The spectrum of reversible minimizers

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ABSTRACT. Poincaré [13, §355]) and Carathéodory [4, §412-413] showed that the Floquet multipliers of 1-dimensional periodic curves minimizing the lagrangian action are real and positive. This result can be generalized to higher-dimensional systems which are reversible in time.

1 Introduction

The motivation of this paper goes back to Poincaré, 130 years ago. At the end of section 355 of the third volume of his celebrated book *Les Méthodes Nouvelles de la Mécanique Céleste* [13], while studying the dynamics of closed trajectories of autonomous planar lagrangian systems, he writes:

Donc, pour qu'une courbe fermée corresponde à une action moindre que toute courbe fermée infiniment voisine, il faut que cette courbe fermée corresponde à une solution périodique instable de la première catégorie.[1]

In modern-day terms, we may paraphrase this assertion as follows: the Floquet multipliers of action-minimizing closed curves in the plane are real and positive. Thus, such action-minimizing curves cannot be elliptic, and they must be either parabolic (the degenerate case, where $\lambda = 1$ is the only Floquet multiplier), or hyperbolic (if there are two Floquet multipliers $0 < \lambda_1 < 1 < \lambda_2$).

In 1935 Carathéodory [4, §412] reinterpreted Poincaré's theorem for time-periodic 1-dimensional lagrangian systems. In this context, he showed that, as in the autonomous planar case, the Floquet multipliers of periodic action minimizers are real and positive (and consequently, the same parabolic/hyperbolic alternative appears). He also provided an example [4, §411] showing that this result is not true in dimension 2. See also Moser [9, pp. 77-78].

In this paper we extend Carathéodory's 1-dimensional theorem to higher dimensions under time reversibility. More precisely, let the lagrangian function

[1] Hence, for a closed curve to correspond to a lower action than any infinitely neighboring closed curve, it is necessary that this closed curve corresponds to an unstable periodic solution of the first category. (Our translation)

$L : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R},\ L = L(t, x, \dot{x})$, be given by

$$L(t, x, \dot{x}) = \frac{1}{2}\langle P(t)\dot{x}, \dot{x}\rangle + \langle R(t)x, \dot{x}\rangle + \frac{1}{2}\langle Q(t)x, x\rangle\,, \tag{1}$$

for some continuous functions $P, R, Q : \mathbb{R}/\mathbb{Z} \to \mathcal{M}_N(\mathbb{R})$ satisfying

$$P(t) = P(-t) = P(t)^* > 0, \quad Q(t) = Q(-t) = Q(t)^*, \quad R(-t) = -R(t)\,. \tag{2}$$

Notice that these conditions correspond to the time-reversibility assumption $L(-t, x, -\dot{x}) = L(t, x, \dot{x})$. The Euler-Lagrange equations associated to L are linear, and given by

$$\frac{d}{dt}\Big[P(t)\dot{x}(t) + R(t)x(t)\Big] = R(t)^*\dot{x}(t) + Q(t)x(t)\,, \tag{3}$$

and it is well-known that its 1-periodic solutions coincide with the critical points of the periodic action functional

$$\mathcal{A} : C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \to \mathbb{R}\,, \qquad x \mapsto \int_{-1/2}^{1/2} L(t, x(t), \dot{x}(t))dt\,.$$

The main result of these notes is the following

Theorem 1.1. *Assume that $\mathcal{A}[x] \geq 0$ for any $x \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. Then the Floquet multipliers for the period 1 associated to the trivial solution $x_* \equiv 0$ are real and positive.*

Let us briefly continue our historical review. The second half of the twentieth century saw a renewed interest on the related question of the extension of Sturm theory to systems of differential equations. This issue was analyzed, e.g., by Hartman [7, Chapter XI, §10], Morse [8, Chapters 5 and 6] and Arnold [1], and all this background will have an important influence on the present paper.

