zero mean

$$g \in \mathrm{Lip}(\mathbb{R}/T\mathbb{Z}) \qquad \text{for some } T > 0;$$
$$\int_0^T g(u)\,du = 0\,,$$

   the latter hypothesis being not restrictive (otherwise, simply subtract its medium value from both sides of the equation in (2.1)). Also, we assume it is not the constant zero function: our problem is **not linear**. Finally, we call $G_1$, $G_2$, $G_3$,... the successive, periodic primitives of $g$ with zero mean.

4. Respecting the forcing term $h = h(x)$, it is assumed to be Lipschitz in $\Omega$. We decompose it in the form

$$h = \bar{h}\varphi + \tilde{h}\,,$$

   where $\bar{h} \in \mathbb{R}$, and

$$\tilde{h} \in \widetilde{Lip}(\Omega) := \left\{ h \in Lip(\Omega) : \int_\Omega h(x)\varphi(x)dx = 0 \right\}.$$

The following geometrical assumption on the domain $\Omega$ in relation with the operator $-\Delta$ will stand throughout this chapter:

**[C$_2$]** $\varphi$ has an unique critical point, which is not degenerate.

Here, the expression 'not degenerate' means that the Hessian matrix of $\varphi$ at this point is assumed to be inversible. It follows from results in [8, 45], that all regular, convex, bounded domains verify **[C$_2$]**. Regular and bounded domains which are Steiner symmetric with respect to all $N$ coordinate hyperplanes, i.e.

$$(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_N x_N) \in \Omega \quad \forall x = (x_1, x_2, \ldots, x_N) \in \Omega, \ \forall \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in [-1, 1]^N ,$$

also verify **[C$_2$]**, (see [7]), and domains $\Omega$ which result by means of small, smooth perturbations of others verifying **[C$_2$]** do verify this same assumption. On the other hand, **[C$_2$]** implies severe restrictions on the topology of the domain; it forces $\bar{\Omega}$ to be $C^1$-diffeomorphic to a closed ball, as it follows from well known arguments of Morse's theory.

We observe here that this hypothesis on the domain $\Omega$ is not merely required by our method of proof, since, for different types of domains, (which include, for instance, anular domains whose inner and outer radius satisfy certain relations) there are results in the literature (see, for instance, [23]) displaying a different qualitative behaviour of the several aspects of the problem. In particular, most of our results fail to hold if **[C$_2$]** is simply removed.

Observe that, in case $N = 1$, $\Omega = ]0, \pi[$, we obtain the same problem which was discussed throughout our previous chapter (for $\alpha = 0$), and whose solvability, in case $\bar{h} = 0$, had already been shown by Ward ([85]). Shortly after Ward's paper, his result was generalized by Solimini [78], and Lupo and Solimini [52], for resonant problems in arbitrary domains of $\mathbb{R}^N$ and higher eigenvalues $\lambda_s$, $s \geq 1$. Thus, in case $\bar{h} = 0$, problem (2.1) is solvable.

Using methods from global bifurcation theory, Schaaf and Schmitt studied the multiplicity of solutions of Ward's problem. In [73] they showed this problem to have infinitely many positive and infinitely many negative solutions. Using a similar approach, they also studied the case of $\Omega$ being a convex subset of the plane. When $\bar{h} = 0$, they showed ([74]) that problem (2.1) is not only solvable, but has infinitely many positive and infinitely many negative solutions. Numerical experiments were provided ([23, 74]) indicating that *'the latter result does not hold for $\Omega$ a ball in dimensions greater than 3'* ([74], pg. 1120).

All these partial results lead us to our first question: Do Schaaf-Schmitt results hold for problem (2.1) and dimensions $N \geq 3$?. With a certain degree of generality, we answer to this question in Theorem 2.1.1 below. In particular, it is shown the answer to be 'yes' in case $N = 3$ and 'certainly not always' in case $N \geq 5$. We also give a new proof of the two dimensional case.

**Theorem 2.1.1.** *Choose $h = \tilde{h} \in \widetilde{Lip}(\Omega)$. Then, in case $N = 2$ or $N = 3$, problem* (2.1) *has infinitely many positive and infinitely many negative solutions. Indeed,*

$$\forall p \in C^1(\bar{\Omega}) \text{ with } p(\Omega) = 0 \ \forall \Omega \in \partial\Omega \text{ there exist solutions } u_1 \text{ and } u_2 \text{ of } (2.1) \text{ with } u_1 \leq p \leq u_2 .$$

*In case $N \geq 5$ and $\Omega$ is convex, for any bounded set $\mathcal{B} \subset \widetilde{Lip}(\Omega)$ there exists a nonempty open set $\mathcal{O}_{\mathcal{B}} \subset \{g \in \mathrm{Lip}(\mathbb{R}/T\mathbb{Z}) : \int_0^T g(u)du = 0\}$ such that, if $\tilde{h} \in \mathcal{B}$ and $g \in \mathcal{O}_{\mathcal{B}}$, the set of solutions of* (2.1) *is bounded.*

Thus, *it remains an open problem to decide whether, in case $N \geq 5$, the set of solutions of* (2.1) *is always bounded, regardless of $g$ and $h$, or might be unbounded. The fourth dimensional case escapes our treatment and remains also open.*

A second related question was motivated by the work of Cañada [11]. Here, it was proved that, in the one-dimensional Ward's problem, in case the forcing term $h = \tilde{h}$ belongs to the range of the linear problem, the associated action functional, (which turns out to be non-coercive and bounded from below), does attain its minimum. In this chapter we also show how this result remains true for dimensions $N = 2$ and $N = 3$. For higher dimensions $N \geq 4$ this continues to hold for generic $g$ and all $\tilde{h}$:

**Theorem 2.1.2.** *Assume $N = 2$ or $N = 3$, or $N \geq 4$ and $G_2(0) \neq 0$, or $G_2(0) = 0$ but $\Omega$ is convex and $G_3(0) < 0$, and let $h = \tilde{h} \in \widetilde{Lip}(\Omega)$. Then, the minimum of the action functional corresponding to* (2.1) *is attained. Furthermore, this minimum is strictly lower than the minimum of the action functional corresponding to the linear problem $-\Delta u - \lambda_1 u = \tilde{h}$, $u \in H_0^1(\Omega)$.*

Finally (following a historic account of the facts), the nondegeneracy problem received a separate attention. As it happened with the ODE problem we studied in chapter 1, it follows from the lower and upper solutions method, the Riemann-Lebesgue Lemma and Solimini results ([78]) that, for any $\tilde{h} \in \widetilde{Lip}(\Omega)$ there exist real numbers $\varepsilon_-(\tilde{h}) \leq 0 \leq \varepsilon_+(\tilde{h})$ such that problem (2.1) is solvable if and only if $\varepsilon_-(\tilde{h}) \leq \bar{h} \leq \varepsilon_+(\tilde{h})$. In case this closed interval always contains a neighborhood of zero $\big($that is, $\varepsilon_-(\tilde{h}) < 0 < \varepsilon_+(\tilde{h}) \quad \forall \tilde{h} \in \widetilde{Lip}(\Omega)\big)$, the nonlinearity $g$ is said to be *nondegenerate*, and the nondegeneracy of $g(u) = A \sin u$ when $N = 1$, $\Omega = ]0, \pi[$, was already shown by Dancer [25]. It was extended for general periodic nonlinearities by Cañada and Roca ([15]).

Some related nondegeneracy results for the PDE problem (2.1) when $\|\tilde{h}\|_{L^2(\Omega)}$ is small were also established in [17], and, for $N = 2$, the nondegeneracy of every nonlinearity $g$ was implicit in [74]. The arguments in this latter paper do not extend to the case $N \geq 3$, and the main contribution of this chapter refers precisely to the three dimensional case. If $N = 3$, we show that every nonlinearity $g$ is not degenerate. In case $N \geq 4$ we show that, generically, nonlinearities are nondegenerate. Finally, we give a new proof of the two-dimensional case.

**Theorem 2.1.3.** *Assume $N = 2$ or $N = 3$. Then, problem (2.1) is not degenerate.*

*Assume $N \geq 4$ and that $G_2(0) \neq 0$, or $G_2(0) = 0$ but $\Omega$ is convex and $G_3(0) < 0$. Then, problem (2.1) is nondegenerate.*

Thus, for convex domains and nonlinearities $g$ of the form, say, $g(u) = A \sin(u) + B \cos(u)$, where $B \neq 0$ or $B = 0$ and $A < 0$, problem (2.1) is not degenerate. *It remains an open problem to decide if nondegeneracy continues to hold for arbitrary dimensions $N \geq 4$ and arbitrary nonlinearities $g$.*

Finally, a whole world of open problems appears when hypothesis $[\mathbf{C}_2]$ is skipped. There exist some previous work in this direction, and several nonconvex domains such as annulus have been considered, (see [23]), but the problem is far from closed.

## 2.2 A variational approach

The splitting

$$H_0^1(\Omega) = \langle \varphi \rangle \oplus \tilde{H}_0^1(\Omega)\,, \tag{2.2}$$

where

$$\tilde{H}_0^1(\Omega) = \left\{ \tilde{u} \in H_0^1(\Omega) : \int_\Omega \tilde{u}(x)\varphi(x)dx = 0 \right\}\,,$$

let us to write any function $u \in H_0^1(\Omega)$ as $u = \bar{u}\varphi + \tilde{u}$, with $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in \tilde{H}_0^1(\Omega)$. Calling $\tilde{u}_{\tilde{h}}$ the only solution in $\tilde{H}_0^1(\Omega)$ of the linear equation $-\Delta\tilde{u} - \lambda_1\tilde{u} = \tilde{h}$, the classical change of variables $v = u - \tilde{u}_{\tilde{h}}$ transforms problem (2.1) into the equivalent one:

$$-\Delta v - \lambda_1 v + g(v + \tilde{u}_{\tilde{h}}) = \bar{h}\varphi\,, \qquad\qquad v \in H_0^1(\Omega) \tag{2.3}$$

We consider the associated action functional $\Phi_h : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\Phi_h(v) := \frac{1}{2}\int_\Omega \|\nabla v(x)\|^2 dx - \frac{\lambda_1}{2}\int_\Omega v(x)^2 dx + \int_\Omega G_1(v(x) + \tilde{u}_{\tilde{h}}(x))dx - \bar{h}\int_\Omega \varphi(x)v(x)dx \tag{2.4}$$

where $G_1$ denotes, as before, the primitive of $g$ with zero mean. It follows that the solutions of (2.3) coincide with the critical points of $\Phi_h$. Observe that

$$\begin{aligned}
&\Phi_h(\bar{v}\varphi + \tilde{v}) = \\
&\quad = \left[\frac{1}{2}\int_\Omega \|\nabla\tilde{v}(x)\|^2 dx - \frac{\lambda_1}{2}\int_\Omega \tilde{v}(x)^2 dx\right] + \int_\Omega G_1\big(\bar{v}\varphi(x) + \tilde{v}(x) + \tilde{u}_{\tilde{h}}(x)\big)dx - \bar{h}\|\varphi\|_2^2\bar{v} = \\
&\quad = \Psi(\tilde{v}) + \Upsilon_{\tilde{h}}(\bar{v}, \tilde{v}) - \bar{h}\|\varphi\|_2^2\bar{v} \ \forall\tilde{v} \in \tilde{H}_0^1(\Omega),\ \forall\bar{v} \in \mathbb{R}.
\end{aligned}$$

On the other hand, $\Phi_{\tilde{h}}$ coincides, up to a constant and a translation in $H_0^1(\Omega)$, with the action functional corresponding to (2.1). Consequently, it attains its global minimum in $H_0^1(\Omega)$ if and only if the action functional corresponding to (2.1) has the same property, and the minimum of this latter functional is strictly lower than the minimum of the associated linear problem if and only if $\min_{H_0^1(\Omega)} \Phi_{\bar{h}} < 0$.

The functional $\Psi : \tilde{H}_0^1(\Omega) \to \mathbb{R}$ is coercive while $\Upsilon_{\bar{h}} : \mathbb{R} \times \tilde{H}_0^1(\Omega) \to \mathbb{R}$ is bounded. On the other hand, $\Psi$ is weak lower semicontinuous, while, for any fixed $\bar{v} \in \mathbb{R}$, $\Upsilon_{\bar{h}}(\bar{v}, \cdot) : \tilde{H}_0^1(\Omega) \to \mathbb{R}$ is easily seen to be sequentially weak lower semicontinuous. Consequently, for any $\bar{v} \in \mathbb{R}$, the minimum of the functional $\tilde{H}_0^1(\Omega) \to \mathbb{R}$, $\tilde{v} \mapsto \Phi_h(\bar{v}\varphi + \tilde{v})$, is attained, and, further, there exists some $R > 0$ such that $\|\tilde{v}\|_{H_0^1(\Omega)} \leq R \, \forall (\bar{v}, \tilde{v}) \in \mathbb{R} \times \tilde{H}_0^1(\Omega)$ with $\Phi_h(\bar{v}\varphi + \tilde{v}) = \min_{\tilde{w} \in \tilde{H}_0^1(\Omega)} \Phi_h(\bar{v}\varphi + \tilde{w})$.

We consider the continuous function

$$m_h : \mathbb{R} \to \mathbb{R}; \qquad m_h(\bar{v}) := \min_{\tilde{w} \in \tilde{H}_0^1(\Omega)} \Phi_h(\bar{v}\varphi + \tilde{w})$$

Observe that $m_h(\bar{v}) = m_{\bar{h}}(\bar{v}) - \bar{h}\|\varphi\|_2^2 \, \bar{v}$. It follows from the Riemann-Lebesgue Lemma (see [78]) that $\Upsilon_{\bar{h}}(\bar{v}, \tilde{v}) \to 0$ as $|\bar{v}| \to \infty$ uniformly with respect to $\tilde{v} \in \tilde{H}_0^1(\Omega)$, $\|\tilde{v}\|_{H_0^1(\Omega)} \leq R$. It means that $\lim_{|\bar{v}| \to \infty} m_{\bar{h}}(\bar{v}) = 0 = \min_{\tilde{H}_0^1(\Omega)} \Psi$. And a $C^1$ function $\xi : \mathbb{R} \to \mathbb{R}$ which has the same limits at $-\infty$ and $+\infty$, if nonconstant, has at least a critical point in $\mathbb{R}$ and the same thing happens to little perturbations of the type $\bar{v} \mapsto \xi(\bar{v}) - \delta \bar{v}$ with $|\delta|$ small. In our case, $m_{\bar{h}}$ is probably not necessarily $C^1$ (it will happen if $g$ is assumed to have a small Lipschitz constant), but anyway, we may easily obtain the same conclusion via the Mountain Pass Theorem.

**Lemma 2.2.1.** *Assume that $m_{\bar{h}}(\bar{v}) \neq 0$ for some $\bar{v} \in \mathbb{R}$. Then, problem* (2.1) *is not degenerate.*

*Proof.* This is an immediate consequence of the direct methods of the calculus of variations if $m_{\bar{h}}(\bar{v}) < 0$ for some $\bar{v} \in \mathbb{R}$ and a consequence of the Mountain Pass theorem in case $m_{\bar{h}}(\bar{v}) > 0$ for some $\bar{v} \in \mathbb{R}$. (Indeed, observe that the Palais-Smale condition for $\Phi_{\bar{h}}$ holds at all levels but 0, and at all levels for $\Phi_h$ when $\bar{h} \neq 0$, see also [6]). In the first case it is clear that the non coercive action functional $\Phi_{\bar{h}}$ attains its minimum in $\mathbb{R}$ while, if the latter possibility would hold for any $\bar{v} \in \mathbb{R}$, there will be no global minima of $\Phi_{\bar{h}}$ in $H_0^1(\Omega)$. □

We devote the first part of this chapter to look for sufficient conditions implying $m_{\bar{h}}$ to be not constantly zero. And maybe the most elementary one follows from the observation

$$\Upsilon_{\bar{h}}(\bar{v}_*, 0) < 0 \Rightarrow m_{\bar{h}}(\bar{v}_*) \leq \Phi_{\bar{h}}(\bar{v}_*\varphi) = \Upsilon_{\bar{h}}(\bar{v}_*, 0) < 0 \,,$$

since $\Psi(\bar{v}_*\varphi) = 0$.

A second possibility in order to show that $m_{\bar{h}}$ is not constant is given below:

$$\Upsilon_{\bar{h}}(\bar{v}_*, \tilde{v}) > 0 \; \forall \tilde{v} \in \tilde{H}_0^1(\Omega) \text{ with } \Phi_{\bar{h}}(\bar{v}_*, \tilde{v}) = \min_{\tilde{v} \in \tilde{H}_0^1(\Omega)} \Phi_{\bar{h}}(\bar{v}_*, \tilde{v}) \Rightarrow m(\bar{v}_*) > 0 \,.$$

We will explore both strategies in our context. Both will give us sufficient conditions for (2.1) to be nondegenerate. Observe that, in case this second one would happen for all $\bar{v}_* \in \mathbb{R}$, the action functional will not attain its minimum in $H_0^1(\Omega)$.

Let us take a sequence $\{\bar{v}_n\}_n$ of real numbers with

$$|\bar{v}_n| \geq 1 \quad \forall n \in \mathbb{N} \,, \qquad\qquad |\bar{v}_n| \to \infty \,,$$

and, for each $n \in \mathbb{N}$, choose $\tilde{v}_n \in \tilde{H}_0^1(\Omega)$ with

$$\Phi_{\bar{h}}(\bar{v}_n\varphi + \tilde{v}_n) = \min_{\tilde{v} \in \tilde{H}_0^1(\Omega)} \Phi_{\bar{h}}(\bar{v}_n\varphi + \tilde{v}) \,.$$

Observe that this element $\tilde{v}_n$ satisfies the so-called *auxiliary equation* in $\Omega$

$$-\Delta \tilde{v}_n - \lambda_1 \tilde{v}_n + g\big(\bar{v}_n\varphi(x) + \tilde{v}_n + \tilde{u}_{\bar{h}}(x)\big) - \frac{1}{\|\varphi\|_2^2} \left[ \int_\Omega g\big(\bar{v}_n\varphi(y) + \tilde{v}_n(y) + \tilde{u}_{\bar{h}}(y)\big)\varphi(y) \, dy \right] \varphi(x) = 0 \,, \quad (2.5)$$

together with the homogeneous Dirichlet boundary conditions $\tilde{v}_n|_{\partial\Omega} = 0$. Thus, regularity arguments (see, for instance, [1],[10] (pp. 197-198), [37]) show that

$$\Big\{\tilde{v}_n : n \in \mathbb{N}\Big\} \subset W^{3,r}(\Omega) \qquad\qquad \forall r \in ]1, \infty[,$$

and $\{\tilde{v}_n\}$ is bounded in the $W^{2,r}(\Omega)$ topology for any $1 < r < \infty$. In particular, the sequence $\{\tilde{v}_n\}_n$ is contained in $C^2(\bar{\Omega})$, though we are not able to ensure (unlike what happens in the ODE problem we studied in the previous chapter) that it is bounded here. In any case, the Riemann-Lebesgue Lemma ([85]) together with (2.5) imply that $\{\tilde{v}_n\} \to 0$ weakly in $W^{2,r}(\Omega)$ for any $1 < r < \infty$; in particular,

$$\{\tilde{v}_n\}_n \to 0 \text{ in } C^1(\bar{\Omega})\,.$$

Derivate with respect to $x_i$ in equation (2.5) to obtain

$$\begin{aligned} -\,\Delta\partial_{x_i}\big(\tilde{v}_n/\bar{v}_n\big) - \lambda_1\partial_{x_i}\big(\tilde{v}_n/\bar{v}_n\big) &= -\big(\partial_{x_i}\varphi + \partial_{x_i}\tilde{v}_n/\bar{v}_n + \partial_{x_i}\tilde{u}_{\tilde{h}}/\bar{v}_n\big)g'\big(\bar{v}_n\varphi + \tilde{v}_n + \tilde{u}_{\tilde{h}}\big) + \\ + \frac{1}{\bar{v}_n}\frac{1}{\|\varphi\|_2^2}&\left[\int_\Omega g(\bar{v}_n\varphi(y) + \tilde{v}_n(y) + \tilde{u}_{\tilde{h}}(y))\varphi(y)\,dy\right]\partial_{x_i}\varphi\,, \quad 1 \le i \le N\,. \end{aligned} \tag{2.6}$$

Since $\{\tilde{v}_n/\bar{v}_n\} \to 0$ in $W^{2,r}(\Omega)$, the Riemann-Lebesgue Lemma [78] together with (2.6) imply that $\{\tilde{v}_n/\bar{v}_n\} \to 0$ weakly in $W^{3,r}(\Omega)$ for any $1 < r < \infty$. In particular,

$$\Big\{\tilde{v}_n/\bar{v}_n\Big\} \to 0 \text{ in } C^2(\bar{\Omega})\,. \tag{2.7}$$

For each $n \in \mathbb{N}$, we write

$$v_n := \bar{v}_n\varphi + \tilde{v}_n + \tilde{u}_{\tilde{h}}\,; \qquad\qquad d_n := \frac{\bar{v}_n}{|\bar{v}_n|}\max_\Omega |v_n|\,. \tag{2.8}$$

Observe that

$$\lim_{n\to\infty}\Big\{d_n - \bar{v}_n\Big\} = \tilde{u}_{\tilde{h}}(\Omega_0)\,,$$

being $\Omega_0$ the unique point in $\Omega$ where $\varphi$ attains its maximum. In particular, if $n$ is taken big enough, $d_n \neq 0$ and the sequence $\{\varphi_n\} \subset C^2(\bar{\Omega})$ defined by

$$\varphi_n := \frac{1}{d_n}v_n, \quad n \in \mathbb{N}, \tag{2.9}$$

converges to $\varphi$ in the $C^2(\bar{\Omega})$ norm, while it is bounded in the $W^{3,r}(\Omega)$ topology for any $1 < r < \infty$. We will just need later to use that it is bounded in $H^3(\Omega)$.

Our hypothesis $[\mathbf{C}_2]$ on the geometry of $\Omega$ says that the gradient of $\varphi$ vanishes only at $\Omega_0$ and $D^2\varphi(\Omega_0)$ is negative definite. Consequently, for $n$ sufficiently big, $\nabla\varphi_n$ will vanish only at one point $\Omega_n \in \Omega$ and $D^2\varphi_n(\Omega_n)$ will be negative definite -we will indeed assume this is true for any $n \in \mathbb{N}$-. The co-area formula (see, for instance [33]) may be used then to find that

$$\Upsilon_{\tilde{h}}(\bar{v}_n, \tilde{v}_n) = \int_\Omega G_1(d_n\varphi_n(x))dx = \int_0^1 G_1(d_nt)\left(\int_{\{\varphi_n(x)=t\}}\frac{1}{\|\nabla\varphi_n(x)\|}ds_x\right)dt = \int_0^1 G_1(d_nt)p_n(t)dt \tag{2.10}$$

where

$$p_n(t) := \int_{\{\varphi_n(x)=t\}}\frac{1}{\|\nabla\varphi_n(x)\|}ds_x, \qquad 0 \le t < 1, \qquad n \in \mathbb{N} \tag{2.11}$$

This idea of using the co-area formula to tackle this problem by means of a careful study of the asymptotic behaviour of some oscillating integrals was already suggested in [25] and used in [74]. It will be shown (Lemma 2.2.2) that $\{p_n\}$ converges to

$$p(t) := \int_{\{\varphi(x)=t\}}\frac{1}{\|\nabla\varphi(x)\|}ds_x \tag{2.12}$$

in $C^1([0, 1-\epsilon])$, while it is contained and bounded in $H^2[0, 1-\epsilon]$ for any $0 < \epsilon < 1$. In order to do that, we need some deeper knowledge on the level sets of $\varphi_n$. With this aim, we use adequate changes of variables carrying copies of $\partial\Omega$ into the level sets of $\varphi_n$ or $\varphi$. The details are shown below.

**Lemma 2.2.2.** *Let $\{u_n\} \to u$ be any convergent sequence in $C^2(\bar\Omega)$ and let $0 < \epsilon < 1$ be given. Assume that $u_n(\Omega) = u(\Omega) = 0$ for all $n \in \mathbb{N}$ and all $\Omega \in \partial\Omega$, that $\max_{\bar\Omega} u = 1$ and that $\nabla u(\Omega) \neq 0$ $\forall \Omega \in \bar\Omega$ with $u(\Omega) < 1$. If $n \in \mathbb{N}$ is big enough, $\nabla u_n(\Omega) \neq 0$ $\forall \Omega \in \bar\Omega$ with $u_n(\Omega) \leq 1-\epsilon$; let us assume that this happens indeed for any $n$. Then, the sequence $\{\varpi_n\}$ defined by*

$$\varpi_n(t) := \int_{\{u_n(x)=t\}} \frac{1}{\|\nabla u_n(x)\|} ds_x \tag{2.13}$$

*converges in $C^1\big([0,1-\epsilon]\big)$ to the function $\varpi$ given by*

$$\varpi(t) := \int_{\{u(x)=t\}} \frac{1}{\|\nabla u(x)\|} ds_x\,. \tag{2.14}$$

*Finally, if, further, $\{u_n\}_n$ is contained and bounded in $H^3(\Omega)$, then $\{\varpi_n\}$ is bounded in $H^2[0,1-\epsilon]$, and, in case $\{u_n\}$ converges in $H^3(\Omega)$, $\{\varpi_n\}$ converges in $H^2[0,1-\epsilon]$.*

*Proof.* For any $n \in \mathbb{N}$, let us consider the mapping $\Theta_n : (\partial\Omega) \times [0,1-\epsilon] \to \mathbb{R}^N$ defined as the solution of the initial value problem

$$\begin{aligned} \Theta_n(x,0) &= x\,; && x \in \partial\Omega \\ \frac{\partial \Theta_n}{\partial t}(x,t) &= \frac{\nabla u_n\big(\Theta_n(x,t)\big)}{\|\nabla u_n\big(\Theta_n(x,t)\big)\|^2}\,; && (x,t) \in (\partial\Omega)\times[0,1-\epsilon] \end{aligned}$$

Analogously, define $\Theta : (\partial\Omega) \times [0,1-\epsilon] \to \mathbb{R}^N$ as the solution of

$$\begin{aligned} \Theta(x,0) &= x\,; && x \in \partial\Omega \\ \frac{\partial \Theta}{\partial t}(x,t) &= \frac{\nabla u\big(\Theta(x,t)\big)}{\|\nabla u\big(\Theta(x,t)\big)\|^2}\,; && (x,t) \in (\partial\Omega)\times[0,1-\epsilon] \end{aligned}$$

Then, for each $t \in [0,1-\epsilon]$,

$$\Theta_n\big[(\partial\Omega)\times\{t\}\big] = \Big\{\Omega \in \Omega : u_n(\Omega) = t\Big\} \quad \forall n \in \mathbb{N}\,; \qquad \Theta\big[(\partial\Omega)\times\{t\}\big] = \Big\{\Omega \in \Omega : u(\Omega) = t\Big\}\,,$$

and the change of variables theorem gives

$$\varpi_n(t) = \int_{\partial\Omega} \Big|J\Theta_n(x,t)\Big|\, ds_x\,, \qquad \varpi(t) = \int_{\partial\Omega} \Big|J\Theta(x,t)\Big|\, ds_x\,, \qquad t \in [0,1-\epsilon]$$

being $J\Theta_n$ and $J\Theta$ the Jacobian determinants of $\Theta_n$ and $\Theta$ respectively. Since these are $C^{0,1}$ mappings (indeed, $C^1$ mappings,) which do not vanish on $(\partial\Omega)\times[0,1-\epsilon]$, we deduce that $\varpi_n$ and $\varpi$ are $C^1$ mappings on $[0,1-\epsilon]$ and we have, for any $0 \leq t \leq 1-\epsilon$,

$$\varpi_n'(t) = \int_{\partial\Omega} \frac{\Big|J\Theta_n\Big|}{\Big|\nabla u_n \circ \Theta_n\Big|^2} \left[\Delta u_n \circ \Theta_n - 2\frac{\big(\nabla u_n \circ \Theta_n\big)^T\big(D^2 u_n \circ \Theta_n\big)\big(\nabla u_n \circ \Theta_n\big)}{\Big|\nabla u_n \circ \Theta_n\Big|^2}\right] ds_x\,, \tag{2.15}$$

$$\varpi'(t) = \int_{\partial\Omega} \frac{\Big|J\Theta\Big|}{\Big|\nabla u \circ \Theta\Big|^2} \left[\Delta u \circ \Theta - 2\frac{\big(\nabla u \circ \Theta\big)^T\big(D^2 u \circ \Theta\big)\big(\nabla u \circ \Theta\big)}{\Big|\nabla u \circ \Theta\Big|^2}\right] ds_x\,. \tag{2.16}$$

being $D^2u_n$ and $D^2u$ the Hessian matrices of $u_n$ and $u$ respectively. Since $\Theta_n \to \Theta$ in the $C^1\big((\partial\Omega)\times[0,1-\epsilon]\big)$ topology, and $u_n \to u$ in $C^2(\bar\Omega)$, we deduce that $\{\varpi_n\} \to \varpi$ in $C^1[0,1-\epsilon]$.

We also deduce from (2.15) and (2.16) that, in case that $\{u_n\}_n$ is, further, contained in $C^3(\bar\Omega)$, and it converges to some function $u \in C^2(\bar\Omega) \cap H^3(\Omega)$ in both spaces $C^2(\bar\Omega)$ and $H^3(\Omega)$, the corresponding sequence $\{\varpi_n\}$ is $H^2[0,1-\epsilon]$-Cauchy. Since it already converges to $\varpi$ in the $C^1[0,1]$ topology, we deduce that $\varpi \in H^2[0,1-\epsilon]$ whenever $u \in H^3(\Omega)$ and also that the mapping $H^3(\Omega) \to H^2[0,1-\epsilon]$, $u \mapsto \varpi = \varpi_u$, is continuous. Now we see that, in case $\{u_n\} \to u$ in $C^2(\bar\Omega)$ is, in addition, bounded in the $H^3(\Omega)$ topology, $\{\varpi_n\}$ is bounded in $H^2[0,1-\epsilon]$. The Lemma is proven.

□

Due to the singularity of $\nabla\varphi/\|\nabla\varphi\|^2$ at $\Omega_0$, our just developed change of variables behaves nicely only far away from $\{1\}\times\mathbb{S}^{N-1}$. To know more about the convergence of $\{p_n\}$ to $p$ (particularly, near 1), we need different changes of variables, also carrying spheres into level sets, but being regular up to $\{1\}\times\mathbb{S}^{N-1}$. To develop these changes of variables is the aim of the next section.

## 2.3 A suitable change of variables

We start this section by an auxiliary lemma which will be used repeatedly through the proof of our next theorem.

**Lemma 2.3.1.** *Let $\mathcal{C}$ be an open subset of the cylinder $[0,\infty[\times\mathbb{S}^{N-1}$, let $m \in L^\infty[0,1]$ be given and let $1<r<\infty$. We assume that $\{0\}\times\mathbb{S}^{N-1}\subset\mathcal{C}$ and, moreover, it is star-shaped with respect to $\{0\}\times\mathbb{S}^{N-1}$ in the following sense:*

$$(t\rho,\theta)\in\mathcal{C}\ \forall(\rho,\theta)\in\mathcal{C},\ \forall t\in[0,1]\,.$$

*For any $z\in L^r(\mathcal{C})$, we consider the mapping*

$$Z:\mathcal{C}\to\mathbb{R}\,,\qquad\qquad Z(\rho,\theta):=\int_0^1 m(t)z(t\rho,\theta)\,dt$$

*Then, the following hold:*

1. *$Z\in L^r(\mathcal{C})$ and*
$$\|Z\|_{L^r}\le\frac{r}{r-1}\|m\|_\infty\|z\|_{L^r}\,. \tag{2.17}$$
*In particular, the mapping $L^r(\mathcal{C})\to L^r(\mathcal{C})$, $\quad z\mapsto Z$, is continuous.*
2. *If $z$ is continuous, $Z$ is continuous.*
3. *If $z\in W^{1,r}(\mathcal{C})$, then $Z\in W^{1,r}(\mathcal{C})$ and*
$$\partial_\rho Z(\rho,\theta)=\int_0^1 m(t)\,t\,\partial_\rho z(t\rho,\theta)\,dt\,,\qquad\qquad \nabla_\theta Z(\rho,\theta)=\int_0^1 m(t)\nabla_\theta z(t\rho,\theta)\,dt\,. \tag{2.18}$$
*In particular, the mapping $W^{1,r}(\mathcal{C})\to W^{1,r}(\mathcal{C})$, $\quad z\mapsto Z$, is continuous.*

*Proof.* To prove 1., it is not restrictive to assume $\mathcal{C}=[0,\infty[\times\mathbb{S}^{N-1}$. (Otherwise, simply extend $z$ by zero). Then, Fubini's Theorem ensures that, for almost every $\theta\in\mathbb{S}^{N-1}$, the mapping $z_\theta:[0,\infty[\to\mathbb{R}$ defined by $z_\theta(\rho):=z(\rho,\theta)$ belongs to $L^r[0,\infty[$; in particular, $Z(\rho,\theta)$ is defined for a.e. $\theta\in\mathbb{S}^{N-1}$ and all $\rho\ge0$. Further, given $\theta\in\mathbb{S}^{N-1}$ such that $z_\theta\in L^r[0,\infty[$, we have

$$\big|Z(\rho,\theta)\big|=\left|\int_0^1 m(t)z(t\rho,\theta)\,dt\right|\le\int_0^1\big|m(t)z(t\rho,\theta)\big|\,dt\le\|m\|_\infty\int_0^1\big|z(t\rho,\theta)\big|\,dt\,.$$

Thus, Hardy's inequality (see, for instance, [72], pp. 72), shows that for such a $\theta\in\mathbb{S}^{N-1}$, $Z_\theta:[0,\infty[\to\mathbb{R}$ defined by $\rho\mapsto Z(\rho,\theta)$, belongs to $L^r[0,\infty[$ and verifies

$$\|Z_\theta\|_{L^r}\le\frac{r}{r-1}\|m\|_\infty\|z_\theta\|_{L^r}\,.$$

Estimation (2.17) follows now from Fubini's Theorem.

