A counterexample for singular equations with indefinite weight

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Abstract

We construct a second-order equation $\ddot{x} = h(t)/x^p$, with $p > 1$ and the sign-changing, periodic weight function h having negative mean, which does not have periodic solutions. This contrasts with earlier results which state that, in many cases, such periodic problems are solvable.

Key Words: Periodic solutions; singular equations; indefinite weight.

1 Introduction

In various kinds of boundary value problems associated to differential equations there is an obvious necessary condition for the existence of a solution, and this necessary condition turns out to be also sufficient. This the case of the well-known Landesman-Lazer conditions for scalar equations of the second order. Another instance of this phenomenon comes from the work of Lazer and Solimini [4], who studied the T-periodic problem associated to the equation

$$\ddot{x} \pm \frac{1}{x^p} = h(t)\,, \qquad x > 0\,,$$

where $p \geq 1$ is a real number and $h : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. Integration of both sides of the equation shows that a necessary condition for the existence of a solution is that $\pm \int_0^T h(t)dt > 0$, and the main result of [4] states that this condition is also sufficient.

In this paper we are concerned with the T-periodic problem associated to equations of the form

$$\ddot{x} = \frac{h(t)}{x^p}\,, \qquad x > 0\,. \tag{1}$$

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Here, $p \geq 1$, and $h : \mathbb{R} \to \mathbb{R}$ is a given (smooth) T-periodic function, which we assume to be not constantly zero. By a solution of this problem we mean a T-periodic function $x : \mathbb{R} \to \mathbb{R}$ of class C^2, with $x(t) > 0$ for all $t \in \mathbb{R}$, and satisfying the equation. Multiplying both sides of the equation by x^p and integrating by parts in the left one arrives to a necessary condition for the existence of a solution:

$$\int_0^T x(t)^p \ddot{x}(t)dt = -p \int_0^T x(t)^{p-1} \dot{x}(t)^2 \, dt = \int_0^T h(t)dt \ \Rightarrow \int_0^T h(t)dt < 0 \, .$$

In addition, a second necessary condition for the existence of a T-period solution is that h must change sign, leading to the words 'indefinite weight' in the title of this paper. It motivates the question: *is it true that if the smooth and T-periodic function $h : \mathbb{R} \to \mathbb{R}$ changes sign and has negative mean then (1) has a T-periodic solution?* This question was hinted in [3] and answered affirmatively in [8] under the additional assumptions that $p \geq 2$ and the zeroes of h are simple. The main result of this paper shows that these additional conditions cannot be simultaneously removed:

Theorem 1.1. *There is a sign-changing, T-periodic function $h : \mathbb{R} \to \mathbb{R}$ of class C^∞ with $\int_0^T h(t)dt < 0$, such that the equation*

$$\ddot{x} = \frac{h(t)}{x^{5/3}} \, , \qquad x > 0 \, , \tag{2}$$

does not have T-periodic solutions.

The function h in our example will be even, i.e., it will satisfy $h(-t) = h(t)$ for any $t \in \mathbb{R}$. Consequently, the Neumann boundary value problem associated to (2) on the time interval $[0, T/2]$ is not solvable either, as any solution would give rise to a solution of the T-periodic problem. In Corollary 2 of [2], Boscaggin and Zanolin have studied the solvability of some Neumann problems for singular equations with indefinite weight. When their result is particularized to equations of the form (1) one obtains the following

Theorem 1.2 (Boscaggin and Zanolin, [2])**.** *Let $h : [0, \hat{T}] \to \mathbb{R}$ be a Lebesgue integrable function with $\int_0^{\hat{T}} h(t)dt < 0$. Assume that*

(i) There exists some $\tau \in]0, \hat{T}[$ with $0 \not\equiv h(t) \geq 0$ on $[0, \tau]$ and $0 \not\equiv h(t) \leq 0$ on $[\tau, d]$.

(ii) There are numbers $\alpha \in]0, p[$ and $c \in]0, +\infty[$ such that $(1/t^\alpha) \int_0^t h(s)ds \to c$ as $t \to 0$.

Then, (1) has an (increasing) solution $x = x(t)$ with $\dot{x}(0) = \dot{x}(\hat{T}) = 0$.

Letting $p = 5/3$ and $\hat{T} = T/2$, the function h which we construct in Theorem 1.1 satisfies all these assumptions with the exception of *(ii)*; indeed, one easily checks that $(1/t^p) \int_0^t h(s)ds \to 0$ as $t \to 0$. Thus, our example also shows that assumption *(ii)* above cannot be dropped from the Boscaggin-Zanolin result without any replacement. Indeed, this idea has somehow guided us, and our proof of Theorem 1.1 can be roughly divided in two steps. On one hand, we see that for the even function h which we construct, any periodic solution of (2) must be even and hence a solution of the Neumann problem on $[0, T/2]$. On the other hand, we check that this Neumann problem is not solvable.

The Dirichlet problem associated to equations of the form (1) and related ones has been treated by many authors which would be too long to list here; we refer instead to Section 2 of the review paper [1] and the references therein. Let us just mention that an important landmark in this field came with the work of Taliaferro [7], who characterized solvability when h is negative, and this paper motivated a big deal of subsequent research. By contrast, the periodic problem has received much less attention, perhaps because it forces to work with indefinite weight functions from the beginning. See [2, 3, 8] for some existence results in this field.

This paper is organized as follows. Section 2 gives an intuitive approach to some of the key steps of our construction. The rigorous treatment is more delicate and will be completed only in Section 7, by the combining the auxiliary results collected in sections 3-6. Section 6 is independent from the others, and studies the regularity of the Poincaré map associated to general second order equations as they are compressed to a zero-length time interval. On the other hand, sections 3-5 are more specific and treat different aspects of equation (2) with $h(t) = \epsilon t^2$. Section 3 is self-contained, while sections 4,5 are based on the properties obtained in the Appendix (Section 8) for (2) with $h(t) = 3t^2/4$.

A couple of comments about the non-standard notation which we use throughout this paper. We denote by $\pi_1 z = z_1$ and $\pi_2 z = z_2$ to the first and second components, respectively, of the point $z = (z_1, z_2) \in \mathbb{R}^2$. Correspondingly, given a set A and a map $F : A \to \mathbb{R}^2$ we denote by $\pi_1 F, \pi_2 F : A \to \mathbb{R}$ to its components. We also denote by $R : \mathbb{R}^2 \to \mathbb{R}^2$ to the reflection map $Rz := (\pi_1 z, -\pi_2 z)$.

2 Towards the example: a heuristic overview

The picking of the coefficient $p = 5/3$ in (2) may seem strange at first glance, and indeed, we believe that Theorem 1.1 keeps its validity for equations of the form (1) regardless of the value of $p \geq 1$. Our choice is motivated by the fact that the Emden-Fowler equation

$$\ddot{x} = \epsilon \, \frac{t^2}{x^{5/3}} \,, \qquad -1 \leq t \leq 1, \quad x > 0 \,, \tag{3}$$

can be solved explicitly. This fact will help us to deal with some difficulties which seem to require a deeper treatment for other, non-solvable choices of the parameters. (Precisely, we do not know how to extend Lemma 4.1 when the coefficient 5/3 in (3) is replaced by some number $p \geq 1$ and t^2 by $|t|^q$ for some $q > p$. However, we believe that it should be true). Of course, the function $h(t) = \epsilon \, t^2$ is not periodic, but our example will consist in taking h to be instead a convenient approximation of

$$h_\epsilon(t) := \epsilon \sum_{m=-\infty}^{+\infty} (t - 2m)^2 \chi_{2m}(t) - 2 \sum_{m=-\infty}^{+\infty} \delta_{2m+1}(t) \,, \tag{4}$$

where each χ_{2m} stands for the characteristic function of $]2m-1, 2m+1[$ and δ_{2m+1} is the Dirac measure at $2m+1$, see Fig. 1 below. Observe that h_ϵ is 2-periodic and its mean value is $\epsilon/3 - 1$, which is negative if $0 < \epsilon < 3$.

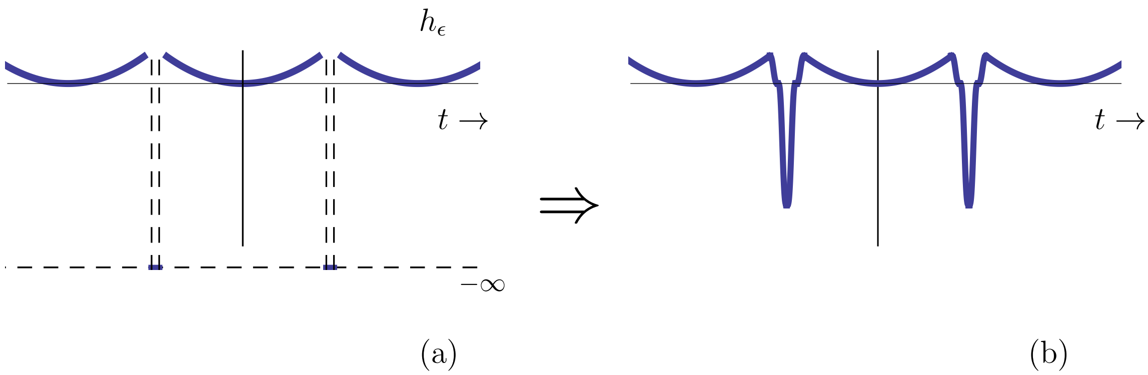


Figure 1: (a): The 'graph' of h_ϵ, and (b): a 'true function' approximation.

For this argument to work, an important step will consist in showing that (2) with $h = h_\epsilon$ does not have 2-periodic solutions for $\epsilon > 0$ small enough. Or, what is the same, that for small $\epsilon > 0$ there are not solutions of (3) satisfying

$$x(-1) = x(1)\,, \qquad \dot{x}(-1) - \dot{x}(1) = -\frac{2}{x(1)^{5/3}}. \tag{5}$$

This will be done in Proposition 4.2, and it is precisely in this step where we shall exploit in a more critical way the explicit form of the solutions of (3). Let us now describe an alternative, heuristic (and somewhat incomplete) argument which may nevertheless shed some light on the situation. In the limit as $\epsilon \to 0$, the solutions of our differential equation become straight lines as long as they are positive, and bounce back, in the way a beam of light would do, if they hit the 'mirror' $x = 0$. See Fig. 2(a). Thus, in some sense, the limit equation is

$$\begin{cases} \ddot{x}(t) = 0 & \text{if } x(t) > 0 \\ \dot{x}(t_+) = -\dot{x}(t_-) & \text{if } x(t) = 0 \end{cases}. \tag{6}$$

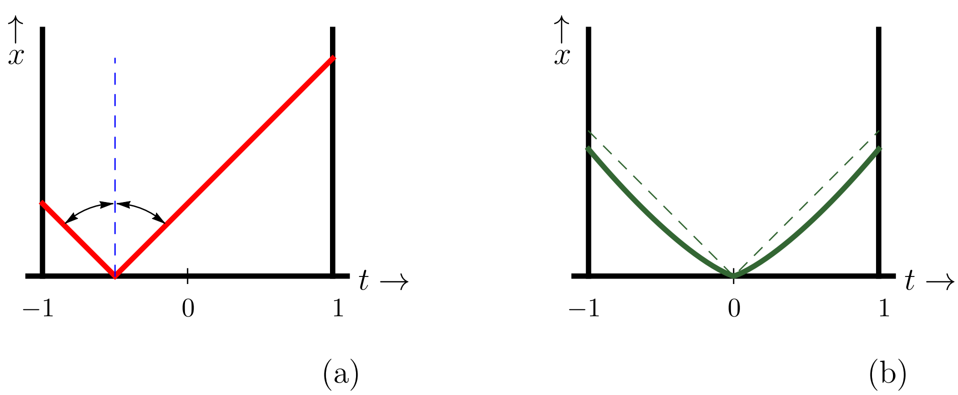


Figure 2: (a) The solutions of the limit equation (6). (b) Continuating x_* to solutions of (3)-(5) for small $\epsilon > 0$.