Almost thirty years ago, Bolotin [2] and Treschev [15] (see also [3, p. 232]) used Hill's formula for the infinite determinant of the Hessian of the action functional to show that nondegenerate closed gedesics of minimal length in even-dimensional orientable manifolds are exponentially unstable. This result is complementary to ours: it removes the reversibility assumption but works only for even dimensions. In addition, it does not eliminate the possibility of some elliptic Floquet multipliers.

More recently, Offin [10] has used techniques from symplectic topology to show hyperbolicity for nondegenerate natural systems (i.e., $P(t) \equiv I_N$, $R(t) \equiv 0_N$), under reversibility assumption. Thus, Theorem 1.1 can be seen as a generalization of Offin's theorem; we shall give a simpler proof of the hyperbolicity assertion in Section 2.

We emphasize that Theorem 1.1 loses its validity if the reversibility assumptions (2) are skipped. See the (already referred) examples in [4, §411] and [9, p. 78]. For instance, in this latter reference it is shown that the Floquet multipliers corresponding to the period 1 associated to the trivial extremal $x_* \equiv 0$ of the lagrangian $L_\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L_\alpha(x, \dot{x}) = \frac{1}{2}|\dot{x} - \alpha J x|^2 , \qquad x, \dot{x} \in \mathbb{R}^2 ,$$

are $e^{\pm i\alpha}$, each one with multiplicity 2 (here, $0 < \alpha < \pi$ is a constant and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ stands for the 90° rotation in the positive sense). On the other hand, for the related lagrangian $\mathcal{L}_\epsilon : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{L}_\epsilon(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}|x|^2 + (1 + \epsilon)\langle x, J\dot{x}\rangle , \qquad x, \dot{x} \in \mathbb{R}^2 ,$$

it can be checked that, if $\epsilon > 0$ is small enough, the trivial extremal $x_* \equiv 0$ is a nondegenerate minimum of the 1-periodic action functional, while the associated Floquet multipliers are four different complex numbers on the unit circle. See [19, Section 3, example *5.*]

In addition to being real and positive, the spectrum associated to (3) (or any Hamiltonian system) remains invariant by the inversion $\lambda \mapsto 1/\lambda$. More precisely, if λ is a Floquet multiplier, then $1/\lambda$ is again a Floquet multiplier with the same algebraic multiplicity (see, e.g., [6, Corollary 6, p. 5]). Thus, Theorem 1.1 can be rephrased by saying that the set of Floquet multipliers associated to a periodic minimizer has the form

$$\left\{ 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N \le 1 \le 1/\lambda_N \le 1/\lambda_{N-1} \le \ldots \le 1/\lambda_1 < +\infty \right\} ,$$

each multiplier being counted one or several times, according to its multiplicity. Conversely, it is clear that any set of $2N$ positive numbers as above is the spectrum of some Lagrangian system (3) under the conditions of Theorem 1.1, as it suffices to take $P(t) \equiv I_N$, $R(t) \equiv 0_N$, and $Q(t)$ as the constant diagonal matrix having in its diagonal the squares of the logarithms of the λ_i. I owe this observation to R. Ortega.

We point out that Theorem 1.1 gives also information for *nonlinear* problems. More precisely, let the lagrangian $\mathscr{L} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\mathscr{L} = \mathscr{L}(t, x, \dot{x})$ be continuous and twice-continuously differentiable with respect to $(x, \dot{x})$, but not necessarily of the form (1). Let it further satisfy the Legendre convexity condition: the $N \times N$ matrix $\mathscr{L}_{\dot{x}\dot{x}}(t, x, \dot{x})$ positive definite for any $(t, x, \dot{x})$. Finally, let $\mathscr{L}$ be 1-periodic in the time variable and time-reversible, i.e.

$$\mathscr{L}(t + 1, x, \dot{x}) = \mathscr{L}(t, x, \dot{x}) = \mathscr{L}(-t, x, -\dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N .$$

The periodic action functional $\mathscr{A} : C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \to \mathbb{R}$ is defined by $\mathscr{A}[x] := \int_{-1/2}^{1/2} \mathscr{L}(t, x(t), \dot{x}(t))dt$. As an immediate consequence of Theorem 1.1 one has the following

Corollary 1.2. *Let $x_* : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a local minimizer of the action functional $\mathscr{A}$. Assume further that x_* is even, i.e. $x_*(-t) = x_*(t)$ for any t. Then, the associated eigenvalues are real and positive.*

Concerning this corollary, we remark that: *(a):* local minimizers are to be understood in the sense of the $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ topology; *(b):* if one assumes that x_* is a *global* minimizer, then it is automatically even, as a well-known argument shows (see, e.g., [14, Proposition 2.2]).