On the other hand, statement 2. is a consequence of the Theorem of continuous dependence of integrals with respect to parameters.

To prove 3. simply observe that, in case $z \in C^1(\mathcal{C})$, the theorem of derivation of integrals with respect to parameters gives that the associated mapping $Z$ also belongs to $C^1(\mathcal{C})$ and its partial derivates are given by (2.18). Given an arbitrary function $z \in W^{1,r}(\mathcal{C})$, take a sequence $\{z_n\}_n \subset C^1(\mathcal{C}) \cap W^{1,r}(\mathcal{C})$ with $\{z_n\} \to z$ in $W^{1,r}(\mathcal{C})$. The corresponding sequence $\{Z_n\} \subset C^1(\mathcal{C})$ defined by

$$Z_n(\rho,\theta) := \int_0^1 m(t) z_n(t\rho,\theta)\, dt\,, \qquad (\rho,\theta) \in \mathcal{C}\,, \tag{2.19}$$

verifies

$$\partial_\rho Z_n(\rho,\theta) = \int_0^1 t\, m(t) \partial_\rho z_n(t\rho,\theta)\, dt\,, \qquad \nabla_\theta Z_n(\rho,\theta) = \int_0^1 m(t) \nabla_\theta z_n(t\rho,\theta)\, dt\,. \tag{2.20}$$

It follows from 1., (2.19) and (2.20) that

$$\begin{aligned} Z_n &\to Z\,, \\ \{\nabla_\theta Z_n\} &\to \left[(\rho,\theta) \mapsto \int_0^1 m(t) \nabla_\theta z(t\rho,\theta)\, dt\right]\,, \\ \{\partial_\rho Z_n\} &\to \left[(\rho,\theta) \mapsto \int_0^1 t\, m(t) \partial_\rho z(t\rho,\theta)\, dt\right] \end{aligned}$$

in $L^r(\mathcal{C})$. We deduce here that $Z_n$ is $W^{1,r}(\mathcal{C})$-Cauchy, and, consequently, it is $W^{1,r}(\mathcal{C})$-convergent. Thus, $Z \in W^{1,r}(\mathcal{C})$ and (2.18) holds.

□

We construct now the promised change of variables carrying $N-1$-dimensional spheres into level sets around nondegenerate critical points.

**Theorem 2.3.2.** *Let $\Omega$ be an open and bounded, regular subset of $\mathbb{R}^N$, and let $u \in C^2(\bar{\Omega})$ be given. We assume that*

*(a)* $u(\Omega) = 0\ \forall \Omega \in \partial\Omega$

*and, for some $\Omega_0 \in \Omega$,*

*(b)* $u(\Omega_0) = 1$

*(c)* $D^2 u(\Omega_0)$ *is strictly negative definite*

*(d)* $\langle \nabla u(\Omega), \Omega - \Omega_0\rangle < 0\ \forall \Omega \in \bar{\Omega},\ \Omega \neq \Omega_0$.

*Then, there exists an unique $C^2$-change of variables $\Gamma : \left]0, \sqrt{2}\right] \times \mathbb{S}^{N-1} \to \bar{\Omega}\backslash\{\Omega_0\}$ such that*

$$u(\Gamma(\rho,\theta)) = 1 - \frac{\rho^2}{2}\ \forall (\rho,\theta) \in \left]0, \sqrt{2}\right] \times \mathbb{S}^{N-1} \tag{2.21}$$

$$\Gamma(\rho,\theta) \in \Omega_0 + \mathbb{R}^+\theta\ \forall (\rho,\theta) \in \left]0, \sqrt{2}\right] \times \mathbb{S}^{N-1} \tag{2.22}$$

*Furthermore, $\Gamma \in C^1\left(\left[0, \sqrt{2}\right] \times \mathbb{S}^{N-1}\right)$ and the mappings*

$$\mathcal{F} : \left]0, \sqrt{2}\right] \times \mathbb{S}^{N-1} \to \mathcal{M}_{N\times(N-1)}(\mathbb{R})\,, \qquad (\rho,\theta) \mapsto \frac{\partial^2 \Gamma}{\partial\rho\partial\theta}(\rho,\theta) \tag{2.23}$$

$$\mathcal{G} : \left]0, \sqrt{2}\right] \times \mathbb{S}^{N-1} \to \mathbb{R}^N\,, \qquad (\rho,\theta) \mapsto \rho\frac{\partial^2 \Gamma}{\partial\rho^2}(\rho,\theta) \tag{2.24}$$

*are continuously extensible on* $\left[0,\sqrt{2}\right]\times\mathbb{S}^{N-1}$. *The following hold:*

$$\left\langle\frac{\partial\Gamma}{\partial\rho}(0,\theta),\theta\right\rangle>0,\qquad \frac{\partial\Gamma}{\partial\theta}(0,\theta)=0,\qquad \left(\frac{\partial\Gamma}{\partial\rho},\frac{\partial^2\Gamma}{\partial\rho\partial\theta}\right)(0,\theta)\ \ \textit{is inversible}\ \ \forall\theta\in\mathbb{S}^{N-1}. \tag{2.25}$$

*Finally, if, in addition* $u\in H^3(\Omega)$, *then* $\Gamma\in H^2\left(\left[0,\sqrt{2}\right]\times\mathbb{S}^{N-1}\right)$, *and both* $\mathcal{F}$ *and* $\mathcal{G}$ *are* $H^1$ *mappings on* $\left[0,\sqrt{2}\right]\times\mathbb{S}^{N-1}$.

*Proof.* We deduce from (a) and (d) that $\Omega$ is star-shaped with respect to $\Omega_0$ and

$$0\le u(\Omega)<1=u(\Omega_0)\ \forall\Omega\in\bar{\Omega},\ \Omega\neq\Omega_0\,.$$

Let us consider the set

$$\mathcal{C}:=\left\{(\rho,\theta)\in[0,+\infty[\times\mathbb{S}^{N-1}:\Omega_0+\rho\theta\in\bar{\Omega}\right\}.$$

It is a closed, bounded subset of the semi-infinite cylinder $[0,+\infty[\times\mathbb{S}^{N-1}$, and $\{0\}\times\mathbb{S}^{N-1}\subset\mathcal{C}$. Moreover, for any $\theta\in\mathbb{S}^{N-1}$,

$$\mathcal{C}_\theta:=\{\rho\in[0,+\infty[:(\rho,\theta)\in\mathcal{C}\}=\{\rho\in[0,+\infty[:\Omega_0+\rho\theta\in\bar{\Omega}\}$$

is a compact interval which starts at $0$.

We next consider the $C^2$ mapping

$$v:\mathcal{C}\to\mathbb{R},\qquad\qquad(\rho,\theta)\mapsto u(\Omega_0+\rho\theta)$$

We observe that, for any $\theta\in\mathbb{S}^{N-1}$, $v(\cdot,\theta):\mathcal{C}_\theta\to[0,1]$, $\rho\mapsto v(\rho,\theta)$ is strictly decreasing,

$$\partial_\rho v(\rho,\theta)=\langle\nabla u(\Omega_0+\rho\theta),\theta\rangle<0\ \forall\rho\in\mathcal{C}_\theta,\ \rho>0$$

and surjective. We call $w(\cdot,\theta):[0,1]\to\mathbb{R}$ its inverse, and the implicit function theorem says that

$$w:]0,1[\times\mathbb{S}^{N-1}\to\mathbb{R}$$

is a $C^2$ mapping. On the other hand, $w:[0,1]\times\mathbb{S}^{N-1}\to\mathbb{R}$ is continuously defined.

Now, conditions (2.21) and (2.22) can be rewritten in the form

$$\Gamma(\rho,\theta)=\Omega_0+k(\rho,\theta)\theta,\ \ (\rho,\theta)\in\left]0,\sqrt{2}\right]\times\mathbb{S}^{N-1} \tag{2.26}$$

where $k:\left]0,\sqrt{2}\right]\times\mathbb{S}^{N-1}\to\mathbb{R}_0^+$ should verify

$$k(\rho,\theta)\in\mathcal{C}_\theta,\qquad v(k(\rho,\theta),\theta)=1-\frac{\rho^2}{2},\qquad\forall\rho\in\left]0,\sqrt{2}\right],\ \theta\in\mathbb{S}^{N-1}, \tag{2.27}$$

that is,

$$k(\rho,\theta)=w\left(1-\frac{\rho^2}{2},\theta\right),\qquad(\rho,\theta)\in[0,\sqrt{2}]\times\mathbb{S}^{N-1} \tag{2.28}$$

At this point observe that, if $k$ is defined by (2.28), the mapping $\Gamma$ in (2.26) is a truly $C^2$ diffeomorphism from $\left]0,\sqrt{2}\right]\times\mathbb{S}^{N-1}$ into $\bar{\Omega}\backslash\{\Omega_0\}$, its inverse being given by

$$\bar{\Omega}\backslash\{\Omega_0\}\to\left]0,\sqrt{2}\right]\times\mathbb{S}^{N-1},\qquad\qquad\Omega\mapsto\left(\sqrt{2}\sqrt{1-u(\Omega)},\frac{\Omega-\Omega_0}{\|\Omega-\Omega_0\|}\right).$$

Further, $k\in C\big([0,\sqrt{2}]\times\mathbb{S}^{N-1}\big)$, and, consequently, $\Gamma(\rho,\theta)=\Omega_0+k(\rho,\theta)\theta$ itself is continuously defined on $[0,\sqrt{2}]\times\mathbb{S}^{N-1}$.

Next, let us show that $\Gamma$ is indeed continuously derivable on this set. Of course, it will suffice to check that $k\in C^1([0,\sqrt{2}]\times\mathbb{S}^{N-1})$. In order to do this, we will simply derivate in the equality

$$\sqrt{2}\sqrt{1-v(k(\rho,\theta),\theta)}=\rho\qquad\forall(\rho,\theta)\in\left]0,\sqrt{2}\right]\times\mathbb{S}^{N-1} \tag{2.29}$$

which follows from the definition of $k$ in (2.28). First of all, let us define

$$s:\mathcal{C}\to\mathbb{R}, \qquad (\rho,\theta)\mapsto\sqrt{1-v(\rho,\theta)}=\sqrt{1-u(\Omega_0+\rho\theta)}$$

Since $v\in C^2(\mathcal{C})$, $s\in C^2\big(\mathcal{C}\backslash(\{0\}\times\mathbb{S}^{N-1})\big)$. Straightforward computations show, for $(\rho,\theta)\in\mathcal{C},\ \rho\neq 0$,

$$\frac{\partial s}{\partial\rho}(\rho,\theta)=-\frac{\langle\nabla u(\Omega_0+\rho\theta),\theta\rangle}{2\sqrt{1-u(\Omega_0+\rho\theta)}}=-\frac{\left\langle\frac{\nabla u(\Omega_0+\rho\theta)}{\rho},\theta\right\rangle}{2\sqrt{\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}}} \tag{2.30}$$

$$\nabla_\theta s(\rho,\theta)=-\frac{\rho\;\Pi_\theta[\nabla u(\Omega_0+\rho\theta)]}{2\sqrt{1-u(\Omega_0+\rho\theta)}}=-\frac{\Pi_\theta[\nabla u(\Omega_0+\rho\theta)]}{2\sqrt{\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}}}=-\rho\frac{\Pi_\theta\left[\frac{\nabla u(\Omega_0+\rho\theta)}{\rho}\right]}{2\sqrt{\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}}} \tag{2.31}$$

being, for each $\theta\in\mathbb{S}^{N-1}$, $\Pi_\theta:\mathbb{R}^N\to\mathbb{R}^N$ the orthogonal projection onto $\langle\theta\rangle^\perp$, that is, $\Pi_\theta(y):=y-\langle y,\theta\rangle\theta$. We consider the mappings

$$\beta:\mathcal{C}\to\mathbb{R}^N; \qquad (\rho,\theta)\mapsto\frac{\nabla u(\Omega_0+\rho\theta)}{\rho}=\int_0^1 H^2u(\Omega_0+t\rho\theta)\theta\,dt \tag{2.32}$$

$$\alpha:\mathcal{C}\to\mathbb{R}; \qquad (\rho,\theta)\mapsto\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}=-\int_0^1 t\langle\beta(t\rho,\theta),\theta\rangle\,dt \tag{2.33}$$

which are, as a consequence of Lemma 2.3.1, continuous on $\mathcal{C}$. Furthermore, $\alpha(\rho,\theta)>0$ for every $(\rho,\theta)\in\mathcal{C}$, and, in case $u\in H^3(\Omega)$, $\beta\in H^1(\mathcal{C},\mathbb{R}^N)$ and $\alpha\in H^1(\mathcal{C})$. Now, (2.30) and (2.31) read

$$\frac{\partial s}{\partial\rho}(\rho,\theta)=-\frac{\langle\beta(\rho,\theta),\theta\rangle}{2\sqrt{\alpha(\rho,\theta)}} \tag{2.34}$$

$$\nabla_\theta s(\rho,\theta)=-\rho\frac{\Pi_\theta\big(\beta(\rho,\theta)\big)}{2\sqrt{\alpha(\rho,\theta)}} \tag{2.35}$$

for any $(\rho,\theta)\in\mathcal{C},\ \rho\neq 0$. We deduce that

$$s\in C^1(\mathcal{C}), \qquad \frac{\partial s}{\partial\rho}(\rho,\theta)>0\ \forall(\rho,\theta)\in\mathcal{C}\,,$$

and, since $\alpha$ is bounded away from 0, in case $u\in H^3(\Omega)$ we further have

$$s\in H^2(\mathcal{C})\,.$$

Thus, derivating in (2.29), which we may rewrite as

$$\sqrt{2}\,s(k(\rho,\theta),\theta)=\rho\,, \qquad (\rho,\theta)\in\,]0,\sqrt{2}]\times\mathbb{S}^{N-1},$$

we obtain

$$\frac{\partial k}{\partial\rho}(\rho,\theta)=\frac{1}{\sqrt{2}\;\frac{\partial s}{\partial\rho}(k(\rho,\theta),\theta)}\,, \qquad \nabla_\theta k(\rho,\theta)=-\frac{\nabla_\theta s(k(\rho,\theta),\theta)}{\frac{\partial s}{\partial\rho}(k(\rho,\theta),\theta)}\,, \qquad (\rho,\theta)\in\,]0,\sqrt{2}]\times\mathbb{S}^{N-1}. \tag{2.36}$$

Consequently,

$$k\in C^1\left([0,\sqrt{2}]\times\mathbb{S}^{N-1}\right), \qquad \frac{\partial k}{\partial\rho}(\rho,\theta)>0\ \forall(\rho,\theta)\in[0,\sqrt{2}[\,\times\mathbb{S}^{N-1}\,,$$

that is,

$$\Gamma\in C^1\left([0,\sqrt{2}]\times\mathbb{S}^{N-1},\mathbb{R}^N\right), \qquad \left\langle\frac{\partial\Gamma}{\partial\rho}(\rho,\theta),\theta\right\rangle>0\ \forall(\rho,\theta)\in[0,\sqrt{2}]\times\mathbb{S}^{N-1}\,,$$

and, since $\frac{\partial s}{\partial \rho}$ is bounded away from 0, in case $u \in H^3(\Omega)$, the function $k$ belongs to $H^2\left(\left[0,\sqrt{2}\right] \times \mathbb{S}^{N-1}\right)$ and we conclude

$$\Gamma \in H^2\left(\left]0,\sqrt{2}\right] \times \mathbb{S}^{N-1}, \mathbb{R}^N\right).$$

On the other hand, derivating in (2.30) and (2.31), for $(\rho,\theta) \in \mathcal{C}$, $\rho \neq 0$, we obtain

$$\begin{aligned}
\mathcal{L}(\rho,\theta) := \frac{\partial \nabla_\theta s}{\partial \rho}(\rho,\theta) = & \\
= -\frac{\Pi_\theta[\nabla u(\Omega_0+\rho\theta)] + \rho\Pi_\theta[D^2u(\Omega_0+\rho\theta)\theta]}{2\sqrt{1-u(\Omega_0+\rho\theta)}} & - \frac{\rho\langle \nabla u(\Omega_0+\rho\theta),\theta\rangle \Pi_\theta[\nabla u(\Omega_0+\rho\theta)]}{4[1-u(\Omega_0+\rho\theta)]^{3/2}} = \\
= -\frac{\Pi_\theta\left[\frac{\nabla u(\Omega_0+\rho\theta)}{\rho}\right] + \Pi_\theta[D^2u(\Omega_0+\rho\theta)\theta]}{2\sqrt{\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}}} & - \frac{\left\langle \frac{\nabla u(\Omega_0+\rho\theta)}{\rho},\theta\right\rangle \Pi_\theta\left[\frac{\nabla u(\Omega_0+\rho\theta)}{\rho}\right]}{4\left[\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}\right]^{3/2}} = \\
= -\frac{\Pi_\theta\big(\beta(\rho,\theta)\big) + \Pi_\theta[D^2u(\Omega_0+\rho\theta)\theta]}{2\sqrt{\alpha(\rho,\theta)}} & - \frac{\langle \beta(\rho,\theta),\theta\rangle\, \Pi_\theta\big(\beta(\rho,\theta)\big)}{4\,[\alpha(\rho,\theta)]^{3/2}} \qquad (2.37)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}(\rho,\theta) := \rho\frac{\partial^2 s}{\partial \rho^2}(\rho,\theta) = -\frac{\rho\langle D^2u(\Omega_0+\rho\theta)\theta,\theta\rangle}{2\sqrt{1-u(\Omega_0+\rho\theta)}} - \frac{\rho\langle \nabla u(\Omega_0+\rho\theta),\theta\rangle^2}{4[1-u(\Omega_0+\rho\theta)]^{3/2}} = \\
= -\frac{\langle D^2u(\Omega_0+\rho\theta)\theta,\theta\rangle}{2\sqrt{\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}}} - \frac{\left\langle \frac{\nabla u(\Omega_0+\rho\theta)}{\rho},\theta\right\rangle^2}{4\left[\frac{1-u(\Omega_0+\rho\theta)}{\rho^2}\right]^{3/2}} = -\frac{\langle D^2u(\Omega_0+\rho\theta)\theta,\theta\rangle}{2\sqrt{\alpha(\rho,\theta)}} - \frac{\langle\beta(\rho,\theta),\theta\rangle^2}{4\,[\alpha(\rho,\theta)]^{3/2}} \qquad (2.38)
\end{aligned}$$

where $\beta$ and $\alpha$ are the mappings defined in (2.32) and (2.33) respectively. Thus, both $\mathcal{L}$ and $\mathcal{M}$ are continuously defined on $\mathcal{C}$ and, in case $u \in H^3(\Omega)$, these functions are in $H^1(\mathcal{C})$. Derivating again in (2.36), we find

$$\begin{aligned}
\frac{\partial \nabla_\theta k}{\partial \rho}(\rho,\theta) = \frac{\partial k}{\partial \rho}(\rho,\theta)\frac{\left[k(\rho,\theta)\frac{\partial^2 s}{\partial \rho^2}(k(\rho,\theta),\theta)\right]\frac{\nabla_\theta s(k(\rho,\theta),\theta)}{k(\rho,\theta)} - \frac{\partial s}{\partial \rho}(k(\rho,\theta),\theta)\frac{\partial \nabla_\theta s}{\partial \rho}(k(\rho,\theta),\theta)}{\frac{\partial s}{\partial \rho}(k(\rho,\theta),\theta)^2} = \\
= \frac{\partial k}{\partial \rho}(\rho,\theta)\frac{\mathcal{M}\big(k(\rho,\theta),\theta\big)\int_0^1 \mathcal{L}\big(t\,k(\rho,\theta),\theta\big)\,dt - \frac{\partial s}{\partial \rho}\big(k(\rho,\theta),\theta\big)\mathcal{L}\big(k(\rho,\theta),\theta\big)}{\frac{\partial s}{\partial \rho}(k(\rho,\theta),\theta)^2} \qquad (2.39)
\end{aligned}$$

$$\rho\frac{\partial^2 k}{\partial \rho^2}(\rho,\theta) = -\frac{\frac{\rho}{k(\rho,\theta)}\left[k(\rho,\theta)\frac{\partial^2 s}{\partial \rho^2}(k(\rho,\theta),\theta)\right]}{2\frac{\partial s}{\partial \rho}\big(k(\rho,\theta),\theta\big)^3} = -\frac{\mathcal{M}\big(k(\rho,\theta),\theta\big)}{2\frac{\partial s}{\partial \rho}\big(k(\rho,\theta),\theta\big)^3\int_0^1 \frac{\partial k}{\partial \rho}(t\rho,\theta)\,dt} \qquad (2.40)$$

and these functions are continuously defined on $\left[0,\sqrt{2}\right] \times \mathbb{S}^{N-1}$. On the other hand, in case $H^3(\Omega)$, it follows from expressions (2.39) and (2.40) above, together with Lemma 2.3.1 and the fact that both $\frac{\partial s}{\partial \rho}$ and $\frac{\partial k}{\partial \rho}$ are bounded away from zero, that both $(\rho,\theta) \mapsto \rho\frac{\partial^2 k}{\partial \rho^2}(\rho,\theta)$, $(\rho,\theta) \mapsto \frac{\partial \nabla_\theta k}{\partial \rho}(\rho,\theta)$ are $H^1$ mappings on $\left[0,\sqrt{2}\right] \times \mathbb{S}^{N-1}$. Thus, the definitions of $\mathcal{F}$ and $\mathcal{G}$ at (2.23) and (2.24) imply that these functions may be continuously extended to $\left[0,\sqrt{2}\right] \times \mathbb{S}^{N-1}$ and, in case $u \in H^3(\Omega)$, they belong to $H^1\left(\left[0,\sqrt{2}\right] \times \mathbb{S}^{N-1}\right)$.

It remains to check (2.25). To do this, simply observe that

$$\left\langle \frac{\partial \Gamma}{\partial \rho}(0,\theta),\theta\right\rangle = \left\langle \frac{\partial k}{\partial \rho}(\rho,\theta)\theta,\theta\right\rangle = \frac{\partial k}{\partial \rho}(\rho,\theta) > 0 \quad \forall(\rho,\theta) \in [0,\epsilon] \times \mathbb{S}^{N-1},$$

$$\frac{\partial \Gamma}{\partial \theta}(\rho,\theta)w = \left(\frac{\partial k}{\partial \theta}(\rho,\theta)w\right)\theta + k(\rho,\theta)w \quad \forall(\rho,\theta) \in [0,\epsilon] \times \mathbb{S}^{N-1}, \forall w \in \mathbb{R}^N \text{ with } \langle w,\theta\rangle = 0,$$

$$\frac{\partial^2\Gamma}{\partial\rho\partial\theta}(0,\theta)w = \left(\frac{\partial^2 k}{\partial\rho\partial\theta}(0,\theta)w\right)\theta + \frac{\partial k}{\partial\rho}(0,\theta)w \quad \forall\theta\in\mathbb{S}^{N-1}, \forall w\in\mathbb{R}^N \text{ with } \langle w,\theta\rangle = 0\,,$$

and, consequently,

$$det\left(\frac{\partial\Gamma}{\partial\rho}(0,\theta), \frac{\partial\Gamma}{\partial\theta}(0,\theta)\right) = \frac{\partial k}{\partial\rho}(0,\theta)^N > 0 \quad \forall\theta\in\mathbb{S}^{N-1}\,.$$

The Theorem is complete. □

At this stage, we want to explore the continuity of the mapping $u\mapsto\Gamma=\Gamma_u$. Thus, assume that we have a sequence $\{u_n\}\to u$ in $C^2(\bar\Omega)$. Assume also that $\max_\Omega u_n = 1\ \forall n\in\mathbb{N}$, that $u$ verifies the hypothesis *(a), (b), (c)* of Lemma 2.3.2 and that $u(\Omega)<1\ \forall\Omega\in\Omega$ with $\Omega\neq\Omega_0$. Then, *(d)* may not hold, but it is possible to find some $0<\epsilon<1$ such that

$$\langle\nabla u(\Omega), \Omega-\Omega_0\rangle < 0 \quad \forall\Omega\in\Omega\backslash\{\Omega_0\} \text{ with } u(\Omega)\geq 1-\frac{\epsilon^2}{2}\,.$$

and thus, we have an associated $C^2$-diffeomorphism

$$\Gamma: ]0,\epsilon[\times\mathbb{S}^{N-1}\to\tilde\Omega\backslash\{\Omega_0\} \qquad\qquad \text{where } \tilde\Omega := \left\{\Omega\in\Omega : u(\Omega) > 1-\frac{\epsilon^2}{2}\right\}\,,$$

which is built in the following form: for any $(\rho,\theta)\in]0,\epsilon[\times\mathbb{S}^{N-1}$, let $\Gamma(\rho,\theta)$ be the only point $z\in\tilde\Omega\cap(\Omega_0+\mathbb{R}^+\theta)$ such that $u(z)=1-\frac{\rho^2}{2}$. For any $n$, denote $\tilde\Omega_n := \{\Omega\in\Omega : u_n(\Omega) > 1-\frac{\epsilon^2}{2}\}$. If $n$ is big enough, $u_n$ attains the value 1 only at one single point $\Omega_n$, which belongs to $\tilde\Omega_n$, and the sequence $\{\Omega_n\}$ converges to $\Omega_0$. Indeed, for big indexes $n$, $D^2u_n(\Omega_n)$ will be negative definite and $\langle\nabla u_n(\Omega), \Omega-\Omega_n\rangle < 0\ \forall\Omega\in\tilde\Omega_n$, $\Omega\neq\Omega_n$. Thus, we may also consider, for each $n\in\mathbb{N}$ big enough, the associated $C^2$-diffeomorphism $\Gamma_n: ]0,\epsilon[\times\mathbb{S}^{N-1}\to\tilde\Omega_n\backslash\{\Omega_n\}$, together with the related mappings $\mathcal{F}_n : [0,\epsilon]\times\mathbb{S}^{N-1}\to\mathcal{M}_{N\times(N-1)}(\mathbb{R})$ and $\mathcal{G}_n : [0,\epsilon]\times\mathbb{S}^{N-1}\to\mathbb{R}^N$; $\mathcal{F} : [0,\epsilon]\times\mathbb{S}^{N-1}\to\mathcal{M}_{N\times(N-1)}(\mathbb{R})$ and $\mathcal{G} : [0,\epsilon]\times\mathbb{S}^{N-1}\to\mathbb{R}^N$. We arrive at the following continuity result:

**Corollary 2.3.3.** *Under the assumptions above, $\{\Gamma_n\}\to\Gamma$ in $C^2(K)$ for any compact set $K\subset\left]0,\epsilon\right]\times\mathbb{S}^{N-1}$. The sequence converges indeed in the $C^1\left(\left[0,\epsilon\right]\times\mathbb{S}^{N-1}\right)$ topology and, moreover,*

$$\Big\{\mathcal{F}_n\Big\}_n\to\mathcal{F}, \qquad\qquad \Big\{\mathcal{G}_n\Big\}_n\to\mathcal{G} \tag{2.41}$$

*uniformly on $[0,\epsilon]\times\mathbb{S}^{N-1}$. Finally, if $\{u_n\}$ is contained and bounded in $H^3(\Omega)$, the sequence $\{\Gamma_n\}_n$ is contained and bounded in $H^2([0,\epsilon]\times\mathbb{S}^{N-1})$, while $\{\mathcal{F}_n\}_n$ and $\{\mathcal{G}_n\}_n$ are contained and bounded in $H^1([0,\epsilon]\times\mathbb{S}^{N-1})$. In case $\{u_n\}\to u$ in $H^3(\Omega)$, $\{\Gamma_n\}\to\Gamma$ in $H^2([0,\epsilon]\times\mathbb{S}^{N-1})$, while $\mathcal{F}_n\to\mathcal{F}$ and $\mathcal{G}_n\to\mathcal{G}$ in $H^1([0,\epsilon]\times\mathbb{S}^{N-1})$.*

This result follows from our proof of Lemma 2.3.2. All objects we constructed there depended continuously on $u$ in the adequate topologies.

□

## 2.4 From integrals on the domain $\Omega$ to one-dimensional integrals

At this stage, we plan to use the results obtained in the previous section in orden to continue the work initiated in Section 2.2 and rewrite both the sequence $\{p_n\}$ and its limit $p$ considered there in a more convenient form. This procedure will likely provide further results on the convergence of $\{p_n\}$ to $p$. As established in Sections 2.1 and 2.2, we call $\varphi$ the first eigenfunction of $-\Delta$ when acting on $H_0^1(\Omega)$, we choose some $\tilde h\in\widetilde{Lip}(\Omega)$, and we denote by $\tilde u_{\tilde h}$ the only solution in $\tilde H_0^1(\Omega)\cap C^1(\bar\Omega)\cap H^3(\Omega)$ of the linear problem $-\Delta u-\lambda_1 u=\tilde h$. We choose a sequence $\{\bar v_n\}$ of real numbers with $|\bar v_n|\geq 1\ \forall n\in\mathbb{N}$ and an arbitrary bounded sequence $\{\tilde v_n\}\subset\tilde H_0^1(\Omega)\cap C^2(\bar\Omega)$. It will be assumed that $\{\tilde v_n\}\to 0$ in $C^1(\bar\Omega)$, and that the related sequence $\{\varphi_n\}_n$ defined by expressions (2.8) and (2.9) converges to $\varphi$ in $C^2(\bar\Omega)$, while it is contained and bounded in $H^3(\Omega)$.

Since our hypothesis $[\mathbf{C}_2]$ guarantees that the eigenfunction $\varphi$ has an unique critical point $\Omega_0$ and its second derivative is not degenerate there, as detailed at the end of last section it is possible to find $\epsilon > 0$ such that

$$\langle \nabla\varphi(\Omega), \Omega - \Omega_0\rangle < 0 \ \forall \Omega \in \Omega \text{ with } \Omega \neq \Omega_0 \text{ and } \varphi(\Omega) \geq 1 - \frac{\epsilon^2}{2} .$$

If $n$ is big enough, also the maximum of $\varphi_n$ is attained at a single point $\Omega_n$ and $\langle \nabla\varphi_n(\Omega_n), \Omega - \Omega_n\rangle < 0 \ \forall \Omega \in \Omega\backslash\{\Omega_n\}$ with $\varphi_n(\Omega) \geq 1 - \frac{\epsilon^2}{2}$. After possibly skipping a finite number of terms of the sequence, it will be assumed that these things happen for all $n \in \mathbb{N}$. Using the change of variables theorem we obtain, for $1 - \frac{\epsilon^2}{2} < t < 1$,

$$p_n(t) = \frac{1}{\sqrt{2-2t}} \int_{\mathbb{S}^{N-1}} \left| J\Gamma_n\left(\sqrt{2-2t}, \theta\right)\right| ds_\theta, \quad p(t) = \frac{1}{\sqrt{2-2t}} \int_{\mathbb{S}^{N-1}} \left| J\Gamma\left(\sqrt{2-2t}, \theta\right)\right| ds_\theta$$

being $\Gamma_n, \Gamma : [0,\epsilon] \times \mathbb{S}^{N-1} \to \Omega \subset \mathbb{R}^N$ the changes of variables associated to $\varphi_n$ and $\varphi$ respectively and $J\Gamma_n, \ J\Gamma$ their Jacobian determinants. This leads us to consider the sequence $\{\sigma_n\}_n$ of $C^1[0, \epsilon/\sqrt{2}]$ functions defined by

$$\sigma_n(\rho) := \frac{1}{\sqrt{2}} \int_{\mathbb{S}^{N-1}} \left| J\Gamma_n\left(\sqrt{2}\,\rho, \theta\right)\right| ds_\theta, \quad 0 \leq \rho \leq \epsilon/\sqrt{2}\,, \tag{2.42}$$

and whose relation with $\{p_n\}$ is given by:

$$p_n(t) = \frac{\sigma_n\big(\sqrt{1-t}\big)}{\sqrt{1-t}}\,, \qquad t \in \left]1 - \frac{\epsilon^2}{2}, 1\right[ \tag{2.43}$$

Expression (2.43) above suggests how to extend each function $\sigma_n$ to a $C^1[0,1]$ function. We define

$$\sigma_n(\rho) := \rho\, p_n(1-\rho^2)\,, \qquad 0 < \rho < 1\,,$$

so that

$$p_n(t) = \frac{\sigma_n(\sqrt{1-t})}{\sqrt{1-t}}\,, \qquad 0 < t < 1\,.$$

Thus, (2.43) may be rephrased by saying that the new definition of the sequence $\{\sigma_n\}$ agrees with the old one on $[0,\epsilon]$. It follows now from Lemma 2.2.2 together with (2.41) in Corollary 2.3.3 that $\{\sigma_n\}$ converges in $C^1[0,1]$ to the $H^2[0,1]$ function $\sigma$ defined by

$$\sigma(\rho) := \rho p(1-\rho^2)\,, \qquad 0 \leq \rho \leq 1\,.$$

Moreover, $\{\sigma_n\}$ is contained and bounded in $H^2[0,1]$.