There is exactly one solution satisfying the boundary conditions (5), namely

$$x_*(t) \equiv |t| .$$

We shall see that this solution can be continued for small $\epsilon > 0$, giving rise to a unique 'solution' of (3)-(5). However, it turns out that these 'solutions' actually vanish at $t = 0$, so that they cannot be properly considered solutions, see Fig. 2(b). Consequently, for small $\epsilon > 0$, problem (3)-(5) is not solvable.

The argumentation above cannot be considered accurate, mainly because we did not make clear the sense in which (3) converges to (6) as $\epsilon \to 0$. For this reason, an alternative (and complete) discussion is presented next. To this aim, we shall need, in first place, some properties of the solutions of (3), which are collected in the next three sections.

The analogous of Theorem 1.1 for equations of the form (1) with $0 < p < 1$ is also true, and is indeed easier to obtain. In this case one checks that, taking $h := 1 - K \sum_{n=-\infty}^{+\infty} \delta_n$, equation (1) does not have 1-periodic solutions if $K > 0$ is sufficiently large. Later on, one can use this 'degenerate' case to construct examples where h is a true function instead of a measure. We shall not go back to this problem in the present paper.

3 Bouncing back from the singularity

We shall begin our study of (3) by having a look at the solutions which, at time $t = -1$ depart from a given position (say, $x = 1$), and head towards the origin at a big (negative) speed. Intuition says that such a solution will rebound at some time slightly bigger than -1 on some positive position, subsequently continuing upwards, to arrive at a given higher position (say, $x = 2$) before time $t = 1$. The point is that all this holds uniformly with respect to the parameter ϵ. More precisely, the main result of this section is the following.

Proposition 3.1. *There exists some $M > 0$ (not depending on ϵ or x) such that, whenever $x : [-1, 1] \to \mathbb{R}$ is a solution of (3) for some $\epsilon > 0$ satisfying $x(-1) = 1$ and $\dot{x}(-1) < -M$, then $x(1) > 2$.*

Proof. We use a contradiction argument and assume instead the existence of a sequence $\{\epsilon_n\}_n$ of positive numbers, and, for each n, a solution $x_n = x_n(t)$ of (3) with $\epsilon = \epsilon_n$ such that $x_n(-1) = 1$, $\dot{x}_n(-1) \to -\infty$, and $x(1) \le 2$. After possibly passing to a subsequence we may assume that, either *(a):* $\dot{x}_n(t) < 0$ for any $t \in [-1, -1/2]$ and any $n \in \mathbb{N}$, or *(b):* for each n there exists some $t_n \in [-1, -1/2]$ with $\dot{x}(t_n) = 0$. We study each case separately:

(a) Let the function $\mathcal{G} :]0, 1] \to \mathbb{R}$ be defined by $\mathcal{G}(x) := (3/2)\left(1/x^{2/3} - 1\right)$. Differentiation shows that the functions $t \mapsto \dot{x}_n(t)^2/2 + \epsilon_n \mathcal{G}(x_n(t))$ are increasing on $[-1, -3/4]$, and hence

$$\dot{x}_n(-1)^2/2 \le \dot{x}_n(-3/4)^2/2 + \epsilon_n \mathcal{G}(x_n(-3/4)) , \qquad n \in \mathbb{N} , \tag{7}$$

(observe that $\mathcal{G}(x_n(1)) = \mathcal{G}(1) = 0$). On the other hand, each x_n is convex, and we deduce that

$$\dot{x}_n(t) \le \dot{x}_n(-3/4) \quad \text{if } t \in [-1, -3/4] ,$$

and integrating both sides of this inequality we find that

$$0 > \dot{x}_n(-3/4) \geq 4\big(x_n(-3/4) - x_n(-1)\big) \geq -4\,, \tag{8}$$

for every $n \in \mathbb{N}$. We combine this information with (7), to obtain

$$\frac{\epsilon_n}{x_n(-3/4)^{2/3}} > \epsilon_n(2/3)\mathcal{G}(x_n(-3/4)) \geq \dot{x}_n(-1)^2/3 - 16/3 \to +\infty \text{ as } t \to +\infty\,.$$

We observe now that

$$\ddot{x}_n(t) = \frac{\epsilon_n t^2}{x_n(t)^{5/3}} \geq \frac{\epsilon_n/4}{x_n(-3/4)^{5/3}} \geq \frac{\epsilon_n/4}{x_n(-3/4)^{2/3}}\,, \qquad t \in [-3/4, -1/2]\,,$$

and consequently, $\ddot{x}_n(t) \to +\infty$ as $n \to +\infty$, uniformly with respect to $t \in [-3/4, -1/2]$. Integration shows that $\dot{x}_n(-1/2) - \dot{x}_n(-3/4) \to +\infty$, or, what is the same, (by (8)), that $\dot{x}_n(-1/2) \to +\infty$. This contradicts our assumption that $\dot{x}_n(t) < 0$ for any $t \in [-1, -1/2]$.

(b) We consider the function $\mathcal{G}$ defined as above. Differentiation shows that each function $t \mapsto \dot{x}_n(t)^2/2 + \epsilon_n \mathcal{G}(x_n(t))$ is increasing on $[-1, t_n]$, and therefore,

$$\epsilon_n \mathcal{G}(x_n(t_n)) \geq \dot{x}_n(-1)^2/2 \to +\infty\,.$$

On the other hand, we observe that

$$\ddot{x}_n(t) \geq \frac{\epsilon_n/16}{x_n^{5/3}} \qquad \text{for any } t \in [t_n, t_n + 1/4] \subset [-1, -1/4]\,.$$

This allows us to apply Lemma 3.3 of [8] to the translated sequence

$$x_n^*(t) := x_n(t_n + t), \qquad t \in [0, 1/4]\,,$$

(take $t_0 = 0$, $t_1 = 1/4$, $\rho_n = x_n(t_n)$, $\bar{h}_n = \epsilon_n/16$, and $g(x) := 1/x^{5/3}$). It follows that $x_n(t_n + 1/4) \to +\infty$, and, since each x_n is increasing on $[t_n, 1]$, we deduce

$$x_n(1) \geq x_n(t_n) \to +\infty \qquad \text{as } n \to +\infty\,,$$

contradicting the right boundary condition in (3). □

The homogeneity of our equation and a rescaling argument immediately lead us to corollaries 3.2 and 3.3 below. Here, the constant $M > 0$ is given by Proposition 3.1.

Corollary 3.2. *If $x : [-1, 1] \to \mathbb{R}$ is a solution of (3) for some $\epsilon > 0$, and $\dot{x}(-1) < -Mx(1)$, then $x(1) > 2x(-1)$.*

Proof. The function $y(t) := x(t)/x(1)$ solves again an equation of the form (3). The result follows by applying Proposition 3.1. □

Corollary 3.3. *There is some $\rho_0 > 0$ such that, whenever $x : [-1, 1] \to \mathbb{R}$ is a solution of (3) for some $\epsilon > 0$ and $x(-1) = x(1) < \rho_0$, then $\dot{x}(-1) > -1$ and $\dot{x}(1) < 1$.*

Proof. Let $\rho_0 := 1/M$, the constant $M > 0$ being as given by Proposition 3.1. It suffices to apply Corollary 3.2 to the solutions $x(t)$ and $\tilde{x}(t) := x(-t)$. □

4 Singular equations with a small parameter

We turn now our attention to certain properties of equation (3) which hold only for small $\epsilon > 0$. An important tool will be the change of variables

$$x(t) = (4\epsilon/3)^{3/8} v(t)\,, \tag{9}$$

which transforms (3) into

$$\ddot{v} = \left(\frac{3}{4}\right) \frac{t^2}{v^{5/3}}\,, \qquad -1 \le t \le 1\,, \quad v > 0\,. \tag{10}$$

This equation will be studied in the Appendix (Section 8). In particular, we shall see that its solutions are explicit, and this fact will be exploited to obtain some of its delicate properties. One of them will be Proposition 8.7; in combination with the change of variables (9) it immediately yields the following symmetry-type result:

Lemma 4.1. *If $\epsilon > 0$ is small enough, any solution $x : [-1, 1] \to \mathbb{R}$ of (3) satisfying $x(-1) = x(1) = 1$ is even.*

Remark. *The even solutions of (10) (or (3)) are explicit (see Corollary 8.4), and combining this fact with Lemma 4.1 one might prove a stronger result: if $\epsilon > 0$ is small enough, then (3) has a unique solution satisfying $x(\pm 1) = 1$. However, this is not needed in this paper.*

We are now ready to prove the main result of this section. It says that for $\epsilon > 0$ small enough, the 2-periodic problem associated to (2) with $h = h_\epsilon$ (as in (4)), is not solvable.

Proposition 4.2. *If $\epsilon > 0$ is small enough, then every solution $x : [-1, 1] \to \mathbb{R}$ of (3) with $x(-1) = x(1)$ satisfies that $\dot{x}(1) - \dot{x}(-1) < 2/x(1)^{5/3}$.*

Proof. We choose some $N > 0$ as given by Corollary 8.14, some $\epsilon_0 > 0$ such that Lemma 4.1 holds for all $0 < \epsilon < \epsilon_0$, and some $0 < \rho_0 < 1$ as given by Corollary 3.3. Choose now $0 < \epsilon < \min\{\rho_0^{8/3}\epsilon_0, 3/(4N)\}$ and some solution $x : [-1, 1] \to \mathbb{R}$ of (3) with $x(-1) = x(1)$. We distinguish two cases, depending on whether $x(\pm 1)$ is smaller or greater than ρ_0:

$\boxed{\textit{(i)}\text{: } x(\pm 1) < \rho_0.}$ Then, Corollary 3.3 implies that

$$\dot{x}(-1) > -1 > -1/x(1)^{5/3}\,, \qquad \dot{x}(1) < 1 < 1/x(1)^{5/3}\,,$$

and the result follows.

