Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign

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Abstract

New criteria for the existence of a maximum or antimaximum principle of a general second order operator with periodic conditions, as well as conditions for nonresonance, are provided and compared with the related literature.

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1 Introduction

The purpose of this paper is to study some qualitative properties of the second order linear operator

$$\mathcal{L}[p,q]u \equiv u'' + p(t)u' + q(t)u$$

with periodic conditions, where $p, q \in L\big([0,\omega]; R\big)$ are given Lebesgue integrable functions. More precisely, we are interested in sufficient conditions for the operator $\mathcal{L}[p,q]$ to be non-degenerate, inversely positive or inversely negative. This question and related ones have focused the attention of many researchers [1, 2, 5, 6, 8, 10, 13–15, 17] because it plays the crucial role in the implementation of different methods of proof for the existence of a periodic solution of an abstract nonlinear second order equation $\mathcal{L}u = Nu$, where N is the so-called Nemitskii operator. We will come back to this connection later.

In order to get a precise description of our objective, it is convenient introduce the following definitions. Here, $AC^1\big([a,b]; R\big)$ stands for the set of functions $u : [a,b] \to R$ which are absolutely continuous together with their first derivative. For brevity, sometimes we will omit the dependence of the operator $\mathcal{L}[p,q]$ on p, q and write simply $\mathcal{L}$.

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Definition 1.1. The operator $\mathcal{L}[p,q]$ belongs to the set V^- if and only if every function $u \in AC^1\big([0,\omega];R\big)$ satisfying

$$\mathcal{L}[p,q]u(t) \geq 0 \qquad \text{for a. e. } t \in [0,\omega], \tag{1.1}$$

$$u(0) = u(\omega), \qquad u'(0) \geq u'(\omega) \tag{1.2}$$

verifies also

$$u(t) \leq 0 \qquad \text{for } t \in [0,\omega]. \tag{1.3}$$

The operator $\mathcal{L}[p,q]$ belongs to the set V_S^- if and only if every function $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1)-(1.2) verifies also

$$u \equiv 0 \quad \text{ or } \quad u(t) < 0 \qquad \text{for } t \in [0,\omega]. \tag{1.4}$$

Definition 1.2. The operator $\mathcal{L}[p,q]$ belongs to the set V^+ if and only if every function $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1)-(1.2) verifies also

$$u(t) \geq 0 \qquad \text{for } t \in [0,\omega]. \tag{1.5}$$

The operator $\mathcal{L}[p,q]$ belongs to the set V_S^+ if and only if every function $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1)-(1.2) verifies also

$$u \equiv 0 \quad \text{ or } \quad u(t) > 0 \qquad \text{for } t \in [0,\omega]. \tag{1.6}$$

Definition 1.3. The operator $\mathcal{L}[p,q]$ is said to be nonresonant if and only if the homogeneous problem

$$\mathcal{L}u = 0, \tag{1.7}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{1.8}$$

has only the trivial solution.

It is convenient to state some comments concerning these concepts. According to Lemma 3.1 (established in Section 3), $V^- \equiv V_S^-$. In the related literature, it is often called a maximum principle. On the other hand, in general we only have the inclusion $V_S^+ \subset V^+$. Therefore, we can distinguish between antimaximum and strong antimaximum principle. Finally, it is easy to realize that an operator $\mathcal{L}$ belonging to V^- or V^+ is automatically nonresonant.

The aim of this paper is to derive sufficient conditions on p, q such that $\mathcal{L}[p,q]$ holds a maximum or antimaximum principle. Such sufficient conditions generalize or complement previous published results in a sense to be detailed in due time. The structure of the paper is as follows: Section 2 contains our main results, which are proved in Section 3. Finally, Section 4 is devoted to compare our results with those published in the related literature.

2 Main Results

Our first result is a characterization of the set V^-.

Theorem 2.1. *$\mathcal{L}[p,q] \in V^-$ holds if and only if there exists a function $\beta \in AC^1\big([0,\omega];R\big)$ satisfying*

$$\mathcal{L}[p,q]\beta(t) \le 0 \qquad \text{for a. e. } t \in [0,\omega], \tag{2.1}$$

$$\beta(t) > 0 \qquad \text{for } t \in [0,\omega], \tag{2.2}$$

$$\beta(0) = \beta(\omega), \qquad \beta'(0) \le \beta'(\omega), \tag{2.3}$$

and

$$\operatorname{meas}\big\{t \in [0,\omega] : \mathcal{L}[p,q]\beta(t) < 0\big\} + \beta'(\omega) - \beta'(0) > 0. \tag{2.4}$$

Such a theoretical result can be applied in order to get a practical criterion shown in the next corollary. At this moment, it is convenient to fix some notation: $[x]_+ = \max\big\{x,0\big\}$, $[x]_- = -\min\big\{x,0\big\}$. Given $p \in L\big([0,\omega];R\big)$, we define $\hat{p}$ as the periodic extension to $[0,2\omega]$

$$\hat{p}(t) = \begin{cases} p(t) & \text{for a. e. } t \in [0,\omega], \\ p(t-\omega) & \text{for a. e. } t \in]\omega, 2\omega]. \end{cases}$$

Let us define the operators $\sigma, \sigma_1 : L\big([a,b];R\big) \to AC\big([a,b];R\big)$ as

$$\sigma(p)(t) = \exp\left(\int_a^t p(s)ds\right) \qquad \text{for } t \in [a,b],$$

$$\sigma_1(p)(t) = \sigma(p)(b)\int_a^t \sigma(p)(s)ds + \int_t^b \sigma(p)(s)ds \qquad \text{for } t \in [a,b].$$