At this point, we are ready to study the asymptotic behavior of sequences of the kind of $\Upsilon_{\tilde{h}}(\bar{u}_n, \tilde{u}_n)$ that we considered in Section 2.2. We remember from (2.10) that

$$\Upsilon_{\tilde{h}}(\bar{u}_n, \tilde{u}_n) = \int_0^1 G_1(d_n t) p_n(t) dt\,.$$

Of course, the Riemann-Lebesgue Lemma implies $\Upsilon_{\tilde{h}}(\bar{u}_n, \tilde{u}_n) \to 0$ as $n \to \infty$. Thus, in orden to obtain some information on the sign of $\Upsilon_{\tilde{h}}(\bar{u}_n, \tilde{u}_n)$ for $n$ big, we must take care of higher terms in the asymptotic expansion of this function around infinity. By writing the Jacobian determinants as finite sums of finite products, we see that

$$0 < \lim_{\rho\to 0} \frac{\sigma_n(\rho)}{\rho^{N-1}} < +\infty \ \forall n \in \mathbb{N}\,; \qquad 0 < \lim_{\rho\to 0} \frac{\sigma(\rho)}{\rho^{N-1}} < +\infty. \tag{2.44}$$

In particular, since $N \geq 2$, $\sigma_n(0) = 0 \ \ \forall n \in \mathbb{N}$. It motivates the Lemma below.

**Lemma 2.4.1.** *Let $g \in C(\mathbb{R}/\mathbb{Z})$ have zero mean, and let $\{\xi_n\} \subset H^2[0,1]$ be a bounded sequence with $\xi_n(0) = 0 \ \forall n \in \mathbb{N}$. We assume that it converges in $C^1[0,1]$ to some $C^1[0,1]$ function $\xi$. Let, finally, $\{d_n\}$ be a sequence of real numbers with $|d_n| \to \infty$. Then,*

$$\lim_{n\to\infty} \left[ d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt - G_1(d_n)\xi'(0) + G_1(0)\xi(1) \right] = 0\,,$$

*being $G_1$ the primitive of $g$ with zero mean.*

*Proof.* Integration by parts gives

$$d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt - G_1(d_n)\xi_n'(0) + G_1(0)\xi_n(1) = -\frac{1}{2}\int_0^1 G_1(d_n t) \frac{\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} - \xi_n'(\sqrt{1-t})}{1-t}\, dt\,.$$

We consider the sequence $\{\zeta_n\} \subset L^1[0,1]$ defined by the rule

$$\zeta_n(x) = \frac{\frac{\xi_n(x)}{x} - \xi_n'(x)}{x} = \int_0^1 \left[ t \int_0^1 \xi_n''(stx)\, ds \right] dt - \int_0^1 \xi_n''(tx)\, dt\,; \quad 0 < x < 1\,,$$

so that

$$-\int_0^1 G_1(d_n t) \frac{\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} - \xi_n'(\sqrt{1-t})}{1-t}\, dt = -\int_0^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt\,.$$

Observe that $\{\zeta_n\} \to 0$ uniformly on $[\epsilon, 1]$ for any $0 < \epsilon < 1$. Further, we have

$$|\zeta_n(x)| \le \int_0^1 t \int_0^1 \left|\xi_n''(stx)\right| ds\, dt - \int_0^1 \left|\xi_n''(tx)\right| dt \le \int_0^1 \int_0^1 \left|\xi_n''(stx)\right| ds\, dt - \int_0^1 \left|\xi_n''(tx)\right| dt$$

Thus, Hardy's inequality (see, for instance, [72], pp. 72), implies that $\{\zeta_n\}$ is, indeed, a bounded sequence in $L^2[0,1]$. Now, given any $0 < a < 1$ which we momentarily fix, we have

$$\left| \int_0^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt \right| \le \left| \int_0^a G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt \right| + \left| \int_a^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt \right|\,.$$

The first term in the sum above converges to 0 as $n \to \infty$. This is a consequence of the Riemann-Lebesgue Lemma (see [85]). Concerning the second, we have

$$\left| \int_a^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt \right| \le \int_a^1 \frac{|G_1(d_n t)|}{(1-t)^{1/4}} \frac{|\zeta_n(\sqrt{1-t})|}{(1-t)^{1/4}}\, dt \le$$

$$\le \sqrt{\int_a^1 \frac{G_1(d_n t)^2}{\sqrt{1-t}}\, dt}\, \sqrt{\int_a^1 \frac{\zeta_n(\sqrt{1-t})^2}{\sqrt{1-t}}\, dt} \le \|G_1\|_\infty \sqrt{\int_a^1 \frac{1}{\sqrt{1-t}}\, dt}\, \sqrt{\int_{\sqrt{1-a}}^1 \zeta_n(x)^2\, dx} \le$$

$$\le \|G_1\|_\infty \sqrt{2}\sqrt[4]{1-a}\, \|\zeta_n\|_{L^2[0,1]}$$

and we obtain

$$\limsup_{n\to\infty} \left| \int_0^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt \right| \le \|G_1\|_\infty \sqrt{2}\sqrt[4]{1-a}\, \sup_n \|\zeta_n\|_{L^2[0,1]}\,.$$

Since it is valid for any $0 < a < 1$, we conclude

$$\lim_{n\to\infty} \int_0^1 G_1(d_n t) \frac{\zeta_n(\sqrt{1-t})}{\sqrt{1-t}}\, dt = 0\,,$$

proving the lemma.

□

## 2.5 Does the action functional attain its minimum?

We are now ready to start to obtain consequences of the work carried out in previous sections. Under the framework established in Section 2.4, we choose a divergent sequence $\{\bar{v}_n\}$ of real numbers,

$$|\bar{v}_n| \to \infty\,,$$

and we take $\tilde{v}_n := 0\ \forall n \in \mathbb{N}$. Thus, the related sequence $\{\varphi_n\}_n$, defined in expressions (2.8) and (2.9), converges to $\varphi$ both in $C^2(\bar{\Omega})$ and $H^3(\Omega)$, and it implies that the related sequence $\{\sigma_n\}$, defined as in (2.42), converges to $\sigma$ both in $C^1[0,1]$ and $H^2[0,1]$. Using Lemma 2.4.1 we deduce that

$$\lim_{n\to\infty}\Big[\bar{v}_n\Upsilon_{\tilde{h}}(\bar{v}_n,0) - G_2(\bar{v}_n+\tilde{u}_{\tilde{h}}(\Omega_0))\sigma'(0) + G_2(0)\sigma(1)\Big] =$$
$$= \lim_{n\to\infty}\Big[\bar{v}_n\int_\Omega G_1(\bar{v}_n\varphi+\tilde{u}_{\tilde{h}})dx - G_2(\bar{v}_n+\tilde{u}_{\tilde{h}}(\Omega_0))\sigma'(0) + G_2(0)\sigma(1)\Big] = 0\,, \quad (2.45)$$

and thus, if the sequence $\bar{v}_n$ is taken in such a way that $G_2(\bar{v}_n+\tilde{u}_{\tilde{h}}(\Omega_0)) = 0\ \forall n \in \mathbb{N}$, we really have

$$\lim_{n\to\infty}\bar{v}_n\Upsilon_{\tilde{h}}(\bar{v}_n,0) = -G_2(0)\sigma(1)\,. \quad (2.46)$$

Of course, for any dimension $N \in \mathbb{N}$, we have

$$\sigma(1) = p(0) = \int_{\partial\Omega}\frac{1}{\|\nabla\varphi(x)\|}ds_x \in ]0,\infty[\,,$$

and we conclude

**Theorem 2.5.1.** *Assume that $G_2(0) \neq 0$, and take any $\tilde{h}:\Omega\to\mathbb{R}$ Lipschitz with*

$$\int_\Omega \tilde{h}(x)\varphi(x)dx = 0\,,$$

*and let $\bar{h}=0$, $h=\tilde{h}$. Then, the action functional $\Phi_h$ in (2.4) attains its global minimum in $H_0^1(\Omega)$, which is negative.*

*Proof.* Expression (2.46) implies in particular that, for $n$ big, $\Upsilon_{\tilde{h}}(\bar{v}_n,0)$ has the same sign as $-G_2(0)\sigma(1)$ if $\bar{v}_n\to+\infty$ and the opposite if $\bar{v}_n\to-\infty$. The theorem follows.

□

In the case $N=2$, (2.44) means that $\sigma'(0)\neq 0$. We immediately conclude:

**Theorem 2.5.2.** *Assume*

$$N = 2\,.$$

*Take any $\tilde{h}:\Omega\to\mathbb{R}$ Lipschitz with*

$$\int_\Omega \tilde{h}(x)\varphi(x)dx = 0\,,$$

*and let $\bar{h}=0$, $h=\tilde{h}$. Then, the action functional in (2.4) attains its global minimum in $H_0^1(\Omega)$. This minimum is strictly negative.*

*Proof.* In case $G_2(0)\neq 0$, the thesis is given by Theorem 2.5.1. In case $G_2(0)=0$, it follows from (2.45) that

$$\lim_{n\to\infty}\Big[\bar{u}_n\Upsilon_{\tilde{h}}(\bar{u}_n,0) - G_2(\bar{u}_n+\tilde{u}_{\tilde{h}}(\Omega_0))\sigma'(0)\Big] = 0\,.$$

The result follows by taking the sequence $\{\bar{u}_n\}\to\infty$ with $G_2(\bar{u}_n+\tilde{u}_{\tilde{h}}(\Omega_0)) = \min_{\mathbb{R}} G_2 < 0\ \forall n\in\mathbb{N}$.

□

In case $N \geq 3$, at least for $n$ big enough, $p_n \in W^{1,1}[0,1]$, and $p_n(1) = 0 \ \forall n \in \mathbb{N}$. Thus, if $G_2(0) = 0$, integration by parts gives

$$d_n \Upsilon_{\tilde{h}}(\bar{u}_n + \tilde{u}_n) = d_n \int_0^1 G_1(d_n t) p_n(t) dt = d_n \int_0^1 G_1(d_n t) \frac{\sigma_n(\sqrt{1-t})}{\sqrt{1-t}} dt =$$
$$= \int_0^1 G_2(d_n t) \frac{\sigma_n'(\sqrt{1-t})/\sqrt{1-t} - \sigma_n(\sqrt{1-t})/(1-t)}{\sqrt{1-t}} \, dt \quad (2.47)$$

This leads us to consider the sequence $\{\Theta_n\}_n$ of $L^2[0,1]$ mappings defined by

$$\Theta_n(\rho) := \frac{\sigma_n'(\rho)}{\rho} - \frac{\sigma_n(\rho)}{\rho^2}, \qquad 0 < \rho < 1,$$

so that

$$\bar{u}_n \Upsilon_{\tilde{h}}(\bar{u}_n + \tilde{u}_n) = \int_0^1 G_2(d_n t) \frac{\Theta_n(\sqrt{1-t})}{\sqrt{1-t}}$$

It follows from the expression of the Jacobian determinant as sum of products of partial derivatives, Corollary 2.3.3 and (2.42) that the sequence $\Theta_n$ is indeed contained and bounded in $H^1[0,1]$. Further, it converges uniformly to the $H^1[0,1]$ function

$$\Theta(\rho) := \frac{\sigma'(\rho)}{\rho} - \frac{\sigma(\rho)}{\rho^2}, \qquad 0 < \rho < 1.$$

On the other hand, using (2.44) we find that, if $N = 3$,

$$\Theta(0) > 0.$$

All this motivates the proposition below

**Proposition 2.5.3.** *Let $g \in C(\mathbb{R}/T\mathbb{Z})$, $g \not\equiv 0$, have zero mean, and let $\{\xi_n\} \subset H^1[0,1]$ be a bounded sequence. We assume that $\{\xi_n\}$ converges uniformly to some function $\xi \in H^1[0,1]$ with $\xi(0) > 0$. We take a sequence $\{d_n\}_n$ with $\{d_n\} \to +\infty$ and $G_1(d_n) = \max_{\mathbb{R}} G_1$ for any $n \in \mathbb{N}$, $G_1$ being a primitive of $g$. Then,*

$$\lim_{n\to\infty} \left\{ d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} dt \right\} = +\infty. \qquad (2.48)$$

*Proof.* Integrating by parts we obtain:

$$d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} dt = [\max_{\mathbb{R}} G_1 - G_1(0)]\xi_n(0) + \int_0^1 [\max_{\mathbb{R}} G_1 - G_1(d_n t)] \frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right] dt.$$

We observe that

$$\frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right] = \frac{\xi_n(\sqrt{1-t}) - \sqrt{1-t}\,\xi_n'(\sqrt{1-t})}{2[1-t]^{\frac{3}{2}}}$$

for any $0 \leq t < 1$ and all $n \in \mathbb{N}$. We choose $K_0 > 0$, $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\xi_n\big(\sqrt{1-t}\big) \geq K_0 \ \forall t \in [a,1].$$

For $n \geq n_0$, we define

$$A_n := \left\{ t \in ]a,1[ : \xi_n\big(\sqrt{1-t}\big) - \sqrt{1-t}\,\xi_n'\big(\sqrt{1-t}\big) < 0 \right\} = \left\{ t \in ]a,1[ : \frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right] < 0 \right\},$$

and

$$B_n := \left\{ \sqrt{1-t} : t \in A_n \right\} \subset ]0,1[ \ \forall n \in \mathbb{N}.$$

Observe that $\xi_n'(x) > \frac{K_0}{x}$ $\forall x \in B_n$, $\forall n \geq n_0$. Since $\{\xi_n'\}$ is bounded in $L^2[0,1]$, it implies that the sequence of real numbers

$$\int_{B_n} \frac{1}{x^2}\,dx\,, \quad n \in \mathbb{N}$$

is bounded. Thus, the sequence

$$\int_{A_n} \left| \frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right]\right| dt = \int_{A_n} \left| \frac{\xi_n(\sqrt{1-t}) - \sqrt{1-t}\,\xi_n'(\sqrt{1-t})}{2[1-t]^{\frac{3}{2}}}\,dt \right| =$$

$$= \int_{B_n} \left| \frac{\xi_n(x) - x\,\xi_n'(x)}{x^2} \right| dx \leq \left(\max_{[0,1]} |\xi_n|\right) \int_{B_n} \frac{1}{x^2}\,dx + \sqrt{\int_{B_n} \frac{1}{x^2}\,dx}\sqrt{\int_0^1 \xi_n'(x)^2\,dx}\,. \quad (2.49)$$

is also bounded. In another words, it is possible to find a constant $0 < C < \infty$ such that

$$\int_{A_n} \left| \frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right]\right| dt \leq C \qquad \forall n \in \mathbb{N}, \quad n \geq n_0\,,$$

and we deduce that, for $n \geq n_0$,

$$\int_a^1 [\max_{\mathbb{R}} G_1 - G_1(d_n t)] \frac{d}{dt}\left(\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right) dt \geq -2\|G\|_\infty \int_{A_n} \left| \frac{d}{dt}\left[\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right]\right| dt \geq -2\|G\|_\infty C$$

is bounded by below. On the other hand,

$$\int_0^a [\max_{\mathbb{R}} G_1 - G_1(d_n t)] \frac{d}{dt}\left(\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right) dt =$$

$$= \max_{\mathbb{R}} G_1 \left(\frac{\xi_n(\sqrt{1-a})}{\sqrt{1-a}} - \xi_n(0)\right) - \int_0^a G_1(d_n t) \frac{d}{dt}\left(\frac{\xi_n(t)}{\sqrt{1-t}}\right) dt$$

Thus, by the Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty} \int_0^a [\max_{\mathbb{R}} G_1 - G_1(d_n t)] \frac{d}{dt}\left(\frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}}\right) dt = \max_{\mathbb{R}} G_1 \left(\frac{h(\sqrt{1-a})}{\sqrt{1-a}} - h(0)\right)$$

and consequently,

$$\liminf_{n\to\infty} \left\{ d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} dt \right\} \geq [\max_{\mathbb{R}} G_1 - G_1(0)]h(0) - 2\|G\|_\infty C + \max_{\mathbb{R}} G_1 \left(\frac{h(\sqrt{1-a})}{\sqrt{1-a}} - h(0)\right)$$

so that, letting $a \to 1$,

$$\lim_{n\to\infty} \left\{ d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} dt \right\} = +\infty\,,$$

as stated.

□

**Note 2.5.4.** *By replacing $g$ by $-g$, we observe that, in case the sequence $\{d_n\}$ is chosen in such a way that $G_1(d_n) = \min_{\mathbb{R}} G_1$ $\forall n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} d_n \int_0^1 g(d_n t) \frac{\xi_n(\sqrt{1-t})}{\sqrt{1-t}} dt = -\infty\,.$$

We immediately conclude:

**Corollary 2.5.5.** *Assume $N = 3$. Then, the action functional in (2.4) attains its global minimum in $H_0^1(\Omega)$. This minimum is strictly negative.*

*Proof.* It is an immediate consequence of our study above: In case $G_2(0) \neq 0$ it follows from Theorem 2.5.1 and, otherwise, from Proposition 2.5.3 and the preceding discussion. $\square$

Derivate now in (2.43) to find

$$p_n'(t) = \frac{\frac{\sigma_n(\sqrt{1-t})}{\sqrt{1-t}} - \sigma_n'(\sqrt{1-t})}{2(1-t)}\,, \qquad 0 < t < 1\,. \tag{2.50}$$

Thus, from (2.42) and the expression of the determinant as finite sum of finite products, we see that, if $N \geq 3$, $\{p_n'\} \to p'$ in $L^1[0,1]$. This fact was already stated in [74]. Derivating again in (2.50) we obtain

$$p_n''(t) = \frac{2\frac{\sigma_n(\sqrt{1-t})}{\sqrt{1-t}} - 2\sigma_n'(\sqrt{1-t}) + \sigma_n''(\sqrt{1-t})\sqrt{1-t}}{(1-t)^2}\,, \qquad 0 < t < 1\,, \tag{2.51}$$

and, again thanks to (2.42) and the expression of the determinant as finite sum of finite products, we see that, since $\sigma_n \to \sigma$ in $H^2[0,1]$, in case $N \geq 4$, the sequence $\{p_n\}$ converges to $p$ in $W^{2,1}[0,1]$.

First of all, let us recall (2.16) in order to estimate the value $p'(0)$. We have

$$p'(0) = \int_{\partial\Omega} \frac{\left|J\Upsilon(x,0)\right|}{\left\|\nabla\varphi(x)\right\|^2}\left[\Delta\varphi(x) - 2\frac{\nabla\varphi(x)^T H^2\varphi(x)\nabla\varphi(x)}{\left\|\nabla\varphi(x)\right\|^2}\right] ds_x = -2\int_{\partial\Omega} \frac{\partial^2\varphi/\partial\nu^2(\theta)}{\|\nabla\varphi(\theta)\|^3} ds_\theta$$

being $\partial^2\varphi/\partial\nu^2$ the second derivative of $\varphi$ with respect to the unit normal of $\partial\Omega$.

However, in case $\Omega$ is convex,

$$\int_{\partial\Omega} \frac{\partial^2\varphi/\partial\nu^2(\theta)}{\|\nabla\varphi(\theta)\|^3} ds_\theta > 0\,. \tag{2.52}$$

To see this, fix an arbitrary point $\theta_0 \in \partial\Omega$, a tangent vector $w_0 \in T_\theta(\partial\Omega)$ and a $C^2$ curve $\gamma_0 : ]-1,1[ \to \partial\Omega$ with $\gamma(0) = \theta_0$, $\gamma_0'(0) = w_0$. The convexity of $\Omega$ implies that

$$\langle \nabla\varphi(\theta_0), x - \theta_0\rangle \geq 0 \quad \forall x \in \bar\Omega\,,$$

and, consequently,

$$\langle \nabla\varphi(\theta_0), \gamma(t) - \theta_0\rangle \geq 0 \quad \forall t \in ]-1,1[\,.$$

However,

$$\langle \nabla\varphi(\theta_0), \gamma_0(0) - \theta_0\rangle = \langle \nabla\varphi(\theta_0), 0\rangle = 0 = \langle \nabla\varphi(\theta_0), w_0\rangle = \langle \nabla\varphi(\theta_0), \gamma_0'(0)\rangle\,,$$

which implies that

$$\langle \nabla\varphi(\theta_0), \gamma_0''(0)\rangle \geq 0\,.$$

On the other hand, derivating twice in the equality

$$\varphi(\gamma(t)) = 0\,,$$

which holds for any $t \in ]-1,1[$, we find

$$w_0^T D^2\varphi(\theta)w_0 + \langle \nabla\varphi(\theta_0), \gamma_0''(0)\rangle = 0$$

or, what is the same,

$$w_0^T D^2\varphi(\theta_0)w_0 = -\langle \nabla\varphi(\theta_0), \gamma_0''(0)\rangle \leq 0\,. \tag{2.53}$$

Since $\Delta\varphi(\theta_0) = 0$, we deduce that

$$\frac{\partial^2\varphi}{\partial\nu^2}(\theta_0) \geq 0\,,$$

and the equality holds if and only if $w^T D^2\varphi(\theta_0)w = 0\ \forall w \in T_{\theta_0}(\partial\Omega)$. Since $\theta_0 \in \partial\Omega$ was taken arbitrary, we deduce that,

$$\int_{\partial\Omega} \frac{\partial^2\varphi/\partial\nu^2(\theta)}{\|\nabla\varphi(\theta)\|^3} ds_\theta \geq 0\,,$$

and the equality holds, if and only if,

$$w^T\varphi''(\theta)w = 0 \;\; \forall\theta \in \partial\Omega,\; \forall w \in T_\theta(\partial\Omega)\,.$$

But, remembering (2.53), it would mean that all curvature functions of $\partial\Omega$ vanish at every point of $\partial\Omega$. Thus, $\partial\Omega$ would be contained in a affine hyperplane of $\mathbb{R}^N$, which is not possible. It proves (2.52).

Assume now $N \geq 4$ and $\Omega$ is convex. Assume also $G_2(0) = 0$ and $G_3(0) < 0$. Integrating by parts twice, we find

$$d_n^2\Upsilon_{\tilde h}(\bar v_n, 0) = d_n^2\int_\Omega G_1(\bar v_n\varphi + \tilde u_{\tilde h})dx = d_n^2\int_0^1 G_1(d_nt)p_n(t)dt = G_3(0)p_n'(0) + \int_0^1 G_3(d_nt)p_n''(t)dt\,.$$

Using the Riemann-Lebesgue lemma we conclude that $\lim_{n\to\infty} d_n^2\Upsilon_{\tilde h}(\bar v_n, 0) = -G_3(0)p'(0) < 0$. It implies that, if $n$ is big enough, $\Upsilon_{\tilde h}(\bar v_n, 0) < 0$, and the minimum of the action functional (2.4) is negative. It finishes the proof of Theorem 2.1.2, and Theorem 2.1.3 becomes now a consequence of Lemma 2.2.1 .

## 2.6 The multiplicity problem: A topological approach

In the final section of this chapter we will study the boundedness (or unboundness) of the set of solutions of (2.1) for given $h \in \mathrm{Lip}(\Omega)$ and $g \in \mathrm{Lip}(\mathbb{R}/T\mathbb{Z})$. In case $g$ is real analytic of a small Lipschitz constant, this boundedness is equivalent to its finiteness, as it is well known. Straightforward arguments, using the Riemann-Lebesgue Theorem show that the solution set is indeed bounded (in the $W^{2,r}(\Omega)$ topology for any $1 < r < +\infty$) if $\bar h \neq 0$, so that we may well concentrate in the case

$$h = \tilde h \in \widetilde{Lip}(\Omega)\,.$$

As announced in Section 2.1, it turns out that the space dimension $N$ plays a key role in the answer of this problem, and, while for $N = 2$ or $3$ this set is always unbounded, for $N \geq 5$ it may be bounded. We see this below.

Consider the linear differential operator

$$\mathcal{L} : H_0^1(\Omega)\cap H^2(\Omega) \to L^2(\Omega), \qquad \mathcal{L}u := -\Delta u - \lambda_1 u,\; \forall u \in H_0^1(\Omega)\cap H^2(\Omega), \tag{2.54}$$

and the Nemytskii operator associated with $g$

$$\mathcal{N} : L^2(\Omega) \to L^2(\Omega), \qquad \big(\mathcal{N}u\big)(x) := g(u(x)) \;\; \forall x \in \Omega\,, \;\; \forall u \in L^2(\Omega)\,, \tag{2.55}$$

so that (2.3) is equivalent to the functional equation

$$\mathcal{L}v + \mathcal{N}(v + \tilde u_{\tilde h}) = 0\,, \qquad v \in H_0^1(\Omega)\cap H^2(\Omega)\,. \tag{2.56}$$

Let $Q : L^2(\Omega) \to L^2(\Omega)$ be the linear projection given by

$$Q(h) = \left(\frac{1}{\|\varphi\|_2^2}\int_\Omega h(x)\varphi(x)dx\right)\varphi$$

We observe that $\ker Q = \tilde L^2(\Omega) = \operatorname{im}\mathcal{L}$, $\ker\mathcal{L} = \langle\varphi\rangle$. Now, (2.56) may be rewritten as the so called *Lyapunov-Schmidt system*

$$\begin{aligned}\mathcal{L}(v) + [I - Q]\mathcal{N}(v + \tilde u_{\tilde h}) = 0\\ 0 = Q\mathcal{N}(v + \tilde u_{\tilde h})\end{aligned} \tag{2.57}$$

where $I$ stands for the identity operator in $L^2(\Omega)$. We call $\mathcal{K} : \tilde L^2(\Omega) \to \tilde H_0^1(\Omega)\cap H^2(\Omega)$ the inverse isomorphism of $\mathcal{L} : \tilde H_0^1(\Omega)\cap H^2(\Omega) \to \tilde L^2(\Omega)$, (so that $\tilde u_{\tilde h} = \mathcal{K}\tilde h$), and (2.57) adopts the form

$$\tilde v + \mathcal{K}[I - Q]\mathcal{N}(\bar v\varphi + \tilde v + \tilde u_{\tilde h}) = 0 \tag{2.58}$$

$$0 = \int_\Omega g(\bar v\varphi(x) + \tilde v(x) + \tilde u_{\tilde h}(x))\varphi(x)\,dx \tag{2.59}$$

where we have used (2.2) in order to write $v$ as $v = \bar{v}\varphi + \tilde{v}$, $\bar{v} \in \mathbb{R}, \tilde{v} \in \tilde{H}_0^1(\Omega)$. Let us call $\Sigma$ the set of solutions of the *auxiliary equation* (2.58), which is indeed the same equation which we considered in (2.5),

$$\Sigma := \left\{ (\bar{v}, \tilde{v}) \in \mathbb{R} \times \tilde{L}^2(\Omega) : \tilde{v} = \mathcal{K}[I - Q]\mathcal{N}(\bar{v}\varphi + \tilde{v} + \tilde{u}_{\tilde{h}}) \right\} .$$

On the first hand, as seen in Section 2.2, regularity theory shows that $\Sigma \subset \mathbb{R} \times W^{3,r}(\Omega)$ for any $1 < r < +\infty$. On the other, it follows from the Schauder fixed point theorem -note that $\mathcal{N}$ is completely continuous- that for any $\bar{v} \in \mathbb{R}$ there exists some $\tilde{v} \in \tilde{L}^2(\Omega)$ such that $(\bar{v}, \tilde{v}) \in \Sigma$. This is something we already know from Section 2.2, where we arrived at this same fact from a different argument. Let $\{(\bar{v}_n, \tilde{v}_n)\}$ be any sequence in $\Sigma$ with $|v_n| \geq 1$ $\forall n \in \mathbb{N}$, $|\bar{v}_n| \to +\infty$. As in previous sections, we consider the sequences $\{d_n\}$ and $\{\varphi_n\}$ defined by (2.8) and (2.9). And following a similar reasoning we used in Sections 2.2 and 2.4, the co-area formula and the change of variables theorem give us, for $n$ big enough, the relation

$$\int_\Omega g(\bar{v}\varphi(x) + \tilde{v}(x) + \tilde{u}_{\tilde{h}}(x))\varphi(x)\, dx = \int_\Omega g(d_n\varphi_n(x))\varphi(x)\, dx = \int_0^1 g(d_n t) q_n(t) dt \qquad \forall n \in \mathbb{N},$$

where the sequence $\{q_n\} \subset C[0,1]$ is defined by

$$\begin{aligned} q_n(t) &:= \int_{\{\varphi_n(x)=t\}} \frac{\varphi(x)}{\|\nabla\varphi_n(x)\|} ds_x = \\ &= \frac{1}{\sqrt{2-2t}} \int_{\mathbb{S}^{N-1}} \varphi\left(\Gamma_n(\sqrt{2-2t}, x)\right) \left|J\Gamma_n(\sqrt{2-2t}, x)\right| ds_x = \frac{\tau_n(\sqrt{1-t})}{\sqrt{1-t}}, \qquad 0 < t < 1, \end{aligned} \tag{2.60}$$

being

$$\tau_n(\rho) := \frac{1}{\sqrt{2}} \int_{\mathbb{S}^{N-1}} \varphi\left(\Gamma_n(\sqrt{2}\,\rho, x)\right) \left|J\Gamma_n(\sqrt{2}\,\rho, x)\right| ds_x, \qquad 0 < \rho < 1. \tag{2.61}$$

Using the same type of arguments we displayed in Section 2.4 with $\{\sigma_n\}$ and $\sigma$, one checks that $\{\tau_n\}$ converges in $C^1[0,1]$ to the function $\tau : [0,1] \to \mathbb{R}$ defined by

$$\tau(\rho) := \frac{1}{\sqrt{2}} \int_{\mathbb{S}^{N-1}} \varphi\left(\Gamma(\sqrt{2}\,\rho, x)\right) \left|J\Gamma(\sqrt{2}\,\rho, x)\right| ds_x = (1-\rho^2)\sigma(\rho), \qquad 0 < \rho < 1. \tag{2.62}$$

Moreover, $\{\tau_n\}$ is bounded in $H^2[0,1]$. On the other hand, $\tau_n(0) = 0$ $\forall n \in \mathbb{N}$. Thus, we are in position to apply Lemma 2.4.1 to deduce that

$$\begin{aligned} &\lim_{n\to\infty} \left[ d_n \int_\Omega g(d_n\varphi_n(x))\varphi(x)\, dx - G_1(d_n)\tau'(0) + G_1(0)\tau(1) \right] = \\ &\qquad = \lim_{n\to\infty} \left[ d_n \int_\Omega g(d_n\varphi_n(x))\varphi(x)\, dx - G_1(\bar{v}_n + \tilde{u}_{\tilde{h}}(\Omega_0))\sigma'(0) \right] = 0, \end{aligned}$$

since $\lim_{n\to\infty}\left[G_1(d_n) - G_1(\bar{v}_n + \tilde{u}_{\tilde{h}}(\Omega_0))\right]\sigma'(0) = 0$. In case $N = 2$, as seen in Section 2.4, $\sigma'(0) > 0$. It means that, if $\{\bar{v}_n^+\}$ is chosen in such a way that $G_1(\bar{v}_n^+ + \tilde{u}_{\tilde{h}}(\Omega_0)) = \max_{\mathbb{R}} G_1$ $\forall n \in \mathbb{N}$,

$$\lim_{n\to\infty} d_n^+ \int_\Omega g(d_n^+\varphi_n(x))\varphi(x)\, dx = \left(\max_{\mathbb{R}} G_1\right)\sigma'(0) > 0,$$

while, in case $G_1(\bar{v}_n^- + \tilde{u}_{\tilde{h}}(\Omega_0)) = \min_{\mathbb{R}} G_1$ $\forall n \in \mathbb{N}$,

$$\lim_{n\to\infty} d_n^- \int_\Omega g(d_n^-\varphi_n(x))\varphi(x)\, dx = \left(\min_{\mathbb{R}} G_1\right)\sigma'(0) < 0.$$

Thus, continuation results based upon the continuity property of the Leray-Schauder topological degree, may be used, in a similar way as in the proof of Theorem 1.1.2 in the previous chapter in order to show Theorem 2.1.1 in the two dimensional case.

In case $N = 3$, $\sigma'(0) = 0$, but the sequence $\{q_n\}_n$ is contained in $W^{1,1}[0,1]$, (in a similar way as happened with $\{p_n\}$), and, integrating by parts, we find

$$d_n^2 \int_\Omega g(d_n\varphi_n(x))\varphi(x)\,dx = d_n^2 \int_0^1 g(d_n t)\frac{\tau_n(\sqrt{1-t})}{\sqrt{1-t}}dt =$$
$$= d_n \int_0^1 G_1(d_n t)\frac{\left(\tau_n'(\sqrt{1-t}) - \frac{\tau_n(\sqrt{1-t})}{\sqrt{1-t}}\right)/\sqrt{1-t}}{2\sqrt{1-t}}\,dt\,,$$

so that, in case $\{\bar u_n^+\}$ has been chosen in such a way that $G_2(d_n^+) = \max_{\mathbb{R}} G_2$, Proposition 2.5.3 implies that

$$\lim_{n\to\infty}(d_n^+)^2\int_\Omega g(d_n^+\varphi_n(x))\varphi(x)\,dx = +\infty\,,$$

while, in case $\{\bar u_n^-\}$ is taken with $G_2(d_n^-) = \min_{\mathbb{R}} G_2$,

$$\lim_{n\to\infty}(d_n^-)^2\int_\Omega g(d_n\varphi_n(x))\varphi(x)\,dx = -\infty\,,$$

implying Theorem 2.1.1 in the three dimensional case.