$\boxed{\textit{(ii)}\text{: } x(\pm 1) \ge \rho_0.}$ We observe that $y(t) := x(t)/x(1)$ satisfies (3) with $\tilde{\epsilon} = \epsilon/x(1)^{8/3}$ in the place of ϵ; moreover, $y(\pm 1) = 1$. Since $\tilde{\epsilon} \le \epsilon/\rho_0^{8/3} < \epsilon_0$, Lemma 4.1 implies that y is even, or, what is the same, that x is even. Then, $v(t) := (3/(4\epsilon))^{3/8} x(t)$ is an even solution of (10), and Corollary 8.14 implies that

$$v(1)^{5/3}\dot{v}(1) = \frac{3}{4\epsilon} x(1)^{5/3}\dot{x}(1) \le N\,,$$

or, what is the same, $x(1)^{5/3}\dot{x}(1) \le 4N\epsilon/3 < 1$. Since x is even, we also have that $x(-1)^{5/3}\dot{x}(-1) \ge -4N\epsilon/3 > -1$ and the result follows. $\square$

5 The Poincaré map for small $\epsilon > 0$.

In this section we continue our study of equation (3), and in particular, we are interested in the associated Poincaré maps $P_\epsilon : (x(-1), \dot{x}(-1)) \mapsto (x(1), \dot{x}(1))$. We notice that not all solutions are defined on the whole time interval $[-1, 1]$, because some of them collide with the singularity $x = 0$ at time $t = 0$. We denote by Γ_ϵ to the set of initial conditions in $]0, +\infty[\times\mathbb{R}$ where P_ϵ is not defined, i.e.,

$$\Gamma_\epsilon := \left\{ \begin{pmatrix} x(-1) \\ \dot{x}(-1) \end{pmatrix} : x : [-1, 0[\to]0, +\infty[\text{ solves (3) and } \lim_{t\to 0} x(t) = 0 \right\}.$$

As observed in the previous section, the change of variables (9) transforms (3) into (10). We use the results of Subsection 8.3, and let $\gamma_\epsilon(a) = \big(4\epsilon/3\big)^{3/8}\gamma(a)$, so that

$$\Gamma_\epsilon = \big\{\gamma_\epsilon(a) : a \geq 0\big\} = \left(\frac{4\epsilon}{3}\right)^{3/8} \Gamma \qquad \text{for any } \epsilon > 0\,.$$

Moreover,

$$P_\epsilon\left(\left(\frac{4\epsilon}{3}\right)^{3/8} z\right) = \left(\frac{4\epsilon}{3}\right)^{3/8} P(z)\,, \qquad z \in (]0, +\infty[\times\mathbb{R})\backslash\Gamma_\epsilon\,, \tag{11}$$

(the set Γ and the map P are defined in (47) and (46) respectively). We extend these maps to $]0, +\infty[\times\mathbb{R}$ by setting

$$P_\epsilon(z) := Rz \qquad \text{if } z \in \Gamma_\epsilon\,, \tag{12}$$

and it follows from Lemma 8.11 that each P_ϵ is a homeomorphisms when seen as a map from $]0, +\infty[\times\mathbb{R}$ into itself, and a C^∞-diffeomorphisms if seen from $(]0, +\infty[\times\mathbb{R})\backslash\{Rz_\epsilon^*\}$ into $(]0, +\infty[\times\mathbb{R}^2)\backslash\{z_\epsilon^*\}$. Here,

$$z_\epsilon^* = \gamma_\epsilon(0) = \left(\frac{4\epsilon}{3}\right)^{3/8} z^* \to 0 \text{ as } \epsilon \to 0\,,$$

and $z^* = (1, 3/2)$ (see (34)). We also consider the map $Q_* :]0, +\infty[\times\mathbb{R} \to]0, +\infty[\times\mathbb{R}$ defined by

$$Q_*(z) := \begin{pmatrix} \pi_1 z \\ \pi_2 z - 1/(\pi_1 z)^{5/3} \end{pmatrix},$$

and Proposition 4.2 can be reformulated as saying that $Q_*^2 \circ P_\epsilon$ does not have fixed points on $\big(]0, +\infty[\times\mathbb{R}\big)\backslash\Gamma_\epsilon$ for small $\epsilon > 0$. In order to check the situation on Γ_ϵ we consider the curve $\beta_\epsilon : [0, +\infty[\times\mathbb{R}^2$ defined by

$$\beta_\epsilon(a) := \Big[Q_*^2 \circ P_\epsilon - \mathrm{Id}\Big]\big(\gamma_\epsilon(a)\big) = 2\left(\frac{4\epsilon}{3}\right)^{3/8} \begin{pmatrix} 0 \\ -\pi_2\gamma(a) - \frac{3}{4\,\epsilon\,(\pi_1\gamma(a))^{5/3}} \end{pmatrix}, \tag{13}$$

(to check the second equality above recall that $P_\epsilon = R$ on Γ_ϵ). The main result of this section is the following

Lemma 5.1. *For $\epsilon > 0$ small enough, $Q_*^2 \circ P_\epsilon$ has a unique fixed point $z_\epsilon \in]0, +\infty[\times \mathbb{R}$. Moreover, $z_\epsilon = \gamma_\epsilon(a_\epsilon) \in \Gamma_\epsilon$ and $z_\epsilon \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as $\epsilon \to 0$.*

Proof. In view of (13), one has to show that there is exactly one solution $a = a_\epsilon \geq 0$ of the equation

$$\beta_\epsilon(a) = 0 \iff (\pi_1\gamma)(a)^{5/3}(\pi_2\gamma)(a) = -\frac{3}{4\epsilon}\,.$$

This follows from the fact that, by (48), the function $a \mapsto (\pi_1\gamma)(a)^{5/3}(\pi_2\gamma)(a)$ is decreasing for big values of a. Moreover,

$$a_\epsilon \to +\infty \text{ as } \epsilon \to 0\,. \tag{14}$$

The assertion above can be made more precise. Indeed, it is shown in (48) that

$$\frac{(\pi_1\gamma)(a)^{5/3}(\pi_2\gamma)(a)}{(3\sqrt{a})^{8/3}} \to -1 \qquad \text{as } a \to +\infty\,,$$

and it follows that $\lim_{\epsilon\to 0} \big(4\epsilon/3\big)^{3/4} a_\epsilon = 1/9$. Thus, since $z_\epsilon = (4\epsilon/3)^{3/8}\gamma(a_\epsilon)$, again by (48) we conclude that $\lim_{\epsilon\to 0} z_\epsilon = (1, -1)$, as claimed. □

In particular, for small $\epsilon > 0$ one has that $z_\epsilon \neq z_\epsilon^*$ and P_ϵ is differentiable at z_ϵ. The lemma below explores some properties of the associated derivative.

Lemma 5.2. *For $\epsilon > 0$ small enough,*

(i) $0 \neq \beta_\epsilon'(a_\epsilon) \in \{0\} \times \mathbb{R}$,

(ii) *z_ϵ is nondegenerate as a fixed point of $Q_*^2 \circ P_\epsilon$.*

Proof. *(i)*: We differentiate in (13) at $a = a_\epsilon$, to find

$$\beta_\epsilon'(a_\epsilon) = \Big[(Q_*^2 \circ P_\epsilon)'\big(z_\epsilon\big) - \text{ Id}\Big]\gamma'(a_\epsilon) = 2\begin{pmatrix} 0 \\ -\pi_2\gamma'(a_\epsilon) + \frac{5\,\pi_1\gamma'(a_\epsilon)}{4\,\epsilon\,(\pi_1\gamma(a_\epsilon))^{8/3}} \end{pmatrix}.$$

Now, (14) and (48) imply that, if $\epsilon > 0$ is small, $\pi_2\gamma'(a_\epsilon) < 0 < \pi_1\gamma'(a_\epsilon)$, and hence, $\pi_2\beta_\epsilon'(a_\epsilon) > 0$.

(ii): In view of *(i)*, we only have to check that the gradient of the first component of $Q_*^2 \circ P_\epsilon$ at z_ϵ is not $(1, 0)$ for small $\epsilon > 0$. In view of (11)-(12), this assertion becomes $\nabla(\pi_1 P)\big(\gamma(a_\epsilon)\big) \neq (1, 0)$ for small $\epsilon > 0$, something which follows from (14) and Lemma 8.12. The proof is now complete. □

6 Regularity of some Poincaré maps for small time

In this section we leave aside the world of equations with singularities and consider a problem related with the so-called averaging method (see, e.g. [5]). Let $I \subset \mathbb{R}$ be an open interval and let $f : [0, \infty[\times[0,1] \times I \to \mathbb{R}$, $f = f(\delta, s, x)$, be continuous in all three variables, and continuously differentiable with respect to x. For any $\delta > 0$ we consider the equation

$$\delta\, \ddot{x} = f(\delta, t/\delta, x), \qquad\qquad t \in [0, \delta]\,. \tag{X_δ}$$

Let us denote by Ω to the set of triples $(\delta, x_0, \dot{x}_0)$ such $\delta > 0$, $x_0 \in I$, and the solution $x = x(t)$ of (X_δ) with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ is defined for $t = \delta$. This set is the natural domain of the Poincaré map $(\mathcal{X}, \dot{\mathcal{X}}) : \Omega \to \mathbb{R}^2$, defined by

$$\mathcal{X}\big(\delta, x(0), \dot{x}(0)\big) := x(\delta), \qquad \dot{\mathcal{X}}\big(\delta, x(0), \dot{x}(0)\big) := \dot{x}(\delta)\,,$$

for any solution $x : [0, \delta] \to I$ of (X_δ).

The usual continuous and differentiable dependence theorems state that Ω is open subset in $\mathbb{R}^3$, moreover, $\mathcal{X}, \dot{\mathcal{X}}$ are continuously differentiable with respect to their second and third variables. We are concerned with the behaviour of these functions as $\delta \to 0$ and with this aim we consider the 'augmented set'

$$\widehat{\Omega} := \big(\{0\} \times I \times \mathbb{R}\big) \cup \Omega\,,$$

and we extend $(\mathcal{X}, \dot{\mathcal{X}})$ to $\widehat{\Omega}$ by setting

$$\mathcal{X}(0, x_0, \dot{x}_0) := x_0\,, \quad \dot{\mathcal{X}}(0, x_0, \dot{x}_0) := \dot{x}_0 + \int_0^1 f(0, s, x_0)ds\,. \tag{15}$$

The main result of this section is the following

Lemma 6.1. *(a) $\widehat{\Omega}$ is open relative to $[0, +\infty[\times\mathbb{R}^2$; (b) $\mathcal{X}, \dot{\mathcal{X}} : \widehat{\Omega} \to \mathbb{R}$ are continuous in all three variables; (c) $\mathcal{X}, \dot{\mathcal{X}} : \widehat{\Omega} \to \mathbb{R}$ are continuously differentiable with respect to $(x_0, \dot{x}_0)$.*

Proof. We introduce the change of independent variable $s = t/\delta$ in (X_δ), to obtain

$$\ddot{y} = \delta f(\delta, s, y), \qquad\qquad s \in [0, 1]\,, \tag{Y_δ}$$

where $y(s) = x(\delta s)$. It motivates us to consider the set $\widehat{\Omega}_Y$ of triples $(\delta, y_0, \dot{y}_0) \in [0, +\infty[\times I \times \mathbb{R}$ such that the solution $y = y(s)$ of (Y_δ) with $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$ is defined for $s = 1$. Observe that $\{0\} \times I \times \{0\} \subset \hat{\Omega}_Y$. We also consider the Poincaré map $(\mathcal{Y}, \dot{\mathcal{Y}}) : \hat{\Omega}_Y \to \mathbb{R}^2$, given by

$$\mathcal{Y}\big(\delta, y(0), \dot{y}(0)\big) := y(1), \qquad \dot{\mathcal{Y}}\big(\delta, y(0), \dot{y}(0)\big) := \dot{y}(1)\,,$$

for any solution $y : [0, 1] \to I$ of (Y_δ). The usual continuous dependence theorems state that $\widehat{\Omega}_Y$ is open relative to $[0, +\infty[\times\mathbb{R}^2$, and it follows from our changes of variables that

$$\Omega = \Big\{ \big(\delta, y_0, \dot{y}_0/\delta\big) : \big(\delta, y_0, \dot{y}_0\big) \in \Omega_Y \Big\}\,, \tag{16}$$

where $\Omega_Y := \widehat{\Omega}_Y \cap (]0, +\infty[\times\mathbb{R}^2)$. Furthermore,

$$\mathcal{X}\big(\delta, y_0, \dot{y}_0/\delta\big) = \mathcal{Y}(\delta, y_0, \dot{y}_0), \qquad \dot{\mathcal{X}}\big(\delta, y_0, \dot{y}_0/\delta\big) = \dot{\mathcal{Y}}(\delta, y_0, \dot{y}_0)/\delta\,, \tag{17}$$

for every $(\delta, y_0, \dot{y}_0) \in \Omega_Y$.