Corollary 2.1. *Let $q \not\equiv 0$ and*

$$\Pi^+ \le \left(1 - \frac{\Phi}{4}\right)\Pi^- \tag{2.5}$$

where

$$\Pi^- = \int_0^\omega [q(s)]_-\sigma(p)(s)\sigma_1(-p)(s)ds, \tag{2.6}$$

$$\Pi^+ = \int_0^\omega [q(s)]_+\sigma(p)(s)\sigma_1(-p)(s)ds, \tag{2.7}$$

$$\Phi = \sup\left\{\int_t^{t+\omega}\sigma(-\hat{p})(s)ds\int_t^{t+\omega}[\hat{q}(s)]_+\sigma(\hat{p})(s)ds : t \in [0,\omega]\right\}. \tag{2.8}$$

Then $\mathcal{L}[p,q] \in V^-$.

Concerning the antimaximum principle, the following one is our main result.

Theorem 2.2. *Let us assume $q \not\equiv 0$ and the following conditions hold*

$$\int_0^\omega q(s)\sigma(p)(s)\sigma_1(-p)(s)ds \ge 0 \tag{2.9}$$

and

$$\Phi \le 4 \tag{2.10}$$

where Φ is given by (2.8). Then $\mathcal{L}[p,q] \in V_S^+$.

Theorem 2.3. *Let us assume $q \not\equiv 0$, (2.9) holds, and*

$$\Phi \leq 16 \tag{2.11}$$

where Φ is given by (2.8). Then $\mathcal{L}[p,q]$ is nonresonant.

In the case when

$$\int_0^\omega p(s)ds = 0 \tag{2.12}$$

one can easily verified that the constants Π^-, Π^+ and Φ defined by (2.6)–(2.8) have the following form:

$$\Pi^- = \int_0^\omega \sigma(-p)(s)ds \int_0^\omega [q(s)]_-\sigma(p)(s)ds,$$
$$\Pi^+ = \Phi = \int_0^\omega \sigma(-p)(s)ds \int_0^\omega [q(s)]_+\sigma(p)(s)ds.$$

Therefore, the consequences established below immediately follow from Corollary 2.1 and Theorems 2.2 and 2.3.

Corollary 2.2. *Let $q \not\equiv 0$, (2.12) holds, and let*

$$\int_0^\omega [q(s)]_+\sigma(p)(s)ds < \frac{4}{\int_0^\omega \sigma(-p)(s)ds},$$
$$\frac{\int_0^\omega [q(s)]_+\sigma(p)(s)ds}{1 - \frac{1}{4}\int_0^\omega \sigma(-p)(s)ds \int_0^\omega [q(s)]_+\sigma(p)(s)ds} \leq \int_0^\omega [q(s)]_-\sigma(p)(s)ds.$$

Then $\mathcal{L}[p,q] \in V^-$.

Corollary 2.3. *Let $q \not\equiv 0$, (2.12) holds, and let*

$$\int_0^\omega q(s)\sigma(p)(s)ds \geq 0, \qquad \int_0^\omega [q(s)]_+\sigma(p)(s)ds \leq \frac{4}{\int_0^\omega \sigma(-p)(s)ds}.$$

Then $\mathcal{L}[p,q] \in V_S^+$.

Corollary 2.4. *Let $q \not\equiv 0$, (2.12) holds, and let*

$$\int_0^\omega q(s)\sigma(p)(s)ds \geq 0, \qquad \int_0^\omega [q(s)]_+\sigma(p)(s)ds \leq \frac{16}{\int_0^\omega \sigma(-p)(s)ds}.$$

Then $\mathcal{L}[p,q]$ is nonresonant.

For the important special case when $p \equiv 0$, i.e. when $\mathcal{L}u := u'' + q(t)u$ is the Hill's operator, the following assertions can be immediately derived from the obtained results.

Corollary 2.5. *Let $q \not\equiv 0$, and let*

$$\int_0^\omega [q(s)]_+ds < \frac{4}{\omega}, \qquad \frac{\int_0^\omega [q(s)]_+ds}{1 - \frac{\omega}{4}\int_0^\omega [q(s)]_+ds} \leq \int_0^\omega [q(s)]_-ds.$$

Then $\mathcal{L}[0,q] \in V^-$.

Corollary 2.6. *Let $q \not\equiv 0$, and let*

$$\int_0^\omega q(s)ds \geq 0, \qquad \int_0^\omega [q(s)]_+ ds \leq \frac{4}{\omega}\,.$$

Then $\mathcal{L}[0,q] \in V_S^+$.

Corollary 2.7. *Let $q \not\equiv 0$, and let*

$$\int_0^\omega q(s)ds \geq 0, \qquad \int_0^\omega [q(s)]_+ ds \leq \frac{16}{\omega}\,.$$

Then $\mathcal{L}[0,q]$ is nonresonant.