Assume now that $N \geq 5$ and $\Omega$ is convex, and let $\mathcal{B} \subset \tilde L^2(\Omega) \cap \mathrm{Lip}(\Omega)$ be a bounded set. Let us see that there exists some positive number $\epsilon = \epsilon_{\mathcal{B}} > 0$ such that, if $\tilde h \in \mathcal{B}$ and

$$\hat g \in \mathcal{O}_\epsilon := \left\{ g \in Lip(\mathbb{R}/T\mathbb{Z},\mathbb{R}) : \int_0^T g(u)du = 0,\ |G_1(0)| > \frac{1}{2}\|G_1\|_\infty,\ \|g\|_\infty < \epsilon,\ \|g'\|_\infty < \epsilon \right\}\,,$$

then, the set of solutions of (2.1) $\big($or (2.3)$\big)$, is bounded. Otherwise, it would be possible to find sequences $\{g_m\} \subset \mathrm{Lip}(\mathbb{R}/T\mathbb{Z},\mathbb{R})$ with $g_m \in \mathcal{O}_{1/m}$ $\forall m \in \mathbb{N}$, and $\{\tilde h_m\}_m \subset \mathcal{B}$, such that, for each $m \in \mathbb{N}$, the set of solutions of the equation

$$-\Delta v_m - \lambda_1 v_m + g_m(v_m + \tilde u_{\tilde h_m}) = 0, \qquad v_m \in H_0^1(\Omega)\,, \tag{2.63}$$

is unbounded. Let us take, for each $m$, a solution $v_m = \bar v_m + \tilde v_m$ with $|\bar v_m| \geq 1$ and $\tilde v_m \in \tilde H_0^1(\Omega)\cap C^2(\bar\Omega)\cap H^3(\Omega)$ of (2.63). Relation (2.58) reads

$$-\Delta\tilde v_m - \lambda_1\tilde v_m + g_m\big(\bar v_m\varphi(x) + \tilde v_m + \tilde u_{\tilde h}\big) - \left[\int_\Omega g_m\big(\bar v_m\varphi(y) + \tilde v_m(y) + \tilde u_{\tilde h}(y)\big)\varphi(y)\,dy\right]\varphi(x) = 0$$

for any $m \in \mathbb{N}$.

It follows that $\{\tilde v_m\} \to 0$ both in $C^1(\bar\Omega)$ and in $H^2(\Omega)$. Derivating in the equality above we find

$$-\Delta\partial_{x_i}\big(\tilde v_m/\bar v_m\big) - \lambda_1\partial_{x_i}\big(\tilde v_m/\bar v_m\big) = -\big(\partial_{x_i}\varphi + \partial_{x_i}\tilde v_m/\bar v_m + \partial_{x_i}\tilde u_{\tilde h}/v_m\big)g'\big(\bar v_m\varphi + \tilde v_m + \tilde u_{\tilde h}\big) +$$
$$+\frac{1}{\bar v_m}\frac{1}{\|\varphi\|_2^2}\left[\int_\Omega g(\bar v_m\varphi(y) + \tilde v_m(y) + \tilde u_{\tilde h}(y))\varphi(y)\,dy\right]\partial_{x_i}\varphi\,, \qquad 1 \leq i \leq N\,,$$

and we deduce that the sequence $\{\tilde v_m/\bar v_m\}$ converges to 0 in $W^{3,r}(\Omega)$ for any $1 < r < \infty$. In particular, it converges to 0 in $C^2(\bar\Omega)\cap H^3(\Omega)$. Consequently, the sequence $\{\varphi_m\}$ defined as in (2.9),(2.8), converges to $\varphi$ both in $C^2(\Omega)$ and $H^3(\Omega)$ and the sequence $\{\tau_m\}$, defined as in (2.61) converges to $\tau$ in $H^2[0,1]$. Thus, a similar reasoning to the one carried out in the previous section in order to show Theorem 2.1.2 for $N \geq 5$, shows that the sequence $\{q_m\}$ defined by (2.60) converges in $W^{2,1}[0,1]$ to the function $q : [0,1] \to \mathbb{R}$ defined by

$$q(t) = \frac{t}{\sqrt{2-2t}}\int_{\mathbb{S}^{N-1}}|J\Gamma(\sqrt{2-2t},\theta)|ds_\theta = t\int_{\{\varphi(x)=t\}}\frac{1}{\|\nabla\varphi(x)\|}dx = tp(t)\,, \qquad 0 < t < 1.$$

We have proved that, given $\epsilon > 0$, there exists some index $m_0 \in \mathbb{N}$ such that, for any $v = \bar v\varphi + \tilde v$ solution of the equation

$$-\Delta v - \lambda_1 v + g_{m_0}(v + \tilde{u}_{\tilde{h}_{m_0}}) = 0, \qquad\qquad v \in H_0^1(\Omega)\,,$$

with $|\bar{v}| \geq 1$, we have

$$\|q_v'' - q''\|_{L^1[0,1]} < \epsilon\,.$$

We choose $\epsilon = -q'(0)/4 = -p'(0)/4 > 0$. Since the set of solutions $v_{m_0}$ of equation (2.63) (for $m = m_0$) is $H_0^1(\Omega)$-unbounded, it is possible to find a sequence of solutions $\{v_n\}_n$ with $v_n = \bar{v}_n\varphi + \tilde{v}_n$, $\{|\bar{v}_n|\} \to \infty$. We call $g := g_{m_0}$, $\tilde{h} := \tilde{h}_{m_0}$. Since the corresponding sequence $\{q_n\}$ is contained in $W^{2,1}[0,1]$, we may integrate twice by parts, to find

$$\begin{aligned} d_n^2 \int_\Omega g(v_n + \tilde{u}_{\tilde{h}})\varphi(x)\,dx &= d_n^2 \int_\Omega g(d_n\varphi_n(x))\varphi(x)\,dx = d_n^2 \int_0^1 g(d_n t) q_n(t) dt = \\ &= -d_n \int_0^1 G_1(d_n t) q_n'(t) dt = G_2(0) q_m'(0) + \int_0^1 G_2(d_n t) q_n''(t) dt\,, \end{aligned}$$

so that

$$\begin{aligned} \left| d_n^2 \int_\Omega g(v_n + \tilde{u}_{\tilde{h}})\varphi(x)\,dx - G_2(0)q'(0) \right| &\leq \left| G_2(0) \right| \left| q_m'(0) - q'(0) \right| + \\ &+ \left| \int_0^1 G_2(d_n t) q''(t) dt \right| + \int_0^1 \left| G_2(d_n t) \right| \left| q_n''(t) - q''(t) \right| dt \leq \left| G_2(0) \right| \left| q_m'(0) - q'(0) \right| + \\ &+ \left| \int_0^1 G_2(d_n t) q''(t) dt \right| + \|G_2\|_\infty |q'(0)|/4\,, \end{aligned}$$

and then,

$$\limsup \left| d_n^2 \int_\Omega g(v_n + \tilde{u}_{\tilde{h}})\varphi(x)\,dx - G_2(0)q'(0) \right| \leq \|G_2\|_\infty |q'(0)|/4\,.$$

It implies in particular that, for $n$ big enough

$$\limsup \left| d_n^2 \int_\Omega g(v_n + \tilde{u}_{\tilde{h}})\varphi(x)\,dx \right| \geq \left| G_2(0) q'(0) \right| - \|G_2\|_\infty |q'(0)|/4 > \|G_2\|_\infty |q'(0)|/4\,,$$

a contradiction. It finishes the proof of Theorem 2.1.1.

# Chapter 3

# On the multiplicity of periodic solutions for pendulum-type equations

## 3.1 Introduction

In this chapter we turn back to ordinary, resonant, pendulum-like equations. We deal with boundary value problems of the type

$$\begin{cases} u'' + cu' + g(u) = e(t) = \bar{e} + \tilde{e}(t) \\ u(T) - u(0) = k; \quad u'(T) - u'(0) = k' \end{cases} \tag{3.1}$$

where the following hypothesis are made:

**[H$_3$]**

1. $k,\ k',\ T,\ c$ are given real constants with $T > 0$,
2. $g \in C^1(\mathbb{R}/2\pi\mathbb{Z})$ is a continuous, $2\pi$-periodic function with zero mean, i.e.,
$$\int_0^{2\pi} g(u)du = 0,$$
3. $e \in L^1(\mathbb{R}/T\mathbb{Z})$ is decomposed as $e = \bar{e} + \tilde{e}$, where
$$\bar{e} \in \mathbb{R}, \qquad \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) := \left\{ e \in L^1(\mathbb{R}/T\mathbb{Z}) : \frac{1}{T}\int_0^T e(s)ds = \frac{k' + ck}{T} \right\}.$$

Simply integrate both sides of the differential equation in (3.1) to check that, in case $g \equiv 0$, a necessary condition for the linear problem (3.1) to have a solution, is $e = \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$, which is easily shown to be also sufficient. Further, the whole set of solutions can be obtained by adding all constant functions to any particular solution. This case being completely understood, we will always assume that $g$ is nontrivial in what follows. On the other hand, the simple change of variables $\hat{u}(t) := u(T - t),\ 0 \le t \le T$ shows that it is not restrictive to assume $c \ge 0$.

Observe also that, in case $u$ is a solution of (3.1), $u + 2\pi$ is again a solution. These solutions are called *geometrically equal* (they coincide when seen in the circumference $\mathbb{R}/2\pi\mathbb{Z}$), and our objective in this chapter is, for given $T, k, k', c, g$, to find external forcing terms $e$ such that (3.1) has at least, or exactly, a prefixed even number $2n$ of *geometrically different* solutions.

This problem, which contains in particular the *periodic problem* ($k = k' = 0$) for the dissipative *pendulum equation* ($g(u) = \Lambda \sin(u)$), has therefore a long history that may be found, for instance, in [57]. As a consequence, many aspects of this problem are known even though also many important and profound questions remain still open.

Most results in literature in connection with this problem deal with the periodic setting:

$$\begin{cases} u'' + cu' + g(u) = e(t) = \bar{e} + \tilde{e}(t) \\ u(T) - u(0) = 0; \quad u'(T) - u'(0) = 0 \end{cases} \tag{3.2}$$

In this framework, it was proved in the 1984 work by Mawhin and Willem [62] that, if the problem is conservative, $(c = 0)$, for any given $e = \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) = \{h \in L^1(\mathbb{R}/T\mathbb{Z}) : \int_0^T h(s)ds = 0\}$, problem (3.2) has, at least, two different solutions. This result, which turns out to be false for the nonconservative case, (just remember the first counterexample, given by Ortega [64], showing that, if $c \neq 0$, (3.2) may not have solutions at all even for $e = \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$), was attained through the use of variational arguments.

More recently, it was proved by Donati [28] that, in the periodic problem for the conservative, forced pendulum equation, $\big(g(u) = \Lambda \sin(u)\big)$, it is always possible to find forcing terms $e = \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$ such that (3.2) has, at least, four geometrically different solutions. This result was extended by Ortega [65], who established that, in the same framework, it is possible to change 4 by any number. Independently, it was shown by Katriel ([42]) that, in case $g$ is not a trigonometric polynomial, has $C^2$ regularity and verifies $g(x + \pi) = -g(x)\ \forall x \in \mathbb{R}$, for arbitrary damping $c$ the number of geometrically different solutions of (3.2) is not bounded as $e = \tilde{e}$ varies in $\tilde{L}^1(\mathbb{R}/T\mathbb{Z})$.

All these partial results lead to the following question: Are additional assumptions for $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean, essential to find, for each $n \in \mathbb{N}$, forcing terms $e \in L^1(\mathbb{R}/T\mathbb{Z})$ such that problem (3.1) has, at least, $n$ geometrically different periodic solutions? In this chapter we complete the work initiated in [80] to show the answer to be 'no':

**Theorem 3.1.1.** *Assume $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ is not trivial. Then, for each $n \in \mathbb{N}$,*

$$\mathcal{S}_n := \big\{e \in L^1(\mathbb{R}/T\mathbb{Z}) \text{ such that (3.1) has at least } n \text{ geometrically different solutions}\big\}$$

*has nonempty interior in $L^1(\mathbb{R}/T\mathbb{Z})$. Moreover,*

1. *$\overset{\circ}{\mathcal{S}_n} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$ if $c = 0$ or $g$ is not a trigonometric polynomial.*

2. $$\overset{\circ}{\mathcal{S}_n} \cap \big\{e = \tilde{e} + \bar{e} \in L^1(\mathbb{R}/T\mathbb{Z}) : \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z}),\ -\epsilon < \bar{e} < 0\big\} \neq \emptyset \neq$$ $$\neq \overset{\circ}{\mathcal{S}_n} \cap \big\{e = \bar{e} + \tilde{e} \in L^1(\mathbb{R}/T\mathbb{Z}) : \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z}),\ 0 < \bar{e} < \epsilon\big\}$$

   *for every $\epsilon > 0$ in case $g$ is a trigonometric polynomial and $c \neq 0$.*

As a consequence, there are (infinitely many) analytic functions $e \in C^{\Omega}(\mathbb{R}/T\mathbb{Z})$ such that problem (3.1) has at least $n$ solutions. On the other hand, in view of Theorem 3.1.1, the following question arises:

*Is it true that $\overset{\circ}{\mathcal{S}_n} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$ independently of $g, c$?*

We do not give the answer to this question, which seems likely to be positive.

Theorem 3.1.1 is proved in two stages. In the first one, we start by considering as forcing term $e$, a constant function

$$e = \bar{e}\,.$$

For a certain value $\bar{e} = \bar{e}_{\alpha,T}$ of this constant, (which is zero if $\alpha = 0$) the differential equation in (3.1) has a closed orbit, which is $T$-periodic in the cylinder $\mathbb{R}/2\pi\mathbb{Z}$, and this orbit generates a continuum of solutions $u$ of this equation. These are solutions of (3.1) for $k = 2\pi$, $k' = 0$, and, thus, $T$-periodic solutions for the equation

$$u'' + cu' + g(u) = \bar{e}_{\alpha,T} - 2\pi\delta'\,,$$

being $\delta'$ the derivative of the usual Dirac delta function at an arbitrary, given instant of time. At this stage, bifurcation results which follow from the implicit function theorem, allow us to obtain, for suitable curves of $L^1(\mathbb{R}/T\mathbb{Z})$ functions bifurcating from $\bar{e}_{c,T} - 2\pi\delta'$, many corresponding curves of solutions bifurcating from this continuum.

However, in order to ensure this argument to work, we further need (together with regularity) some additional nondegeneracy hypothesis. It was shown by Ortega [65] that these hypothesis are met in case $g(u) = A\sin u$ and $c = 0$. This is not the case for general nonlinearities $g$, not even for $c = 0$, as it is shown in Remark 3.3.2. However, we show here that the hypothesis always hold when $g$ is the restriction to the real line of an entire function on the complex plane. This allows us to prove Theorem 3.1.1 in this particular case. It is also possible to ensure nondegeneracy hypothesis in another cases; this gives rise to Theorems 3.1.2 and 3.1.3 below.

In a second stage, we take a nonlinearity function $g$ which is assumed to be **not** a trigonometric polynomial. The main idea here comes from Katriel's work [42]. For the limit case of zero period ($T = 0$), it could be thought, of course, in a heuristic way, that the forced pendulum-type equation $u'' + cu' + g(u) = 0$ has the following curve of 'periodic solutions': for any $a \in \mathbb{R}$, we may consider the 'solution' which remains still at $a$ along this zero-length time period. Under some regularity, symmetry and nondegeneracy hypothesis on $g$, Katriel was able to bifurcate, for small positive time period $T$, forcing terms $e$ with many associated periodic solutions.

Here, we modify Katriel's argument so that regularity and symmetry hypothesis are no longer needed. And, in the second hand, we manage to bifurcate forcing terms with many associated ordered branches of *strictly lower and upper solutions*, so that we are in the appropriate framework to use topological arguments in order to obtain open sets of forcing terms with the same properties.

Subsequently, we devote ourselves to the study of the particular interesting case of conservative, pendulum-type systems:

$$\begin{cases} u'' + g(u) = e(t) = \bar{e} + \tilde{e}(t) \\ u(T) - u(0) = k; \quad u'(T) - u'(0) = k' \end{cases} \tag{3.3}$$

This time we may use our better knowledge of the problem to explore *exact multiplicity* results. To get a feeling of what we should expect, observe that, in case $g$ is $\frac{2\pi}{p}$-periodic for some $p \in \mathbb{N}$, the number of geometrically different solutions of (3.3) (or (3.1)), if finite, is always a multiple of $p$. Consequently, we impose a new assumption on $g$ implying, in particular, that its minimal period is $2\pi$.

$[\mathbf{G}_3]$ $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$ has a primitive $G$ which attains its maximum only once in $[0, 2\pi[$.

Then, if the time period $T$ is big enough, it is possible to show the existence of forcing terms $e = \tilde{e} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$ such that problem (3.3) has exactly a prefixed even number $2n$ of solutions.

**Theorem 3.1.2.** *Assume* $[\mathbf{G}_3]$. *Then, for each given* $n \in \mathbb{N}$ *there exists* $T_0 = T_0(n) > 0$ *such that, for any* $T > T_0(n)$, *there exists an open set* $\mathcal{O}_{n,T} \subset L^1(\mathbb{R}/T\mathbb{Z})$ *with* $\mathcal{O}_{n,T} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, *and with the property that for any* $e \in \mathcal{O}_{n,T}$, *problem (3.3) has exactly* $2n$ *geometrically different solutions.*

In particular cases, say, in the case of the pendulum equation, we are able to estimate the quantity $T_0(n)$. We obtain:

**Theorem 3.1.3.** *Assume* $g(x) = \Lambda \sin(x)$, $\Lambda \neq 0$, *and let* $n \in \mathbb{N}$ *be given. If*

$$T \geq 12 \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \frac{n}{\sqrt{|\Lambda|}} \tag{3.4}$$

*then, there exists an open set* $\mathcal{O}_{n,T} \subset L^1(\mathbb{R}/T\mathbb{Z})$ *with* $\mathcal{O}_{n,T} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, *such that for any* $e \in \mathcal{O}_{n,T}$, *problem (3.3) has exactly* $2n$ *geometrically different solutions.*

Thus, *it remains an open problem to decide whether this result continues to hold without assuming (3.4).*

Next result will follow from Theorem 3.1.2 above.

**Corollary 3.1.4.** *Assume* $[\mathbf{G}_3]$. *Then, for each given* $n \in \mathbb{N}$, *there exists a discrete and closed set* $F_n \subset \mathbb{R}^+$, *such that, for any* $T \in \mathbb{R}^+ \backslash F_n$, *there exists an open set* $\mathcal{O}_{n,T} \subset L^1(\mathbb{R}/T\mathbb{Z})$ *with* $\mathcal{O}_{n,T} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, *and with the property that, for any* $e \in \mathcal{O}_{n,T}$, *problem (3.3) has exactly* $2n$ *geometrically different solutions.*

*In particular, there exists a countable subset* $F$ *of* $\mathbb{R}^+$ *such that, for any* $T \in \mathbb{R}^+ \backslash F$ *and for any* $n \in \mathbb{N}$, *there exists an open set* $\mathcal{O}_{n,T} \subset L^1(\mathbb{R}/T\mathbb{Z})$ *with* $\mathcal{O}_{n,T} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, *with the property that, for any* $e \in \mathcal{O}_{n,T}$, *problem (3.3) has exactly* $2n$ *geometrically different solutions.*

Some remarks on the notation. Through this chapter, a function of several variables, $S = S(x_1, x_2, \dots, x_p)$, defined on a open subset of the cartesian product of the Banach spaces $X_1, X_2, \dots, X_p$, will be called $C^1$ (or continuously differentiable) with respect to $x_i$ if it is continuous and the partial derivative $\partial_{x_i} S$ is continuously defined on the whole domain of $S$. We write $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$, (so that $L^1(\mathbb{T}) \equiv L^1(\mathbb{R}/T\mathbb{Z})$, $C(\mathbb{T}) \equiv \{f \in C(\mathbb{R}/T\mathbb{Z}) : f(T) - f(0) = 0\}$, $C^1(\mathbb{T}) \equiv \{f \in C^1(\mathbb{R}/T\mathbb{Z}) : f(T) - f(0) = 0 = f'(T) - f'(0)\}$, $W_{1,1}(\mathbb{T}) \equiv \{f \in W_{1,1}(\mathbb{R}/T\mathbb{Z}) : f(0) = f(T)\}$, etc). Given $s \in \mathbb{R}$ we call $\tau_s$ the associated translation operator $\big($defined by $\tau_s f(x) := f(s+x)\big)$. A (real) trigonometric polynomial of degree $r \in \mathbb{N}$ on $\mathbb{T}$ is a function $P : \mathbb{T} \to \mathbb{R}$ of the form $P(t) = p_0 + \sum_{j=1}^{r} [p_j \cos(j\frac{2\pi}{T}t) + q_j \sin(j\frac{2\pi}{T}t)]$ for some real coefficients $p_j, q_j$ with $p_j^2 + q_j^2 \neq 0$, or, in complex notation, $P(t) = \sum_{j=-r}^{r} \Omega_j e^{ij\frac{2\pi}{T}t}$ for some complex coefficients $\Omega_j$ with $\Omega_{-j} = \overline{\Omega_j}$ and $\Omega_r \neq 0$.

## 3.2 The abstract framework: A bifurcation result

The implicit function theorem may be used to obtain the existence of nontrivial branches of solutions bifurcating from a trivial one. There are many results of this type in the literature, see, for instance, [3], [22]. This section is devoted to recall some general bifurcation arguments, which we will need later.

Let $X, Y$ be real Banach spaces, let $U \subset X$, $V \subset Y$ be open and $y_0 \in V$; let $I \subset \mathbb{R}$ be an open interval with $0 \in I$; finally, let $\mathcal{H} : I \times U \times V \to X$, $(\lambda, x, y) \mapsto \mathcal{H}(\lambda, x, y)$ be a $C^1$ mapping. We think of $\lambda, x, y$ as being the bifurcation parameter, the variable, and an extra perturbation parameter respectively.

We are interested in the solutions of the equation

$$\mathcal{H}(\lambda, x, y) = 0; \qquad \lambda \in I,\ x \in U,\ y \in V, \tag{3.5}$$

for $\lambda \neq 0$.

We assume that for $(\lambda, y) = (0, y_0)$ there exists a trivial branch of solutions given by the $C^1$ curve $\gamma : \mathbb{R} \to U \subset X$

$$\mathcal{H}(0, \gamma(s), y_0) = 0\ \forall s \in \mathbb{R} \tag{3.6}$$

The curve $\gamma$ is further assumed to have the following property: There exists some closed, linear hyperplane $\tilde{X} \subset X$ such that

$$\gamma'(s) \notin \tilde{X}\ \forall s \in \mathbb{R} \tag{3.7}$$

(in particular, $\gamma$ should be injective and $\gamma'(s) \neq 0\ \forall s \in \mathbb{R}$). Derivating (3.6) with respect to $s$, we obtain

$$\partial_x \mathcal{H}(0, \gamma(s), y_0)\gamma'(s) = 0\ \forall s \in \mathbb{R}, \tag{3.8}$$

and consequently,

$$0 \neq \gamma'(s) \in \ker \partial_x \mathcal{H}(0, \gamma(s), y_0)\ \forall s \in \mathbb{R}$$

We further assume that

- $(a)$ $\partial_x \mathcal{H}(0, \gamma(s), y_0) : X \to X$ is a Fredholm operator of zero index for every $s \in \mathbb{R}$
- $(b)$ $\dim \ker \partial_x \mathcal{H}(0, \gamma(s), y_0) = 1$, $\big($that is, $\ker \partial_x \mathcal{H}(0, \gamma(s), y_0) = \langle \gamma'(s) \rangle\big)$ $\forall s \in \mathbb{R}$.

Hypothesis $(a)$ implies the equality

$$\Big[\operatorname{im} \partial_x \mathcal{H}(0, \gamma(s), y_0)\Big]^{\perp} = \ker \partial_x \mathcal{H}(0, \gamma(s), y_0)^*\ \forall s \in \mathbb{R}, \tag{3.9}$$

($(\cdot)^*$ denoting adjoint operator), which allows us to use the implicit function theorem to obtain the existence of a continuous curve[1] $\sigma : \mathbb{R} \to X^*$ such that

$$\|\sigma(s)\|_* = 1, \quad \langle \sigma(s) \rangle = \Big[\operatorname{im} \partial_x \mathcal{H}(0, \gamma(s), y_0)\Big]^{\perp} \forall s \in \mathbb{R}. \tag{3.10}$$

[1]Observe that, due to hypothesis $(b)$, -after, possibly, a reparametrization- $\gamma$ will be of class $C^{p+1}(\mathbb{R})$, $p \geq 1$, provided $U \to X$, $x \mapsto \mathcal{H}(0, x, y_0)$ has class $C^{p+1}$. In this case, $\sigma$ will be of class $C^p(\mathbb{R})$.

Using a partition of the unity argument, it is not difficult to show the existence of a $C^\infty$ curve $m : \mathbb{R} \to X$ such that

$$m(t) \notin \operatorname{im} \partial_x \mathcal{H}(t, \gamma(t), y_0) \quad \forall t \in \mathbb{R}$$

Thus, for any $t \in \mathbb{R}$, the space $X$ splits as $X = \langle m(t) \rangle \oplus \operatorname{im} \partial_x \mathcal{H}(t, \gamma(t), y_0)$ and also as $X = \langle \gamma'(t) \rangle \oplus \tilde{X}$. We use this latter splitting together with the inverse function theorem to uniquely write each element $x \in X$ in a small ('tubular') open neighborhood of $\gamma(\mathcal{J})$ as $x = \gamma(t) + \tilde{x},\ t \in \mathcal{J},\ \tilde{x} \in \tilde{X}$ near 0, and we call $\Pi_t : X \to \langle m(t) \rangle \equiv \mathbb{R}$ the linear projection associated with the first one. Observe that $\Pi_t(x) = \frac{\langle x, \sigma(t) \rangle}{\langle m(t), \sigma(t) \rangle} m(t)\ \forall t \in \mathbb{R},\ \ \forall x \in X$.

With this notation, equation (3.5) can be rewritten as the system

$$\big(I_X - \Pi_t\big)\mathcal{H}(\lambda, \gamma(t) + \tilde{x}, y) = 0 \tag{3.11}$$

$$\Pi_t \mathcal{H}(\lambda, \gamma(t) + \tilde{x}, y) = 0 \tag{3.12}$$

This is the so-called *Lyapunov-Schmidt* system for (3.5). Usually, (3.11) is referred to as the auxiliary equation and (3.12) as the bifurcation equation of the system.

Let us fix instants $-\infty < a < b < +\infty$ and denote $\mathcal{J} := ]a, b[$. Our task will be to study the bifurcation branches, alongside with $\lambda$, of solutions of equation (3.5) emanating from the curve $\gamma_{|\mathcal{J}} : \mathcal{J} \to X$. Using the implicit function theorem we may solve equation (3.11) near $\{0\} \times \gamma(\mathcal{J}) \times \{0\}$, obtaining:

**Lemma 3.2.1.** *There exist open sets $\mathcal{U} \subset X$ with $\gamma(\mathcal{J}) \subset \mathcal{U} \subset U$, $\mathcal{I} \subset \mathbb{R}$ with $0 \in \mathcal{I} \subset I$, $\mathcal{V} \subset Y$ with $y_0 \in \mathcal{V} \subset V$, and a $C^1$ mapping $\Psi : \mathcal{I} \times \mathcal{J} \times \mathcal{V} \to \tilde{X}$ such that*

$$\big\{(\lambda, x, y) \in \mathcal{I} \times \mathcal{U} \times \mathcal{V} : (I_X - \Pi_t)\mathcal{H}(\lambda, x, y) = 0\big\} = \\ = \big\{(\lambda, \gamma(t) + \Psi(\lambda, t, y), y) : (\lambda, t, y) \in \mathcal{I} \times \mathcal{J} \times \mathcal{V}\big\}$$

This means that, on $\mathcal{I} \times \mathcal{U} \times \mathcal{V}$, equation (3.5) reads

$$\langle \mathcal{H}(\lambda, \gamma(t) + \Psi(\lambda, t, y), y), \sigma(t) \rangle = 0, \qquad (\lambda, t, y) \in \mathcal{I} \times \mathcal{J} \times \mathcal{V}$$

We start by exploring the structure of the solution set of this equation for $y = y_0$. We define

$$\xi : \mathcal{I} \times \mathcal{J} \subset \mathbb{R}^2 \to \mathbb{R}, \qquad (\lambda, t) \mapsto \langle \mathcal{H}(\lambda, \gamma(t) + \Psi(\lambda, t, y_0), y_0), \sigma(t) \rangle$$

Of course, $\xi$ is a $C^1$ mapping and verifies $\xi(0, t) = 0\ \forall t \in \mathcal{J}$. Further,

$$\partial_\lambda \xi(0, t) = \langle \partial_\lambda \mathcal{H}(0, \gamma(t), y_0) + \partial_x \mathcal{H}(0, \gamma(t), y_0) \partial_\lambda \Psi(0, t, y_0), \sigma(t) \rangle = \\ = \langle \partial_\lambda \mathcal{H}(0, \gamma(t), y_0), \sigma(t) \rangle \quad \forall t \in \mathcal{J}$$

Therefore, the mapping $\vartheta : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ defined by the rule

$$(\lambda, t) \mapsto \begin{cases} \frac{1}{\lambda}\xi(\lambda, t) = \frac{1}{\lambda}\langle \mathcal{H}(\lambda, \gamma(t) + \Psi(\lambda, t, y_0), y_0), \sigma(t) \rangle & \text{if } \lambda \neq 0 \\ \langle \partial_\lambda \mathcal{H}(0, \gamma(t), y_0), \sigma(t) \rangle & \text{if } \lambda = 0 \end{cases}$$

is continuous. We recall that equation (3.5) with $y = y_0,\ \lambda \in \mathcal{I}\backslash\{0\},\ x \in \mathcal{U}$, reduces to $\vartheta(\lambda, t) = 0$, $t \in \mathcal{J}$.

Thus, we are lead to consider the real-valued, continuous curve:

$$\Gamma : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \langle \partial_\lambda \mathcal{H}(0, \gamma(t), y_0), \sigma(t) \rangle \tag{3.13}$$

A remarkable fact of this formula is that no explicit mention to $\Psi$ appears in the right hand side, even though it was built using this function. In particular, **the curve $\Gamma : \mathcal{J} \to \mathbb{R}$ does not depend on the particular choices of $\tilde{X}$, $m$.**

It does not seem strange now that, under suitable nondegeneracy hypothesis, zeroes of $\Gamma$ could be bifurcated to zeroes of $\xi(\lambda, \cdot)$ and, consequently, to zeroes of $\mathcal{H}(\lambda, \cdot, y_0)$ for $|\lambda|$ small. This is shown below.

**Lemma 3.2.2.** *Let $\mathcal{U}_*$ be any open subset of $U$ with $\mathcal{U}_* \supset \gamma(\mathcal{J})$, and let $a < c_0 < c_1 < \ldots < c_p < b$ verify*

$$(-1)^i \Gamma(c_i) > 0 \qquad i = 0, \ldots, p \tag{3.14}$$

*Then, there exists some $\epsilon_* > 0$ with $\mathcal{I}_* := ]0, \epsilon_*[ \subset I$ such that $\mathcal{H}(\lambda, \gamma(c_i) + \Psi(\lambda, c_i, y_0), y_0) \in (-1)^i \mathbb{R}^+ m(c_i)$ $\forall \lambda \in \mathcal{I}_*$, $\forall i : 0, \ldots, p$. In particular, for any $\lambda \in \mathcal{I}_*$, equation (3.5) with $y = y_0$ has, at least, $p$ different solutions $x \in \mathcal{U}$ for all $\lambda \in \mathcal{I}_*$.*

*Furthermore, for any $\tilde{\lambda} \in \mathcal{I}_*$, there exist an open interval $\tilde{\mathcal{I}} \subset \mathcal{I}_*$ with $\tilde{\lambda} \in \tilde{\mathcal{I}}$ and an open set $\tilde{\mathcal{V}} \subset V$ with $y_0 \in \tilde{\mathcal{V}}$ such that $\mathcal{H}(\lambda, \gamma(c_i) + \Psi(\lambda, c_i, y), y) \in (-1)^i \mathbb{R}^+ m(c_i)$ $\forall \lambda \in \mathcal{I}_*$, $\forall y \in \tilde{\mathcal{V}}$, $\forall i : 0, \ldots, p$. In particular, equation (3.5) has at least $p$ different solutions $x \in \mathcal{U}_*$ for all $\lambda \in \tilde{\mathcal{I}}$, $y \in \tilde{\mathcal{V}}$.*

Of course, all this is a simple consequence of the continuity of $\vartheta$; if it is positive somewhere, it remains positive in a neighborhood, and, whenever $\vartheta(\lambda, \cdot)$ has different sign at two instants $c_i$, $c_{i+1}$, it vanishes somewhere between them.