Proof of (a): Since Ω is open in $]0,+\infty[\times\mathbb{R}^2$, we only need to check that the set $\{0\}\times I\times\mathbb{R}$ is contained in the interior of $\widehat{\Omega}$ relative to $[0,+\infty[\times\mathbb{R}^2$. With this goal, choose some point $x_0 \in I$ and some constant $M > 0$. The point $(0, x_0, 0)$ belongs to $\widehat{\Omega}_Y$ and hence, there is some $0 < \delta < 1$ such that $[0, \delta[\times]z_0 - \delta, z_0 + \delta[\times] - M\delta, M\delta[\subset \widehat{\Omega}_Y$. It follows from (16) that $]0, \delta[\times]z_0 - \delta, z_0 + \delta[\times] - M, M[\subset \Omega$, implying the result.

Proof of (b)-(c): We rewrite (17) as

$$\mathcal{X}\big(\delta, x_0, \dot{x}_0\big) = \mathcal{Y}(\delta, x_0, \delta\dot{x}_0), \qquad \dot{\mathcal{X}}\big(\delta, x_0, \dot{x}_0\big) = \dot{\mathcal{Y}}(\delta, x_0, \delta\dot{x}_0)/\delta, \tag{18}$$

In principle, this holds for $(\delta, x_0, \dot{x}_0) \in \Omega$. However, in view of (15), the left equality of (18) holds actually for $(\delta, x_0, \dot{x}_0) \in \widehat{\Omega}$, and it follows that $\mathcal{X}(\delta, x_0, \dot{x}_0)$ is C^1-smooth with respect to $(x_0, \dot{x}_0)$ up to $\delta = 0$. Concerning to $\dot{\mathcal{X}}$ one has to check that

$$\lim_{\delta\to 0} \frac{\dot{\mathcal{Y}}(\delta, x_0, \delta\dot{x}_0)}{\delta} = \dot{x}_0 + \int_0^1 f(0, s, x_0)\, ds\,,$$

in the $C^1(x_0, \dot{x}_0)$ sense and uniformly with respect to $(x_0, \dot{x}_0)$ in compact subsets of $I \times \mathbb{R}$. With this aim we choose some converging sequence $(\delta_n, x_{0,n}, \dot{x}_{0,n}) \to (0, x_0, \dot{x}_0)$ in $\widehat{\Omega}$. Let $y_n : [0, 1] \to \mathbb{R}$ be the solution of (Y_{δ_n}) satisfying $y_n(0) = x_{0,n}$ and $\dot{y}_n(0) = \delta_n\dot{x}_{0,n}$; by continuous dependence, $y_n(s) \to x_0$ uniformly with respect to $s \in [0, 1]$. Consequently,

$$\frac{\dot{\mathcal{Y}}(\delta_n, x_{0,n}, \delta_n x_{0,n})}{\delta_n} = \frac{\dot{y}_n(1)}{\delta_n} = \frac{\dot{y}_n(0)}{\delta_n} + \int_0^1 f(\delta_n, s, y_n(s))\, ds \to \dot{x}_0 + \int_0^1 f(0, s, x_0) ds \quad \text{as } n \to \infty\,.$$

This establishes the continuity of $\dot{\mathcal{X}}$ up to $\delta = 0$. The remainig of the proof follows in an analogous way from the differentiable dependence theorem and the continuous dependence theorem (applied to the linearized equation). □

We shall use Lemma 6.1 in two particular cases. The first one is concerned with equations of the form

$$\ddot{x} = \frac{\eta(\delta, t/\delta)}{x^p}, \qquad t \in [0, \delta]\,. \tag{19}$$

Here, $p > 0$ is a constant, $\delta > 0$ is a parameter, and $\eta : [0, +\infty[\times[0, 1] \to \mathbb{R}$, $(\delta, s) \mapsto \eta(\delta, s)$, is continuous. This equation becomes (X_δ) by setting $I :=]0, +\infty[$ and $f(\delta, s, x) := \delta\, \eta(\delta, s)/x^p$. Let Ω_1 be the set of points $(\delta, x_0, \dot{x}_0)$ such that $\delta, x_0 > 0$ and the solution $x = x(t)$ of (19) with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ is defined for $t = \delta$, and let $J : \Omega_1 \to \mathbb{R}^2$, $(\delta, x(0), \dot{x}(0)) \mapsto (x(\delta), \dot{x}(\delta))$, be the associated Poincaré map. Lemma 6.1 gives:

Corollary 6.2. $\widehat{\Omega}_1 := \big(\{0\}\times]0, +\infty[\times\mathbb{R}\big) \cup \Omega_1$ *is open relative to* $[0, +\infty[\times\mathbb{R}^2$, *and the extension of* J *to* $\widehat{\Omega}_1$ *given by*

$$J(0, x_0, \dot{x}_0) := (x_0, \dot{x}_0)\,, \tag{20}$$

is continuous in all three variables and continuously differentiable with respect to $(x_0, \dot{x}_0)$.

The second case of Lemma 6.1 that we are interested in involves equations of the form

$$\ddot{x} = \frac{\zeta(t/\delta)/\delta}{x^p}\,, \qquad t \in [0, \delta]\,,$$

where $p > 0$ is a constant, $\delta > 0$ is a parameter, and $\zeta : [0,1] \to \mathbb{R}$, $s \mapsto \zeta(s)$, is a given continuous function. It becomes (X_δ) by setting $I :=]0,+\infty[$ and $f(\delta, s, x) := \zeta(s)/x^p$. Let now Ω_2 be the set of points $(\delta, x_0, \dot{x}_0)$ such that $\delta, x_0 > 0$ and the solution $x = x(t)$ of (19) with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ is defined for $t = \delta$, and let $Q : \Omega_2 \to \mathbb{R}^2$, $(\delta, x(0), \dot{x}(0)) \mapsto (x(\delta), \dot{x}(\delta))$, be the associated Poincaré map. Lemma 6.1 gives:

Corollary 6.3. *$\widehat{\Omega}_2 := \big(\{0\}\times]0,+\infty[\times\mathbb{R}\big) \cup \Omega_2$ is open relative to $[0,+\infty[\times\mathbb{R}^2$, and the extension of Q to $\widehat{\Omega}_2$ given by*

$$Q(0, x_0, \dot{x}_0) := \left(x_0, \dot{x}_0 + \frac{1}{x_0^p} \int_0^1 \zeta(s) ds \right) , \tag{21}$$

is continuous in all three variables and continuously differentiable with respect to $(x_0, \dot{x}_0)$.

7 The construction

In this section we finally complete the proof of Theorem 1.1. Pick some C^∞ function $\eta : [0,+\infty[\times[0,1] \to \mathbb{R}$, $\eta = \eta(\delta, s)$, with

$$\eta(\delta, 0) = 1, \qquad \partial_s \eta(\delta, 0) = 2\delta, \qquad \partial_s^{(2)} \eta(\delta, 0) = 2\delta^2, \qquad \partial_s^{(r)} \eta(\delta, 0) = 0 \text{ if } r \geq 3,$$

and

$$\partial_s^{(r)} \eta(\delta, 1) = 0 \qquad \text{if } r \geq 0.$$

Choose also some C^∞, even function $\zeta : [-1,1] \to \mathbb{R}$ with

$$\zeta^{(r)}(\pm 1) = 0 \text{ for any } r \geq 0 \,, \qquad \int_0^1 \zeta(t) dt = -1 \,,$$

and define, for any $\epsilon, \delta > 0$, an even function $h_{\epsilon,\delta} : [-1-2\delta, 1+2\delta] \to \mathbb{R}$ as follows:

$$h_{\epsilon,\delta}(t) := \begin{cases} \epsilon\, t^2 & \text{if } |t| \leq 1 \,, \\ \epsilon\, \eta\big(\delta, (|t|-1)/\delta\big) & \text{if } 1 < |t| \leq 1+\delta \,, \\ \zeta\big((|t|-1-2\delta)/\delta\big)/\delta & \text{if } 1+\delta < |t| \leq 1+2\delta \,. \end{cases}$$

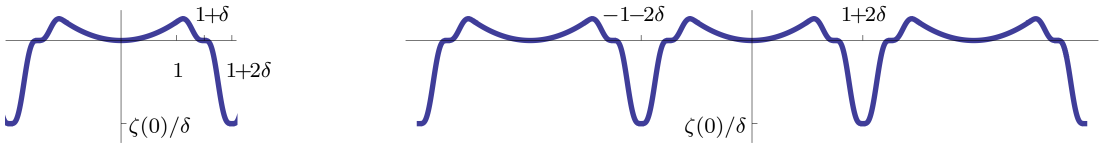
Finally, we extend $h_{\epsilon,\delta}$ by periodicity to the whole real line, see Fig. 3 below. This function is C^∞, even, and $(2+4\delta)$-periodic. One can check that

$$\int_0^{2+4\delta} h_{\epsilon,\delta}(t) dt = -2 + \frac{2\epsilon}{3} + 2\epsilon\delta \int_0^1 \eta(\delta, s) ds \,. \tag{22}$$

Hence, for $\epsilon, \delta > 0$ small, $h_{\epsilon,\delta}$ has negative mean. However, we shall also see that, roughly speaking, for small $\epsilon, \delta > 0$ the equation

$$\ddot{x} = \frac{h_{\epsilon,\delta}(t)}{x^{5/3}} \,, \qquad x > 0 \,, \tag{23}$$

does not have $(2+4\delta)$-periodic solutions. More precisely, we shall prove the following result, which obviously implies Theorem 1.1:



(a) (b)

Figure 3: (a): The function $h_{\epsilon,\delta}$ on the interval $[-1-2\delta, 1+2\delta]$, and (b): repeated by periodicity.