3 Auxiliary Propositions

Lemma 3.1. *Let $p, q \in L\big([a,b];R\big)$, $t_0 \in [a,b]$. Let, moreover, $u \in AC^1\big([a,b];R\big)$ satisfy*

$$\mathcal{L}[p,q]u(t) \leq 0 \qquad \text{for a. e. } t \in [a,b], \tag{3.1}$$

$$u(t) \geq 0 \qquad \text{for } t \in [a,b], \tag{3.2}$$

$$u(t_0) = 0, \qquad u'(t_0) = 0. \tag{3.3}$$

Then

$$u(t) = 0 \qquad \text{for } t \in [a,b]. \tag{3.4}$$

Proof. Put

$$w(t) = \max\big\{u(s) : (t-s)(s-t_0) \geq 0\big\} \qquad \text{for } t \in [a,b],$$
$$A = \big\{t \in [a,b] : w(t) = u(t)\big\}.$$

Then, obviously, $w \in AC\big([a,b];R\big)$,

$$w(t) \geq u(t) \qquad \text{for } t \in [a,b], \tag{3.5}$$

$$w'(t)\operatorname{sgn}(t-t_0) \geq 0 \qquad \text{for a. e. } t \in [a,b], \tag{3.6}$$

$$w(t_0) = 0, \tag{3.7}$$

and

$$w'(t) = \begin{cases} u'(t) & \text{for a. e. } t \in A, \\ 0 & \text{for } t \in [a,b] \setminus A. \end{cases} \tag{3.8}$$

From (3.1) we obtain

$$\big[u'(t)\sigma(p)(t)\big]' + q(t)\sigma(p)(t)u(t) \leq 0 \qquad \text{for a. e. } t \in [a,b]. \tag{3.9}$$

The integration of (3.9) from t_0 to t (from t to t_0), in view of (3.2) and (3.3), yields

$$u'(t)\sigma(p)(t)\operatorname{sgn}(t-t_0) \leq \operatorname{sgn}(t-t_0)\int_{t_0}^t [q(s)]_-\sigma(p)(s)u(s)ds \qquad \text{for } t \in [a,b]. \tag{3.10}$$

Now the inequality (3.10), with respect to (3.5), (3.6), and (3.8), results in

$$w'(t)\operatorname{sgn}(t-t_0) \le w(t)\sigma(-p)(t)\operatorname{sgn}(t-t_0)\int_{t_0}^{t}[q(s)]_-\sigma(p)(s)ds \quad \text{for a. e. } t\in[a,b]. \tag{3.11}$$

However, according to Gronwall–Bellman Lemma and (3.7), from (3.11) it follows that

$$w(t)\le 0 \qquad \text{for } t\in[a,b]. \tag{3.12}$$

Now (3.2), (3.5), and (3.12) yield (3.4). □

Lemma 3.2. *Let $p,h\in L\big([a,b];R\big)$, and let $u\in AC^1\big([a,b];R\big)$ be such that*

$$u''(t)+p(t)u'(t)=h(t) \qquad \textit{for a. e. } t\in[a,b], \tag{3.13}$$

$$u(a)=u(b). \tag{3.14}$$

Then

$$\begin{aligned}u(t)=u(a)-\frac{1}{\int_a^b\sigma(-p)(s)ds}\Bigg[&\int_t^b\sigma(-p)(s)ds\int_a^t h(s)\sigma(p)(s)\int_a^s\sigma(-p)(\xi)d\xi ds\\&+\int_a^t\sigma(-p)(s)ds\int_t^b h(s)\sigma(p)(s)\int_s^b\sigma(-p)(\xi)d\xi ds\Bigg] \qquad \textit{for } t\in[a,b]\end{aligned} \tag{3.15}$$

and

$$\big[u'(b)-u'(a)\big]\int_a^b\sigma(-p)(s)ds=\int_a^b h(s)\sigma(p)(s)\sigma_1(-p)(s)ds. \tag{3.16}$$

Proof. From (3.13) we get

$$\big[u'(t)\sigma(p)(t)\big]'=h(t)\sigma(p)(t) \qquad \text{for a. e. } t\in[a,b]. \tag{3.17}$$

Multiplying both sides of (3.17) by $\int_a^t\sigma(-p)(s)ds$, resp. by $\int_t^b\sigma(-p)(s)ds$, and integrating it from a to t, resp. from t to b, we obtain

$$u'(t)\sigma(p)(t)\int_a^t\sigma(-p)(s)ds-u(t)+u(a)=\int_a^t h(s)\sigma(p)(s)\int_a^s\sigma(-p)(\xi)d\xi ds, \tag{3.18}$$

resp.

$$-u'(t)\sigma(p)(t)\int_t^b\sigma(-p)(s)ds+u(b)-u(t)=\int_t^b h(s)\sigma(p)(s)\int_s^b\sigma(-p)(\xi)d\xi ds. \tag{3.19}$$

Now, multiplying (3.18) by $\int_t^b\sigma(-p)(s)ds$ and (3.19) by $\int_a^t\sigma(-p)(s)ds$, we get

$$\begin{aligned}u'(t)\sigma(p)(t)\int_a^t\sigma(-p)(s)ds\int_t^b\sigma(-p)(s)ds+\big[u(a)-u(t)\big]\int_t^b\sigma(-p)(s)ds\\=\int_t^b\sigma(-p)(s)ds\int_a^t h(s)\sigma(p)(s)\int_a^s\sigma(-p)(\xi)d\xi ds,\end{aligned} \tag{3.20}$$

$$- u'(t)\sigma(p)(t)\int_a^t \sigma(-p)(s)ds\int_t^b \sigma(-p)(s)ds + \big[u(b)-u(t)\big]\int_a^t \sigma(-p)(s)ds$$
$$= \int_a^t \sigma(-p)(s)ds\int_t^b h(s)\sigma(p)(s)\int_s^b \sigma(-p)(\xi)d\xi ds. \quad (3.21)$$