To conclude this introduction, let us mention that the related problem of the Lyapunov-instability of action minimizers in possibly degenerate nonlinear problems has also been studied in many works, including Dancer and Ortega [5], Ortega [11],[12], and the author [16], [17] (for the 1-dimensional case), and [18] in the higher-dimensional situation.

2 Hyperbolicity of symmetric minimizers

We devote this section to show the hyperbolicity of nondegenerate symmetric minimizers of the action functional $\mathcal{A}$. Our results here generalize a previous theorem by Offin [10, Proposition 2.2], who obtains a similar conclusion for natural lagrangians $L(t, x, \dot{x}) = |\dot{x}|^2/2 - V(x)$. Precisely, we shall prove the result below, which can be considered as a first step towards Theorem 1.1. Here, $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ stands for the unit circle in the complex plane.

Proposition 2.1. *Under the assumptions of Theorem 1.1, the Floquet multipliers (corresponding to the period 1) associated to x_* belong to $(\mathbb{C}\backslash\mathbb{S}^1) \cup \{1\}$.*

Even though Proposition 2.1 can be deduced immediately from Theorem 1.1 we prefer to give a direct proof here, firstly because of its simplicity, but also because Corollary 2.3, which will be obtained in our way, will play an important role later. We shall start our arguments with the following result, which uses a well-known argument for reversible problems (see, e.g. [14, Proposition 2.2]).

Lemma 2.2. *Under the assumptions of Theorem 1.1,*

$$\mathcal{A}_{1/2}[x] := \int_0^{1/2} L(t, x(t), \dot{x}(t))dt \geq 0 \text{ for any } x \in C^1([0, 1/2], \mathbb{R}^N)\,. \tag{4}$$

Moreover, the equality holds if and only if x is a solution of (3) with $\dot{x}(0) = \dot{x}(1/2) = 0$.

Proof. We start by observing that the action functional $\mathcal{A}$ can be continuously extended to the Sobolev space $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. Since $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ is dense there, we deduce that

$$\mathcal{A}[x] \geq 0 \text{ for any } x \in H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)\,. \tag{5}$$

On the other hand, by (2), for any $x \in C^1([0,1/2],\mathbb{R}^N)$ one has

$$\mathcal{A}_{1/2}[x] = \frac{1}{2}\,\mathcal{A}[x_\sharp] \geq 0\,, \tag{6}$$

where $x_\sharp(-t) = x_\sharp(t)$ denotes the even extension of x to $[-1/2,1/2]$. Notice that $x_\sharp$ may not be continuously differentiable; however, its 1−periodic extension to the real line belongs to $H^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^N)$. Inequality (4) follows.

On the other hand, the equality in (5) holds if and only if x is a 1-periodic solution of (3). It means that the equality in (6) holds if and only if $x_\sharp$ is a 1-periodic solution of (3), proving the second part of the lemma. □

We immediately obtain the following result:

Corollary 2.3. *Under the assumptions of Theorem 1.1, for any natural number $q \geq 1$ one has*

$$\mathcal{A}_q[x] := \int_{-q/2}^{q/2} L(t,x(t),\dot{x}(t))dt \geq 0 \text{ for any } x \in C^1([-q/2,q/2],\mathbb{R}^N)\,, \tag{7}$$

with the equality holding if and only if x is a solution of (3) with $\dot{x}(k/2)=0$ for any integer $k \in [-q,q]$. In particular, $\mathcal{A}_1[x] > 0$ for any $x \in C^1([-1/2,1/2],\mathbb{R}^N)$ with

$$|x(1/2) - x(-1/2)| + |\dot{x}(-1/2)| + |\dot{x}(1/2)| \neq 0.$$

We recall that the system of equations (3) is said to be *disconjugate* if every nontrivial solution vanishes at most once on $\mathbb{R}$. The disconjugacy of periodic minimizers was studied by Carathéodory [4, §412] in the 1-dimensional setting and, in higher dimensions by Offin [10, Proposition 2.2] assuming that $P(t) \equiv I_N$ and $R(t) \equiv 0$.

Corollary 2.4. *Under the assumptions of Theorem 1.1, system (3) is disconjugate.*

Proof. Assume, by a contradiction argument, that there exists a nonzero solution $x : \mathbb{R} \to \mathbb{R}^N$ of (3) with $x(t_-) = 0 = x(t_+)$ for some $t_- < t_+$. Multiplying both sides of (3) by $x(t)$ and integrating by parts on the left we find that

$$\int_{t_-}^{t_+} L(t,x(t),\dot{x}(t))dt = 0\,.$$

Choose some integer $q \geq 1$ such that $[t_-,t_+] \subset]-q/2,q/2[$, and define $\hat{x} : [-q/2,q/2] \to \mathbb{R}^N$ by setting $\hat{x}(t) := \begin{cases} x(t) & \text{if } t \in [t_-,t_+] \\ 0 & \text{otherwise} \end{cases}$. Using the argument at the beginning of the proof of Lemma 2.2 we see that the action functional $\mathcal{A}_N$ can be continuously extended to the Sobolev space $H^1(]-q/2,q/2[,\mathbb{R}^N)$ and is nonnegative. Thus, $\mathcal{A}_N$ attains its minimum at $\hat{x}$, implying that it must be a solution of (3). In particular, $\dot{x}(t_\pm) = 0 = x(t_\pm)$, and, by uniqueness, $x \equiv 0$. This contradiction concludes the proof. □

Proof of Proposition 2.1. Assume, by a contradiction argument, that $e^{i\theta}$ were a Floquet multiplier for some $\theta \in]0, 2\pi[$. This implies the existence of a complex-valued solution $x + iy : \mathbb{R} \to \mathbb{C}^N$ of (3) satisfying

$$\begin{cases} x(m+1/2) + iy(m+1/2) = e^{i\theta}(x(m-1/2) + iy(m-1/2)) \,, \\ \dot{x}(m+1/2) + i\dot{y}(m+1/2) = e^{i\theta}(\dot{x}(m-1/2) + i\dot{y}(m-1/2)) \,, \end{cases}$$

for any integer m. After possibly replacing $x + iy$ by $-y + ix$ we may further assume that $x \not\equiv 0$. Then, some computations imply that $|x(m+1/2) - x(m-1/2)| + |\dot{x}(m-1/2)| + |\dot{x}(m+1/2)| \neq 0$ for any $m \in \mathbb{Z}$, and, by the final assertion of Corollary 2.3 we see that

$$\frac{1}{2}\int_{m-1/2}^{m+1/2} \Big(\langle P(t)\dot{x}(t), \dot{x}(t)\rangle + 2\langle R(t)x(t), \dot{x}(t)\rangle + \langle Q(t)x(t), x(t)\rangle \Big) dt > 0 \,, \quad m \in \mathbb{Z} \,.$$

We rewrite the integrand as $\langle P(t)\dot{x}(t) + R(t)x(t), \dot{x}(t)\rangle + \langle R(t)^*\dot{x}(t) + Q(t)x(t), x(t)\rangle$, and integrate by parts in the first term. Combining the facts that x is a solution of (3), that $P(-1/2) = P(1/2)$ and $Q(-1/2) = Q(1/2)$ (by periodicity), and that $R(\pm 1/2) = 0_N$ (by periodicity and reversibility), we obtain that the sequence of real numbers

$$a_m := \langle P(1/2)\dot{x}(m+1/2), x(m+1/2)\rangle \,,$$

is strictly increasing.