Proposition 7.1. *For $\epsilon > 0$ small enough there is some $\delta_0(\epsilon) > 0$ such that (23) does not have $(2+4\delta)$-periodic solutions if $0 < \delta < \delta_0(\epsilon)$.*

It will be convenient to 'freeze' ϵ and study the associated Poincaré map as a function of the parameter δ and the initial condition. Thus, from this moment on we fix the number $\epsilon > 0$ small enough to fit the requirements of lemmas 5.1 and 5.2, and we consider the Poincaré map

$$\mathcal{P}_\epsilon\big(\delta, x(-1), \dot{x}(-1)\big) := \big(x(1+4\delta), \dot{x}(1+4\delta)\big)\,,$$

where $x = x(t)$ is a solution of (23). In principle, this map is naturally defined on an open subset of $]0,+\infty[\times\big(]0,+\infty[\times\mathbb{R}\big)$ which does not intersect $]0,+\infty[\times\Gamma$. However, we observe that

$$\mathcal{P}_\epsilon(\delta,\cdot) = R \circ J(\delta,\cdot)^{-1} \circ Q(\delta,\cdot)^{-1} \circ R \circ Q(\delta,\cdot) \circ J(\delta,\cdot) \circ P_\epsilon\,, \tag{24}$$

where P_ϵ, $Q(\delta,\cdot)$ and $J(\delta,\cdot)$ stand for the Poincaré maps $(x(-1),\dot{x}(-1)) \mapsto (x(1),\dot{x}(1))$, $(x(1),\dot{x}(1)) \mapsto (x(1+\delta),\dot{x}(1+\delta))$ and $(x(1+\delta),\dot{x}(1+\delta)) \mapsto (x(1+2\delta),\dot{x}(1+2\delta))$ respectively. By extending P_ϵ to $]0,+\infty[\times\mathbb{R}$ (as in (12)) and J, Q to respective relatively open subsets of $[0+\infty[\times]0,+\infty[\times\mathbb{R}$ (as in (20) and (21)), we obtain a continuous extension of $\mathcal{P}_\epsilon$ to a relatively open set $\Omega \subset [0,+\infty[\times\mathbb{R}$ containing $\{0\}\times]0,+\infty[\times\mathbb{R}$. By Corollaries 6.2 and 6.3, this extension is continuously differentiable with respect to z and satisfies

$$\mathcal{P}_\epsilon(0,\cdot) = R \circ Q_*^{-1} \circ R \circ Q_* \circ P_\epsilon = Q_*^2 \circ P_\epsilon \text{ on }]0,+\infty[\times\mathbb{R}\,. \tag{25}$$

Thus, Lemma 5.1 states that $\mathcal{P}_\epsilon(0,\cdot)$ has an unique fixed point on $]0,+\infty[\times\mathbb{R}$, and this fixed point is z_ϵ. Our next result studies how to obtain fixed points for small $\delta > 0$.

Lemma 7.2. *There exists an open set $V \subset]0,+\infty[\times\mathbb{R}$ with $z_\epsilon \in V$, and some $\delta_0 > 0$, such that for any $0 < \delta < \delta_0$, $\mathcal{P}_\epsilon(\delta,\cdot)\big|_V$ has a unique fixed point $z(\delta)$. Moreover, $z(\delta) \in \Gamma_\epsilon$ for any $0 < \delta < \delta_0$.*

Proof. The first part of the result follows immediately from (25) and part *(ii)* of Lemma 5.2 via the implicit function theorem. To check the 'moreover' part we observe that, by (24),

$$\mathcal{P}_\epsilon\big[\delta, \gamma_\epsilon(a)\big] = S(\delta,\cdot)^{-1} \circ R \circ S(\delta,\cdot)\gamma_\epsilon(a)\,,$$

where $S(\delta,\cdot) := Q(\delta,\cdot) \circ J(\delta,\cdot) \circ R$. Thus, $\gamma_\epsilon(a)$ is a fixed point of $\mathcal{P}_\epsilon(\delta,\cdot)$ if and only if $s(\delta,a) := \pi_2 S\big(\delta, \gamma_\epsilon(a)\big)$ vanishes. This function $s = s(\delta,a)$ is continuously defined on an open subset of $[0, +\infty[\times]0, +\infty[$ containing $\{0\}\times]0, +\infty[$, and it is continuously differentiable with respect to its second variable a. Moreover, since

$$s(0, a) = \pi_2\beta_\epsilon(a)\,, \qquad a > 0\,,$$

one has that $s(0, a_\epsilon) = 0$ (corresponding to the fixed point z_ϵ of $\mathcal{P}_\epsilon(0, \cdot)$), and, by Lemma 5.2 *(i)*, this zero is nondegenerate. The implicit function theorem applies again and shows the existence of a continuous function

$$\alpha : [0, \delta_0[\to \mathbb{R}\,, \qquad \alpha = \alpha(\delta)\,,$$

with $\alpha(0) = a_\epsilon$ and $s(\delta, \alpha(\delta)) = 0$. With other words, $\gamma_\epsilon(a)$ is a fixed point of $\mathcal{P}_\epsilon(\delta, \cdot)$ for $0 \le \delta \le \delta_0$. Moreover, since $\gamma_\epsilon(\alpha(0)) \in V$, after possibly replacing δ by a smaller quantity we may assume that $\gamma_\epsilon(\alpha(\delta)) \in V$ for any $\delta \in [0, \delta_0[$. Hence, by uniqueness, $z(\delta) = \gamma_\epsilon(\alpha(\delta))$, proving the statement. □

Proof of Proposition 7.1. We fix $0 < \epsilon < 1$ small enough to satisfy the assumptions of lemmas 5.1 and 5.2, and we claim that for $\delta > 0$ small enough, equation (23) does not have $(2+4\delta)$-periodic solutions. We use a contradiction argument and assume instead the existence of a sequence $\delta_n \to 0$ such that, for each n, there is a solution x_n of (23) with $\delta = \delta_n$.

Step 1: We claim that there is some $\rho_1 > 0$ such that $\max\{x_n(-1), x_n(1)\} \ge \rho_1$ for every $n \in \mathbb{N}$. Indeed otherwise, after possibly passing to a subsequence, we may assume that $x_n(\pm 1) \to 0$. But each x_n is convex on $[-1, 1]$, and we deduce that $x_n(t) \to 0$ as $n \to +\infty$, uniformly with respect to $t \in [-1, 1]$. In particular, $x_n(t) \to 0$ as $n \to +\infty$, uniformly with respect to $t \in [-1, -1/2]$. But on this interval one has $\ddot{x}_n \ge 1/(4x_n^{5/3})$, and hence, $\ddot{x}_n(t) \to +\infty$, uniformly with respect to $t \in [-1, -1/2]$, leading easily to a contradiction.

Step 2: We claim that $\max_{[1,1+4\delta_n]} x_n - \min_{[1,1+4\delta_n]} x_n \to 0$. To see this we consider the functions $y_n : [0, 1] \to \mathbb{R}$ defined by

$$y_n(t) := x_n(1 + 4\,\delta_n t)\,, \qquad t \in [0, 1]\,.$$

Then,

$$\ddot{y}_n(t) = \frac{\varphi_n(t)}{y_n^{5/3}}\,, \tag{26}$$

where

$$\varphi_n(t) = 16\,\delta_n^2\, h_n(1 + \delta_n t) \to 0 \text{ as } n \to +\infty, \text{ uniformly with respect to } t \in [0, 1]\,. \tag{27}$$

On the other hand, each x_n attains its maximum at some point in $[1, 1 + 4\delta_n]$ (actually, the point must be in $[1+\delta_n, 1+3\delta_n]$), and we see that each y_n attains its maximum at some point in $]0, 1[$. The value of this maximum is at least $\max\{y_n(0), y_n(1)\} = \max\{x_n(1), x_n(-1)\}$, and Step 1 implies that $\max_{[0,1]} y_n \ge \rho_1$ for each n. The result follows now from (26), (27) and the fact that $\max_{[1,1+4\delta_n]} x_n - \min_{[1,1+4\delta_n]} x_n = \max_{[0,1]} y_n - \min_{[0,1]} y_n$.

It follows from Step 2 that $x_n(1) - x_n(-1) = x_n(1) - x_n(1+4\delta_n) \to 0$ as $n \to +\infty$. Hence, we may replace the constant ρ_1 in Step 1 by a possibly smaller one so that

$$x_n(\pm 1) \geq \rho_1 \text{ for every } n. \tag{28}$$

The next step gives upper bounds for the sequences $x_n(\pm 1)$. Actually, we shall prove a slightly stronger result.

Step 3: There is a constant $\rho_2 > 0$ such that $\max_{[0,2+4\delta_n]} x_n \leq \rho_2$ for every n. To check this fact we multiply both sides of (23) by $x_n^{5/3}$ and integrate by parts in the left side, to obtain

$$\int_0^{2+4\delta_n} \ddot{x}_n x_n^{5/3} dt = -\frac{5}{3} \int_0^{2+4\delta_n} x_n(t)^{2/3} \dot{x}_n(t)^2 dt = \int_0^{2+4\delta_n} h_{\epsilon,\delta_n}(t)\, dt\,.$$

The right hand side above is a bounded sequence of n (by (22)). On the other hand, $\int_0^{2+4\delta_n} x_n(t)^{2/3} \dot{x}_n(t)^2\, dt = (9/16)\|\dot{y}_n\|^2_{L^2(0,2+4\delta_n)}$, where $y_n(t) := x_n(t)^{4/3}$. It follows that $\|\dot{y}_n\|_{L^2(0,2+4\delta_n)}$ is bounded, and we deduce that $\max_{[0,2+4\delta_n]} y_n - \min_{[0,2+4\delta_n]} y_n$ is bounded.

We use now a contradiction argument and assume instead that, after possibly passing to a subsequence, $\max_{[0,2+4\delta_n]} x_n \to +\infty$. Then, also $\max_{[0,2+4\delta_n]} y_n \to +\infty$, and consequently $\max_{[0,2+4\delta_n]} y_n / \min_{[0,2+4\delta_n]} y_n \to 1$, or, what is the same,

$$\max_{[0,2+4\delta_n]} x_n / \min_{[0,2+4\delta_n]} x_n \to 1\,.$$

On the other hand,

$$0 = x_n(-1)^{5/3} \int_0^{2+4\delta_n} \ddot{x}_n(t)\, dt = \int_0^{2+4\delta_n} \big(x_n(-1)/x_n(t)\big)^{5/3} h_{\epsilon,\delta_n}(t)\, dt\,, \qquad n \in \mathbb{N}\,,$$

and taking limits we obtain (by (22)),

$$-2 + \frac{2\epsilon}{3} = 0 \;\Rightarrow\; \epsilon = 3\,,$$

a contradiction since, by assumption $0 < \epsilon < 1$.

Combining (28) and steps 2-3 we see that, after possibly passing to a subsequence,

$$x_n(\pm 1) \to p_0 \text{ as } n \to +\infty\,, \tag{29}$$

for some number $p_0 > 0$. In our next step we observe that also $\dot{x}_n(-1)$ has a convergent subsequence

Step 4: $\dot{x}_n(-1)$ is a bounded subsequence. Indeed, each x_n being convex on $[-1, 1]$,

$$\dot{x}_n(t) \geq \dot{x}_n(-1)\,, \qquad t \in [-1, 1]\,,$$

and integrating, we obtain

$$x_n(1) - x_n(-1) \geq 2\dot{x}_n(-1)\,, \qquad n \geq 0\,.$$

Thus, (29) implies that $\dot{x}_n(-1)$ is bounded from above. On the other hand, (29) also implies that,

$$x_n(1) < x_n(-1) \text{ for } n \text{ big enough},$$

and hence, by Corollary 3.2,

$$\dot{x}_n(-1) \geq -Mx_n(-1) \text{ for } n \text{ big enough},$$

which, in view of (29), proves the result.

The end of the proof: Combining (29) and Step 4 we see that, after possibly passing to a subsequence we may assume that

$$(x_n(-1), \dot{x}_n(1)) \to (p_0, v_0) \in]0, +\infty[\times\mathbb{R}.$$

On the other hand, $(x_n(-1), \dot{x}_n(1))$ is a fixed point of $\mathcal{P}_\epsilon(\delta_n, \cdot)$ for each $n \in \mathbb{N}$, and passing to the limit we see that (p_0, v_0) is a fixed point of $\mathcal{P}_\epsilon(0, \cdot) = Q_*^2 \circ P_\epsilon$. Thus, Lemma 5.1 implies that $(p_0, v_0) = z_\epsilon$, and we conclude that $(x_n(-1), x_n(1)) \in V$ for sufficiently big n. Here, V is the open neighborhood of z_ϵ given by Lemma 7.2. By uniqueness, $(x_n(-1), x_n(1)) = z(\delta_n) \in \Gamma$ for n big enough. In particular, $x_n(0) = 0$ for n big enough, which is a contradiction and completes the proof. $\square$

8 Appendix: On an Emden-Fowler equation with negative exponent

This Appendix is dedicated to the study of equation (10), which we rewrite here for the reader's convenience:

$$\ddot{v} = \left(\frac{3}{4}\right) \frac{t^2}{v^{5/3}}, \qquad -1 \leq t \leq 1, \quad v > 0. \tag{10}$$

We devote Subsection 8.1 to obtain an explicit description of the solutions of (10), and take advantage of this knowledge in the subsequent subsections 8.2, 8.3 and 8.4. We point out that the results of Subsection 8.1 are not completely new; indeed, Lemma 8.3 can be found, in a somewhat less precise form and without a proof, in [6], paragraph 2.3.1-2, §9-2°.