Summing the corresponding sides of (3.20) and (3.21), on account of (3.14), we arrive at (3.15). Further, from (3.15) we obtain

$$u'(t) = \frac{\sigma(-p)(t)}{\int_a^b \sigma(-p)(s)ds}\Bigg[\int_a^t h(s)\sigma(p)(s)\int_a^s \sigma(-p)(\xi)d\xi ds - \int_t^b h(s)\sigma(p)(s)\int_s^b \sigma(-p)(\xi)d\xi ds\Bigg] \qquad \text{for } t\in[a,b]$$

whence we get (3.16). $\square$

Lemma 3.3. *Let $p, q \in L\big([a,b];R\big)$, and let $u \in AC^1\big([a,b];R\big)$ satisfy*

$$\mathcal{L}[p,q]u(t) \geq 0 \qquad \text{for a. e. } t\in[a,b], \quad (3.22)$$
$$u(a)=0, \qquad u(b)=0. \quad (3.23)$$

Moreover, let us assume that there exists a function $v \in AC^1\big([a,b];R\big)$ such that

$$\mathcal{L}[p,q]v(t) \leq 0 \qquad \text{for a. e. } t\in[a,b], \quad (3.24)$$
$$v(t) > 0 \qquad \text{for } t\in[a,b]. \quad (3.25)$$

Then

$$u(t) \leq 0 \qquad \text{for } t\in[a,b].$$

Proof. Suppose on the contrary that u has a positive value. Put

$$\lambda = \max\left\{\frac{u(t)}{v(t)} : t\in[a,b]\right\},$$
$$w(t) = \lambda v(t) - u(t) \qquad \text{for } t\in[a,b]. \quad (3.26)$$

Then $w \in AC^1\big([a,b];R\big)$,

$$\lambda > 0, \quad (3.27)$$
$$w(t) \geq 0 \qquad \text{for } t\in[a,b], \quad (3.28)$$

and, because of (3.23) and (3.27), there exists $t_0 \in]a,b[$ such that

$$w(t_0)=0, \qquad w'(t_0)=0. \quad (3.29)$$

Moreover, in view of (3.22), (3.24), (3.26), and (3.27), we have

$$\mathcal{L}[p,q]w(t) \leq 0 \qquad \text{for a. e. } t\in[a,b]. \quad (3.30)$$

Now, on account of (3.28)–(3.30), w satisfies assumptions of Lemma 3.1. Therefore, $w(t)=0$ for $t\in[a,b]$, i.e.,

$$\lambda v(t) = u(t) \qquad \text{for } t\in[a,b].$$

In particular, we have

$$\lambda v(a) = u(a). \quad (3.31)$$

However, (3.31) together with (3.25) and (3.27) contradicts (3.23). $\square$

Lemma 3.4. *Let $p, q \in L\big([a,b];R\big)$, and let $u \in AC^1\big([a,b];R\big)$ satisfy* (3.22), (3.23), *and*

$$u(t) > 0 \qquad \text{for } t \in]a,b[\,. \tag{3.32}$$

Then

$$4 < \int_a^b \sigma(-p)(s)ds \int_a^b [q(s)]_+ \sigma(p)(s)ds. \tag{3.33}$$

Proof. Put

$$h(t) = \mathcal{L}[p,q]u(t) \qquad \text{for a. e. } t \in [a,b]. \tag{3.34}$$

Then, in view of (3.22),

$$h(t) \geq 0 \qquad \text{for a. e. } t \in [a,b]. \tag{3.35}$$

According to Lemma 3.2, on account of (3.34), we have

$$\begin{aligned} u(t) = -\frac{1}{\int_a^b \sigma(-p)(s)ds} \Bigg[\int_t^b \sigma(-p)(s)ds \int_a^t [-q(s)u(s) + h(s)]\sigma(p)(s) \int_a^s \sigma(-p)(\xi)d\xi ds \\ + \int_a^t \sigma(-p)(s)ds \int_t^b [-q(s)u(s) + h(s)]\sigma(p)(s) \int_s^b \sigma(-p)(\xi)d\xi ds \Bigg] \qquad \text{for } t \in [a,b]. \end{aligned} \tag{3.36}$$

Put

$$M = \max\big\{u(t) : t \in [a,b]\big\} \tag{3.37}$$

and choose $t_0 \in]a,b[$ such that

$$u(t_0) = M. \tag{3.38}$$

In view of (3.32) we have

$$M > 0. \tag{3.39}$$

From (3.36), on account of (3.32), (3.35), (3.37), and (3.38), it follows that

$$\begin{aligned} M \leq \frac{M}{\int_a^b \sigma(-p)(s)ds} \Bigg[\int_{t_0}^b \sigma(-p)(s)ds \int_a^{t_0} [q(s)]_+ \sigma(p)(s) \int_a^s \sigma(-p)(\xi)d\xi ds \\ + \int_a^{t_0} \sigma(-p)(s)ds \int_{t_0}^b [q(s)]_+ \sigma(p)(s) \int_s^b \sigma(-p)(\xi)d\xi ds \Bigg]. \end{aligned} \tag{3.40}$$