To proceed, we will need some extra regularity on $\mathcal{H}$. Namely, let us assume that both mappings

$$I \times U \to X, \quad (\lambda, x) \mapsto \partial_\lambda \mathcal{H}(\lambda, x, y_0) \qquad \text{and} \qquad I \times U \to L(X), \quad (\lambda, x) \mapsto \partial_x \mathcal{H}(\lambda, x, y_0)$$

are $C^1$ with respect to $x$. If this is the case, $\sigma$ is a $C^1$ curve and $\vartheta$ is itself continuously differentiable with respect to $t$. In particular, $\Gamma : \mathbb{R} \to \mathbb{R}$ is $C^1$.

Let us call $U_{a,b}$ the open subset of $U$ delimited by the (affine) hyperplanes $\gamma(a) + \tilde{X}$ and $\gamma(b) + \tilde{X}$. We further assume:

($c$) $\partial_y \mathcal{H} : I \times U_{a,b} \times V \to L(Y, X)$ and $\partial_\lambda \mathcal{H} : I \times U_{a,b} \times V \to X$ are bounded.

($d$) For any sequence $\{x_n\}_n \subset U_{a,b}$ such that $\{\mathcal{H}(0, x_n, y_0)\} \to 0$, $\big\{\text{dist}\big(x_n, \gamma(\mathbb{R})\big)\big\} \to 0$.

The purpose of these two hypothesis is to guarantee that given any open subset $O$ of $X$ containing $\gamma([a, b])$ there exist open sets $\mathcal{I}_* \subset I$ and $\mathcal{V}_*$ containing 0 and $y_0$ respectively, such that equation (3.5) has no solutions $x \in U_{a,b} \backslash O$ for any $(\lambda, y) \in \mathcal{I}_* \times \mathcal{V}_*$. In this way, under hypothesis ensuring the nondegeneracy of the zeroes of $\Gamma$, the implicit function theorem may be used to obtain precise results on the number of solutions of (3.5) for $x \in U_{a,b}$.

**Lemma 3.2.3.** *Assume $a < c_0 < c_1 < \ldots < c_p < b$ verify*

$$\Gamma(c_i) = 0, \ \Gamma'(c_i) \neq 0, \quad i = 0, \ldots, p \qquad\qquad \Gamma(t) \neq 0 \ \forall t \in [a, b] \backslash \{c_0, c_1, \ldots, c_p\}$$

*Then, there exist $\epsilon_* > 0$ with $]-\epsilon_*, \epsilon_*[ := \mathcal{I}_* \subset I$, and continuous curves $\gamma_1, \ldots, \gamma_p : \mathcal{I}_* \to U_{a,b} \subset X$ which are, further, $C^1$ on $\mathcal{I}_* \backslash \{0\}$, such that $\gamma_i(0) = \gamma(c_i)$, $1 \leq i \leq p$, and*

$$\{(\lambda, x) \in \mathcal{I}_* \times U_{a,b} : \lambda \neq 0, \ \mathcal{H}(\lambda, x, y_0) = 0\} = \bigcup_{i=1}^{p} \{(\lambda, \gamma_i(\lambda)) : \lambda \in \mathcal{I}_*, \ \lambda \neq 0\}.$$

*Moreover, given any $\tilde{\lambda} \in \mathcal{I}_*$, $\tilde{\lambda} \neq 0$, there exist an open interval $\tilde{\mathcal{I}} \subset I$ containing $\tilde{\lambda}$, an open subset $\tilde{\mathcal{V}} \subset Y$ with $y_0 \in \tilde{\mathcal{V}} \subset V$, and $C^1$ mappings $\tilde{s}_1, \ldots, \tilde{s}_p : \tilde{\mathcal{I}} \times \tilde{\mathcal{V}} \to U \subset X$ such that*

$$\tilde{s}_i(\lambda, y_0) = \gamma_i(\lambda) \ \forall \lambda \in \tilde{\mathcal{I}}$$

$$\Big\{(\lambda, x, y) \in \tilde{\mathcal{I}} \times U_{a,b} \times \tilde{\mathcal{V}} : \mathcal{H}(\lambda, x, y) = 0\Big\} = \bigcup_{i=1}^{p} \Big\{(\lambda, \tilde{s}_i(\lambda, y), y) : (\lambda, y) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{V}}\Big\}.$$

## 3.3 A functional framework for the periodic pendulum

The goal of this section is to establish the needed functional setting in order to reformulate problem (3.1) as a fixed point one for a regular mapping on a Banach space and apply the results in last section.

Denoting by $\varphi$ the only solution to the linear problem

$$\begin{cases} \varphi'' + c\varphi' = \tilde{e}(t) \\ \varphi(0) = 0; \quad \varphi(T) = k; \quad \varphi'(T) - \varphi'(0) = k' \end{cases}$$

the standard change of variables $v = u - \varphi$ transforms problem (3.1) into the periodic problem

$$\begin{cases} v'' + cv' + g(v + \varphi(t)) = \bar{e} \\ v(T) - v(0) = 0; \quad v'(T) - v'(0) = 0 \end{cases} \tag{3.15}$$

It will be more convenient to work directly on this problem rather than with the original one. Namely, for any and given $\varphi \in L^1(\mathbb{T})$ and $\bar{e} \in \mathbb{R}$ we may consider the problem

$$v'' + cv' + g(v + \varphi(t)) = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}) \tag{3.16}$$

We define the linear differential operator

$$\mathcal{L}^0 : W_{2,1}(\mathbb{T}) \to L^1(\mathbb{T}), \quad \mathcal{L}^0(v) := v'' \ \forall v \in W_{2,1}(\mathbb{T}),$$

and the Nemytskii operator associated with $g$

$$\mathcal{N} : L^1(\mathbb{T}) \to L^1(\mathbb{T}),$$
$$[\mathcal{N}(v)](x) := g(v(x)) \ \ \forall x \in \mathbb{T}, \ \ \forall v \in L^1(\mathbb{T}),$$

so that (3.16) is equivalent to the functional equation

$$\mathcal{L}^0(v) + cv' + \mathcal{N}(v + \varphi) = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}) \tag{3.17}$$

The operator $\mathcal{L}^0$ is not injective, but (3.17) is not changed if the same quantity $v$ is subtracted and added, to get the equality

$$[\mathcal{L}^0(v) - v] + [\mathcal{N}(v + \varphi) + v + cv'] = \bar{e}, \quad v \in W_{2,1}(\mathbb{T}) \tag{3.18}$$

whose first term is invertible. We denote by $\mathcal{K}$ the inverse operator of $v \mapsto \mathcal{L}^0(v) - v$, which is a compact operator when seen from $L^1(\mathbb{T})$ to $W_{1,1}(\mathbb{T})$. We also observe that $\mathcal{K}$ is 'self-adjoint' in the sense that

$$\int_0^T [\mathcal{K}(f)](x)g(x)dx = \int_0^T f(x)[\mathcal{K}(g)](x)dx \qquad \forall f, g \in \mathbb{L}^1(\mathbb{T}) \tag{3.19}$$

In this way, equation (3.18) can be rewritten as a fixed point problem

$$v = -\mathcal{K}[\mathcal{N}(v + \varphi) + v + cv' - \bar{e}] = -\mathcal{K}[\mathcal{N}(v + \varphi) + v + cv'] - \bar{e}, \qquad v \in W_{1,1}(\mathbb{T}) \tag{3.20}$$

We fix $\psi_0$ in $W_{1,1}(\mathbb{T})$ (which will be determined later) and define

$$\mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times \Big[\mathbb{R} \times L^1(\mathbb{T})\Big] \to W_{1,1}(\mathbb{T}), \quad \big(\lambda, v, \bar{e}, \varphi\big) \mapsto v + \mathcal{K}[\mathcal{N}(v + \lambda\psi_0 + \varphi) + v + cv'] + \bar{e} \tag{3.21}$$

It is easily checked that $\mathcal{H}$ is $C^1$ and the continuous, linear operator $\partial_v \mathcal{H}(\lambda, v, \bar{e}, \varphi) : W_{1,1}(\mathbb{T}) \to W_{1,1}(\mathbb{T})$ has the form identity minus compact for any $(\lambda, v, \bar{e}, \varphi)$, so that $(a)$ is automatically satisfied. Furthermore, the partial derivatives $\partial_\lambda \mathcal{H}, \partial_{\bar{e}} \mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times \Big[\mathbb{R} \times L^1(\mathbb{T})\Big] \to W_{1,1}(\mathbb{T})$, and $\partial_\varphi \mathcal{H} : \mathbb{R} \times W_{1,1}(\mathbb{T}) \times \Big[\mathbb{R} \times L^1(\mathbb{T})\Big] \to L\big(L^1(\mathbb{T}), W_{1,1}(\mathbb{T})\big)$ are clearly bounded, as required in $(c)$. Finally, it is easily checked that, in case $g \in C^2(\mathbb{R})$, both mappings

$$\mathbb{R} \times W_{1,1}(\mathbb{T}) \to W_{1,1}(\mathbb{T}), \qquad (\lambda, v) \mapsto \partial_\lambda \mathcal{H}(\lambda, v, \bar{e}, \varphi)$$
$$\mathbb{R} \times W_{1,1}(\mathbb{T}) \to L\big(W_{1,1}(\mathbb{T})\big), \qquad (\lambda, v) \mapsto \partial_v \mathcal{H}(\lambda, v, \bar{e}, \varphi)$$

are continuously differentiable with respect to $v$ for any $(\bar{e}, \varphi) \in \mathbb{R} \times L^1(\mathbb{T})$.

In orden to position ourselves in the abstract framework studied in previous section we still have to find $\bar{e} \in \mathbb{R}$ and $\varphi \in L^1(\mathbb{T})$ such that (3.20) has a whole nontrivial curve of solutions. Alternatively, we may try to find $e = \bar{e} + \tilde{e} \in L^1(\mathbb{T})$, $k, k' \in \mathbb{R}$ such that problem (3.1) has a curve of solutions.

The following proposition has interest by its own.

**Proposition 3.3.1.** *There exists an unique constant external force* $\bar{e} = \bar{e}_{c,T} \in \mathbb{R}$ *such that*

$$\begin{cases} u'' + cu' + g(u) = \bar{e} \\ u(0) = 0; \quad u(t+T) = 2\pi + u(t) \ \forall t \in \mathbb{R}; \end{cases} \tag{3.22}$$

*has solution. This solution is unique (we will call it* $u_{c,T}$*) and verifies*

$$u'_{c,T}(t) > 0 \ \forall t \in \mathbb{R}, \qquad u_{c,T}\Big(\frac{T}{2\pi}t\Big) \underset{T\to 0}{\longrightarrow} t \text{ uniformly w.r.t. } t \in \mathbb{R}, \text{ for } c > 0 \text{ fixed.} \tag{3.23}$$

*Finally,*

$$\bar{e}_{0,T} = 0 \ \forall T > 0 \qquad \bar{e}_{c,T} > \frac{2\pi}{T}c \ \forall c, T > 0 \qquad \bar{e}_{c,T} - \frac{2\pi}{T}c \underset{T\to 0}{\longrightarrow} 0 \text{ for } c > 0 \text{ fixed.} \tag{3.24}$$

*Proof.* Observe that condition $u(0) = 0$, which appears in (3.22), is nothing but a normalization condition. By this, we mean that, since our equation is autonomous and every solution to

$$\begin{cases} u'' + cu' + g(u) = \bar{e} \\ u(t+T) = 2\pi + u(t) \ \forall t \in \mathbb{R}; \end{cases} \tag{3.25}$$

verifies $\lim_{t\to+\infty} u(t) = +\infty$; $\lim_{t\to-\infty} u(t) = -\infty$, solutions to (3.25) are, up to translations in the time variable $t$, solutions to (3.22). Therefore, in order to find $\bar{e} \in \mathbb{R}$ such that (3.22) has at least one solution, it suffices to show the existence of $\bar{e} \in \mathbb{R}$ such that (3.25) has some solution. At this point we introduce the change of variables $v(t) := u(t) - \frac{2\pi}{T}t$, which transforms (3.25) into

$$\begin{cases} v'' + cv' + g(\frac{2\pi}{T}t + v) = \bar{e} - \frac{2\pi c}{T} \\ v(t+T) = v(t) \ \forall t \in \mathbb{R}; \end{cases} \tag{3.26}$$

and the existence of the constant $\bar{e}$ we were looking for, follows now from Schauder's fixed point theorem. Thus, we may fix such an $\bar{e}_{c,T} \in \mathbb{R}$ and a corresponding solution $u_{c,T}$ to (3.22) for $\bar{e} = \bar{e}_{c,T}$; $v_{c,T}(t) := u_{c,T}(t) - \frac{2\pi}{T}t$. Now, for $\bar{e} = \bar{e}_{c,T}$, $t \mapsto u_{c,T}(t+s)$ is a solution of (3.25) for every $s \in \mathbb{R}$ and, consequently, $t \mapsto u_{c,T}(t+s) - \frac{2\pi}{T}t = v_{c,T}(t+s) + \frac{2\pi}{T}s$ is a solution to (3.26) for every $s \in \mathbb{R}$ .

Second orden, periodic problems such as (3.26), having a nontrivial curve

$$\begin{aligned} \gamma : \mathbb{R} &\to W_{1,1}(\mathbb{T}) \\ s &\mapsto \tau_s v_{c,T} + \frac{2\pi}{T}s \end{aligned} \tag{3.27}$$

of solutions for some value $\bar{e}_{c,T}$ of $\bar{e}$ are usually called degenerate, and have been extensively studied in the literature. In particular, it is known ([77], see also [67]) that system (3.26) cannot have solutions for $\bar{e} \neq \bar{e}_{c,T}$ and not other solutions than $\{\gamma(s) : s \in \mathbb{R}\}$ for $\bar{e} = \bar{e}_{c,T}$. We shortly recall the argument for completeness. Let us take $\bar{e} \in \mathbb{R}$ such that (3.26) has a solution $u$. We consider the quantities

$$\alpha := \min\{s \in \mathbb{R} : \exists t \in \mathbb{R} \text{ with } u(t) = [\gamma(s)](t)\} \tag{3.28}$$

$$\beta := \max\{s \in \mathbb{R} : \exists t \in \mathbb{R} \text{ with } u(t) = [\gamma(s)](t)\} \tag{3.29}$$

Then, there exist $t_\alpha, t_\beta \in \mathbb{R}$ such that

$$\begin{array}{lll} [\gamma(\alpha)](t_\alpha) = u(t_\alpha); & [\gamma(\alpha)]'(t_\alpha) = u'(t_\alpha); & [\gamma(\alpha)](t) \le u(t) \ \forall t \in \mathbb{R} \\ [\gamma(\beta)](t_\beta) = u(t_\beta); & [\gamma(\beta)]'(t_\beta) = u'(t_\beta); & [\gamma(\beta)](t) \ge u(t) \ \forall t \in \mathbb{R} \end{array}$$

and we obtain

$$\bar{e}_{c,T} - \frac{2\pi}{T}c = [\gamma(\alpha)]''(t_\alpha) + c[\gamma(\alpha)]'(t_\alpha) + g\left([\gamma(\alpha)](t_\alpha) + \frac{2\pi}{T}t_\alpha\right) \le$$
$$\le u''(t_\alpha) + cu'(t_\alpha) + g\left(u(t_\alpha) + \frac{2\pi}{T}t_\alpha\right) = \bar{e} - \frac{2\pi}{T}c$$

so that

$$\bar{e}_{c,T} \leq \bar{e}$$

and similarly, comparing $u$ and $\gamma(\beta)$ in a neighborhood of $t_\beta$, we get

$$\bar{e}_{c,T} \geq \bar{e}$$

obtaining the equality $\bar{e} = \bar{e}_{c,T}$. Now,

$$[\gamma(\alpha)](t_\alpha) = u(t_\alpha); \qquad [\gamma(\alpha)]'(t_\alpha) = u'(t_\alpha)$$

so that $\gamma(\alpha) = u$. Similarly, $\gamma(\beta) = u$.

A similar reasoning shows indeed that the curves $\gamma(a)$ and $\gamma(b)$ do not intersect as soon as $a \neq b$. Otherwise, there would exist $a, b \in \mathbb{R}$ with $a < b$ and $\hat{t} \in \mathbb{R}$ such that $[\gamma(a)](\hat{t}) = [\gamma(b)](\hat{t})$. We may define $u := \gamma(b)$, and $\alpha$ as in (3.28), and the argument above shows that $\gamma(b) = u = \gamma(\alpha)$, which is a contradiction since $\alpha \leq a < b$ and consequently, $\gamma(\alpha)$ and $\gamma(b)$ have different mean. And we conclude that

$$a < b \to [\gamma(a)](t) < [\gamma(b)](t) \ \forall t \in \mathbb{T}$$

It means also that no different solutions to system (3.22) ($\bar{e} = \bar{e}_{c,T}$) intersect. On the other hand, as $u_{c,T}(t+T) = u_{c,T}(t) + 2\pi$, there exists some point $t_0 \in \mathbb{R}$ such that $u'_{c,T}(t_0) > 0$. Let us assume that the same inequality did not hold always and let $t_1$ be the minimum of those $t > t_0$ such that $u'_{c,T}(t) = 0$. Being $u_{c,T}$ a solution of the autonomous equation (3.22) ($\bar{e} = \bar{e}_{c,T}$), which is not an equilibrium, $u''_{c,T}(t_1) \neq 0$, and we deduce $u''_{c,T}(t_1) < 0$.

In this way, for $s \neq 0$ small, $u_{c,T}$ and $t \mapsto u_{c,T}(t+s)$ are different solutions to (3.22) -they are different at $t_0$- but intersecting near $t_1$, which is a contradiction.

Being $g$ bounded, it follows from (3.26) that, for fixed $c > 0$,

$$\left\| v_{c,T}(\cdot) - \frac{1}{T}\int_0^T v_{c,T}(s)ds \right\|_{L^\infty(\mathbb{R}/T\mathbb{Z})} \longrightarrow 0 \text{ as } T \to 0,$$

so that, as stated, $u_c(\frac{T}{2\pi}t) \to t$ uniformly with respect to $t \in \mathbb{R}$ as $T \to 0$. Finally, to prove (3.24), just multiply equation (3.26) by $\frac{2\pi}{T} + v'$ and integrate on $(\mathbb{R}/T\mathbb{Z})$, to get:

$$\frac{c}{2\pi}\int_0^T v'_{c,T}(s)^2 ds = \bar{e}_{c,T} - \frac{2\pi c}{T}$$

so that $\bar{e}_{0,T} = 0$, $\bar{e}_{c,T} > \frac{2\pi}{T}c \ \forall c, T > 0$. Furthermore,

$$\bar{e}_{c,T} - \frac{2\pi}{T}c = \frac{1}{T}\int_0^T g(\frac{2\pi}{T}s + v_{c,T}(s))ds = \frac{1}{2\pi}\int_0^{2\pi} g(s + v_{c,T}(\frac{T}{2\pi}s))ds \underset{T\to 0}{\longrightarrow} \frac{1}{2\pi}\int_0^{2\pi} g(s)ds = 0.$$

□

**Remark 3.3.2.** *Assume now $c \in \mathbb{R}$ is fixed. The mapping:*

$$\Psi : \left\{ g \in C^1(\mathbb{R}/2\pi\mathbb{Z})\backslash\{0\} : \int_0^T g(s)ds = 0 \right\} \to \left\{ v \in C^3(\mathbb{T}) : v(0) = 0, \frac{2\pi}{T} + v'(t) > 0 \ \forall t \in \mathbb{R} \right\}$$

*mapping $g$ into the only solution $v$ to (3.26) with $\bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c$ verifying $v(0) = 0$ is continuous. Furthermore, it is clearly bijective, its inverse being given by the rule*

$$v \mapsto -(v'' + cv') \circ \left[\frac{2\pi}{T}\iota + v\right]^{-1} + \frac{1}{T}\int_0^T (v'' + cv') \circ \left[\frac{2\pi}{T}\iota + v\right]^{-1}(x)dx$$

*($\iota(t) := t \ \forall t \in \mathbb{R}$), which is also continuous. Then, both laws are homeomorphisms and it is easily checked that they conserve regularity:*

$$\Psi(g) \in C^{n+2}(\mathbb{T}) \iff g \in C^n(\mathbb{R}/2\pi\mathbb{Z}) \qquad \forall n \geq 1$$

*In particular, for any trigonometric polynomial $P(t) = p_0 + \sum_{j=1}^r [p_j \cos(j\frac{2\pi}{T}t) + q_j \sin(j\frac{2\pi}{T}t)]$ with $P'(t) > -\frac{2\pi}{T} \ \forall t \in \mathbb{T}$, there exists $g \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ whose associated curve $\Psi(g)$ is exactly $P$.*

## 3.4 Many periodic solutions bifurcating from a closed loop at a constant external force

Thus, we have found that the equation

$$\mathcal{H}(\lambda, v, \bar{e}, \varphi) = 0 \qquad\qquad v \in W_{1,1}(\mathbb{T})$$

with $\mathcal{H}$ given in (3.21), has a nontrivial curve $\gamma$ (given in (3.27)), of solutions for $\lambda = 0,\ \bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c,\ \varphi = \varphi_0(t) := \frac{2\pi}{T}t - \text{Ent}\left(\frac{2\pi}{T}t\right)$. To set ourselves under the framework of section 2, we still have to check

(b) $\dim\left[\ker \partial_x\mathcal{H}(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0)\right] = 1$, (that is, $\langle\gamma'(s)\rangle = \ker\left(\partial_x\mathcal{H}(0, \gamma(s), \bar{e}_{c,T}, \varphi_0)\right)$ for every $s \in \mathbb{R}$.

This is to say that the only solutions of the linear problem

$$\begin{cases} w'' + cw' + g'(u_{c,T}(t+s))w = 0 \\ w(t+T) = w(t)\ \forall t \in \mathbb{R} \end{cases} \tag{3.30}$$

should be the scalar multiples of $\tau_s u'_{c,T}$, for every $s \in \mathbb{R}$. Equivalently, the only $T-$periodic solutions of Hill's equation

$$w'' + cw' + g'(u_{c,T}(t))w = 0 \tag{3.31}$$

should be the scalar multiples of $u'_{c,T}$. To see this we apply the reduction of order method; we already know that $u'_{c,T}$ is a solution to (3.31) and we conclude that

$$w_{c,T}(t) := u'_{c,T}(t)\int_0^t \frac{e^{-cr}dr}{u'_{c,T}(r)^2}$$

is another. Of course, this latter is not $T-$periodic,

$$w_{c,T}(0) = 0; \qquad\qquad w_{c,T}(T) > 0.$$

We next establish (d) for any $a < b \in \mathbb{R}$. With this aim, take any sequence $\{v_n\} \subset W_{1,1}(\mathbb{T})$ such that

$$\left\{\mathcal{H}(0, v_n, \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0)\right\}_n \to 0$$

$$\left\{\frac{1}{T}\int_0^T v_n(t)dt\right\}_n \text{ bounded.}$$

For any $n \in \mathbb{N}$, write $v_n := \bar{v}_n + \tilde{v}_n$, $\bar{v}_n := \frac{1}{T}\int_0^T v_n(t)dt$, $\tilde{v}_n \in \tilde{X}$. By hypothesis, $\{\bar{v}_n\}$ is bounded, so that it has some convergent subsequence. Let us check that the same thing happens also for $\{\tilde{v}_n\}$. We call, for each $n \in \mathbb{N}$, $\theta_n := \mathcal{H}(0, v_n, \bar{e} - \frac{2\pi}{T}c, \varphi_0)$, so that

$$\tilde{v}_n + \mathcal{K}\left[\tilde{v}_n + c\tilde{v}_n'\right] = -\mathcal{K}\left[\mathcal{N}(\bar{v}_n + \tilde{v}_n + c\tilde{v}_n' + \varphi_0)\right] - \bar{e}_{c,T} + \frac{2\pi}{T}c + \theta_n \quad \forall n \in \mathbb{N} \tag{3.32}$$

The sequence $\{\mathcal{N}(\bar{v}_n + \tilde{v}_n + c\tilde{v}_n' + \varphi_0)\}$ being bounded in $L^\infty(\mathbb{T})$, there exists a subsequence $\{v_{\sigma(n)}\}$ of $\{v_n\}$ such that $\left\{\mathcal{K}\left[\mathcal{N}(\bar{v}_{\sigma(n)} + \tilde{v}_{\sigma(n)} + c\tilde{v}'_{\sigma(n)} + \varphi_0)\right]\right\}_n$ is convergent in $W_{1,1}(\mathbb{T})$. As the operator $v \mapsto v + \mathcal{K}\left[v + cv'\right]$ is a linear homeomorphism when seen from $\tilde{X}$ to its image (endowed with the $W_{1,1}(\mathbb{T})$ topology), we deduce from (3.32) that $\{\tilde{v}_{\sigma(n)}\}$ itself converges in $\tilde{X}$. Thus, there exists a convergent subsequence of $\{v_n\}$ and the limit must be a zero of $\mathcal{H}(0, \cdot, \bar{e} - \frac{2\pi}{T}c, \varphi_0)$. However, the set of zeroes of this mapping, as shown in Proposition 3.3.1, reduces to $\gamma(\mathbb{R})$, implying (d). We finally note that hypothesis (c) holds as soon as $g \in C^2(\mathbb{R})$.

To proceed further with the scheme of Section 2., let us pick a nonzero T-periodic solution $\nu_{c,T}$ of the adjoint equation of (3.31),

$$\Omega'' - c\Omega' + g'(u_{c,T}(t))\Omega = 0 \tag{3.33}$$

In the conservative case, problem (3.30) is self-adjoint and $\nu_{0,T}$ can be taken as $u'_{0,T}$. Consequently, $\nu_{0,T}$ does not change sign on $\mathbb{T}$. Let us see that the same thing happens for $\nu_{c,T}$ when $c \in \mathbb{R}$ is arbitrary.

**Lemma 3.4.1.** *For any $c \in \mathbb{R}$, consider the Hill's equation*

$$y'' + cy' + \alpha(t)y = 0 \tag{$E_c$}$$

*where $\alpha : \mathbb{R} \to \mathbb{R}$ is a given locally integrable, $T$-periodic function. Then, $(E_c)$ has a $T$-periodic, positive solution if and only if $(E_{-c})$ has a $T$-periodic, positive solution.*

*Proof.* The solutions of $(E_c)$ are related with those of $(E_{-c})$ by the rule

$$y(t) \text{ is a solution of } (E_c) \iff z(t) = e^{ct}y(t) \text{ is a solution of } (E_{-c}) \tag{3.34}$$

Using a Sturm-Liouville argument we know that, in case $(E_c)$ has a never vanishing solution, the equation is disconjugate, meaning that any other nonzero solution of $(E_c)$ vanishes, at most, at one single point in $\mathbb{R}$. Thanks to (3.34) we know that also $(E_{-c})$ is disconjugate, and therefore, its periodic solution cannot vanish. □

Observe that, for any $s \in \mathbb{R}$, $\tau_s \nu_{c,T}$ is a solution to the adjoint problem of (3.30)

$$\begin{cases} \Omega'' - c\Omega' + g'(u_{c,T}(t+s))\Omega = 0 \\ \Omega(t+T) = \Omega(t) \ \forall t \in \mathbb{R} \end{cases} \tag{3.35}$$

Thus, given $h \in L^1(\mathbb{T})$, the nonhomogeneous, linear problem

$$\begin{cases} w'' + cw' + g'(u_{c,T}(t+s))w = h(t) \\ w(t+T) = w(t) \ \forall t \in \mathbb{R} \end{cases}$$

has solution if and only if $\int_0^T h(t)\nu_{c,T}(s+t)dt = 0$. Using (3.19) we deduce

$$\operatorname{im} \partial_x \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right) = \left\{ v \in W_{1,1}(\mathbb{T}) : \int_0^T v(t)\big[\mathcal{L}^0(\tau_s\nu_{c,T}) - \tau_s\nu_{c,T}\big](t)ds = 0 \right\}$$

so that, thanks to Lemma 3.4.1 above, we may take $m(s) \equiv 1 \ \forall s \in \mathbb{R}$, and

$$\left[\operatorname{im} \partial_x \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right)\right]^\perp = \langle \mathcal{L}^0(\tau_s\nu_{c,T}) - \tau_s\nu_{c,T}\rangle = \langle \tau_s(\mathcal{L}^0(\nu_{c,T}) - \nu_{c,T})\rangle \ \forall s \in \mathbb{R},$$

equality where the identifications $L^2(\mathbb{T}) \equiv L^2(\mathbb{T})^* \subset W_{1,1}(\mathbb{T})^*$ have been utilized. In this way, we obtain a explicit form for the curve $\sigma$ in (3.10):

$$\begin{aligned} \sigma : \mathbb{R} &\to L^2(\mathbb{T}) \subset W_{1,1}(\mathbb{T})^* \\ s &\mapsto \tau_s\big[\mathcal{L}^0(\nu_{c,T}) - \nu_{c,T}\big] \end{aligned}$$

Finally, we are lead to consider the real valued, continuous curve $\Gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} s \mapsto \left\langle \partial_\lambda \mathcal{H}\left(0, \gamma(s), \bar{e}_{c,T} - \frac{2\pi}{T}c, \varphi_0\right), \ \sigma(s) \right\rangle &= \\ = -\int_0^T \Big(\mathcal{K}\big[\mathcal{N}_{g'}(\gamma(s) + \varphi_0)\psi_0\big]\Big)\Big(\mathcal{L}^0(\tau_s\nu_{c,T}) - \tau_s\nu_{c,T}\Big)dt &= \\ = -\int_0^T g'(u_{c,T}(t+s))\nu_{c,T}(t+s)\psi_0(t)dt = \int_0^T \big(\nu''_{c,T}(t+s) - c\nu'_{c,T}(t+s)\big)\psi_0(t)dt \end{aligned} \tag{3.36}$$

that is, the convolution of $\nu''_{c,T} - c\nu'_{c,T}$ and $\psi_0$. In the conservative case, $\nu_{0,T} = u'_{0,T}$ and $\Gamma$ is the convolution of $u'''_{0,T}$ and $\psi_0$.