8.1 Solving explicitly an Emden-Fowler equation

Our starting point will be to consider, for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the functions f_θ, $g_\theta : \mathbb{R} \to \mathbb{R}$ defined by

$$f_\theta(u) := \sinh u + \sin(u + \theta), \qquad g_\theta(u) := \cosh u + \cos(u + \theta).$$

Observe that $f_\theta'(u) = g_\theta(u) > 0$ for any $(\theta, u) \in \big[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}\big] \setminus \{(\pi, 0)\}$. Consequently,

$$\mathfrak{v}_\theta(t) := g_\theta(f_\theta^{-1}(t))^{3/2}, \qquad t \in \mathbb{R}, \tag{30}$$

are positive and C^∞-smooth on the real line, with the exception of $\mathfrak{v}_\pi$, which satisfies

$$\mathfrak{v}_\pi(0) = 0 < \mathfrak{v}_\pi(t) \text{ for } t \neq 0\,,$$

and is C^∞-smooth at all times $t \neq 0$. Fig. 4(a) below shows the graphs of some of the functions $\mathfrak{v}_\theta$.

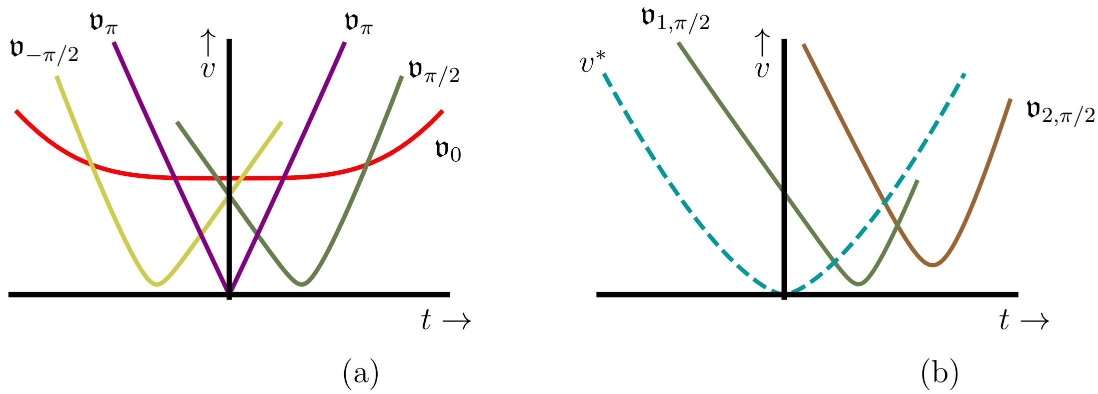


Figure 4: (a): The graphs of some functions $\mathfrak{v}_\theta$. (b): The graphs of v_*, $\mathfrak{v}_{1,\pi/2}$ and $\mathfrak{v}_{2,\pi/2}$.

We set, for $a > 0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $t \in \mathbb{R}$,

$$\mathfrak{v}_{a,\theta}(t) := a^{3/2}\mathfrak{v}_\theta(t/a)\,, \tag{31}$$

(see Fig. 4(b)). We begin our study by showing that these functions are solutions of (10).

Lemma 8.1. *$\mathfrak{v}_{a,\theta}$ is a solution of (10) for any $a > 0$ and $-\pi < \theta < \pi$. For $\theta = \pi$, $\mathfrak{v}_{a,\pi}$ solves (10) on $]-\infty, 0[$ and $]0, +\infty[$.*

Proof. One immediately checks that $\mathfrak{v}_{a,\theta}$ is a solution of (10) if and only if $\mathfrak{v}_\theta = \mathfrak{v}_{1,\theta}$ is another one. Hence, it suffices to prove the result in the case $a = 1$. Recalling that $f_\theta' = g_\theta$ and differentiating in (30) we see that, for $(\theta, t) \neq (\pi, 0)$,

$$\dot{\mathfrak{v}}_\theta(t) = \left(\frac{3}{2}\right) \frac{g_\theta'(f_\theta^{-1}(t))}{\sqrt{g_\theta(f_\theta^{-1}(t))}}\,, \qquad \ddot{\mathfrak{v}}_\theta(t) = \left(\frac{3}{4}\right) \left[\frac{2\,g_\theta''(f_\theta^{-1}(t))g_\theta(f_\theta^{-1}(t)) - g_\theta'(f_\theta^{-1}(t))^2}{g_\theta(f_\theta^{-1}(t))^{5/2}}\right]\,. \tag{32}$$

One easily checks that $2g_\theta''(u)g_\theta(u) - g_\theta'(u)^2 = f_\theta(u)^2$, and the second equality above becomes (10). The result follows. $\square$

Being solutions of (10), it is clear that all functions $\mathfrak{v}_{a,\theta}$ are convex. In the following result we extend our knowledge of these functions by computing the sign of their derivatives at $t = 0$.

Lemma 8.2. $\dot{\mathfrak{v}}_{a,\theta}(0) \begin{cases} < 0 & \textit{if } -\pi < \theta < 0\,, \\ = 0 & \textit{if } \theta = 0\,, \\ > 0 & \textit{if } 0 < \theta < \pi\,. \end{cases}$

Proof. By the first part of (32), $\dot{\mathfrak{v}}_\theta(0)$ has the same sign as $g'_\theta(f_\theta^{-1}(0)) = f_\theta(f_\theta^{-1}(0)) - 2\sin(\theta + f_\theta^{-1}(0)) = -2\sin(\theta + f_\theta^{-1}(0))$. The result follows from the fact that $\theta \mapsto \theta + f_\theta^{-1}(0)$ is an strictly increasing function of θ which coincides with the identity at $\theta = -\pi, 0, \pi$. □

We observe now that there are solutions $v = v(t)$ of (10) which cannot be written as $v = \mathfrak{v}_{a,\theta}$ for some $a > 0$ and $\theta \in \mathbb{R}$. For instance, one checks that

$$v^*(t) := |t|^{3/2}\,, \tag{33}$$

(defined either on $]-\infty, 0[$ or on $]0, +\infty[$) is a solution of (10) satisfying

$$\lim_{t\to 0} v_*(t) = \lim_{t\to 0} \dot{v}_*(t) = 0\,,$$

and it is easy to deduce that $v^* \neq \mathfrak{v}_{a,\theta}$ for any a, θ. However, this is the only exceptional case. Indeed, if one fixes some positive initial time t_0 (say, for instance, $t_0 = 1$), and we let

$$z^* := \begin{pmatrix} v^*(1) \\ \dot{v}^*(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}\,, \tag{34}$$

then the set of initial conditions corresponding to our family $\{\mathfrak{v}_{a,\theta}\}$ is the punctured half-plane $(]0, +\infty[\times\mathbb{R})\backslash\{z^*\}$. A (slightly stronger) version of this fact is stated in the lemma below, which is the main result of this subsection.

Lemma 8.3. *The map* $\Phi :]0, +\infty[\times(\mathbb{R}/2\pi\mathbb{R}) \to (]0, +\infty[\times\mathbb{R})\backslash\{z^*\}$ *defined by*

$$\Phi(a, \theta) := (\mathfrak{v}_{a,\theta}(1), \dot{\mathfrak{v}}_{a,\theta}(1))\,, \tag{35}$$

is a C^∞ *diffeomorphism See Fig. 5 below.*

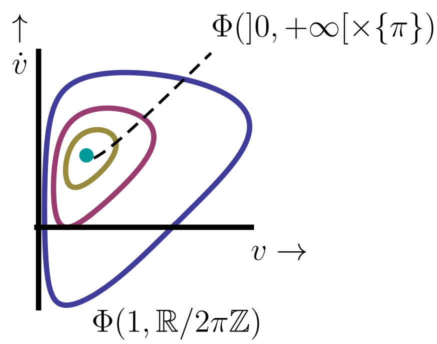


Figure 5: The point z_* (in the center of the picture), the branch $\Phi(]0, +\infty[\times\{\pi\})$ (corresponding to the solutions which collide with the singularity $x = 0$ at time $t = 0$), and (from inner to outer) the closed curves $\Phi(1/4, \mathbb{R}/2\pi\mathbb{Z})$, $\Phi(1/2, \mathbb{R}/2\pi\mathbb{Z})$ and $\Phi(1, \mathbb{R}/2\pi\mathbb{Z})$.

Proof. Remembering the definition of the functions $\mathfrak{v}_\theta$ in (30) and $\mathfrak{v}_{a,\theta}$ in (31) we see that

$$\Phi(a, \theta) = \left(a^{3/2}\mathfrak{v}_\theta\left(\frac{1}{a}\right), a^{1/2}\dot{\mathfrak{v}}_\theta\left(\frac{1}{a}\right)\right) = \left(a^{3/2} g_\theta(f_\theta^{-1}(1/a))^{3/2},\ \frac{3}{2}\sqrt{a}\,\frac{g'_\theta(f_\theta^{-1}(1/a))}{\sqrt{g_\theta(f_\theta^{-1}(1/a))}}\right). \tag{36}$$

This expression can be simplified by letting $a = 1/f_\theta(u)$; we obtain

$$\Phi\left(\frac{1}{f_\theta(u)}, \theta\right) = \left(\left(\frac{g_\theta(u)}{f_\theta(u)}\right)^{3/2}, \ \frac{3}{2}\frac{g'_\theta(u)}{\sqrt{f_\theta(u) g_\theta(u)}} \right), \qquad u > f_\theta^{-1}(0). \tag{37}$$

We shall use this equality in Subsections 8.3 and 8.4 below. On the other hand, straightforward computations show that

$$g_\theta(u) = \sqrt{1 - (f_\theta(u) - \sin(\theta+u))^2} + \cos(u+\theta), \qquad g'_\theta(u) = f_\theta(u) - 2\sin(u+\theta),$$

and therefore, (36) becomes $\Phi(a,\theta) = (\Phi_1(a,\theta), \Phi_2(a,\theta))$, where

$$\begin{cases} \Phi_1(a,\theta) = a^{3/2} \left[\sqrt{1 + \left[1/a - \sin\omega_{1/a}(\theta)\right]^2} + \cos\omega_{1/a}(\theta) \right]^{3/2}, \\ \\ \Phi_2(a,\theta) := \dfrac{3}{2} \left[\dfrac{1 - 2a\sin\omega_{1/a}(\theta)}{\sqrt{\sqrt{a^2 + [1 - a\sin\omega_{1/a}(\theta)]^2} + a\cos\omega_{1/a}(\theta)}} \right], \\ \\ \omega_\epsilon(\theta) = \theta + f_\theta^{-1}(\epsilon). \end{cases} \tag{38}$$