Let now $m_k \to +\infty$ be a sequence of integers with $m_k\theta \to 0 \bmod 2\pi$; then, $e^{im_k\theta} \to 1$ and we deduce that $x(m_k + 1/2) \to x(1/2)$ and $\dot{x}(m_k + 1/2) \to \dot{x}(1/2)$. Consequently, $a_{m_k} \to a_0$, contradicting the fact that the sequence $\{a_m\}$ is strictly increasing. This contradiction concludes the proof.

□

3 Matrix-valued solutions of the Euler-Lagrange equations

We turn our attention to the matrix version of (3):

$$\frac{d}{dt}\Big[P(t)\dot{M}(t) + R(t)M(t)\Big] = R(t)^*\dot{M}(t) + Q(t)M(t) \,, \tag{8}$$

Here, $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{R})$ is a matrix-valued curve. Notice that M is a solution of (8) if and only if each of its columns is a solution of (3). In this case, the matrix

$$(\mathfrak{T}M)(t) := (P(t)\dot{M}(t) + R(t)M(t))^*M(t) - M(t)^*(P(t)\dot{M}(t) + R(t)M(t))$$

does not depend on t, as one easily checks. Following [7, Chapter XI, §10] we shall say that $M = M(t)$ is *self-conjugate* provided that $\mathfrak{T}M = 0_N$.

Remark 3.1. In [8, §18], the same notion is referred to as *conjugate families in the sense of Von Escherich*. On the other hand, using the Legendre transformation $y = P(t)\dot{x} + R(t)x$, the lagrangian system (3) becomes a linear Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = JA(t) \begin{pmatrix} x \\ y \end{pmatrix} .$$

Let $\mathbb{R}^N \times \mathbb{R}^N$ be endowed with the canonical symplectic form. In this context, one inmediately checks that $M = M(t)$ is self-conjugate if and only if the subspace of generated by the columns of $\mathfrak{M}(t) := \begin{pmatrix} M(t) \\ P(t)\dot{M}(t) + R(t)M(t) \end{pmatrix}$ is isotropic.

Proposition 3.2. *Under the conditions of Theorem 1.1, there exists a self-conjugate solution* $M : [-1/2, 1/2] \to \mathcal{M}_N(\mathbb{R})$ *of (8) satisfying:*

(i) M *is even, i.e.,* $M(-t) = M(t)$,

(ii) $M(\pm 1/2) = I_N$,

(iii) $\det M(t) \neq 0\ \forall t \in [-1/2, 1/2]$,

(iv) $P(1/2)\dot{M}(1/2) \geq 0$ *(i.e., is positive semidefinite).*

Proof. Let $S : [-1/2, 1/2] \to \mathcal{M}_N(\mathbb{R})$ be the solution of (8) satisfying

$$S(-1/2) = 0_N\,, \qquad S'(-1/2) = I_N\,.$$

We notice that $\det(S(t)) \neq 0$ for any $t \in]-1/2, 1/2]$. Indeed, if $S(t_0)\xi = 0$ for some $-1/2 < t_0 \leq 1/2$, then, $x(t) := S(t)\xi$ would be a solution of (3) vanishing at $t = -1/2$ and $t = t_0$, and by Corollary 2.4, $x \equiv 0$, so that $\xi = \dot{x}(-1/2) = 0$. This allows us to define

$$M(t) := (S(t) + S(-t))S(1/2)^{-1}, \qquad t \in [-1/2, 1/2]\,.$$

This is a solution of (8) satisfying *(i)* and *(ii)*. In order to see that it is self-conjugate we notice that $K := (\mathfrak{T}M)(t)$ does not depend on t, and yet, $(\mathfrak{T}M)(-t) = -(\mathfrak{T}M)(t)$. Thus, $K = 0$.