The following result is now an straightforward consequence of Lemma 3.2.2

**Proposition 3.4.2.** *Let*

$$A_n := \frac{2}{T}\int_0^T \nu_{c,T}(t)\cos(n\frac{2\pi}{T}t)dt \qquad B_n := \frac{2}{T}\int_0^T \nu_{c,T}(t)\sin(n\frac{2\pi}{T}t)dt,$$

*be the sequences of Fourier coefficients of $\nu_{c,T}$. We assume that, for some $n_0 \in \mathbb{N}$,*

$$A_{n_0}^2 + B_{n_0}^2 \neq 0$$

*Then, given any $\epsilon > 0$, it is possible to find $\varphi \in C^\infty(\mathbb{T})$ and $v_0, \ldots, v_{2n_0} \in C^2(\mathbb{T})$, $\varrho_0, \ldots, \varrho_{2n_0} \in \mathbb{R}^+$ such that:*

$$v_{2n_0}(t) = v_0(t) + 2\pi \ \forall t \in \mathbb{T}, \ \varrho_0 = \varrho_{2n_0} \tag{3.37}$$

$$v_q''(t) + cv_q'(t) + g(v_q(t) + \varphi(t)) = \bar{e}_{c,T} - \frac{2\pi}{T}c + (-1)^q\varrho_q \ \forall t \in \mathbb{T}, \ \forall q = 0, \ldots, 2n_0 \tag{3.38}$$

$$v_{q-1}(t) < v_q(t) \ \forall t \in \mathbb{T}, \ \forall q = 1, \ldots, 2n_0 \tag{3.39}$$

*Proof.* Write $\nu_{c,T}$ as the sum of its Fourier series

$$\nu_{c,T}(t) = \sum_{n=1}^{\infty}\left[A_n\cos(\frac{2\pi}{T}nt) + B_n\sin(\frac{2\pi}{T}nt)\right] + A_0$$

Being $\nu_{c,T} \in C^2(\mathbb{T})$, we are allowed to derivate twice in the infinite sum above to get

$$\nu_{c,T}''(t) + c\nu_{c,T}'(t) =$$
$$= \sum_{n=1}^{\infty}\left[\left[-n^2\left(\frac{2\pi}{T}\right)^2 A_n + cn\frac{2\pi}{T}B_n\right]\cos(\frac{2\pi}{T}nt) + \left[-cn\frac{2\pi}{T}A_n - n^2\left(\frac{2\pi}{T}\right)^2 B_n\right]\sin(\frac{2\pi}{T}nt)\right]$$

Observe now that, if for some $n \in \mathbb{N}$,

$$\begin{matrix} -n^2\left(\frac{2\pi}{T}\right)^2 A_n & +cn\frac{2\pi}{T}B_n & = 0 \\ -cn\frac{2\pi}{T}A_n & -n^2\left(\frac{2\pi}{T}\right)^2 B_n & = 0 \end{matrix}$$

then, $A_n = 0 = B_n$, since the determinant of the linear system is strictly positive. We conclude that

$$C_{n_0} := \left[cn_0\frac{2\pi}{T}B_{n_0} - n_0^2\left(\frac{2\pi}{T}\right)^2 A_{n_0}\right]^2 + \left[-cn_0\frac{2\pi}{T}A_{n_0} - n_0^2\left(\frac{2\pi}{T}\right)^2 B_{n_0}\right]^2 > 0$$

At this point, we choose $\psi_0(t) = \frac{2}{T}\cos(\frac{2\pi}{T}n_0t)$ in (3.36). We obtain:

$$\Gamma(s) = \tilde{A}_{n_0}\cos(n_0\frac{2\pi}{T}s) + \tilde{B}_{n_0}\sin(n_0\frac{2\pi}{T}s)$$

for some $\tilde{A}_{n_0}, \ \tilde{B}_{n_0} \in \mathbb{R}$ with $\sqrt{\tilde{A}_{n_0}^2 + \tilde{B}_{n_0}^2} = \sqrt{C_{n_0}} > 0$. This function has exactly $2n_0$ zeroes in $[0, T[$ and, on each one, its derivative does not vanish. The theorem follows from Lemma 3.2.2 □

From such a scheme of ordered lower-upper-lower-upper... solutions, it follows immediately the existence of at least $n_0$ (geometrically) different solutions for the equation $\{v'' + cv' + g(v + \varphi(t)) = \bar{e}_{c,T} - \frac{2\pi}{T}c\}$ -one between each pair of consecutive ordered lower and upper solutions. The three solutions theorem (see [4]) in fact implies the existence of *at least* $2n_0$ different solutions for this same equation. These solutions turn to come from mappings with nonzero degree so that all this keeps its validity under small perturbations. We state the precise result below:

**Proposition 3.4.3.** *Let $f_0 : (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R},\ (t,x) \mapsto f_0(t,x)$ be continuous. Assume $v_0, v_1, v_2, v_3 \in C^2(\mathbb{T})$ verify:*

1. $v_0(t) < v_1(t) < v_2(t) < v_3(t)\ \forall t \in \mathbb{R}$.
2. $(-1)^i \Big[ v_i''(t) + cv_i'(t) + f_0(t, v_i(t)) \Big] > 0\ \forall t \in \mathbb{T},\ i = 0, \ldots, 3.$

*Then, there exists $\epsilon > 0$ such that, for any Carathéodory function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with*

$$\int_0^T \sup_{x\in\mathbb{R}} \Big( \big\{ |f_0(t,x) - f(t,x)| \big\} \Big) dt < \epsilon, \tag{3.40}$$

*the perturbed problem*

$$\begin{cases} w'' + cw' + f(t,w) = 0 \\ w \in W_{2,1}(\mathbb{T}) \end{cases} \tag{3.41}$$

*has at least three solutions $w_1,\ w_2,\ w_3$ verifying*

1. $v_0(t) < w_1(t) < v_1(t),\ v_2(t) < w_3(t) < v_3(t)\ \forall t \in \mathbb{T}$.
2. $v_1(\hat t) < w_2(\hat t) < v_2(\hat t)$ *for some* $\hat t \in \mathbb{T}$.

*Proof.* Problem (3.41) can be rewritten as

$$w'' + cw' - w + [f(t,w) + w] = 0 \qquad\qquad w \in W_{2,1}(\mathbb{T}) \tag{3.42}$$

The advantage in this reformulation is that the linear operator $W_{2,1}(\mathbb{T}) \to L^1(\mathbb{T}),\ w \mapsto w'' + cw' - w$, is invertible. We call $\mathcal{K} : L^1(\mathbb{T}) \to W_{2,1}(\mathbb{T})$ its inverse, so that problem (3.42) becomes

$$w + \mathcal{K}[\mathcal{N}_f(w) + w] = 0 \qquad\qquad w \in C(\mathbb{T}) \tag{3.43}$$

being $\mathcal{N}_f$ the Nemytskii operator associated with $g$, that is, $\mathcal{N}_f : C(\mathbb{T}) \to L^1(\mathbb{T})$ is the continuous mapping defined by $[N_f(x)](t) = f(t, x(t))\ \forall t \in \mathbb{T},\ \forall x \in C(\mathbb{T})$.

We consider the completely continuous, nonlinear operator

$$\mathcal{T}_f : C(\mathbb{T}) \to C(\mathbb{T}) \qquad w \mapsto -\mathcal{K}[w + \mathcal{N}_f(w)] \tag{3.44}$$

so that the solutions of (3.41) coincide with the fixed points of $\mathcal{T}_f$. At this point it would be desirable to compute the Leray-Schauder degree of $I_{C(\mathbb{T})} - \mathcal{T}_f$ on convenient open sets. We define

$$\mathcal{G}_1 := \{w \in C(\mathbb{T}) : v_0(t) < w(t) < v_1(t)\} \qquad \mathcal{G}_3 := \{w \in C(\mathbb{T}) : v_2(t) < w(t) < v_3(t)\}$$
$$\mathcal{G} := \{w \in C(\mathbb{T}) : v_0(t) < w(t) < v_3(t)\} \qquad \mathcal{G}_2 := \mathcal{G} \backslash (\overline{\mathcal{G}_1 \cup \mathcal{G}_3})$$

Being $v_0$ a strict $C^2$ subsolution and $v_3$ a strict $C^2$ supersolution for problem (3.41) with $f = f_0$, which is continuous, it follows that there may not exist fixed points of $\mathcal{T}_{f_0}$ on $\partial\mathcal{G} = \{w \in C(\mathbb{T}) : v_0(t) \le w(t) \le v_3(t)\ \forall t \in \mathbb{T}\} \backslash \mathcal{G}$. Similarly, there may not exist fixed points of $\mathcal{T}_{f_0}$ on $\partial G_1$, $\partial G_3$ or $\partial G_2 \subset \partial G \cup \partial G_1 \cup \partial G_3$. Indeed, it follows from the method of upper and lower solutions that

$$\deg(I_{C(\mathbb{T})} - \mathcal{T}_{f_0}, \mathcal{G}) = 1 \qquad \deg(I_{C(\mathbb{T})} - \mathcal{T}_{f_0}, \mathcal{G}_1) = 1 \qquad \deg(I_{C(\mathbb{T})} - \mathcal{T}_{f_0}, \mathcal{G}_3) = 1$$

so that, by the aditivity property of the Leray-Schauder degree,

$$\deg(I - \mathcal{T}_{f_0}, \mathcal{G}_2) = -1 \tag{3.45}$$

On the other hand, $\mathcal{T}_{f_0}$ being completely continuous, $I - \mathcal{T}_{f_0}$ is closed. It means in particular that $(I - \mathcal{T}_{f_0})\big(\partial\mathcal{G} \cup \partial\mathcal{G}_1 \cup \partial\mathcal{G}_3\big)$ is a closed subset of $C(\mathbb{T})$ and, as 0 is not in this set, there exist $\delta > 0$ such that $\|w - \mathcal{T}_{f_0}(w)\|_\infty > \delta\ \forall w \in \partial\mathcal{G} \cup \partial\mathcal{G}_1 \cup \partial\mathcal{G}_3$. The continuity of $\mathcal{K}$ as an operator from $L^1(\mathbb{T})$ to $C(\mathbb{T})$ implies that there exists $\epsilon > 0$ such that $\|\mathcal{K}(x)\|_\infty < \delta\ \forall x \in L^1(\mathbb{T})$ with $\|x\|_{L^1(\mathbb{T})} < \epsilon$, and this means that $\|\mathcal{T}_f(w) - \mathcal{T}_{f_0}(w)\|_\infty < \delta\ \forall w \in C(\mathbb{T})$, so that $w \neq \mathcal{T}_f(w)\ \forall w \in \partial\mathcal{G} \cup \partial\mathcal{G}_1 \cup \partial\mathcal{G}_3$ as soon as $f$ verifies (3.40). The invariance by homotopies of the Leray-Schauder degree shows that

$$\deg(I_{C(\mathbb{T})} - \mathcal{T}_f, \mathcal{G}) = 1 \qquad \deg(I_{C(\mathbb{T})} - \mathcal{T}_f, \mathcal{G}_1) = 1 \qquad \deg(I_{C(\mathbb{T})} - \mathcal{T}_f, \mathcal{G}_3) = 1$$

which proves the theorem. □

Along next results, it will be necessary to take into account, not only the time period $T$, which was, so far, fixed, but also all its divisors. Let us call, for any $m \in \mathbb{N}$, $A_{n,m}$ and $B_{n,m}$ the respective quantities $A_n$ and $B_n$ corresponding to the time period $\frac{T}{m}$.

**Corollary 3.4.4.** *Assume that, for some $n, m \in \mathbb{N}$, $A_{n,m}^2 + B_{n,m}^2 \neq 0$. Then, there exists an open set $\mathcal{O} \subset L^1(\mathbb{T})$ with $\mathcal{O} \cap \Big\{e = \tilde{e} + \bar{e} \in L^1(\mathbb{T}) : \bar{e} = \bar{e}_{c,T/m} - 2\pi c\, m/T\Big\} \neq \emptyset$ such that, for any $e \in \mathcal{O}$, problem* (3.1) *has at least $n$ geometrically different solutions.*

*Proof.* From Proposition 3.4.2 we know the existence of $\varphi \in C^\infty(\mathbb{R}/\frac{T}{m}\mathbb{Z})$ and a scheme of lower and upper solutions as given there on the interval $[0, \frac{T}{m}]$. These give rise to a corresponding scheme of ordered lower and upper solutions associated to $\varphi \in C^\infty(\mathbb{T})$ on the interval $(\mathbb{R}/T\mathbb{Z})$. The result follows now from Proposition 3.4.3.

□

**Corollary 3.4.5.** *Let $n \in \mathbb{N}$ be given, and assume that, for infinitely many $m \in \mathbb{N}$, $\nu_{c,T/m}$ is not a trigonometric polynomial of degree strictly lower than $n$. Then, for any $\epsilon > 0$, there exists an open set $\mathcal{O} = \mathcal{O}_{n,\epsilon} \subset L^1(\mathbb{T})$ such that*

$$\mathcal{O} \cap \tilde{L}^1(\mathbb{T}) \neq \emptyset \quad \textit{if } c = 0$$
$$\mathcal{O} \cap \big\{e = \bar{e} + \tilde{e} \in L^1(\mathbb{T}) : -\epsilon < \bar{e} < 0\big\} \neq \emptyset \neq \mathcal{O} \cap \big\{e = \bar{e} + \tilde{e} \in L^1(\mathbb{T}) : 0 < \bar{e} < \epsilon\big\} \quad \textit{if } c \neq 0$$

*and for any $e \in \mathcal{O}$, problem (3.1) has at least $2n$ geometrically different solutions.*

*Proof.* The case $c = 0$ follows directly from Corollary 3.4.4. Concerning the case $c > 0$, observe that it suffices to prove $\mathcal{O} \cap \big\{e = \bar{e} + \tilde{e} \in L^1(\mathbb{T}) : 0 < \bar{e} < \epsilon\big\} \neq \emptyset \ \forall \epsilon > 0$, since the remaining statements follow from the change of variables $\hat{u} = -u,\ \hat{g}(x) := -g(-x)$. In this way, this becomes a consequence of Corollary 3.4.4 and the fact that, as seen in Proposition 3.3.1, $\{\bar{e}_{c,T/m} - \frac{2\pi}{T} mc\}_m$ is a sequence of positive numbers converging to 0 as $m \to +\infty$.

□

*Proof of Theorem 3.1.1 when $g$ is the restriction to the real line of an entire function.* In view of Corollary 3.4.5, we may assume there exist $n_0,\ m_0 \in \mathbb{N}$ such that $\nu_{c,T/m}$ is a trigonometric polynomial of degree not bigger that $n_0$ for all $m \geq m_0$. We choose $\nu_{c,T/m}$ so that $\|\nu_{c,T/m}\|_{L^\infty[0,\frac{T}{m}]} = 1$ and write $\nu_{c,T/m}(t) = \sum_{j=-n_0}^{n_0} \Omega_{m,j} e^{\frac{2\pi m}{T} ijt},\ t \in [0, \frac{2\pi}{m}]\ \forall m \geq m_0$, where the complex coefficients $\{\Omega_{m,j}\}_{-n_0 \leq j \leq n_0}$ verify $\Omega_{m,-j} = \overline{\Omega}_{m,j},\ j : -n_0, \ldots, n_0$. The sequences $\{\Omega_{m,j}\}_{m \geq m_0}$ are bounded for any $j : -n_0, \ldots, n_0$, and, after possibly passing to a subsequence, we may assume $\{\Omega_{m,j}\} \to \Omega_j\ \forall j : -n_0, \ldots, n_0$. Passing to the limit in the inequality $\sum_{j=-n_0}^{n_0} |\Omega_{m,j}| \geq 1\ \forall m \geq m_0$ we deduce that $\sum_{j=-n_0}^{n_0} |\Omega_j| \geq 1$, and the trigonometric polynomial $\nu_c(t) := \sum_{j=-n_0}^{n_0} \Omega_j e^{jit}$ is not the zero polynomial. We recall the differential equation verified by $\nu_{c,T/m}$

$$\nu_{c,T/m}''(t) - c\nu_{c,T/m}'(t) + g'(u_{c,T/m}(t))\nu_{c,T/m}(t) = 0, \quad 0 \leq t \leq \frac{T}{m}$$

or, what is the same,

$$\nu_{c,T/m}''\big(\frac{T}{2\pi m}t\big) - c\nu_{c,T/m}'\big(\frac{T}{2\pi m}t\big) + g'\big(u_{c,T/m}(\frac{T}{2\pi m}t)\big)\nu_{c,T/m}\big(\frac{T}{2\pi m}t\big) = 0, \quad 0 \leq t \leq 2\pi$$

Using the explicit form of $\nu_{c,T/m}$ as a trigonometric polynomial and passing to the limit as $m \to +\infty$, we deduce

$$\nu_c''(t) - c\nu_c'(t) + g'(t)\nu_c(t) = 0, \quad 0 \leq t \leq 2\pi$$

since, as shown in Proposition 3.3.1, $u_{c,T/m}\big(\frac{T}{2\pi m}(t)\big) \to t$ uniformly with respect to $t \in \mathbb{R}$ as $m \to +\infty$. Here, we have an entire function which vanishes on a whole segment. It is, consequently, zero everywhere:

$$\nu_c''(z) - c\nu_c'(z) + g'(z)\nu_c(z) = 0\ \forall z \in \mathbb{C} \tag{3.46}$$

Observe now that both nonzero trigonometric polynomials $\nu_c$ and $\nu_c'' - c\nu_c'$ have the same degree (recall the proof of Proposition 3.4.2 above). Therefore, by similar argument to those carried out in the proof of Theorem 3.5.2, they have the same number of roots, counting multiplicity. However, as given in (3.46), any root of $\nu_c$ is a root of $\nu_c'' - c\nu_c'$, so that they are in fact equal. It means $g'(z) = 1\ \forall z \in \mathbb{C}$, which is a contradiction. The theorem is now proved. $\square$

The proof of Theorem 3.1.1 will be completed in Section 3.6 using a different approach. Now,we have the following consequence of Lemma 3.2.3.

**Theorem 3.4.6.** *Assume $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$, let $A_n$, $B_n$, $n \geq 1$, be the sequences of Fourier coefficients of $\nu_{c,T}$ as defined in Proposition 3.4.2, and fix $k, k' \in \mathbb{R}$. If, for some $n_0 \in \mathbb{N}$, $A_{n_0}^2 + B_{n_0}^2 \neq 0$, then there exists an open set $\mathcal{O} \subset L^1(\mathbb{R}/T\mathbb{Z})$ with $\mathcal{O} \cap \{e = \bar{e} + \tilde{e} \in L^1(\mathbb{R}/T\mathbb{Z}) : \bar{e} = \bar{e}_{c,T} - \frac{2\pi}{T}c\} \neq \emptyset$, such that, for any $e \in \mathcal{O}$, problem (3.1) has exactly $2n_0$ geometrically different solutions.*

## 3.5 The conservative pendulum problem

Theorem 3.4.6 can be criticized on the fact that it may not be easy to explicitly compute the Fourier series of the function $\nu_{c,T}$. In the conservative case, problem (3.30) is self-adjoint and things are simplified.

**Corollary 3.5.1.** *Let $g \in C^2(\mathbb{R}/2\pi\mathbb{Z})$, and let*

$$A_n := \frac{2}{T}\int_0^T u_{0,T}'(t)\cos(n\frac{2\pi}{T}t)dt \quad \text{and} \quad B_n := \frac{2}{T}\int_0^T u_{0,T}'(t)\sin(n\frac{2\pi}{T}t)dt, \quad n \geq 1, \tag{3.47}$$

*be the sequences of Fourier coefficients of $u_{0,T}'$. As before, fix $k, k' \in \mathbb{R}$. If, for some $n \in \mathbb{N}$, $A_n^2 + B_n^2 \neq 0$, then there exists an open set $\mathcal{O} \subset L^1(\mathbb{R}/T\mathbb{Z})$ with $\mathcal{O} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, such that for any $e \in \mathcal{O}$, problem (3.1) has exactly $2n$ geometrically different solutions.*

In [65], it was seen that, in the special case of the conservative, pendulum equation (problem (3.3), $g(u) = \Lambda\sin(u)$), $u_{0,T}'$ cannot be a trigonometric polynomial, and this was used to see that the number of periodic solutions for the forced pendulum equation was not bounded as the forcing term varies in $C^\infty(\mathbb{T})$. In this chapter we have seen (Remark 3.3.2) that the analogous statement is not true for an arbitrary $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ function $g$. However, an improved argument can be used to prove that $u_{0,T}'$ is not a trigonometric polynomial when $g$ belongs to an intermediate class of periodic nonlinearities, namely, those which are restriction to the real line of an entire function.

**Theorem 3.5.2.** *Assume that there exists an entire function whose restriction to the real line is $g$. Then, the number of $n \in \mathbb{N}$ such that $\left|\int_0^T u_{0,T}'(t)e^{in\frac{2\pi}{T}t}dt\right|^2 \neq 0$ is infinite. Consequently, there exists a sequence $\{n_m\}_{m\in\mathbb{N}} \to +\infty$ of natural numbers and, for each $m \in \mathbb{N}$, an open set $\mathcal{O}_{n_m} \subset L^1(\mathbb{R}/T\mathbb{Z})$ with $\mathcal{O}_{n_m} \cap \tilde{L}^1(\mathbb{R}/T\mathbb{Z}) \neq \emptyset$, such that for any $\bar{e} \in \mathcal{O}_{n_m}$, problem (3.1) has exactly $2n_m$ geometrically different solutions.*

*Proof.* To deny the statement of the Theorem above is to say that $u_{0,T}'$ is a trigonometric polynomial. In complex notation, this can be written as $u_{0,T}'(t) = \sum_{j=-p}^{p}\Omega_j e^{\frac{2\pi}{T}ijt}$ for some complex coefficients $\{\Omega_j\}_{j=-p}^{p}$, which should, furthermore, satisfy the relationship $\Omega_{-j} = \overline{\Omega_j}$. Of course, $u_{0,T}' \equiv cte$ is only possible if $g \equiv 0$, and thus, we should have $p \geq 1$, $\Omega_p \neq 0$. On the other hand, the inequality $u_{0,T}'(t) > 0\ \forall t \in \mathbb{R}$ implies $\Omega_0 > 0$. Now, $u_{0,T}(t) = \Omega_0 t + \sum_{j=1}^{p}\frac{T}{2\pi j}i\big(\Omega_{-j}e^{\frac{-2\pi}{T}ijt} - \Omega_j e^{\frac{2\pi}{T}ijt}\big)$, $u_{0,T}'''(t) = -\sum_{j=-p}^{p}(\frac{2\pi j}{T})^2\Omega_j e^{\frac{2\pi}{T}ijt}$, and the equality $u_{0,T}'''(t) = g'(u_{0,T}(t))u_{0,T}'(t)$ becomes

$$-\sum_{j=-p}^{p}(\frac{2\pi j}{T})^2\Omega_j e^{\frac{2\pi}{T}ijt} = g'\Big(\Omega_0 t + \sum_{j=1}^{p}\frac{T}{2\pi j}i\big(\Omega_{-j}e^{\frac{-2\pi}{T}ijt} - \Omega_j e^{\frac{2\pi}{T}ijt}\big)\Big)\sum_{j=-p}^{p}\Omega_j e^{\frac{2\pi}{T}ijt} \quad \forall t \in \mathbb{R}$$

Here, we have two entire functions which coincide on the real line. They are, consequently, equal on the whole complex plane:

$$-\sum_{j=-p}^{p}(\frac{2\pi j}{T})^2\Omega_j e^{\frac{2\pi}{T}ijz} = g'\Big(\Omega_0 z + \sum_{j=1}^{p}\frac{T}{2\pi j}i\big(\Omega_{-j}e^{\frac{-2\pi}{T}ijz} - \Omega_j e^{\frac{2\pi}{T}ijz}\big)\Big)\sum_{j=-p}^{p}\Omega_j e^{\frac{2\pi}{T}ijz} \quad \forall z \in \mathbb{C}$$

We multiply both sides of the equality above by $e^{i\frac{2\pi}{T}pz}$ to get

$$-\sum_{j=0}^{2p}\left(\frac{2\pi(j-p)}{T}\right)^2\Omega_{j-p}e^{\frac{2\pi}{T}ijz} = g'\Big(\Omega_0 z + \sum_{j=1}^{p}\frac{T}{2\pi j}i\big(\Omega_{-j}e^{\frac{-2\pi}{T}ijz} - \Omega_j e^{\frac{2\pi}{T}ijz}\big)\Big)\sum_{j=0}^{2p}\Omega_{j-p}e^{\frac{2\pi}{T}ijt}$$
$$\forall z\in\mathbb{C} \quad (3.48)$$

What is of interest for us in the equality above is the following: there exists an entire function $\vartheta:\mathbb{C}\to\mathbb{C}$ such that

$$-\sum_{j=0}^{2p}\left(\frac{2\pi(j-p)}{T}\right)^2\Omega_{j-p}e^{\frac{2\pi}{T}ijz} = \vartheta(z)\sum_{j=0}^{2p}\Omega_{j-p}e^{\frac{2\pi}{T}ijt} \quad \forall z\in\mathbb{C}$$

We consider the complex polynomials

$$q_1(z) := -\sum_{j=0}^{2p}\left(\frac{2\pi(j-p)}{T}\right)^2\Omega_{j-p}z^j \qquad q_2(z) := \sum_{j=0}^{2p}\Omega_{j-p}z^j$$

Both of them have degree $2p$, so that both of them have $2p$ roots, counting multiplicity. Furthermore, 0 is not a root of either. However, the equality

$$q_1(e^{\frac{2\pi i}{T}z}) = \vartheta(z)q_2(e^{\frac{2\pi i}{T}z}) \quad \forall z\in\mathbb{C}$$

says that every root of $q_2$ is a root of $q_1$ with at least, the same multiplicity. We deduce that there exists $\varsigma\in\mathbb{C}$ such that $q_1=\varsigma q_2$, that is

$$\vartheta(z)=\varsigma \ \forall z\in\mathbb{C}$$

In particular, $\vartheta(t)=g'(u_0(t))=\varsigma \ \forall t\in\mathbb{R}$. Thus, $\varsigma=0$ and $g\equiv 0$, a contradiction.

□

For pendulum-type equations without friction, a conservation of energy argument provides a explicit expression for $u_{0,T}$. Indeed, derivating the sum of kinetic plus potential energy along the trajectory $u_{0,T}$,

$$\mathcal{E}(t)=\frac{1}{2}u_{0,T}'(t)^2+G(u_{0,T}(t))$$

(here, G is any primitive of $g$), we find that the total energy does not change with time; there exists $\mathcal{E}_0\in\mathbb{R}$ (total energy), such that

$$\mathcal{E}_0=\frac{1}{2}u_{0,T}'(t)^2+G(u_{0,T}(t)) \ \forall t\in\mathbb{R}$$

As $u_{0,T}'(t)>0 \ \forall t\in\mathbb{R}$, we find that $\mathcal{E}_0>\max_{\mathbb{R}}G$, and, further,

$$u_{0,T}'(t)=\sqrt{2\big(\mathcal{E}_0-G(u_{0,T}(t))\big)} \ \forall t\in\mathbb{R}.$$

Equivalently,

$$\frac{u_{0,T}'(t)}{\sqrt{2\big(\mathcal{E}_0-G(u_{0,T}(t))\big)}}=1 \ \forall t\in\mathbb{R}.$$

We consider the mapping

$$\mathcal{F}_{\mathcal{E}_0}:\mathbb{R}\to\mathbb{R}$$
$$x\mapsto\frac{1}{\sqrt{2}}\int_0^x\frac{1}{\sqrt{\mathcal{E}_0-G(y)}}dy$$

which is a increasing diffeomorphism in $\mathbb{R}$. Now,

$$\mathcal{F}_{\mathcal{E}_0}(u_{0,T}(t))=t \ \forall t\in\mathbb{R}$$

as it follows by simply derivating both sides of the equality. Therefore,

$$u_{0,T}(t) = \mathcal{F}_{\mathcal{E}_0}^{-1}(t) \ \forall t \in \mathbb{R}^+ \,, \tag{3.49}$$

in particular,

$$T = \mathcal{F}_{\mathcal{E}_0}(2\pi) = \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}} dy \,. \tag{3.50}$$

Previous result motivates the following question: Will it be possible, to find natural numbers $n$ such that, with the notation of (3.47), $A_n^2 + B_n^2 = 0$? That is, may both terms of the same degree $n$ in the Fourier series of $u'_{0,T}$ vanish simultaneously? If the answer were 'no', at least for some 'nice' class of functions $g$, it would imply, as a consequence of Theorem 3.5.1, the existence, for each even number $2n$, of forcing terms $e \in L^1[0, 2\pi]$ such that (3.3) has *exactly* $2n$ solutions.

However, as seen in the introduction, this cannot be true in general, since, in case $g$ is $2\pi/p$-periodic for some entire number $p \geq 2$, the number of geometrically different solutions to (3.1), if finite, is always an entire multiple of $p$. Indeed, what happens here is that the associated curve $u'_{0,T}$ is $\frac{2\pi}{p}$-periodic and consequently, all Fourier coefficients of degree not an integer multiple of $p$ are zero.

On the other hand, numerical experiments carried out by the author seem to indicate that cosine Fourier coefficients of all orders

$$A_n := \int_0^T u'_{0,T}(t) \cos(n\frac{2\pi}{T}t)dt, \quad n \geq 0$$

are positive in the case of the pendulum equation $[g(u) = \Lambda \sin(u)]$. However, we do not know a proof of this fact, and *the question remains open.*

We observe here that all sine Fourier coefficients of $u'_{0,T}$ vanish as soon as $g$ is an odd function. Indeed, if this happens, the uniqueness of $u_{0,T}$ as a solution to (3.22) implies that

$$u_{0,T}(-t) = -u_{0,T}(t) \ \forall t \in \mathbb{R},$$

and, consequently,

$$u'_{0,T}(-t) = u'_{0,T}(t) \ \forall t \in \mathbb{R},$$

so that

$$B_n = \int_0^T u'_{0,T}(t) \sin(n\frac{2\pi}{T}t)dt = 0 \ \forall n \in \mathbb{N}$$

However, cosine Fourier coefficients can be shown to be positive when the time is big enough under our hypothesis $[\mathbf{G}_3]$. Indeed, it follows from (3.49) that

$$u'_{0,T}(t) = \frac{1}{\mathcal{F}'_{\mathcal{E}_0}(\mathcal{F}_{\mathcal{E}_0}^{-1}(t))} \quad \forall t \in \mathbb{R},$$

which implies

$$A_n = \int_0^T u'_{0,T}(t) \cos(n\frac{2\pi}{T}t)dt = \int_0^T \frac{1}{\mathcal{F}'_{\mathcal{E}_0}(\mathcal{F}_{\mathcal{E}_0}^{-1}(t))} \cos(n\frac{2\pi}{T}t)dt = \int_0^{2\pi} \cos(n\frac{2\pi}{T}\mathcal{F}_{\mathcal{E}_0}(x))dx =$$

$$= \int_0^{2\pi} \cos\left(n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}} dy\right) dx \tag{3.51}$$

Assume now that $G$ attains its maximum only once on the interval $[0, 2\pi[$. Furthermore, assume the only point where this maximum is achieved is, precisely, $\pi$. Fix $n \in \mathbb{N}$ and let us make the time $T$ diverge in expression (3.51). Simultaneously, $\mathcal{E}_0$, the energy of the trajectory, whose relation with $T$ is given by (3.50), decreases to $\max_{\mathbb{R}} G$. Thus,

$$n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}} dy \longrightarrow 2\pi n \chi_{]\pi, 2\pi]}(x), \quad 0 \leq x \leq 2\pi$$

uniformly on compact subsets of $[0,\pi[\cup]\pi,2\pi]$. Consequently,

$$A_n = \int_0^{2\pi} \cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 - G(y)}} dy \right) dx \to 1 \text{ as } T \to +\infty$$

We can use now Corollary 3.5.1 to prove Theorem 3.1.2.

*Proof of Theorem 3.1.2.* Of course, the maximum of $G$ may not be attained precisely at $\pi$, but the number of solutions to problem (3.1) is not changed if $g$ is translated on the real line, that is, replaced by $g(w+(\cdot))$, $w \in \mathbb{R}$. The Theorem follows now from the discussion above. □

*Proof of Theorem 3.1.3.* We may well concentrate in the case $\Lambda > 0$, since the number of solutions of problem (3.3), is not changed as the periodic term $g(u)$ is replaced by $g(u+\pi)$. In this way, $G(u) = -\Lambda\cos(u)$ attains its maximum at $\pi$. Now, for any $0 \le x < \frac{2\pi}{3}$ we have

$$0 \le \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy < \int_0^{\frac{2\pi}{3}} \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy < \frac{1}{\sqrt{\Lambda}} \int_0^{\frac{2\pi}{3}} \frac{1}{\sqrt{1+\cos(y)}} dy = \\ = \frac{2\sqrt{2}}{\sqrt{\Lambda}} \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \le \frac{T}{3n\sqrt{2}}$$

and, consequently,

$$\cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy \right) > \cos(\pi/3) = \frac{1}{2} \quad \forall x \in [0, \frac{2\pi}{3}[$$

Therefore,

$$\begin{aligned} A_n = \int_0^{2\pi} \cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy \right) dx = \\ = 2\int_0^{\pi} \cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy \right) dx = \\ = 2\int_0^{\frac{2\pi}{3}} \cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy \right) dx + \\ + 2\int_{2\pi/3}^{\pi} \cos\left( n\frac{2\pi}{T\sqrt{2}} \int_0^x \frac{1}{\sqrt{\mathcal{E}_0 + \Lambda\cos(y)}} dy \right) dx > \frac{2\pi}{3} - 2\frac{\pi}{3} = 0 \end{aligned}$$

□

*Proof of Corollary 3.1.4.* We simply observe that for any $g \in C^1(\mathbb{R}/2\pi\mathbb{Z})$, expression (3.51), which relates $A_n$, $\mathcal{E}_0$ and $T$, is analytic in these variables. Also, we know since (3.50) that $\mathcal{E}_0$ is an analytic function of $T$. An analytic function cannot be zero in an open set unless it is constantly zero, and, thus, Theorem 3.1.2 implies in fact Corollary 3.1.4. □

## 3.6 Many periodic solutions bifurcating from zero period

The key idea across previous sections was the following one: The autonomous, pendulum-type problem (3.25) has, when $\bar{e} = \bar{e}_{c,T}$, a closed orbit, and this orbit generates a continuum of solutions for (3.26). All our results there were on the line of looking for sufficient conditions (nondegeneracy conditions) on the closed orbit $u_{c,T}$ to ensure the existence of many branches of solutions bifurcating from this continuum.

For the limit case of zero time period ($T = 0$), it could also be thought, of course, in a heuristic way, that the autonomous, unforced pendulum-type equation $u'' + cu' + g(u) = 0$ has the following curve of 'solutions': for any $a \in \mathbb{R}$, we may consider the 'solution' which remains still at $a$ along this zero-length time period. Under new nondegeneracy hypothesis on $g$ we will be able to bifurcate from this continuum, for small positive time $T$, many branches of periodic solutions, and these will generate branches of subharmonic, periodic solutions, for big time intervals. The method of lower and upper solutions will be used to find many solutions for non-periodic problems of the type (3.1).

After the change of variables in (3.15), problem (3.1) can be rewritten in the form

$$v'' + cv' + g(v + \varphi(t)) = \bar{e}, \quad v \in W_{2,1}(\mathbb{R}/T\mathbb{Z}), \tag{3.52}$$

where $\varphi \in L^1(\mathbb{R}/T\mathbb{Z})$ and $\bar{e} \in \mathbb{R}$ are data of the problem. Also in this chapter, we will work on this problem in order to study (3.1).