After Φ has been written explicitly, it can be shown to be a diffeomorphism by rewriting it as a composition of diffeomorphisms. Indeed,

$$\Phi = \Phi_{(IV)} \circ \Phi_{(III)} \circ \Phi_{(II)} \circ \Phi_{(I)}, \tag{39}$$

where

$$\begin{cases} \Phi_{(I)} :]0,+\infty[\times (\mathbb{R}/2\pi\mathbb{Z}) \to]0,+\infty[\times (\mathbb{R}/2\pi\mathbb{Z}), & (a,\theta) \mapsto \left(a, \omega_{1/a}(\theta)\right), \\ \Phi_{(II)} :]0,+\infty[\times (\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}^2 \backslash \{(0,0)\}, & (a,\omega) \mapsto \left(a\cos\omega, a\sin\omega\right), \\ \Phi_{(III)} : \mathbb{R}^2 \backslash \{(0,0)\} \to (]0,+\infty[\times \mathbb{R}) \backslash \{(1,1)\}, & (x,y) \mapsto \left(\sqrt{x^2+y^2+(1-y)^2} + x, 1-2y\right), \\ \Phi_{(IV)} : (]0,+\infty[\times \mathbb{R}) \backslash \{(1,1)\} \to (]0,+\infty[\times \mathbb{R}) \backslash \{z^*\}, & (x,y) \mapsto \left(x^{3/2}, \frac{3y}{2\sqrt{x}}\right). \end{cases}$$

One observes without difficulty that all four maps above are diffeomorphisms; in the case of $\Phi_{(I)}$ one needs to differentiate in the expression of ω_ϵ in (38) to check that, for any fixed $\epsilon > 0$,

$$\omega'_\epsilon(\theta) = \frac{\cosh(f_\theta^{-1}(\epsilon))}{g_\theta(f_\theta^{-1}(\epsilon))} > 0, \qquad \theta \in \mathbb{R}. \tag{40}$$

It completes the proof of the lemma. □

The combination of lemmas 8.1, 8.2 and 8.3 leads us to the following result:

Corollary 8.4. *A solution $v : [-1,1] \to \mathbb{R}$ of (10) is even if and only if $v = \mathfrak{v}_{a,0}$ for some $a > 0$.*

In Corollary 8.5 below we point out a couple of facts about the diffeomorphism Φ and the first component $\pi_1\Phi^{-1}$ of its inverse. The proof of *(a)* is immediate from (37), while *(b)* arises from (39) and the fact that $\lim_{z\to z^*}\Phi^{-1}_{(IV)}(z) = (1,1)$, $\lim_{z\to(1,1)}\Phi^{-1}_{(III)}(z) = (0,0)$, $\lim_{z\to(0,0)}\pi_1\Phi^{-1}_{(II)}(z) = 0$, and $\lim_{a\to 0}\Phi^{-1}_{(I)}(a,\omega) = 0$ uniformly with respect to $\omega \in \mathbb{R}/2\pi\mathbb{Z}$.

Corollary 8.5. *The following hold:*

(a) $\lim_{a\to 0}\Phi(a,\theta) = z^*$ *uniformly with respect to* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

(b) $\lim_{z\to z^*}\pi_1\Phi^{-1}(z) = 0$.

At this moment we observe that if $v = v(t)$ is a solution of (10) then $v(-t)$ is also a solution. When $v(t) = \mathfrak{v}_{a,\theta}$ one checks that

$$\mathfrak{v}_{a,\theta}(-t) = \mathfrak{v}_{a,-\theta}(t)\,, \qquad t \in \mathbb{R}\,,\ a > 0,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\,.$$

This leads us to consider the map $\hat{\Phi} :]0,+\infty[\times(\mathbb{R}/2\pi\mathbb{R}) \to \mathbb{R}^2$ defined by

$$\hat{\Phi}(a,\theta) := (\mathfrak{v}_{a,\theta}(-1), \dot{\mathfrak{v}}_{a,\theta}(-1))\,,$$

or, what is the same,

$$\hat{\Phi} = R \circ \Phi \circ R\,, \tag{41}$$

(we recall that $Rz := (\pi_1 z, -\pi_2 z)$ stands for the orthogonal reflection with respect to the first coordinate axis). We arrive to the corollary below, whose proof arises from the combination of (41) and Lemma 8.1.

Corollary 8.6. $\hat{\Phi} :]0,+\infty[\times(\mathbb{R}/2\pi\mathbb{R}) \to (]0,+\infty[\times\mathbb{R})\backslash\{Rz^*\}$ *is a C^∞ diffeomorphism.*

8.2 The solutions of certain non-homogeneous Dirichlet problems are even

The goal of this subsection is a symmetry result for some Dirichlet problems associated to (10):

Proposition 8.7. *If $v_0 > 0$ is big enough, every solution $v : [-1,1] \to \mathbb{R}$ of (10) with $v(-1) = v(1) = v_0$ is even.*

Proof. In view of the definition of Φ in (35) and Corollary 8.4, the statement above can be rewritten in the following way:

$$\pi_1\Phi(a,\theta) = \mathfrak{v}_{a,\theta}(1) \neq \mathfrak{v}_{a,\theta}(-1) = \mathfrak{v}_{a,-\theta}(1) = \pi_1\Phi(a,-\theta) \qquad \text{for } 0 < \theta < \pi \text{ and } a > 0 \text{ big}\,. \tag{42}$$

We can rewrite this inequality still more explicitly by observing that, by (38),

$$\pi_1\Phi(a,\theta)^{2/3} = a\,\psi_{1/a}(\omega_{1/a}(\theta))\,, \tag{43}$$

the functions $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ being defined by

$$\psi_\epsilon(\omega) := \sqrt{1 + (\epsilon - \sin\omega)^2} + \cos\omega\,. \tag{44}$$

It allows us to reformulate (42) in the following way:

$$\psi_\epsilon(\omega_\epsilon(\theta)) \neq \psi_\epsilon(\omega_\epsilon(-\theta)) \qquad \text{for } 0 < \theta < \pi \text{ and } \epsilon > 0 \text{ small}\,. \tag{45}$$

In order to show this inequality we shall need the use some elementary properties of these functions. Two of them concern the functions ω_ϵ and are stated in Lemma 8.8 below:

Lemma 8.8. *The following hold:*

(i) $0 < \omega_\epsilon(\theta) - \omega_\epsilon(-\theta) < 2\pi\,,$ *for all* $\epsilon > 0$ *and* $0 < \theta < \pi\,,$

(ii) $0 < \omega_\epsilon(\theta) + \omega_\epsilon(-\theta) < \pi/2\,,$ *for* $0 < \epsilon$ *small and any* $0 < \theta < \pi\,.$

We turn now our attention to the 2π-periodic functions $\psi_{\pm\epsilon} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$. We observe that

$$\psi_0(\omega) = \sqrt{1 + (\sin\omega)^2} + \cos\omega\,, \qquad \omega \in \mathbb{R}\,.$$

Elementary computations show that ψ_0 is an even function, strictly increasing on $[-\pi, 0]$ and strictly decreasing on $[0, \pi]$. The lemma below studies how this geometry changes for small $\epsilon > 0$.

Lemma 8.9. *Let* $\epsilon > 0$ *be small; then*

(a) $\psi'_\epsilon(\omega) > 0$ *if* $\omega \in [-\pi, -\pi/4]$*, and* $\psi'_\epsilon(\omega) < 0$ *if* $\omega \in [0, 3\pi/4]$.

(b) $\psi_\epsilon(\omega) \leq \psi_\epsilon(-\omega)$ *for any* $\omega \in [0, \pi]$.

We postpone the proof of lemmas 8.8 and 8.9 to the end of this section and continue now with the proof of (45). We consider several cases, depending on the numbers $\omega_\epsilon(\pm\theta)$:

Case I: $\boxed{\omega_\epsilon(\theta) > \pi.}$ Then, by Lemma 8.8 *(i)-(ii)*,

$$\pi < \omega_\epsilon(\theta) < 2\pi + \omega_\epsilon(-\theta) < \frac{5\pi}{2} - \omega_\epsilon(\theta) < \frac{3\pi}{2}\,,$$

and, by Lemma 8.9*(a)*,

$$\psi_\epsilon(\omega_\epsilon(\theta)) < \psi_\epsilon(2\pi + \omega_\epsilon(-\theta)) = \psi_\epsilon(\omega_\epsilon(-\theta))\,.$$

Case II: $\boxed{\omega_\epsilon(-\theta) > 0.}$ Then, again by Lemma 8.8 *(i)-(ii)*,

$$0 < \omega_\epsilon(-\theta) < \omega_\epsilon(\theta) < \frac{\pi}{2} - \omega_\epsilon(-\theta) < \frac{\pi}{2}\,,$$

and, by Lemma 8.9*(a)*,

$$\psi_\epsilon(\omega_\epsilon(-\theta)) > \psi_\epsilon(\omega_\epsilon(\theta))\,.$$

Case III: $\boxed{3\pi/4 \leq \omega_\epsilon(\theta) \leq \pi.}$ Then, by Lemma 8.8 *(ii)*,

$$-\pi \leq -\omega_\epsilon(\theta) < \omega_\epsilon(-\theta) \leq \frac{\pi}{2} - \omega_\epsilon(\theta) \leq -\frac{\pi}{4},$$

so that, by Lemma 8.9 *(a)-(b)*,

$$\psi_\epsilon(\omega_\epsilon(\theta)) \leq \psi_\epsilon(-\omega_\epsilon(\theta)) < \psi_\epsilon(\omega_\epsilon(-\theta)) .$$

Case IV: $\boxed{-\pi/4 \leq \omega_\epsilon(-\theta) \leq 0.}$ Then, by Lemma 8.8 *(ii)*,

$$0 \leq -\omega_\epsilon(-\theta) < \omega_\epsilon(\theta) < -\omega_\epsilon(-\theta) + \frac{\pi}{2} \leq \frac{3\pi}{4} ,$$

so that, by Lemma 8.9 *(a)-(b)*,

$$\psi_\epsilon(\omega_\epsilon(\theta)) < \psi_\epsilon(-\omega_\epsilon(-\theta)) \leq \psi_\epsilon(\omega_\epsilon(-\theta)) .$$

Case V: $\boxed{\omega_\epsilon(-\theta) < -\pi/4 \text{ and } \omega_\epsilon(\theta) < 3\pi/4.}$ Then, by Lemma 8.8 *(ii)*,

$$\frac{\pi}{4} < -\omega_\epsilon(-\theta) < \omega_\epsilon(\theta) < \frac{3\pi}{4} ,$$

and, by Lemma 8.9 *(a)-(b)*

$$\psi_\epsilon(\omega_\epsilon(-\theta)) \geq \psi_\epsilon(-\omega_\epsilon(-\theta)) > \psi_\epsilon(\omega_\epsilon(\theta)) .$$

The proof is now complete. □

We close this section with some comments on the proofs of lemmas 8.8 and 8.9. These results follow quite straightforwardly from the definition of the functions ψ_ϵ and ω_ϵ in (44) and (38), so that we will be quite brief.