According to (3.39), the function $[q]_+$ is not identically equal to zero, and therefore from (3.40) we obtain

$$\int_a^b \sigma(-p)(s)ds < \int_a^{t_0} \sigma(-p)(s)ds \int_{t_0}^b \sigma(-p)(s)ds \int_a^b [q(s)]_+ \sigma(p)(s)ds. \tag{3.41}$$

Now, using the inequality $AB \leq \frac{1}{4}(A+B)^2$, from (3.41) we get (3.33). $\square$

Lemma 3.5. *Let us assume that $q \not\equiv 0$ and* (2.9) *hold. Then there is no positive function $u \in AC^1\big([0,\omega];R\big)$ satisfying*

$$\mathcal{L}[p,q]u(t) \leq 0 \qquad \text{for a. e. } t \in [0,\omega], \tag{3.42}$$

$$u(0) = u(\omega), \qquad u'(0) \leq u'(\omega). \tag{3.43}$$

Proof. Assume on the contrary that there is a positive function $u \in AC^1\big([0,\omega];R\big)$ satisfying (3.42) and (3.43). Put

$$\rho(t) = \frac{u'(t)}{u(t)} \qquad \text{for } t \in [0,\omega]. \tag{3.44}$$

Then

$$\rho'(t) \le -q(t) - p(t)\rho(t) - \rho^2(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{3.45}$$

From (3.45) we obtain

$$\begin{aligned}\big[\rho(t)\sigma(p)(t)\big]'\sigma_1(-p)(t) \le -q(t)\sigma(p)(t)\sigma_1(-p)(t)&\\ - \rho^2(t)\sigma(p)(t)\sigma_1(-p)(t) &\qquad \text{for a. e. } t \in [0,\omega].\end{aligned} \tag{3.46}$$

Further, the integration of (3.46) from 0 to ω, in view of (3.43) and (3.44), results in

$$\begin{aligned}0 \le \big[\rho(\omega) - \rho(0)\big]\int_0^\omega \sigma(-p)(s)ds \le -\int_0^\omega q(s)\sigma(p)(s)\sigma_1(-p)(s)ds&\\ -\int_0^\omega \rho^2(s)\sigma(p)(s)\sigma_1(-p)(s)ds.&\end{aligned} \tag{3.47}$$

Now (2.9) and (3.47) imply

$$\rho(t) = 0 \qquad \text{for } t \in [0,\omega].$$

Consequently, in view of (3.44) we have that u is a positive constant function. This fact together with (3.42) implies

$$q(t) \le 0 \qquad \text{for a. e. } t \in [0,\omega]. \tag{3.48}$$

However, (2.9) and (3.48) yield $q \equiv 0$ which contradicts the assumption of the lemma. $\square$

Lemma 3.6. *Let there exists a function $v \in AC^1\big([0,\omega];R\big)$ satisfying*

$$\mathcal{L}[p,q]v(t) \le 0 \qquad \textit{for a. e. } t \in [0,\omega], \tag{3.49}$$

$$v(t) > 0 \qquad \textit{for } t \in [0,\omega], \tag{3.50}$$

$$v(0) = v(\omega), \qquad v'(0) \le v'(\omega), \tag{3.51}$$

and

$$\operatorname{meas}\big\{t \in [0,\omega] : \mathcal{L}[p,q]v(t) < 0\big\} + v'(\omega) - v'(0) > 0. \tag{3.52}$$

Then there is no non–negative non–trivial function $u \in AC^1\big([0,\omega];R\big)$ satisfying

$$\mathcal{L}[p,q]u(t) \ge 0 \qquad \textit{for a. e. } t \in [0,\omega], \tag{3.53}$$

$$u(0) = u(\omega), \qquad u'(0) \ge u'(\omega). \tag{3.54}$$

Proof. Assume on the contrary that there exists a function $u \in AC^1\big([0,\omega];R\big)$ satisfying (3.53), (3.54), $u \not\equiv 0$, and

$$u(t) \ge 0 \qquad \text{for } t \in [0,\omega]. \tag{3.55}$$

Put

$$\lambda = \max\left\{\frac{u(t)}{v(t)} : t \in [0,\omega]\right\}, \tag{3.56}$$

$$w(t) = \lambda v(t) - u(t) \qquad \text{for } t \in [0,\omega]. \tag{3.57}$$

Then, in view of (3.49)–(3.51), (3.53)–(3.57), we have $w \in AC^1\big([0,\omega];R\big)$,

$$\lambda > 0, \tag{3.58}$$

$$\mathcal{L}[p,q]w(t) \le 0 \qquad \text{for a. e. } t \in [0,\omega], \tag{3.59}$$

$$w(t) \ge 0 \qquad \text{for } t \in [0,\omega], \tag{3.60}$$

$$w(0) = w(\omega), \tag{3.61}$$

and there exists $t_0 \in [0,\omega]$ such that

$$w(t_0) = 0. \tag{3.62}$$

If $t_0 \in]0,\omega[$ then, in view of (3.60), we get

$$w'(t_0) = 0. \tag{3.63}$$

If $t_0 \in \{0,\omega\}$ then, in view of (3.54), (3.57), (3.60)–(3.62), we have

$$\lambda v'(0) \ge u'(0) \ge u'(\omega) \ge \lambda v'(\omega)$$

which together with (3.51) and (3.58) implies (3.63) again. Therefore, according to Lemma 3.1 and (3.59) we find