Let us consider the linear, differential operator

$$\mathcal{L}^c : W_{2,1}(\mathbb{R}/T\mathbb{Z}) \to L^1(\mathbb{R}/T\mathbb{Z}), \quad \mathcal{L}(v) := v'' + cv' \quad \forall v \in W_{2,1}(\mathbb{R}/T\mathbb{Z}),$$

and the Nemytskii operator associated with $g$

$$\mathcal{N} : L^1(\mathbb{R}/T\mathbb{Z}) \to L^1(\mathbb{R}/T\mathbb{Z}),$$
$$[\mathcal{N}(v)](x) := g(v(x)) \quad \forall x \in \mathbb{R}/T\mathbb{Z}, \ \forall v \in L^1(\mathbb{R}/T\mathbb{Z}),$$

so that (3.52) is equivalent to the functional equation

$$\mathcal{L}^c(v) + \mathcal{N}(v + \varphi) = \bar{e}, \quad v \in W_{2,1}(\mathbb{R}/T\mathbb{Z}). \tag{3.53}$$

$\mathcal{L}^c$ is a Fredholm operator of zero index, its kernel been made up by the the set of constant functions (which can be naturally identified with $\mathbb{R}$) and its image by the integrable functions of zero mean. Such a behaviour suggests the following splittings of its domain and codomain

$$W_{2,1}(\mathbb{R}/T\mathbb{Z}) = [\ker \mathcal{L}] \oplus X = \mathbb{R} \oplus X \qquad\qquad L^1(\mathbb{R}/T\mathbb{Z}) = \mathbb{R} \oplus [\operatorname{im} \mathcal{L}] = \mathbb{R} \oplus Z$$

being $X := \{\tilde{u} \in W_{2,1}(\mathbb{R}/T\mathbb{Z}) : 1/T \int_0^T \tilde{u}(s)ds = 0\}$, $Z := \{\tilde{h} \in L^1(\mathbb{R}/T\mathbb{Z}) : 1/T \int_0^T u(s)ds = 0\}$. We use the first splitting to write each $v \in W_{2,1}(\mathbb{R}/T\mathbb{Z})$ in the form $v = \bar{v} + \tilde{v}$, $\bar{v} \in \mathbb{R}$, $\tilde{v} \in X$, and we call $Q : L^1(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}$ the linear projection associated to the latter one, $\big($given by $Q[h] = \frac{1}{T}\int_0^T h(s)ds\big)$. With this notation, problem (3.53) reads

$$\mathcal{L}^c(\tilde{v}) + (I - Q)\big[\mathcal{N}(\bar{v} + \tilde{v} + \varphi)\big] = 0, \tag{3.54}$$
$$Q\big[\mathcal{N}(\bar{v} + \tilde{v} + \varphi)\big] = \bar{e}. \tag{3.55}$$

This is the so-called *Lyapunov-Schmidt decomposition* of problem (3.53). We call $\mathcal{K}_c : Z \to X$ the inverse operator of the topological isomorphism $\mathcal{L}^c : X \to Z$ so that (3.54) becomes

$$\tilde{v} + \mathcal{K}_c\left[(I - Q)\big[\mathcal{N}(\bar{v} + \tilde{v} + \varphi)\big]\right] = 0. \tag{3.56}$$

On the other hand, taking into account the explicit expression for $Q$, (3.55) is nothing but

$$\frac{1}{T}\int_0^T g(\bar{v} + \tilde{v}(t) + \varphi(t))dt = \bar{e}. \tag{3.57}$$

We denote by $\mathcal{S}$ the set of solutions of (3.56), that is,

$$\mathcal{S} := \Big\{(\bar{v}, \tilde{v}) \in \mathbb{R} \times X : \tilde{v} + \mathcal{K}_c\left[(I - Q)\big[\mathcal{N}(\bar{v} + \tilde{v} + \varphi)\big]\right] = 0\Big\}. \tag{3.58}$$

Well-known results based upon the continuity of the Leray-Schauder topological degree, (see, for instance, [25]), show that for any $\bar{v}_- < \bar{v}_+ \in \mathbb{R}$ there exists a connected subset $\mathcal{S}_{[\bar{v}_-, \bar{v}_+]} \subset \mathcal{S} \cap \big([v_-, v_+] \times X\big)$ intersecting $\mathcal{S} \cap (\{v_-\} \times X)$ and $\mathcal{S} \cap (\{v_+\} \times X)$. We study the number of solutions of equation

$$\frac{1}{T}\int_0^T g(\bar{v} + \tilde{v}(t) + \varphi(t))dt = \bar{e}, \qquad (\bar{v}, \tilde{v}) \in \mathcal{S}, \quad 0 \le \bar{v} < 2\pi. \tag{3.59}$$

Observe now that, if $T$ is small, the norm of $\mathcal{K}_c$ when seen as an operator from

$$Y := \left\{ h \in C(\mathbb{R}/T\mathbb{Z}) : \int_0^T h(t)dt = 0 \right\} \subset C(\mathbb{R}/T\mathbb{Z}) \tag{3.60}$$

to itself is small. On the other hand, $\|(I-Q)\mathcal{N}(x)\|_\infty \leq 2\max_{\mathbb{R}}|g|\ \forall x \in L^1(\mathbb{R}/T\mathbb{Z})$ with independence of $T$. We deduce from (3.58) that $\|\tilde{v}\|_\infty$ is small for every $(\bar{v},\tilde{v}) \in \mathcal{S}$, so that

$$\frac{1}{T}\int_0^T g(\bar{v}+\tilde{v}(t)+\varphi(t))dt \approx \frac{1}{T}\int_0^T g(\bar{v}+\varphi(t))dt \tag{3.61}$$

as soon as $T$ is small. The discussion above motivates the following result, which slightly improves Proposition 2 in [42].

**Proposition 3.6.1.** *Let $n_0 \in \mathbb{N},\ \alpha > 2$ be given. Then, there exists a periodic function $\psi_{n_0} \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ such that, for any $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean,*

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(a+\psi_{n_0}(t)+t\right)dt = \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{\alpha}\cos(n_0a)\ \forall a \in \mathbb{R} \tag{3.62}$$

*where $A_{n_0}(g) = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos(n_0x)dx$ and $B_{n_0}(g) = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin(n_0x)dx$ are the ${n_0}^{th}$ coefficients in the Fourier series of $g$.*

*Proof.* Define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = x + \frac{2}{\alpha}\sin x. \tag{3.63}$$

Then, $H'(x) = 1+\frac{2}{\alpha}\cos x > 0\ \forall x \in \mathbb{R}$, so that it is an increasing diffeomorphism in $\mathbb{R}$. Define $\psi_{n_0} : \mathbb{R} \to \mathbb{R}$ by the rule

$$\psi_{n_0}(t) := \frac{1}{n_0}H^{-1}(n_0t) - t + d$$

where $d \in \mathbb{R}$ is a constant which will be fixed later. In this way,

$$\psi_{n_0}(t+2\pi) = \frac{1}{n_0}H^{-1}(n_0t+n_02\pi) - (t+2\pi) + d = \psi_{n_0}(t)\ \forall t \in \mathbb{R}.$$

Also, taken $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean and $a \in \mathbb{R}$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} g\left(a+\psi_{n_0}(t)+t\right)dt &= \frac{1}{2\pi}\int_0^{2\pi} g\left(a+d+\frac{1}{n_0}H^{-1}(n_0t)\right)dt = \\ &= \frac{1}{2\pi}\int_0^{2\pi} g\left(a+d+x\right)H'(n_0x)dx = \frac{1}{2\pi}\int_0^{2\pi} g\left(a+d+x\right)\left(1+\frac{2}{\alpha}\cos(n_0x)\right)dx = \\ &= \frac{1}{\alpha\pi}\int_0^{2\pi} g\left(a+d+x\right)\cos(n_0x)dx = \frac{1}{\alpha\pi}\int_0^{2\pi} g\left(x\right)\cos\left(n_0(x-a-d)\right)dx = \\ &= \frac{1}{\alpha}\Big[A_{n_0}(g)\cos(n_0a+n_0d)+B_{n_0}(g)\sin(n_0a+n_0d)\Big]. \end{aligned}\tag{3.64}$$

It is clear now that $d \in \mathbb{R}$ can be chosen so that (3.62) is satisfied.

□

**Corollary 3.6.2.** *Let $n_0 \in \mathbb{N},\ \alpha > 2$ and $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean be given. Then, for any $\epsilon > 0$, there exists an open set $\mathcal{G} = \mathcal{G}_\epsilon \subset L^1(\mathbb{R}/T\mathbb{Z})$ such that, with the notation from Proposition 3.6.1:*

$$\left|\frac{1}{2\pi}\int_0^{2\pi} g\left(a+h(t)\right)dt - \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{\alpha}\cos(n_0a)\right| < \epsilon\ \forall a \in \mathbb{R},\ \forall h \in \mathcal{G}.$$

*Proof.* By Proposition 3.6.1, it is possible to find $h_0 \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(a+h_0(t)\right)dt = \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{\alpha}\cos(n_0 a)\ \forall a\in\mathbb{R}. \tag{3.65}$$

However, the operator

$$\begin{aligned}\Psi : L^1\left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right) &\to C(\mathbb{R}/2\pi\mathbb{Z})\\ h &\mapsto \left[a\mapsto \frac{1}{2\pi}\int_0^{2\pi} g(a+h(t))dt\right]\end{aligned} \tag{3.66}$$

is continuous. To see this, choose a sequence of $L^1(\mathbb{R}/2\pi\mathbb{Z})$ functions $\{h_n\}\to h$ and a sequence of real numbers $\{a_n\}\subset\mathbb{R}/2\pi\mathbb{Z}$. Take subsequences $\{h_{n_k}\}$ and $\{a_{n_k}\}$ such that $\{a_{n_k}\}_k\to a\in\mathbb{R}/2\pi\mathbb{Z}$ and $\{h_{n_k}\}_k\to h$ a.e. Apply the Lebesgue Theorem to conclude

$$\lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi} g(a_{n_k}+h_{n_k}(t))dt = \frac{1}{2\pi}\int_0^{2\pi} g(a+h(t))dt$$

so that

$$\lim_{k\to\infty}\left[\frac{1}{2\pi}\int_0^{2\pi} g(a_{n_k}+h_{n_k}(t))dt - \frac{1}{2\pi}\int_0^{2\pi} g(a_{n_k}+h(t))dt\right]=0,$$

in another words,

$$\lim_k\Big[[\Psi h_{n_k}](a_{n_k})-[\Psi h](a_{n_k})\Big]$$

Being $\{a_n\}$ an arbitrary sequence in $\mathbb{R}/2\pi\mathbb{Z}$, it follows that $\{\Psi h_{n_k}\}_k$ converges to $\Psi h$ uniformly. This completes the argument.

□

At this stage, we are ready to use the abstract work carried out above in this section to obtain our next result, on the existence of many ordered lower and upper solutions for some specific pendulum-type equations. This should lead us to complete the proof of Theorem 3.1.1.

**Proposition 3.6.3.** *Let $n_0\in\mathbb{N}$ and $g\in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean be given. We assume that, with the notation from Proposition 3.6.1, $A_{n_0}(g)^2+B_{n_0}(g)^2\neq 0$. Then, for any $T>0$, $\epsilon>0$, it is possible to find $\varphi\in C^\infty(\mathbb{R}/T\mathbb{Z})$ and $0\le v_0<v_1<\ldots<v_{2n_0-1}<v_{2n_0}=v_0\in C^2(\mathbb{R}/T\mathbb{Z})$ such that:*

$$(-1)^q\left[v_q''+cv_q'+g(v_q+\varphi(t))\right] > \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{4}\ \forall q=0,\ldots,2n_0-1, \tag{3.67}$$

$$\left|v_q(t)-\frac{\pi}{n_0}q\right| < \epsilon\ \forall t\in\mathbb{R}/T\mathbb{Z},\ \forall q=0,\ldots,2n_0-1. \tag{3.68}$$

*Proof.* It suffices to prove the result for $T>0$ small, since, given any $p\in\mathbb{N}$, whenever $v_0,\ldots,v_{2n_0-1}\in C^2(\mathbb{R}/T\mathbb{Z}),\ \varphi\in C^\infty(\mathbb{R}/T\mathbb{Z})$ satisfy (3.67), $\tilde v_0\ldots,\tilde v_{2n_0-1}\in C^2(\mathbb{R}/pT\mathbb{Z})$, $\tilde\varphi\in C^\infty(\mathbb{R}/pT\mathbb{Z})$ defined by the rule

$$\tilde v_q(t):=v_q(t),\qquad \tilde\varphi(t)=\varphi(t),\quad t\in\mathbb{R}$$

will do the job for the period $pT$. We may (and we will) also assume $0<\epsilon<\frac{\pi}{n_0}$.

By Corollary 3.6.2, we are able to find some $\varphi_0\in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ and $\delta>0$ such that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} g\left(a+\varphi_0(t)+h(t)\right)dt - \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{3}\cos(n_0 a)\right| < \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{12}$$
$$\forall a\in\mathbb{R},\ \forall h\in L^1(\mathbb{R}/2\pi\mathbb{Z}),\ \|h\|_1<\delta. \tag{3.69}$$

At this point, let us fix $T > 0$ so small that

$$\|\mathcal{K}_c\|_Y < \frac{1}{2\max_{\mathbb{R}}|g|}\min\left\{\frac{\delta}{2\pi},\epsilon\right\}$$

where $\|\mathcal{K}_c\|_Y$ stands for the operator norm of $\mathcal{K}_c$ when seen as an operator from the space $Y$ defined in (3.60) to itself. In this way, we ensure that

$$\|\tilde{v}\|_\infty < \min\left\{\frac{\delta}{2\pi},\epsilon\right\}$$

for any $(\bar{v},\tilde{v})$ in the set $\mathcal{S} := \Big\{(\bar{v},\tilde{v}) \in \mathbb{R}\times X \,:\, \tilde{v} + \mathcal{K}_c\left[(I-Q)\big[\mathcal{N}(\bar{v}+\tilde{v}+\varphi)\big]\right] = 0\Big\}$ $\big($where $\varphi(t) := \varphi_0(2\pi t/T)\big)$. In particular, $\|\tilde{v}\|_1 < \frac{\delta T}{2\pi}$ $\forall(\bar{v},\tilde{v}) \in \mathcal{S}$. Now, a simple change of scale in (3.69) gives

$$\left|\frac{1}{T}\int_0^T g\left(a+\varphi(t)+h(t)\right)dt - \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{3}\cos(n_0 a)\right| < \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{12}$$
$$\forall a\in\mathbb{R},\ \forall h\in L^1(\mathbb{R}/T\mathbb{Z}),\ \|h\|_1 < \frac{\delta T}{2\pi},$$

and we conclude

$$\left|\frac{1}{T}\int_0^T g\left(\bar{v}+\tilde{v}(t)+\varphi(t)\right)dt - \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{3}\cos(n_0 \bar{v})\right| < \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{12}$$
$$\forall(\bar{v},\tilde{v})\in\mathcal{S}. \quad (3.70)$$

Inequality (3.70) implies, for $\bar{v} = \bar{v}_q = \frac{\pi}{n_0}q$, $\big(q = 0,\ldots,2n_0-1\big)$, that

$$(-1)^q\frac{1}{T}\int_0^T g\left(\frac{\pi}{n_0}q+\tilde{v}(t)+\varphi(t)\right)dt > \frac{\sqrt{A_{n_0}(g)^2+B_{n_0}(g)^2}}{4} \quad \forall\tilde{v}\in X \text{ with } \left(\frac{\pi}{n_0}q,\tilde{v}\right)\in\mathcal{S}.$$

□

We have shown:

**Corollary 3.6.4.** *Let $n_0 \in \mathbb{N}$ and $g \in C(\mathbb{R}/2\pi\mathbb{Z})$ with zero mean be given. We assume that, with the notation from Proposition 3.6.1, $A_{n_0}(g)^2 + B_{n_0}(g)^2 \neq 0$. Then, there exists an open set $\mathcal{O} \subset L^1(\mathbb{R}/T\mathbb{Z})$ with $\mathcal{O}\cap\{h \in L^1(\mathbb{R}/T\mathbb{Z}) : \frac{1}{T}\int_0^T h(t)dt = \frac{k'+ck}{T}\} \neq \emptyset$, such that for any $e \in \mathcal{O}$, problem (3.1) has at least $2n_0$ geometrically different solutions.*

*Proof.* It is a consequence from Proposition 3.6.3 above and the three solutions theorem (see [4]). Simply take $0 < \epsilon < \frac{\pi}{2n_0}$ in this Proposition to obtain a ordered scheme of lower and upper solutions. □

*Proof of Theorem 3.1.1.* In case $g$ is a trigonometric polynomial, we already proved it in page 64. In case it is not, apply Corollary 3.6.4 above. □

# Chapter 4

# A Hartman-Nagumo inequality for the vector ordinary $p$-Laplacian and applications to nonlinear boundary value problems

## 4.1 Introduction

In 1960, Hartman [39] (see also [40]) showed that the second order system in $\mathbb{R}^N$

$$u'' = f(t, u, u')\,, \tag{4.1}$$
$$u(0) = u_0, \quad u(1) = u_1, \tag{4.2}$$

with $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ continuous, has at least one solution $u$ such that $\|u(s)\| \le R$ for all $s \in [0,1]$ when there exists $R > 0$, a continuous function $\varphi : [0, +\infty[ \to \mathbb{R}^+$ such that

$$\int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty \tag{4.3}$$

and nonnegative numbers $\gamma, C$ such that the following conditions hold :

$(i)$ $\langle x, f(t,x,y)\rangle + \|y\|^2 \ge 0$ for all $t \in [0,1]$ and $x, y \in \mathbb{R}^N$ such that $\|x\| = R,\ \langle x, y\rangle = 0$.

$(ii)$ $\|f(t,x,y)\| \le \varphi(\|y\|)$ and $\|f(t,x,y)\| \le 2\gamma(\langle x, f(t,x,y)\rangle + \|y\|^2) + C$ for all $t \in [0,1]$ and $x, y \in \mathbb{R}^N$ such that $\|x\| \le R$.

$(iii)$ $\|u_0\|,\ \|u_1\| \le R$.

In 1971, Knobloch [44] proved, under conditions $(i)$ and $(ii)$ on the (locally Lipschitzian in $u, u'$) nonlinearity $f$, the existence of a solution for the periodic problem arising from equation (4.1). The local Lipschitz conditions was shown to be superfluous in [79]. A basic ingredient in those proofs is the so-called *Hartman-Nagumo inequality* which tells that if $x \in C^2([0,1], \mathbb{R}^N)$ is such that

$$\|x(t)\| \le R,\ \|x''(t)\| \le \varphi(\|x'(t)\|),\ \text{and}\ \|x''(t)\| \le \gamma\left(\|x(t)\|^2\right)'' + C, \quad (t \in [0,1]),$$

for some $\varphi$ satisfying (4.3), some $R > 0$, $\gamma \ge 0$, $C \ge 0$, then there exists some $K > 0$, only depending on $\varphi$, $R$, $\gamma$ and $C$, such that

$$\|x'(t)\| < K, \quad (t \in [0,1]).$$

Recently, Mawhin [60, 59] extended the Hartman-Knobloch results to nonlinear perturbations of the ordinary vector $p$-Laplacian of the form

$$(\|u'\|^{p-2}u')' = f(t, u).$$

His approach was based upon the application of the Schauder fixed point theorem to a suitable modification of the original problem whose solutions coincide with those of the original one.

Our aim here is to extend, at the same time, the Hartman-Knobloch results to nonlinear perturbations of the ordinary $p$-Laplacian and Mawhin's results to derivative-depending nonlinearities. In the case of Dirichlet boundary conditions, we use the Schauder fixed point theorem to find solutions to a modified problem, while in dealing with periodic ones, our main tool is the continuation theorem proved in [60]. Both procedures strongly depend on the extension of the Hartman-Nagumo inequality developed in Section 2.

Even though Theorem 4.4.1 exactly yields, when $p = 2$, the Hartman-Knobloch theorem, *this is probably not the best possible extension.* On the other hand, further extensions to more general operators of, say, *$\phi$-Laplacian* type $u \mapsto (\phi(u'))'$ (as considered in [60]) *remain, as far as we know, unexplored.*

For $N \in \mathbb{N}$ and $1 < p < +\infty$ fixed, we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^N$ and by $|\cdot|$ the absolute value in $\mathbb{R}$, while $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product in $\mathbb{R}^N$. By $p'$ we mean the Hölder conjugate of $p$ (given by $\frac{1}{p} + \frac{1}{p'} = 1$). For $q \in \{p, p'\}$, the symbol $\phi_q$ is used to represent the mapping

$$\phi_q : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto \begin{cases} \|x\|^{q-2}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then, it is clear that $\phi_p$ and $\phi_{p'}$ are mutually inverse homeomorphisms from $\mathbb{R}^N$ to itself, and mutually inverse analytic diffeomorphisms from $\mathbb{R}^N \backslash \{0\}$ to itself. Furthermore, an elementary computation shows that

$$\phi_q'(x)v = \|x\|^{q-2} \left( (q-2) \left\langle \frac{x}{\|x\|}, v \right\rangle \frac{x}{\|x\|} + v \right) \tag{4.4}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$, all $v \in \mathbb{R}^N$, and $q = p, p'$.

## 4.2 A Hartman-Nagumo inequality for the $p$-Laplacian

In this section, we extend the Hartman-Nagumo-type inequality [39, 40] associated to the second order differential operator $x \to x''$ to the p-Laplacian case $x \to (\phi_p(x'))'$. We need first a preliminary result giving an estimate on the $L^{p-1}$ norm of $x'$ when $x$ is bounded in the uniform norm and some differential inequalities involving $(\phi_p(x'))'$ hold. Let us call, for brevity, *$p$-admissible* any $C^1$ mapping $x : [0, 1] \to \mathbb{R}^N$ such that $\phi_p(x') : [0, 1] \to \mathbb{R}^N$ is of class $C^1$.

**Lemma 4.2.1.** *Let $B > 0$ be given. Then, there exists a positive number $M > 0$ (depending only on $B$) such that for each p-admissible mapping $x$ verifying the following inequalities, with $r : [0, 1] \to \mathbb{R}$ a $C^1$ convex function :*

(i) $\|x(t)\|, \ |r(t)| \leq B$ *for all* $t \in [0, 1]$*;*

(ii) $\|(\phi_p(x'))'\| \leq r''$ *a.e. on* $[0, 1]$,

*one has*

$$\int_0^1 \|x'(t)\|^{p-1} dt < M.$$

*Proof.* Condition (ii) can be rewritten as

$$\|\phi_p(x'(s)) - \phi_p(x'(t))\| \leq r'(s) - r'(t), \quad (0 \leq t \leq s \leq 1), \tag{4.5}$$

which implies that

$$\|\phi_p(x'(s))\| \leq \|\phi_p(x'(t))\| + r'(s) - r'(t), \quad (0 \leq t \leq s \leq 1), \tag{4.6}$$

$$\|\phi_p(x'(s))\| \le \|\phi_p(x'(t))\| + r'(t) - r'(s), \quad (0 \le s \le t \le 1). \tag{4.7}$$

Integrating inequality (4.6) with respect to $s$, we find

$$\int_t^1 \|x'(s)\|^{p-1} ds \le (1-t)\|x'(t)\|^{p-1} + r(1) - r(t) - (1-t)r'(t)$$
$$\le (1-t)\|x'(t)\|^{p-1} + 2B - (1-t)r'(t), \quad (0 \le t \le 1), \tag{4.8}$$

while integrating inequality (4.7) with respect to $s$ we get

$$\int_0^t \|x'(s)\|^{p-1} ds \le t\|x'(t)\|^{p-1} - r(t) + r(0) + tr'(t)$$
$$\le t\|x'(t)\|^{p-1} + 2B + tr'(t), \quad (0 \le t \le 1). \tag{4.9}$$

Adding expressions (4.8) and (4.9), we find

$$\int_0^1 \|x'(s)\|^{p-1} ds \le \|x'(t)\|^{p-1} + (2t-1)r'(t) + 4B, \quad (0 \le t \le 1), \tag{4.10}$$

and we deduce that

$$\int_0^1 \|x'(s)\|^{p-1} ds \le \|x'(t)\|^{p-1} + |r'(t)| + 4B, \quad (0 \le t \le 1). \tag{4.11}$$

Now, the convexity of $r$ means that $r'$ is increasing. Together with the bound $|r(t)| \le B$ for all $t \in [0,1]$, it implies that

$$|r'(t)| \le 6B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right], \tag{4.12}$$

which, together with (4.11), gives us the inequality

$$\int_0^1 \|x'(s)\|^{p-1} ds \le \|x'(t)\|^{p-1} + 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{4.13}$$

Following a similar process as before but integrating inequalities (4.6) and (4.7) with respect to $t$ instead of $s$, we get

$$\|x'(s)\|^{p-1} \le \int_0^1 \|x'(t)\|^{p-1} dt + |r'(s)| + 4B, \quad (0 \le s \le 1),$$

which, after changing the names of the variables $s$ and $t$, is equivalent to

$$\|x'(t)\|^{p-1} \le \int_0^1 \|x'(s)\|^{p-1} ds + |r'(t)| + 4B, \quad (0 \le t \le 1), \tag{4.14}$$

and, again, using (4.12), gives

$$\|x'(t)\|^{p-1} \le \int_0^1 \|x'(s)\|^{p-1} ds + 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{4.15}$$

The information given by (4.13) and (4.15) can be written jointly as

$$\left| \|x'(t)\|^{p-1} - \int_0^1 \|x'(s)\|^{p-1} ds \right| \le 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right], \tag{4.16}$$

which clearly implies that

$$\left| \|x'(t)\|^{p-1} - \|x'(s)\|^{p-1} \right| \le 20B \quad \text{for all} \quad t, s \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{4.17}$$

Suppose now that the conclusion of Lemma 2.1 is not true. This would imply the existence of sequences $\{x_n\}$ in $C^1([0,1], \mathbb{R}^N)$ and $\{r_n\}$ in $C^1([0,1], \mathbb{R})$ such that $x_n$ is p-admissible and $r_n$ is convex for all $n \in \mathbb{N}$, and, furthermore,

$(\widehat{\text{i}})_n$ $\|x_n(t)\|,\ |r_n(t)| \le B$ for all $t \in [0,1]$,

$(\widehat{\text{ii}})_n$ $\|(\phi_p(x_n'))'(t)\| \le r_n''(t)$ a.e. on $[0,1]$,

$(\widehat{\text{iii}})$ $\int_0^1 \|x_n'(t)\|^{p-1} dt \to +\infty$.

From $(\widehat{\text{iii}})$ and (4.16) we deduce that $\left\|x_n'(\frac{1}{2})\right\|^{p-1} \to +\infty$ as $n \to +\infty$, or what is the same, that $\left\|x_n'(\frac{1}{2})\right\| \to +\infty$ as $n \to +\infty$. In particular, we can suppose, after taking apart a finite set of terms if necessary, that $x_n'(\frac{1}{2}) \neq 0$ for all $n \in \mathbb{N}$. From (4.17) we can conclude now that the sequence of continuous functions $\left\{\frac{\|x_n'(\cdot)\|^{p-1}}{\|x_n'(\frac{1}{2})\|^{p-1}}\right\}$ converges to 1 uniformly on $[\frac{1}{3}, \frac{2}{3}]$ as $n \to \infty$, or, what is the same, that the sequence of continuous functions $\left\{t \mapsto \frac{\|x_n'(t)\|}{\|x_n'(\frac{1}{2})\|}\right\}$ converges to 1 uniformly on $[\frac{1}{3}, \frac{2}{3}]$ as $n \to \infty$.

Going back to (4.5) we can use (4.12) to obtain the inequalities

$$\left\|\phi_p(x_n'(t)) - \phi_p\left(x_n'\left(\frac{1}{2}\right)\right)\right\| \le \left|r_n'(t) - r_n'\left(\frac{1}{2}\right)\right| \le 12B \tag{4.18}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$, and all $n \in \mathbb{N}$, and, if $n$ is large enough so that $x_n'(t) \neq 0$ for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$, dividing inequality (4.18) by $\left\|x_n'(\frac{1}{2})\right\|^{p-1}$ we obtain

$$\left\|\frac{\|x_n'(t)\|^{p-1}}{\|x_n'(\frac{1}{2})\|^{p-1}} \frac{x_n'(t)}{\|x_n'(t)\|} - \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|}\right\| \le \frac{12B}{\|x_n'(\frac{1}{2})\|^{p-1}} \tag{4.19}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$, and we deduce that

$$\frac{x_n'(t)}{\|x_n'(t)\|} - \frac{x_n'\left(\frac{1}{2}\right)}{\|x_n'\left(\frac{1}{2}\right)\|} \to 0 \quad \text{as} \quad n \to \infty \tag{4.20}$$

uniformly on $\left[\frac{1}{3}, \frac{2}{3}\right]$. We can find, therefore, an integer $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$,

$$\left\langle \frac{x_n'(t)}{\|x_n(t)\|}, \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\rangle \ge \frac{1}{2}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$, what is the same as

$$\left\langle x_n'(t), \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\rangle \ge \frac{\|x_n'(t)\|}{2} \tag{4.21}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and all $n \ge n_0$. To end the proof, fix any $n_1 \ge n_0$ such that

$$\|x_{n_1}'(t)\| > 12B \tag{4.22}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$, and verify that, because of (4.21),

$$\left\langle x_{n_1}'(t), \frac{x_{n_1}'(\frac{1}{2})}{\|x_{n_1}(\frac{1}{2})\|} \right\rangle > 6B \tag{4.23}$$

for all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$,. This inequality, integrated between $\frac{1}{3}$ and $\frac{2}{3}$ gives us

$$\left\langle x_{n_1}(\frac{2}{3}) - x_{n_1}(\frac{1}{3}), \frac{x_{n_1}(\frac{1}{2})}{\|x_{n_1}(\frac{1}{2})\|} \right\rangle > 2B. \tag{4.24}$$

Hence, using the Cauchy-Schwartz inequality, we obtain the contradiction

$$\left\langle x_{n_1}(\frac{2}{3}) - x_{n_1}(\frac{1}{3}), \frac{x_{n_1}(\frac{1}{2})}{\|x_{n_1}(\frac{1}{2})\|} \right\rangle \le \left\|x_{n_1}\left(\frac{2}{3}\right) - x_{n_1}\left(\frac{1}{3}\right)\right\| \le 2B. \tag{4.25}$$

□

The following lemma provides an estimate for the uniform norm of $x'$.

**Lemma 4.2.2.** *Let $B > 0$ be given and choose the corresponding $M > 0$ according to Lemma 4.2.1. Let $\varphi : [M, +\infty[\to \mathbb{R}^+$ be continuous and such that*

$$\int_M^{+\infty} \frac{s}{\varphi(s)} ds > M.$$

*Then, there exists a positive number $K > 0$ (depending only on $B$, $M$ and $\varphi$) such that for each p-admissible mapping $x$ satisfying, for some $C^1$ convex function $r : [0,1] \to \mathbb{R}$, the following conditions :*

1. $\|x(t)\|,\ |r(t)| \le B$ *for all* $t \in [0,1]$;
2. $\|(\phi_p(x'))'\| \le r''$ *a.e. on* $[0,1]$;
3. $\|(\phi_p(x'))'(t)\| \le \varphi(\|x'(t)\|^{p-1})$ *for any* $t \in [0,1]$ *with* $\|x'(t)\|^{p-1} \ge M$,

*one has*

$$\|x'(t)\| < K \quad \textit{for all} \quad t \in [0,1].$$

*Proof.* Choose $K > \sqrt[p-1]{M}$ such that

$$\int_M^{K^{p-1}} \frac{s}{\varphi(s)} ds = M.$$

We show that the thesis holds for this $K$. To this aim, fix any $x,\ r$ verifying the hypothesis of the Lemma, and suppose that there exists some $t_0 \in [0,1]$ such that $\|x'(t_0)\| \ge K$, and hence $\|x'(t_0)\|^{p-1} \ge K^{p-1} > M$. However, by definition of the constant $M$, we have $\int_0^1 \|x'(t)\|^{p-1} dt < M$, so that there must exist some $t_1 \in [0,1]$ (we pick the closest one to $t_0$), such that $\|x'(t_1)\|^{p-1} = M$.

Define

$$\Phi : [M, +\infty[\to [0, +\infty[, \quad t \mapsto \int_M^t \frac{s}{\varphi(s)} ds, \tag{4.26}$$

and notice that $\Phi$ is continuous, $\Phi(M) = 0$, $\Phi$ is strictly increasing and $\Phi(K^{p-1}) = M$. Now,

$$\begin{aligned} M = \Phi(K^{p-1}) &\le \Phi(\|x'(t_0)\|^{p-1}) = |\Phi(\|x'(t_0)\|^{p-1})| \\ &= \left| \int_M^{\|x'(t_0)\|^{p-1}} \frac{s}{\varphi(s)} ds \right| = \left| \int_{\|x'(t_0)\|^{p-1}}^{\|x'(t_1)\|^{p-1}} \frac{s}{\varphi(s)} ds \right| = \left| \int_{\|\phi_p(x'(t_0))\|}^{\|\phi_p(x'(t_1))\|} \frac{s}{\varphi(s)} ds \right|. \end{aligned} \tag{4.27}$$

Using the change of variables $s = \|\phi_p(x'(t))\|,\ \ t \in [\min\{t_0, t_1\}, \max\{t_0, t_1\}]$, (which is absolutely continuous because $\phi_p(x')$ is $C^1$ and $\|\cdot\|$ is Lipschitz), we obtain, from hypothesis 3,

$$\begin{aligned} M &\le \left| \int_{\|\phi_p(x'(t_0))\|}^{\|\phi_p(x'(t_1))\|} \frac{s}{\varphi(s)} ds \right| = \left| \int_{t_0}^{t_1} \frac{\|\phi_p(x'(t))\|}{\varphi(\|\phi_p(x'(t))\|)} \frac{\langle \phi_p(x'(t)), (\phi_p(x'))'(t) \rangle}{\|\phi_p(x'(t))\|} dt \right| \\ &\le \left| \int_{t_0}^{t_1} \|\phi_p(x'(t))\| \cdot \frac{\|(\phi_p(x'))'(t)\|}{\varphi(\|\phi_p(x'(t))\|)} dt \right| \le \left| \int_{t_0}^{t_1} \|\phi_p(x'(t))\| dt \right| = \left| \int_{t_0}^{t_1} \|x'(t)\|^{p-1} dt \right| \end{aligned}$$

so that

$$M \le \int_{\min\{t_0,t_1\}}^{\max\{t_0,t_1\}} \|x'(t)\|^{p-1} dt \le \int_0^1 \|x'(t)\|^{p-1} dt < M, \tag{4.28}$$

a contradiction. □

The following elementary result of real analysis is used in the proof of the next theorem.