Concerning *(iii)* we first observe that $M(0) = 2S(0)S(1/2)^{-1}$, so that $\det(M(0)) \neq 0$. Thus, by *(i)* it only remains to show that $\det(M(t)) \neq 0$ for every $t \in]0, 1/2[$. This follows from an argument similar to the one made above for S: if $M(t_0)\xi = 0$, then, $x(t) := M(t)\xi$ would be a solution of (3) vanishing at $\pm t_0$, and by Corollary 2.4, $x \equiv 0$, implying that $\xi = x(\pm 1/2) = 0$.

In order to show *(iv)* we consider, for any $\xi \in \mathbb{R}^N$, the function $x_\xi \in C^1([-1/2, 1/2], \mathbb{R}^N)$ defined by $x_\xi(t) := M(t)\xi$. Since $x_\xi(-1/2) = \xi = x_\xi(1/2)$, x_ξ can be considered as an element of the Sobolev space $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. The action functional $\mathcal{A}$ can be continuously extended to this bigger space, and, the subspace $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ being dense, $\mathcal{A}$ is positive semidefinite on $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. It follows that

$$0 \le \mathcal{A}[x_\xi] = \langle U\xi, \xi\rangle \qquad \forall \xi \in \mathbb{R}^N ,$$

where $U := (1/2)\int_{-1/2}^{1/2} \big(\dot{M}^* P \dot{M} + \dot{M}^* R M + M^* R^* \dot{M} + M^* Q M\big) dt$. Notice that the term inside the integral can be decomposed as $\dot{M}^*(P\dot{M} + RM) + M^*(R^*\dot{M} + QM)$. Integrating by parts in the first summand, using (8), and remembering that, by (2), $P(-1/2) = P(1/2)$ and $R(\pm 1/2) = 0_N$ one finds that $U = P(1/2)\dot{M}(1/2)$. This establishes *(iv)* and concludes the proof.

□

Some comments are in order:

Remark 3.3. In connection with *(iv)*, we observe that the matrix $P(1/2)\dot{M}(1/2)$ is symmetric. This follows from the fact that M is self-conjugate (and $M(1/2) = I_N,\ R(1/2) = 0$).

Remark 3.4. The converse of Proposition 3.2 is also true; if there exists a self-conjugate solution $M : [-1/2, 1/2] \to \mathcal{M}_N(\mathbb{R})$ of (8) satisfying *(i)-(iv)*, then the action functional $\mathcal{A}$ is positive semidefinite on $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. We shall not use this fact, which can be deduced from [7, Theorem 10.3, p.390].

Remark 3.5. With the terminology of Remark 3.1, condition *(iii)* above states that the columns of $\mathfrak{M}(t)$ span a lagrangian subspace of $\mathbb{R}^N \times \mathbb{R}^N$, and this lagrangian subspace is transversal to the 'vertical subspace' $\{0_{\mathbb{R}^N}\} \times \mathbb{R}^N$ for every $t \in [-1/2, 1/2]$.

4 Changing variables

From this moment on, we fix $M : [-1/2, 1/2] \to \mathcal{M}_N(\mathbb{R})$ as given by Proposition 3.2.

Lemma 4.1. *The change of variables $x = M(t)v$ transforms the Euler-Lagrange equations (3) into*

$$\frac{d}{dt}\Big[(M(t)^* P(t) M(t))\dot{v}\Big] = 0\,, \qquad -1/2 \le t \le 1/2\,. \tag{9}$$

Proof. After introducing this change of variables in the Euler-Lagrange equations (3), one gets

$$\frac{d}{dt}\Big[P(\dot{M}y + M\dot{v}) + RMy\Big] = R^*(\dot{M}y + M\dot{v}) + QMy\,,$$

or, what is the same,

$$\frac{d}{dt}\Big[(P\dot{M} + RM)v\Big] + \frac{d}{dt}(PM\dot{v}) = (R^*\dot{M} + QM)v + R^*M\dot{v}\,,$$

and, by (8),

$$(P\dot{M} + RM)\dot{v} + \frac{d}{dt}(PM\dot{v}) = R^*M\dot{v}\,.$$

Multiplying both sides of the equality by M (which has nonzero determinant, by *(iii)*), and using that M is self-conjugate, one obtains

$$(P\dot{M} + RM)^*M\dot{v} + M^*\frac{d}{dt}(PM\dot{v}) = M^*R^*M\dot{v}\,,$$

which, after the subtraction of $M^*R^*M\dot{v}$ from both sides of the equation becomes (9). This completes the proof. □