$$w(t) = 0 \qquad \text{for } t \in [0,\omega]. \tag{3.64}$$

However, from (3.64) on account of (3.49), (3.51), (3.53), (3.54), (3.57), and (3.58) it follows that

$$\mathcal{L}[p,q]v(t) = 0 \qquad \text{for a. e. } t \in [a,b],$$
$$v(0) = v(\omega), \qquad v'(0) = v'(\omega),$$

which contradicts (3.52). $\square$

4 Proofs of Main Results

Proof of Theorem 2.1. Let us assume that there exists $\beta \in AC^1\big([0,\omega];R\big)$ satisfying (2.1)–(2.4). We will show that $\mathcal{L}[p,q] \in V^-$. According to Definition 1.1 it is sufficient to show that every function $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1) and (1.2) is non–positive. Assume on the contrary that there exists $u \in AC^1\big([0,\omega];R\big)$ with positive values satisfying (1.1) and (1.2). According to Lemmas 3.3 and 3.6 there exist $t_1 \in]0,\omega[$ and $t_2 \in]t_1,\omega[$ such that

$$u(t) > 0 \qquad \text{for } t \in [0,t_1[\cup]t_2,\omega], \tag{4.1}$$

$$u(t_1) = 0, \qquad u(t_2) = 0. \tag{4.2}$$

Put

$$\lambda_1 = \max\left\{\frac{u(t)}{\beta(t)} : t \in [0,t_1]\right\}, \tag{4.3}$$

$$\lambda_2 = \max\left\{\frac{u(t)}{\beta(t)} : t \in [t_2,\omega]\right\}, \tag{4.4}$$

$$w_1(t) = \lambda_1\beta(t) - u(t) \qquad \text{for } t \in [0,t_1], \tag{4.5}$$

$$w_2(t) = \lambda_2\beta(t) - u(t) \qquad \text{for } t \in [t_2,\omega]. \tag{4.6}$$

Note that

$$w_1(t) \geq 0 \qquad \text{for } t \in [0, t_1], \qquad w_2(t) \geq 0 \qquad \text{for } t \in [t_2, \omega] \tag{4.7}$$

and

$$w_1(t_1) > 0, \qquad w_2(t_2) > 0. \tag{4.8}$$

We claim that $\lambda_1 = \frac{u(0)}{\beta(0)}$. On the contrary, suppose that $\lambda_1 = \frac{u(t_*)}{\beta(t_*)}$ with $t_* \in]0, t_1]$. By (4.2), $t_* < t_1$. Then, $w_1(t_*) = \lambda_1 \beta(t_*) - u(t_*) = 0$ and $w_1'(t_*) = 0$ as a consequence of (4.7). Then, Lemma 3.1 can be applied to w_1, obtaining that w_1 is identically zero, which contradicts (4.8). Thus, $\lambda_1 = \frac{u(0)}{\beta(0)}$ and therefore $w_1(0) = 0$. By an analogous argument, $w_2(\omega) = 0$.

Again by Lemma 3.1, we necessarily have

$$w_1(0) = 0, \qquad w_2(\omega) = 0, \tag{4.9}$$

$$w_1'(0) > 0, \qquad w_2'(\omega) < 0. \tag{4.10}$$

Now (1.2), (2.2), (2.3), (4.5), (4.6), and (4.9) yield

$$\lambda_1 = \lambda_2. \tag{4.11}$$

Consequently, (1.2), (4.3)–(4.6), (4.10), and (4.11) result in

$$\beta'(0) > \beta'(\omega). \tag{4.12}$$

However, (4.12) contradicts (2.3). Therefore, every function $u \in AC^1\big([0, \omega]; R\big)$ satisfying (1.1) and (1.2) is non–positive. Consequently, $\mathcal{L}[p, q] \in V^-$.

Reciprocally, let us take $\mathcal{L}[p, q] \in V^-$. Then the equation

$$\mathcal{L}[p, q]u = -1 \tag{4.13}$$

has a unique periodic solution u. According to Definition 1.1, $u(t) \geq 0$ for $t \in [0, \omega]$. We will show that u is a positive function. Assume on the contrary that there exists $t_0 \in [0, \omega]$ such that

$$u(t_0) = 0.$$

Therefore, we have also

$$u'(t_0) = 0.$$

Thus, in view of Lemma 3.1 we obtain

$$u(t) = 0 \qquad \text{for } t \in [0, \omega]. \tag{4.14}$$

However, (4.14) contradicts (4.13). Therefore,

$$u(t) > 0 \qquad \text{for } t \in [0, \omega]. \tag{4.15}$$

Put

$$\beta(t) = u(t) \qquad \text{for } t \in [0, \omega]. \tag{4.16}$$

Then β satisfies (2.1)–(2.4). $\square$

Proof of Corollary 2.1. First note that according to the assumptions we have

$$[q]_- \not\equiv 0. \tag{4.17}$$

Therefore, according to Theorem 2.1 we have

$$\mathcal{L}\big[p, [q]_-\big] \in V^-. \tag{4.18}$$

Indeed, it is sufficient to put $\beta(t) = 1$ for $t \in [0, \omega]$. Consequently, without loss of generality we can assume

$$[q]_+ \not\equiv 0. \tag{4.19}$$

Put

$$\begin{aligned} w(t) = &-\frac{1}{\int_0^\omega \sigma(-p)(s)ds} \\ &\times \bigg[\int_t^\omega \sigma(-p)(s)ds \int_0^t \big(\Pi^+[q(s)]_- - \Pi^-[q(s)]_+\big)\sigma(p)(s) \int_0^s \sigma(-p)(\xi)d\xi ds \\ &+ \int_0^t \sigma(-p)(s)ds \int_t^\omega \big(\Pi^+[q(s)]_- - \Pi^-[q(s)]_+\big)\sigma(p)(s) \int_s^\omega \sigma(-p)(\xi)d\xi ds\bigg] \quad \text{for } t \in [0, \omega]. \end{aligned}$$