**Lemma 4.2.3.** *Let $\alpha,\ h : [0,1] \to \mathbb{R}$ be continuous functions, $\alpha$ non decreasing. Suppose that $h'$ exists and is nonnegative in the open set $\{t \in ]0,1[ : h(t) \ne \alpha(t)\}$. Then $h$ is non decreasing on $[0,1]$.*

*Proof.* Suppose, by contradiction, that there exist $s < t$ in $[0,1]$ such that $h(s) > h(t)$. There must be some $x \in ]s,t[$ such that $h(x) = \alpha(x)$ (otherwise, the Lagrange mean value theorem would give us the inequality $h(s) \le h(t)$). Define

$$a := \min\{x \in [s,t] : h(x) = \alpha(x)\},\ b := \max\{x \in [s,t] : h(x) = \alpha(x)\}.$$

Again, by the Lagrange mean value Theorem, we have the inequalities

$$h(s) \le h(a) = \alpha(a) \le \alpha(b) = h(b) \le h(t),$$

a contradiction. □

We can now prove the proposed extension of the Hartman-Nagumo inequality.

**Theorem 4.2.4.** *Let $R > 0, \gamma \ge 0, C \ge 0$ be given and choose $M > 0$ as associated by Lemma 4.2.1 to $B := \max\left\{R, \gamma R^2 + \frac{C}{2}\right\}$. Let $\varphi : [M, +\infty[ \to \mathbb{R}^+$ be continuous and such that*

$$\int_M^{+\infty} \frac{s}{\varphi(s)} ds > M.$$

*Then, there exists a positive number $K > 0$ (depending only on $R$, $p$, $\gamma$, $C$, $M$ and $\varphi$) such that, for any $p$-admissible mapping $x$ satisfying the following conditions :*

*(i)* $\|x(t)\| \le R, \quad (0 \le t \le 1)$;

*(ii)* $\|(\phi_p(x'))'(t)\| \le \gamma \left(\|x(t)\|^2\right)'' + C$ *for all* $t \in [0,1]$ *such that* $x'(t) \ne 0$;

*(iii)* $\|(\phi_p(x'))'(t)\| \le \varphi(\|x'(t)\|^{p-1})$ *for all* $t \in [0,1]$ *such that* $\|x'(t)\|^{p-1} \ge M$,

*one has*

$$\|x'(t)\| < K \quad (t \in [0,1]).$$

*Proof.* From the chain rule we know that $x' = \phi_{p'}(\phi_p(x'))$ is a $C^1$ mapping on the set $\{t \in [0,1] : x'(t) \ne 0\}$. Let us define

$$r : [0,1] \to \mathbb{R}, \quad t \mapsto \gamma m(t) + C\frac{t^2}{2}, \tag{4.29}$$

where $m(t) = \|x(t)\|^2$. It is clear that $r$ is a $C^1$ function. Moreover,

$$r'(t) = 2\gamma\langle x(t), x'(t)\rangle + Ct \quad (t \in [0,1]). \tag{4.30}$$

It means that $x'$ does not vanish on the set $\{t \in [0,1] : r'(t) \ne Ct\}$, and then, on this set, $r$ is $C^2$ and $r''(t) = \gamma m''(t) + C \ge \|(\phi_p(x'))'(t)\| \ge 0$.

By Lemma 4.2.3 we deduce that $r'$ is increasing, what is equivalent to say that $r$ is convex. Also, it is clear that

$$\|x(t)\|,\ |r(t)| \le B \quad (t \in [0,1]), \tag{$\tilde{i}$}$$

and, to be able to apply Lemma 4.2.2 we only have to check that inequality

$$\|(\phi_p(x'))'(t)\| \le r''(t) \tag{$\widetilde{ii}$}$$

holds for almost every $t$ in $[0,1]$.

Notice, firstly, that our hypothesis (ii) says that ($\widetilde{ii}$) is true for all $t$ in $[0,1]$ such that $x'(t) \ne 0$. Secondly, in the interior of the set $\{t \in [0,1] : x'(t) = 0\}$ we have

$$\|(\phi_p(x'))'(t)\| = 0 \le r''(t) = C.$$

It remains to see what happens on $A := \partial(\{t \in [0,1] : x'(t) = 0\})$. We will prove that at every point $t \in A\cap ]0,1[$ such that $(r')'(t) = r''(t)$ exists we have the inequality

$$\|(\phi_p(x'))'(t)\| \le r''(t). \tag{4.31}$$

Pick some point $t_0 \in A\cap ]0,1[$ such that $r''(t)$ exists. If $t_0$ is an isolated point of $A$, there exists some $\epsilon > 0$ such that $]t_0, t_0+\epsilon[ \subset ]0,1[\setminus A$. Then, $r'$ and $\phi_p(x')$ are both of class $C^1$ on $]t_0, t_0+\epsilon[$ and we have the inequality

$$\|(\phi_p(x'))'(t)\| \le r''(t) \quad (t \in ]t_0, t_0+\epsilon[). \tag{4.32}$$

It follows that $\|\phi_p(x'(t)) - \phi_p(x'(s))\| \le r'(t) - r'(s)$ for all $s,\ t$ with $t_0 < s < t < t_0+\epsilon$, and letting $s \to t_0$, that

$$\|\phi_p(x'(t)) - \phi_p(x'(t_0))\| \le r'(t) - r'(t_0) \quad (t \in ]t_0, t_0+\epsilon[), \tag{4.33}$$

from which we deduce that $\|\phi_p(x'))'(t_0)\| \le r''(t_0)$. If, otherwise, $t_0$ is an accumulation point of $A$, there exists a sequence $\{a_n\}$ of points from $A \setminus \{t_0\}$ converging to $t_0$. But $x'(a_n) = 0$ for all $n \in \mathbb{N}$, which implies that $\phi_p(x'(a_n)) = 0$ and $r'(a_n) = Ca_n$ for all $n \in \mathbb{N}$. We conclude then that $(\phi_p(x'))'(t_0) = 0 \le C = r''(t_0)$.

Theorem (4.2.4) is now a simple consequence of Lemma 4.2.2. □

## 4.3 Nonlinear perturbations of the $p$-Laplacian

Let $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be continuous, and consider the following differential equation

$$(\phi_p(x'))' = f(t,x,x'), \quad (0 \le t \le 1). \tag{4.34}$$

Our goal in the remaining part of this chapter is to develop some existence results for the solutions of this equation verifying either the periodic boundary conditions :

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{\mathbf{P}}$$

or the Dirichlet boundary conditions

$$x(0) = x_0, \quad x(1) = x_1, \tag{\mathbf{D}}$$

where $x_0$ and $x_1$ are some given points of $\mathbb{R}^N$.

Next, we state and prove two lemmas that will be needed later:

**Lemma 4.3.1.** *Let $x$ be a $p$-admissible mapping. For each $t_0 \in ]0,1[$ such that $\|x(t_0)\| = \max_{t\in[0,1]} \|x(t)\|$, one has*

$$\langle x(t_0), x'(t_0)\rangle = 0 \quad \text{and} \quad \langle x(t_0), (\phi_p(x'))'(t_0)\rangle + \|x'(t_0)\|^p \le 0.$$

*Furthermore, the same conclusion remains true when $t_0 = 0$ or $1$ if $x$ is assumed to verify the periodic boundary conditions* (**P**).

*Proof.* Suppose first that $t_0 \in ]0,1[$. The equality

$$\|x(t_0)\|^2 = \max_{t\in[0,1]} \|x(t)\|^2 \tag{4.35}$$

implies that

$$2\langle x(t_0), x'(t_0)\rangle = \frac{d}{dt}_{|t=t_0} \|x(t)\|^2 = 0. \tag{4.36}$$

Next, suppose by contradiction that

$$\langle x(t_0), (\phi_p(x'))'(t_0)\rangle + \|x'(t_0)\|^p > 0, \tag{4.37}$$

what is the same as

$$\frac{d}{dt}_{|t=t_0} \langle x(t), \phi_p(x'(t))\rangle > 0. \tag{4.38}$$

As

$$\langle x(0), \phi_p(x'(0))\rangle = \|x'(0)\|^{p-2}\langle x(0), x'(0)\rangle = 0,$$

we deduce the existence of some $\epsilon > 0$ such that $]t_0 - \epsilon, t_0 + \epsilon[ \subset [0,1]$ and

$$\langle x(t), \phi_p(x'(t))\rangle < 0, \quad t \in ]t_0 - \epsilon, t_0[ \tag{4.39}$$

$$\langle x(t), \phi_p(x'(t))\rangle > 0, \quad t \in ]t_0, t_0 + \epsilon[. \tag{4.40}$$

Equivalently,

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle x(t), x'(t)\rangle < 0, \quad t \in ]t_0 - \epsilon, t_0[\,, \tag{4.41}$$

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle x(t), x'(t)\rangle > 0, \quad t \in ]t_0, t_0 + \epsilon[\,, \tag{4.42}$$

which implies that $\|x(t)\|$ attains a strict local minimum at $t = t_0$. Of course, this is not compatible with our hypothesis and this first case is proved.

If now $x$ verifies the periodic boundary conditions (**P**) and

$$\|x(0)\| = \|x(1)\| = \max_{t\in[0,1]} \|x(t)\|,$$

define $y : [0,1] \to \mathbb{R}^N$ by $y(t) := x(t + 1/2)$ if $0 \le t \le \frac{1}{2}$, $y(t) := x(t - 1/2)$ if $\frac{1}{2} \le t \le 1$ and apply the above result to $y$ (at $t_0 = \frac{1}{2}$) to obtain the desired result. □

**Lemma 4.3.2.** *Let $f_i : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $(i = 1, 2, 3, ...)$, be a sequence of continuous mappings, uniformly converging on compact sets to $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$. Suppose that there exist positive numbers $R,\ K > 0$ such that, for every $i \in \mathbb{N}$, there exist a solution $x_i$ of the differential equation*

$$(\phi_p(x'))' = n_i(t, x, x')$$

*with*

$$\|x_i(t)\| \le R,\ \|x_i'(t)\| \le K \quad (t \in [0,1]).$$

*Then there exists a subsequence of $\{x_i\}$ converging in the space $C^1[0,\pi]$ to some $p$-admissible mapping $\bar{x} : [0,1] \to \mathbb{R}^N$, which is a solution of (4.34).*

*Proof.* The two sequences of continuous mappings $\{x_i\}$ and $\{\phi_p(x_i')\}$ are uniformly bounded together with its derivatives, so that, by the Ascolí-Arzelà Lemma, we can find a subsequence $\{z_i\}$ of $\{x_i\}$ uniformly converging on $[0,1]$ and such that the sequence $\{\phi_p(z_i')\}$ is also uniformly converging on $[0,1]$. As $\phi_p$ is an homeomorphism from $\mathbb{R}^N$ to itself, we deduce that both $\{z_i\}$ and $\{z_i'\}$ are uniformly converging on $[0,1]$. Finally, from the equalities

$$(\phi_p(x_i'))' = f_i(t, x_i, x_i') \quad (i = 1, 2, 3, ...,)$$

we deduce that also the sequence $\{(\phi_p(z_i'))'\}$ converges uniformly on $[0,1]$. The result now follows. □

The following set of hypothesis on the nonlinearity $f$ will be widely used in the remaining of this chapter:

**[H$_4$]** There exist $R > 0$, $\gamma \ge 0$, $C \ge 0$, $M > 0$ associated by Lemma 4.2.1 to $B := \max\{R, \gamma R^2 + \frac{C}{2}\}$ and $\varphi : [M, +\infty[ \to \mathbb{R}^+$ continuous with

$$\int_M^{+\infty} \frac{s}{\varphi(s)} ds > M$$

such that

(*a*) For any $t \in [0,1]$, $x, y \in \mathbb{R}^N$ such that $\|x\| = R$, $\langle x, y\rangle = 0$, we have

$$\langle x, f(t, x, y)\rangle + \|y\|^p \ge 0.$$

($b$) For any $t \in [0,1],\ x, y \in \mathbb{R}^N$ such that $\|x\| \leq R$ and $\|y\|^{p-1} \geq M$,

$$\|f(t,x,y)\| \leq \varphi(\|y\|^{p-1}).$$

($c$) For any $t \in [0,1],\ x \in \mathbb{R}^N$ with $\|x\| \leq R$ and $y \in \mathbb{R}^N$,

$$\|y\|^p \|f(t,x,y)\| \\ \leq 2\gamma\big((p'-2)\langle y, f(t,x,y)\rangle\langle x,y\rangle + \|y\|^2\langle x, f(t,x,y)\rangle + \|y\|^{p+2}\big) + C\|y\|^p.$$

As we will see next, these assumptions on $f$ will be sufficient to ensure the existence of a solution for both the periodic and the Dirichlet problems associated to equation (4.34). However, in our approach to these problems, we will have to assume firstly a slightly stronger set of hypothesis, consisting in replacing ($a$) by

($\tilde{a}$) *For any* $t \in [0,1];\ x, y \in \mathbb{R}^N$ *such that* $\|x\| = R,\ \langle x, y\rangle = 0$*, we have*

$$\langle x, f(t,x,y)\rangle + \|y\|^p > 0.$$

The new set of hypothesis will be denoted by $[\tilde{\mathbf{H}}_4]$.

Notice, furthermore, that if there exist numbers $R > 0,\ \gamma \geq 0,\ C \geq 0$ and a continuous function $\varphi : [0, +\infty[ \to \mathbb{R}^+$ verifying the classical Nagumo condition

$$\int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty,$$

such that ($a$), ($b$), and ($c$) are still satisfied, then, the whole set of hypothesis $[\mathbf{H}_4]$ is ensured.

## 4.4 The periodic problem

We prove in this section the existence of a solution for the periodic problem associated to equation (4.34).

**Theorem 4.4.1.** *Let* $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *be a continuous mapping satisfying* $[\mathbf{H}_4]$*. Then, the periodic boundary value problem* ($\mathbf{P}$) *for equation (4.34) has at least one solution* $x : [0,1] \to \mathbb{R}^N$ *such that* $\|x(t)\| \leq R$ *for all* $t \in [0,1]$.

*Proof.* The theorem will be proved in two steps. In the first one, we assume that the set of hypothesis $[\tilde{\mathbf{H}}_4]$ holds. To prove the theorem in this more restrictive case, choose $K > 0$ as given by Theorem 4.2.4 for $R,\ \gamma,\ C,\ M$ and $\varphi$, and define

$$\Omega := \{x \in C^1_T([0,1]) : \|x(t)\| < R,\ \|x'(t)\| < K \quad \text{for all} \quad t \in [0,1]\}. \tag{4.43}$$

Our aim is to apply the continuation theorem 5.1 from [60] in our case. The first thing we have to prove is that for each $\lambda \in ]0,1[$, the problem

$$(P_\lambda) \equiv \begin{cases} (\phi_p(x'))' = \lambda f(t,x,x') \\ x(0) = x(1),\ x'(0) = x'(1) \end{cases} \tag{4.44}$$

has no solutions on $\partial\Omega$. Indeed, notice that

$$\bar{\Omega} = \{x \in C^1_T[0,1] : \|x(t)\| \leq R,\ \|x'(t)\| \leq K \quad \text{for all} \quad t \in [0,1]\}. \tag{4.45}$$

Now, fix any $\lambda \in ]0,1[$ and let $\bar{x} \in \bar{\Omega}$ be a solution of $(P_\lambda)$. Our hypothesis (b) tells us that

$$\|(\phi_p(x'))'(t)\| = \lambda\|f(t,x(t),x'(t))\| \\ \leq \|f(t,x(t),x'(t))\| \leq \varphi(\|x'(t)\|^{p-1}) = \varphi(\|\phi_p(x'(t))\|) \tag{4.46}$$

for every $t \in [0,1]$ such that $\|x'(t)\|^{p-1} \geq M$. That is the third hypothesis needed in Theorem 4.2.4. The first one is obviously satisfied. Let us check the second one. We can use (4.4) to find that

$$z, v \in \mathbb{R}^N,\ z \neq 0 \Rightarrow \phi'_{p'}(\phi_p(z))v = \|z\|^{-p}\big((p'-2)\langle z, v\rangle z + \|z\|^2 v\big). \tag{4.47}$$

In our context, it means that, for each $t \in [0,1]$ such that $x'(t) \neq 0$, $x''(t)$ exists, and furthermore,

$$\begin{aligned} x''(t) &= (\phi_{p'}(\phi_p(x')))'(t) = \phi'_{p'}(\phi_p(x'(t)))(\phi_p(x'))'(t) = \phi'_{p'}(\phi_p(x'(t)))(\lambda f(t, x(t), x'(t))) \\ &= \lambda\|x'(t)\|^{-p}\big((p'-2)\langle x'(t), f(t, x(t), x'(t)\rangle x'(t) + \|x(t)\|^2 f(t, x(t), x'(t))\big), \end{aligned} \tag{4.48}$$

and then,

$$\begin{aligned} 2(\langle x(t), x''(t)\rangle + \|x'(t)\|^2) &\geq 2(\langle x(t), x''(t)\rangle + \lambda\|x'(t)\|^2) \\ &= 2\|x'(t)\|^{-p}\big((p'-2)\langle x'(t), f(t, x(t), x'(t))\rangle\langle x(t), x'(t)\rangle \\ &\quad + \|x'(t)\|^{p+2} + \|x'(t)\|^2\langle x(t), x(t, x(t), x'(t))\rangle\big) \end{aligned} \tag{4.49}$$

for all $t \in [0,1]$ with $x'(t) \neq 0$. It turns out that, if we define $r : [0,1] \to \mathbb{R}$ by $r(t) = \|x(t)\|^2$, for each $t \in [0,1]$ such that $x'(t) \neq 0$, we can write, using hypothesis (c),

$$\begin{aligned} \|(\phi_p(x'))'(t)\| &= \lambda\|f(t, x(t), x'(t))\| \\ &\leq 2\lambda\gamma\|x'(t)\|^{-p}\big((p'-2)\langle x'(t), f(t, x(t), x'(t))\rangle\langle x(t), x'(t)\rangle \\ &\quad + \|x'(t)\|^2\langle x(t), f(t, x(t), x'(t))\rangle + \|x'(t)\|^{p+2}\big) + \lambda C \\ &\leq \gamma r''(t) + \lambda C \leq \gamma r''(t) + C. \end{aligned} \tag{4.50}$$

Now, Theorem 4.2.4 tells us that

$$\|x'(t)\| < K \quad (t \in [0,1]), \tag{4.51}$$

and therefore, in order to see that $x \in \Omega$, it only remains to prove the inequality

$$\|x(t)\| < R \quad (t \in [0,1]). \tag{4.52}$$

Suppose, otherwise, that there exists some point $t_0 \in [0,1]$ such that $\|x(t_0)\| = R$. Then, $\|x(t_0)\| = \max_{t\in[0,1]} \|x(t)\|$, and from Lemma 4.3.1 we should have

$$\langle x(t_0), (\phi_p(x'))'(t_0)\rangle + \|x'(t_0)\|^p = \langle x(t_0), f(t_0, x(t_0), x'(t_0))\rangle + \|x'(t_0)\|^p \leq 0,$$

contradicting our hypothesis $(\tilde{a})$.

Finally, it remains to check that the equation

$$\mathcal{F}(a) := \int_0^1 f(t, a, 0)dt = 0 \tag{4.53}$$

has no solutions on $(\partial\Omega) \cap \mathbb{R}^N = \{a \in \mathbb{R}^N : \|a\| = R\}$, and that the Brouwer degree

$$deg_B(\mathcal{F}, \Omega \cap \mathbb{R}^N, 0) = deg_B(\mathcal{F}, B_R(0), 0)$$

is not zero. But from hypothesis $(\widetilde{a})$ (taking $y = 0$) we deduce

$$\langle a, f(t, a, 0)\rangle > 0 \quad \text{for all} \quad a \in \mathbb{R}^N,\ \|a\| = R, \quad \text{and all} \quad t \in [0,1], \tag{4.54}$$

and, integrating from 0 to 1 we get

$$\langle a, \int_0^1 f(t, a, 0)dt\rangle = \langle a, \mathcal{F}(a)\rangle > 0 \quad \text{for all} \quad a \in B_R(0), \tag{4.55}$$

which, effectively, implies that $deg_B(\mathcal{F}, \mathbb{B}_R(0), 0) = 1$. This concludes our first step. The theorem is proved assuming $(\widetilde{a})$ instead of (a). And the whole theorem follows now from a simple approximation theorem that we sketch below.

Fix some $\epsilon_* > 0$ small enough so that, after defining

$$B := \max\left\{R, \gamma R^2 + \frac{C}{2}\right\}, \; R_* := R, \; C_* := C + \frac{R\epsilon_*}{2},$$

$$B_* := \max\left\{R, \gamma R^2 + \frac{C_*}{2}\right\}, \; M_* := \max\left\{\left(\frac{B_*}{B}\right)^{\frac{1}{p-1}}, \frac{B_*}{B}\right\} M,$$

(where, as the reader can easily check, $M_*$ has been carefully chosen so that it satisfies the conditions of Lemma 4.2.1 for the parameter $B_*\big)$, we still have the inequality

$$\int_{M_*}^{+\infty} \frac{s}{\varphi_*(s)} ds > M_* \tag{4.56}$$

Next, choose a sequence $\{\epsilon_i\}_{i\in\mathbb{N}} \to 0$ with $0 < \epsilon_i < \epsilon_* \quad (i \in \mathbb{N})$, and define

$$f_i : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, \quad (t, x, y) \mapsto f(t, x, y) + \epsilon_i x \quad (i \in \mathbb{N}). \tag{4.57}$$

Now, it is clear that, for each $i \in \mathbb{N}$,

$(a_i)_*$ For any $t \in [0,1]$, $x, y \in \mathbb{R}^N$ such that $\|x\| = R, \ \langle x, y\rangle = 0$, we have $\langle x, f_i(t, x, y)\rangle + \|y\|^p > 0$.

$(b_i)_*$ For any $t \in [0,1], \ x, y \in \mathbb{R}^N$ such that $\|x\| \leq R_*$ and $\|y\|^{p-1} \geq M_*$, $\|f_i(t, x, y)\| \leq \varphi(\|y\|^{p-1})$.

$(c_i)_*$ For any $t \in [0,1], \ x \in \mathbb{R}^N$ such that $\|x\| \leq R_*$ and $y \in \mathbb{R}^N$,

$$\begin{aligned} \|y\|^p\|f_i(t,x,y)\| \leq \|y\|^p\|f(t,x,y)\| + \|y\|^p\epsilon_* R \leq 2\gamma\big((p'-2)\langle y, f(t,x,y)\rangle\langle x,y\rangle \\ + \|y\|^2\langle x, f(t,x,y)\rangle + \|y\|^{p+2}\big) + C_*\|y\|^p = 2\gamma\big((p'-2)\langle y, f_i(t,x,y)\rangle\langle x,y\rangle \\ + \|y\|^2\langle x, f_i(t,x,y)\rangle + \|y\|^{p+2}\big) + C_*\|y\|^p - 2\gamma\big((p'-2)\epsilon_i\langle x,y\rangle^2 + \epsilon_i\|x\|^2\|y\|^2\big) \\ \leq 2\gamma\big((p'-2)\langle y, f(t,x,y)\rangle\langle x,y\rangle + \|y\|^2\langle x, f_i(t,x,y)\rangle + \|y\|^{p+2}\big) + C_*\|y\|^p. \end{aligned} \tag{4.58}$$

(because $p' - 2 > -1$).

We deduce, by the first step proved above, the existence for each $i \in \mathbb{N}$ of a solution $x_i : [0,1] \to \mathbb{R}^N$ of the periodic boundary value problem

$$(P_i) \equiv \begin{cases} (\phi_p(x'))' = f(t, x, x') + \epsilon_i x \\ x(0) = x(1), \quad x'(0) = x'(1), \end{cases} \tag{4.59}$$

verifying $\|x_i(t)\| < R, \ \|x_i'(t)\| < K_*$ for all $t \in [0,1]$. ($K_*$ being given by Theorem 4.2.4 for $R, p, \gamma, C_*$ and $M_*$).

The existence of a solution to our problem is now a consequence of Lemma 4.3.2. □

## 4.5 The Dirichlet problem

Consider now the boundary value problem arising from equation (4.34) together with the Dirichlet boundary conditions (**D**). For the reader's convenience, we reproduce here a result of [54].

**Lemma 4.5.1.** *Let $x_0,\ x_1 \in \mathbb{R}^N$ be fixed. Then, for each $h \in C[0,1]$ there exists a unique solution $x_h \in C^1[0,1]$ to the problem*

$$(D_h) \equiv \begin{cases} (\phi_p(x'))' = h \\ x(0) = x_0, \quad x(1) = x_1 \end{cases} \tag{4.60}$$

*Furthermore, if we define $\mathcal{K} : C[0,1] \to C^1[0,1]$ by $h \mapsto x_h$, the mapping $\mathcal{K}$ is completely continuous.*

*Proof.* Integrating the differential equation in (4.60) from 0 to $t$ we find that a continuous mapping $x : [0,1] \to \mathbb{R}^N$ is a solution to this equation if and only if there exist some $a \in \mathbb{R}^N$ (necessarily unique) such that

$$\phi_p(x'(t)) = a + \mathcal{H}(h)(t) \quad (t \in [0,1]), \tag{4.61}$$

where $\mathcal{H}(h)(t) := \int_0^t h(s)ds$. This formula can be rewritten as

$$x'(t) = \phi_p^{-1}(a + \mathcal{H}(h)(t)) \quad (t \in [0,1]). \tag{4.62}$$

Now, the boundary conditions imply that

$$x(t) = x_0 + \int_0^t \phi_p^{-1}(a + \mathcal{H}(h)(s))ds \quad (t \in [0,1]), \tag{4.63}$$

and that

$$\int_0^T \phi_p^{-1}(a + \mathcal{H}(h)(s))ds = x_1 - x_0 \tag{4.64}$$

We therefore conclude that there exists a bijective correspondence between the set of solutions to (4.60) and the set of points $a \in \mathbb{R}^N$ verifying (4.64), given by $x \mapsto \phi_p(x'(0))$.

Following a completely analogous reasoning to that carried out in Proposition 2.2 from [53], we find that

(i) For each $h \in C[0,1]$ there exists an unique solution $a(h)$ of (4.64).

(ii) The function $a : C[0,1] \to \mathbb{R}^N$ defined in (i) is continuous and maps bounded sets into bounded sets.

We deduce that for every $h \in C[0,1]$, there exists a unique solution $\mathcal{K}(h)$ of $(D_h)$, given by the formula

$$\mathcal{K}(h)(t) = x_0 + \int_0^t \phi_p^{-1}(a(h) + \mathcal{H}(h)(s))ds \quad (t \in [0,1]). \tag{4.65}$$

The continuity of the mapping $a$ allows us to deduce the continuity of $\mathcal{K}$. The boundedness of $a$ on bounded sets of $C[0,1]$ has as a consequence the compactness of $\mathcal{K}$ on bounded sets of $C[0,1]$. □

This lemma is now used to prove the following existence theorem for the Dirichlet problem associated to (4.34).

**Theorem 4.5.2.** *Let $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous mapping verifying $[\mathbf{H}_4]$. Let $x_0,\ x_1 \in \mathbb{R}^N$ with $\|x_0\|,\ \|x_1\| \leq R$. Then, the boundary value problem (4.34)$-$($\mathbf{D}$), has at least one solution $x : [0,1] \to \mathbb{R}^N$ such that $\|x(t)\| \leq R$ for all $t \in [0,1]$.*

*Proof.* Define $\mathcal{F} : C^1[0,1] \to C[0,1]$ by

$$\mathcal{F}(x)(t) := f(t, x(t), x'(t)), \quad (t \in [0,1]), \tag{4.66}$$

so that our problem can be rewritten as

$$x = \mathcal{K}\mathcal{F}(x), \quad (x \in C^1[0,1]). \tag{4.67}$$

Notice that $\mathcal{K}\mathcal{F} : C^1[0,1] \to C^1[0,1]$ is a completely continuous mapping, so that if $f$ were bounded, $\mathcal{F}$ and $\mathcal{K}\mathcal{F}$ would be bounded and the Schauder fixed point theorem would give us the existence of a solution of our problem. Thus, our problem now is reduced to finding some $f_* : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ continuous, bounded and such that every solution to the equation

$$(\phi_p(x'))' = f_*(t, x, x') \tag{4.68}$$

verifying the boundary conditions (**D**) is also a solution to (4.34).

The following construction is essentially taken from [40]. As in the periodic case, start assuming that $f$ actually verifies the more restrictive set of hypothesis $[\tilde{\mathbf{H}}_4]$. Let $K > 0$ be as given by Theorem 4.2.4 for $R,\ \gamma,\ C, M$ and $\varphi$. Choose some continuous function

$$\rho : [0, \infty[\to \mathbb{R}^+ \tag{4.69}$$

such that

$$\rho(t) = 1, \quad (0 \le t \le K), \tag{4.70}$$

and

$$\sup\{\rho(\|y\|)\|f(t,x,y)\| : t \in [0,1],\ \|x\| \le R,\ y \in \mathbb{R}^N\} < +\infty. \tag{4.71}$$

For instance, $\rho$ could be chosen as

$$\rho(t) := \begin{cases} 1 \quad \text{if} \quad 0 \le t \le K \\ \frac{1}{1+\max\{\|f(t,x,y)\| : t\in[0,1],\ \|x\|\le R,\ \|y\|\le t\}} \quad \text{if} \quad t \ge K+1 \\ (1+K-t)\rho(K) + (t-K)\rho(1+K) \quad \text{if} \quad K \le t \le K+1 \end{cases}. \tag{4.72}$$

Define

$$f_*(t,x,y) := \begin{cases} \rho(\|y\|)f(t,x,y) \quad \text{if} \quad \|x\| \le R \\ \rho(\|y\|)f(t, R\frac{x}{\|x\|}, y) \quad \text{if} \quad \|x\| \ge R. \end{cases} \tag{4.73}$$

It is easy to check that $f_*$ is still a continuous bounded function satisfying not only the same set $[\tilde{\mathbf{H}}_4]$ of hypothesis (for the same parameters $R,\ \gamma,\ C,\ M$), but, moreover,

$(a_*)$ For any $t \in [0,1],\ x, y \in \mathbb{R}^N$ such that $\|x\| \ge R,\ \langle x, y\rangle = 0$, we have

$$\langle x, f(t,x,y)\rangle + \|y\|^p > 0.$$

Also, it is clear that $f_*(t,x,y) = f(t,x,y)$ if $\|x\| \le R$ and $\|y\| \le K$.

So, let $\bar{x} : [0,1] \to \mathbb{R}^N$ be a solution to (4.68) verifying the boundary conditions (**D**), where $\|x_0\|,\ \|x_1\| \le R$. Let us show that $\|\bar{x}(t)\| \le R,\ \|\bar{x}'(t)\| \le K$ for all $t \in [0,1]$. First suppose that there exist some point $t_0 \in [0,1]$ such that $\|\bar{x}(t_0)\| > R$. This point $t_0$ can be taken so as $\|\bar{x}(t_0)\| = \max_{t\in[0,1]} \|\bar{x}(t)\|$. As $\|\bar{x}(0)\| = \|x_0\| \le R,\ \|\bar{x}(1)\| = \|x_1\| \le R$, we see that $t_0 \in ]0,1[$. Now, using Lemma 4.3.1, we deduce that $\langle \bar{x}(t_0), \bar{x}'(t_0)\rangle = 0$ and

$$\langle \bar{x}(t_0), (\phi_p(\bar{x}'))'(t_0)\rangle + \|x'(t_0)\|^p = \langle \bar{x}(t_0), n_*(t_0, \bar{x}(t_0), \bar{x}'(t_0))\rangle + \|\bar{x}'(t_0)\| \le 0,$$

which contradicts $(a_*)$. It means that $\|\bar{x}(t)\| \le R$ for all $t \in [0,1]$. And, in the same way as happened in the proof of Theorem 4.4.1, our hypothesis $(b)$ and $(c)$ on $f$ (applied to $f_*$) make $\bar{x}$ verify the second and third hypothesis of Theorem 4.2.4. Applying it we obtain that $\|\bar{x}'(t)\| \le K$ for all $t \in [0,1]$, so that $\bar{x}$ is in fact solution to the system (4.34-(**D**)). □

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