Let now $x : \mathbb{R} \to \mathbb{R}^N$ be a solution of (3). For any $m \in \mathbb{Z}$ we define $v_m : [-1/2, 1/2] \to \mathbb{R}^N$ by

$$x(m+t) = M(t)v_m(t)\,, \qquad -1/2 \le t \le 1/2\,.$$

Lemma 4.2. *The following hold, for every integer $m \in \mathbb{Z}$:*

(a) $v_{m+1}(-1/2) = v_m(1/2) = x(m+1/2)\,.$

(b) $\dot{v}_{m+1}(-1/2) = \dot{v}_m(1/2) + 2\dot{M}(1/2)x(m+1/2)\,.$

Proof. Assertion *(a)* follows from the fact that $M(\pm 1/2) = I_N$. Item *(b)* follows after comparing the value of $\dot{x}(m+1/2)$ obtained by differentiating on the equalities $x(m+t) = M(t)v_m(t)$ and $x(m+1+t) = M(t)v_{m+1}(t)$ and using that $\dot{M}(-1/2) = -\dot{M}(1/2)$ (because M is even). □

At this moment, we consider the linear map $\mathfrak{P} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ defined by

$$\mathfrak{P}(v_m(1/2), w_m(1/2)) := (v_{m+1}(1/2), w_{m+1}(1/2))\,,$$

where

$$w_m(t) := P(t)\dot{v}_m(t), \qquad -1/2 \le t \le 1/2\,, \quad m \in \mathbb{Z}\,.$$

(Observe that every point of $\mathbb{R}^N \times \mathbb{R}^N$ can be uniquely written as $(v_m(1/2), w_m(1/2))$ for a solution $x = x(t)$ of (3)). Since $v_m(1/2) = x(m+1/2)$ and $\dot{v}_m(1/2) = -\dot{M}(1/2)x(m+1/2) + \dot{x}(m+1/2)$, we see that $\mathfrak{P}$ is (linearly) conjugate with the Poincaré map

$$(x(m+1/2), \dot{x}(m+1/2)) \mapsto (x(m+1+1/2), \dot{x}(m+1+1/2))$$

associated to (3); therefore, they have the same eigenvalues. In other words, the Floquet multipliers of (3) are the eigenvalues of $\mathfrak{P}$.

Lemma 4.3. *The matrix of $\mathfrak{P}$ is structured in four $N \times N$ blocks, as follows:*

$$\mathfrak{P} = \left(\begin{array}{c|c} I_N + S_1S_2 & S_1 \\ \hline S_2 & I_N \end{array} \right),$$

where $S_1, S_2 \in \mathcal{M}_N(\mathbb{R})$ are symmetric, with S_1 positive definite and S_2 positive semidefinite.

Proof. Set

$$S_1 := \int_{-1/2}^{1/2} (M(t)^* P(t) M(t))^{-1} dt \,, \qquad S_2 := 2P(1/2)\dot{M}(1/2) \,.$$

Then, it is clear that S_1 is symmetric and positive definite, while the fact that S_2 is symmetric and positive semidefinite has already been noticed in Proposition 3.2-*(iv)* and Remark 3.3.