Then, obviously, $w \in AC^1\big([0, \omega]; R\big)$, and

$$w''(t) = \Pi^+[q(t)]_- - \Pi^-[q(t)]_+ - p(t)w'(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{4.20}$$
$$w(0) = w(\omega). \tag{4.21}$$

Therefore, according to Lemma 3.2, in view of (2.6) and (2.7), we have

$$w'(0) = w'(\omega). \tag{4.22}$$

Let $t_1 \in [0, \omega[$ and $t_2 \in]t_1, t_1 + \omega[$ be such that

$$\hat{w}(t_1) = m, \qquad \hat{w}(t_2) = M, \tag{4.23}$$

where

$$m = \min\big\{w(t) : t \in [0, \omega]\big\}, \qquad M = \max\big\{w(t) : t \in [0, \omega]\big\}, \tag{4.24}$$

and

$$\hat{w}(t) = \begin{cases} w(t) & \text{for } t \in [0, \omega], \\ w(t - \omega) & \text{for } t \in]\omega, 2\omega]. \end{cases} \tag{4.25}$$

According to Lemma 3.2, (4.23), and (4.25), we have

$$\begin{aligned} M = m &- \frac{1}{\int_{t_1}^{t_1+\omega} \sigma(-\hat{p})(s)ds} \\ &\times \bigg[\int_{t_2}^{t_1+\omega} \sigma(-\hat{p})(s)ds \int_{t_1}^{t_2} \big(\Pi^+[\hat{q}(s)]_- - \Pi^-[\hat{q}(s)]_+\big)\sigma(\hat{p})(s) \int_{t_1}^s \sigma(-\hat{p})(\xi)d\xi ds \\ &+ \int_{t_1}^{t_2} \sigma(-\hat{p})(s)ds \int_{t_2}^{t_1+\omega} \big(\Pi^+[\hat{q}(s)]_- - \Pi^-[\hat{q}(s)]_+\big)\sigma(\hat{p})(s) \int_s^{t_1+\omega} \sigma(-\hat{p})(\xi)d\xi ds\bigg]. \end{aligned} \tag{4.26}$$

In view of (2.8), (4.17), (4.19), and the inequality $AB \le \frac{1}{4}(A+B)^2$, from (4.26) it follows that

$$M < m + \frac{\Phi\Pi^-}{4}. \tag{4.27}$$

Now put

$$\beta(t) = \Pi^+ + w(t) - m \qquad \text{for } t \in [0,\omega]. \tag{4.28}$$

Then, in view of (4.19)–(4.24) we have (2.2), (2.3), and

$$\beta''(t) = \Pi^+[q(t)]_- - \Pi^-[q(t)]_+ - p(t)\beta'(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{4.29}$$

On the other hand, on account of (2.5), (4.24), and (4.27), from (4.28) it follows that

$$\Pi^+ \le \beta(t) < \Pi^- \qquad \text{for } t \in [0,\omega]. \tag{4.30}$$

Thus, using (4.30) in (4.29) we find that (2.1) holds and, with respect to (4.19), also (2.4) is satisfied. Now the conclusion of the corollary follows from Theorem 2.1. □

Proof of Theorem 2.2. According to Definition 1.2 it is sufficient to show that every function $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1) and (1.2) is positive. Take, therefore, $u \in AC^1\big([0,\omega];R\big)$ satisfying (1.1), (1.2), and suppose on the contrary that u assumes nonpositive values. According to Lemma 3.5, the function u has a zero; according to Lemma 3.1, u takes also positive values. Thus there exist $t_1, t_2 \in [0,\omega]$ such that $t_1 < t_2$,

$$u(t_1) = 0, \qquad u(t_2) = 0, \tag{4.31}$$

and either

$$u(t) > 0 \qquad \text{for } t \in]t_1, t_2[\,, \tag{4.32}$$

or

$$u(t) > 0 \qquad \text{for } t \in [0, t_1[\cup]t_2, \omega]. \tag{4.33}$$

If (4.32) holds true, then according to Lemma 3.4 we have

$$4 \;<\; \int_{t_1}^{t_2} \sigma(-p)(s)ds \int_{t_1}^{t_2} [q(s)]_+ \sigma(p)(s)ds \;\le\; \int_0^{\omega} \sigma(-p)(s)ds \int_0^{\omega} [q(s)]_+ \sigma(p)(s)ds. \tag{4.34}$$

However, (4.34) contradicts (2.10).