Proof of Lemma 8.8. It follows from (40) that, for fixed $\epsilon > 0$, both functions $\theta \mapsto \omega_\epsilon(\theta)$ and $\theta \mapsto -\omega_\epsilon(-\theta)$ are strictly increasing. Consequently, also the function $\theta \mapsto \omega_\epsilon(\theta) - \omega_\epsilon(-\theta)$ is strictly increasing on $[0, \pi]$, and hence,

$$0 = \omega_\epsilon(0) - \omega_\epsilon(0) < \omega_\epsilon(\theta) - \omega_\epsilon(-\theta) < \omega_\epsilon(\pi) - \omega_\epsilon(-\pi) = 2\pi , \qquad 0 < \theta < \pi ,$$

showing *(i)*. In order to check *(ii)* we observe that

$$\omega_\epsilon(\theta) + \omega_\epsilon(-\theta) = f_\theta^{-1}(\epsilon) + f_{-\theta}^{-1}(\epsilon) = f_\theta^{-1}(\epsilon) - f_\theta^{-1}(-\epsilon) .$$

Now, the inequality $\omega_\epsilon(\theta) + \omega_\epsilon(-\theta) > 0$ follows from the fact that each function f_θ^{-1} is strictly increasing. On the other hand, the inequality $\omega_\epsilon(\theta) + \omega_\epsilon(-\theta) < \pi/2$ for small $\epsilon > 0$ follows from the uniform continuity of the function

$$\begin{cases} (\epsilon, \theta) \mapsto \omega_\epsilon(\theta) + \omega_\epsilon(-\theta) = f_\theta^{-1}(\epsilon) - f_\theta^{-1}(-\epsilon) & \text{if } \epsilon > 0 , \\ (0, \theta) \mapsto 0 , \end{cases}$$

on $[0, 1] \times (\mathbb{R}/2\pi\mathbb{Z})$. □

Proof of Lemma 8.9. Part *(b)* is immediate from the definition of ψ_ϵ in (44). Concerning the statement *(a)* we compute:

$$\psi_\epsilon'(\omega) = -\frac{\epsilon\cos\omega + (\sin\omega\big(\sqrt{1+(\epsilon-\sin\omega)^2}-\cos\omega\big)}{\sqrt{1+(\epsilon-\sin\omega)^2}}\,.$$

If $\epsilon > 0$ and $0 \le \omega \le \pi/2$, both terms in the numerator are nonnegative. Furthermore, they do not vanish simultaneously. Consequently,

$$\psi_\epsilon'(\omega) < 0 \text{ if } \omega \in [0, \pi/2] \text{ and } \epsilon > 0\,.$$

Moreover, $\psi_0'(\omega) < 0$ for $\pi/2 \le \omega \le 3\pi/2$, and using a uniform continuity argument we see that also $\psi_\epsilon'(\omega) < 0$ for ω in this interval if $\epsilon > 0$ is small. This proves the part of the statement concerning $0 \le \omega \le 3\pi/4$. The proof of the statement when $-\pi \le \omega \le -\pi/4$ is analogous. □

8.3 Regularizing the singularity

Consider the Poincaré map P associated to (10) on the time interval $[-1, 1]$. With other words, we are interested in the map

$$P : (v(-1), \dot v(-1)) \mapsto (v(1), \dot v(1)) \qquad \text{for any solution } v : [-1,1] \to \mathbb{R} \text{ of } (10)\,. \tag{46}$$

A key observation in this paper is that *the domain of P is a proper subset of the half space* $]0,+\infty[\times\mathbb{R}$, not because there are solutions of (10) exploding in finite time (such a thing cannot happen), but because some initial conditions lead to solutions arriving at the singularity $v = 0$ at time $t = 0$. This is for instance the case of the point $Rz^* = (1, -3/2)$, which gives rise to the solution v^*, already considered in (33). We denote by Γ to the subset of $]0,+\infty[\times\mathbb{R}$ where the Poincaré map P is not defined, i.e.

$$\Gamma := \left\{ \begin{pmatrix} v(-1) \\ \dot v(-1) \end{pmatrix} : v : [-1, 0[\to]0, +\infty[\text{ solves (10) and } \lim_{t\to 0_-} v(t) = 0 \right\}. \tag{47}$$

The lemma below uses the explicit form of the solutions of (10) described in the previous subsection to obtain some insight on the set Γ.

Lemma 8.10. *There exists a $C([0,+\infty[,\mathbb{R}^2) \cap C^\infty(]0,+\infty[,\mathbb{R}^2)$ curve $\gamma = \gamma(a)$ with*

$$\gamma(0) = Rz^*\,, \qquad \lim_{a\to+\infty} \frac{1}{3\sqrt{a}}\,\gamma(a) = \lim_{a\to+\infty} \frac{2\sqrt{a}}{3}\,\gamma'(a) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{48}$$

and such that $\Gamma = \{\gamma(a) : a \ge 0\}$.

Proof. It follows from Corollary 8.6 that $\Gamma = \{z^*\} \cup \{\hat\Phi(a,\pi) : a > 0\}$; this leads us to define

$$\gamma(a) := \begin{cases} Rz^* & \text{if } a = 0\,, \\ \hat\Phi(a,\pi) = R\Phi(a,\pi) & \text{if } a > 0\,. \end{cases}$$

It is clear that γ is C^∞ on $]0,+\infty[$, while the continuity of γ at $a=0$ follows from Corollary 8.5 *(a)*. Moreover, by (37) and (41),

$$\gamma\left(\frac{1}{f_\pi(u)}\right)=\left(\left(\frac{g_\pi(u)}{f_\pi(u)}\right)^{3/2},-\frac{3}{2}\frac{g'_\pi(u)}{\sqrt{f_\pi(u)g_\pi(u)}}\right),\qquad u>0\,.$$

We recall that $f_\pi(u)=\sinh u-\sin u$ and $g_\pi(u)=\cosh u-\cos u$. The second and third statements in (48) can now be easily obtained from elementary computations. □

Consequently, the Poincaré map P is naturally defined on $(]0,+\infty[\times\mathbb{R})\backslash\Gamma$, and equation (10) being reversible in time, its image is $(]0,+\infty[\times\mathbb{R})\backslash R(\Gamma)$; moreover, P establishes a diffeomorphism between these sets. We extend it to a map $P:]0,+\infty[\times\mathbb{R}\to]0,+\infty[\times\mathbb{R}$ by setting

$$P(z):=Rz \text{ for any } z\in\Gamma\,.$$

To conclude this subsection we show that (10) can be regularized, in the sense that this extension is smooth, with the possible exception of the point Rz^*, where one has continuity.

Lemma 8.11. *The map $P:]0,+\infty[\times\mathbb{R}\to]0,+\infty[\times\mathbb{R}$ is a homeomorphism. Moreover, it induces a C^∞ diffeomorphism from $(]0,+\infty[\times\mathbb{R})\backslash\{Rz^*\}$ into $(]0,+\infty[\times\mathbb{R})\backslash\{z^*\}$.*

Proof. We observe that

$$P=\Phi\circ\hat{\Phi}^{-1}=\Phi\circ R\circ\Phi^{-1}\circ R\quad\text{on }]0,+\infty[\times\mathbb{R}\,,\tag{49}$$

by checking that it holds both on $(]0,+\infty[\times\mathbb{R})\backslash\Gamma$ and Γ. The result follows now from Lemma 8.3 (in what refers to the diffeomorphism) and Corollary 8.5 (for the homeomorphism statement). □

Lemma 8.12. $\nabla(\pi_1 P)(\gamma(a))\neq\begin{pmatrix}1\\0\end{pmatrix}$ *for $a>0$ big enough.*

Proof. It follows from (49) that $(\pi_1 P)R\Phi=(\pi_1\Phi)R$, and we deduce that

$$\Phi'(a,\pi)^*\circ R\circ\nabla(\pi_1 P)(R\Phi(a,\pi))=R\nabla(\pi_1\Phi)(a,\pi)\,,\qquad a>0\,,$$

or, what is the same,

$$\Phi'(a,\pi)^*\circ R\circ\nabla(\pi_1 P)(\gamma(a))=R\nabla(\pi_1\Phi)(a,\pi)\,,\qquad a>0\,,$$

and the statement of the Lemma may equivalently be rewritten as

$$\Phi'(a,\pi)^*\begin{pmatrix}1\\0\end{pmatrix}\neq R\nabla(\pi_1\Phi)(a,\pi)\ \text{ for } a>0 \text{ big enough}\,,$$

or, with other words,

$$\partial_\theta(\pi_1\Phi)(a,\pi)\neq 0\ \text{ for } a>0 \text{ big enough.}$$

This inequality can be checked by going back to (43) and combining the chain rule with (40); we have to check that

$$\psi'_\epsilon(\omega_\epsilon(\pi))\neq 0\qquad\text{for }\epsilon>0\text{ small.}$$

However, for small $\epsilon>0$ we see that $\omega_\epsilon(\pi)=\pi+f_\pi^{-1}(\epsilon)\in]\pi,3\pi/2[$. The result follows from Lemma 8.9 *(a)*. □

8.4 The collection of even solutions

The family of solutions of (10) colliding with the singularity $v = 0$ at $t = 0$ has played an especial role in the previous subsection. A second set of solutions of (10) with a particular importance in this paper is the collection of even solutions on $[-1, 1]$. The corresponding set of initial conditions is:

$$\Sigma := \left\{ \begin{pmatrix} v(-1) \\ \dot{v}(-1) \end{pmatrix} : v : [-1,1] \to]0, +\infty[\text{ solves (10) and } \dot{v}(0) = 0 \right\} .$$

In view of Corollary 8.4,

$$\Sigma = \left\{ \begin{pmatrix} \mathfrak{v}_{a,0}(-1) \\ \dot{\mathfrak{v}}_{a,0}(-1) \end{pmatrix} : a > 0 \right\} = \left\{ \sigma(a) : a > 0 \right\} ,$$

the parametrized curve $\sigma : [0, +\infty[\to \mathbb{R}^2$ being defined by

$$\sigma(a) := \begin{cases} \hat{\Phi}(a, 0) = R\Phi(a, 0) & \text{if } a > 0 \, , \\ Rz^* & \text{if } a = 0 \, . \end{cases}$$

By (37) and (41),

$$\sigma\left(\frac{1}{f_0(u)}\right) = \left(\left(\frac{g_0(u)}{f_0(u)}\right)^{3/2}, -\frac{3}{2} \frac{g_0'(u)}{\sqrt{f_0(u) g_0(u)}} \right) , \qquad u > 0 \, .$$

We recall that $f_0(u) = \sinh u + \sin u$ and $g_0(u) = \cosh u + \cos u$. It follows from here (or from Corollary 8.5 *(a)*) that σ is continuous at $a = 0$. Elementary computations lead us to Lemma 8.13 below, which is reminiscent of Lemma 8.10 in the previous subsection.

Lemma 8.13. *There exists a $C([0, +\infty[, \mathbb{R}^2) \cap C^\infty(]0, +\infty[, \mathbb{R}^2)$ curve $\sigma = \sigma(a)$ with*

$$\sigma(0) = Rz^* \, , \quad \lim_{a \to +\infty} \frac{\pi_1 \sigma(a)}{\sqrt{a^3}} = \lim_{a \to +\infty} \frac{2\, \pi_1 \sigma'(a)}{3\, \sqrt{a}} = 2\sqrt{2} \, , \quad \lim_{a \to +\infty} \sqrt{a^5}\, \pi_2 \sigma(a) = -\frac{1}{16\sqrt{2}} \, ,$$

and such that $\Sigma = \{\sigma(a) : a > 0\}$.

A consequence of this result is that $\lim_{a \to +\infty} (\pi_1 \sigma(a))^{5/3} \pi_2 \sigma(a) = -1$. It leads us to the following

Corollary 8.14. *There is some $N > 0$ such that $v(1)^{5/3} \dot{v}(1) \le N$ for any even solution $v : [-1, 1] \to \mathbb{R}$ of (10).*

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