Moreover, it follows from Lemma 4.1 that for each integer m there exists some $\xi_m \in \mathbb{R}^N$ such that

$$\dot{v}_m(t) = (M^* P M)(t)^{-1} \xi_m \,, \qquad t \in [-1/2, 1/2] \,. \tag{10}$$

Thus, $w_m(-1/2) = \xi_m = w_m(1/2)$, and one has:

$$w_{m+1}(1/2) = w_{m+1}(-1/2) = w_m(1/2) + P(1/2)(\dot{v}_{m+1}(-1/2) - \dot{v}_m(1/2)) \,,$$

and, in view of Lemma 4.2 we see that

$$w_{m+1}(1/2) = w_m(1/2) + S_2 v_m(1/2) \,. \tag{11}$$

Moreover, again by (10),

$$v_{m+1}(1/2) = v_{m+1}(-1/2) + \int_{-1/2}^{1/2} \dot{v}_{m+1}(t) dt = v_m(1/2) + S_1 w_{m+1}(1/2) \,,$$

which in combination with (11) gives

$$v_{m+1}(1/2) = (I_N + S_1 S_2) v_m(1/2) + S_1 w_m(1/2) \,,$$

and completes the proof.

□

5 The Floquet multipliers are real and positive

In this section we shall complete the proof of Theorem 1.1. As observed in the comments preceding Lemma 4.3, the Floquet multipliers of (3) coincide with the eigenvalues of $\mathfrak{P}$. We shall begin with the following:

Lemma 5.1. $\mathfrak{P}$ *has no real negative eigenvalues.*

Proof. Assume, by a contradiction argument, that $-\lambda < 0$ were an eigenvalue; then, there would be vectors $v, w \in \mathbb{R}^N$, not simultaneoulsly zero, such that

$$\begin{cases} (I_N + S_1 S_2) v + S_1 w = -\lambda v \,, \\ S_2 v + w = -\lambda w \,. \end{cases}$$

From the second equation we see that $w = -S_2 v/(1+\lambda)$. It easily follows that $v \neq 0 \neq w$, and the first equation gives

$$v = -\frac{\lambda}{(1+\lambda)^2} S_1 S_2 v \,,$$

so that

$$S_1^{-1} v = -\frac{\lambda}{(1+\lambda)^2} S_2 v \Rightarrow 0 < \langle S_1^{-1} v, v \rangle = -\frac{\lambda}{(1+\lambda)^2} \langle S_2 v, v \rangle \leq 0 \,,$$

a contradiction. □

We set $\widehat{\mathbb{Q}} := \{p/q : p \in 1 + 2\mathbb{Z}, q \in \mathbb{N}\}$, which is a dense subset of $\mathbb{R}$. The combination of Corollary 2.3 and Lemma 5.1 will lead us to the following

Lemma 5.2. $\mathfrak{P}$ *does not have eigenvalues* $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $\arg(\lambda) \in \pi\, \widehat{\mathbb{Q}}$.

Proof. We use a contradiction argument and assume instead that $\mathfrak{P}$ had some eigenvalue $\lambda = re^{ip\pi/q}$ with $r > 0$, $q \in \mathbb{N}$ and $p \in 1 + 2\mathbb{Z}$. Then, $\mathfrak{P}^q$ would have the eigenvalue $\lambda^q = -r^q < 0$. However, $\mathfrak{P}^q$ is conjugate to the Poincaré map associated (3) and the time period q. Since the action functional $\mathcal{A}_q$ is positive semidefinite (by Corollary 2.3) we can apply Lemma 5.1 above and find a contradiction. □

Proof of Theorem 1.1. For any $0 \leq s \leq 1$ we define

$$L_s : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \,, \quad (t, x, \dot{x}) \mapsto (1-s) L(t, x, \dot{x}) + s(|x|^2 + |\dot{x}|^2) \,.$$

The corresponding action functionals are positive semidefinite on $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ for every $s \in [0, 1]$. The associated Floquet multipliers (associated to the period 1) vary continuously with s and hence, by Lemma 5.2, they must move along rays $\mathcal{R}_\theta = \{z \in \mathbb{C} \backslash \{0\} : \arg(z) = \theta\}$. For $s = 1$, these Floquet multipliers are e and $1/e$ (each repeated N times); thus, they lie on the positive part of the real axis and we deduce that the same must happen for $s = 0$. The proof is complete. □

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