Let, therefore, (4.33) be fulfilled. Define

$$h(t) = \mathcal{L}[p,q]u(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{4.35}$$

By Picard-Lindelöff Theorem, there exists a unique function $v \in AC^1\big([0,\omega];R\big)$ of the initial value problem

$$\mathcal{L}[p,q]v(t) = h(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{4.36}$$

$$v(0) = u(\omega), \qquad v'(0) = u'(\omega). \tag{4.37}$$

Put

$$w(t) = u(t) - v(t) \qquad \text{for } t \in [0,\omega].$$

Then, in view of (1.2), (4.35)–(4.37), we have

$$\mathcal{L}[p,q]w(t) = 0 \qquad \text{for a. e. } t \in [0,\omega],$$
$$w(0) = 0, \qquad w'(0) = u'(0) - u'(\omega).$$

Consequently, according to (1.2), Lemma 3.4, and the assumption (2.10), we get $w(t) \geq 0$ for $t \in [0,\omega]$, i.e., $u(t) \geq v(t)$ for $t \in [0,\omega]$. Thus, in view of (4.31), (4.33), (4.37), there exists $t_3 \in]0,t_1]$ such that

$$v(t_3) = 0, \qquad v(t) > 0 \qquad \text{for } t \in [0,t_3[\,. \tag{4.38}$$

Define $z : [t_2, t_3+\omega] \to R$ as follows:

$$z(t) = \begin{cases} u(t) & \text{for } t \in [t_2,\omega], \\ v(t-\omega) & \text{for } t \in]\omega, t_3+\omega]. \end{cases} \tag{4.39}$$

Then, on account of (1.1), (4.31), (4.33), (4.35)–(4.39) we have $z \in AC^1\big([t_2,t_3+\omega];R\big)$ and

$$\mathcal{L}[\hat{p},\hat{q}]z(t) \geq 0 \qquad \text{for a. e. } t \in [t_2,t_3+\omega],$$
$$z(t) > 0 \qquad \text{for } t \in]t_2,t_3+\omega[\,,$$
$$z(t_2) = 0, \qquad z(t_3+\omega) = 0.$$

Therefore, according to Lemma 3.4, the inequality

$$4 < \int_{t_2}^{t_3+\omega} \sigma(-\hat{p})(s)ds \int_{t_2}^{t_3+\omega} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds \tag{4.40}$$

holds true. However, in view of the fact $t_3 \leq t_1 < t_2$ we have $t_3+\omega < t_2+\omega$, and so from (4.40) we obtain

$$4 < \int_{t_2}^{t_2+\omega} \sigma(-\hat{p})(s)ds \int_{t_2}^{t_2+\omega} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds,$$

which contradicts (2.10). □

Proof of Theorem 2.3. Assume on the contrary that there exists a nontrivial solution u to (1.7)-(1.8). According to Lemma 3.5, u has a zero, and according to Lemma 3.1, u assumes both positive and negative values. Therefore, there exist $t_0 \in [0,\omega[\,$, $t_2 \in]t_0, t_0+\omega]$, and $t_1 \in]t_0,t_2[$ such that

$$\hat{u}(t_0) = 0, \qquad \hat{u}(t_1) = 0, \qquad \hat{u}(t_2) = 0,$$
$$\hat{u}(t) > 0 \qquad \text{for } t \in]t_0,t_1[\,, \qquad \hat{u}(t) < 0 \qquad \text{for } t \in]t_1,t_2[\,,$$

where $\hat{u}$ is given by

$$\hat{u}(t) = \begin{cases} u(t) & \text{for } t \in [0,\omega], \\ u(t-\omega) & \text{for } t \in]\omega, 2\omega]. \end{cases}$$

Consequently, according to Lemma 3.4 we have

$$4 < \int_{t_0}^{t_1} \sigma(-\hat{p})(s)ds \int_{t_0}^{t_1} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds, \tag{4.41}$$

$$4 < \int_{t_1}^{t_2} \sigma(-\hat{p})(s)ds \int_{t_1}^{t_2} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds. \tag{4.42}$$

Now, multiplying the corresponding sides of (4.41) and (4.42), and using the inequality $AB \le \frac{1}{4}(A+B)^2$, we arrive to

$$16 < \int_{t_0}^{t_2} \sigma(-\hat{p})(s)ds \int_{t_0}^{t_2} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds$$
$$\le \int_{t_0}^{t_0+\omega} \sigma(-\hat{p})(s)ds \int_{t_0}^{t_0+\omega} [\hat{q}(s)]_+ \sigma(\hat{p})(s)ds. \quad (4.43)$$

However, (4.43) contradicts (2.11). □

5 Further comments and comparison with related results.

As commented in the Introduction, an operator $\mathcal{L}$ belonging to V^- or V^+ is automatically nonresonant. Therefore, according to the classical Fredholm alternative, the complete problem

$$\mathcal{L}u = h(t) \quad (5.1)$$
$$u(0) - u(\omega) = c_0, \qquad u'(0) - u'(\omega) = c_1 \quad (5.2)$$

with $h \in L\big([0,\omega]; R\big)$, $c_0, c_1 \in R$, has a unique solution u. In terms of the associated Green's function $G(t,s)$, $\mathcal{L} \in V^-$ means that $G(t,s) < 0$, $\mathcal{L} \in V^+$ means that $G(t,s) \ge 0$ and $\mathcal{L} \in V_S^+$ means that $G(t,s) > 0$ for all t, s. In this way, our results can be compared with those available in the related literature concerning the sign of the Green's function of the second order linear operator with periodic boundary conditions.

In our opinion, the main strength of our results is that the coefficients are allowed to change sign. The case of constant coefficients can be fully solved [10]. The case of one-signed and bounded q was considered in [1]. Essentially, Corollaries 2.6 and 2.7 are known since the times of Lyapunov. Corollary 2.7 can be found in [8] and more recently in [15, Th.4], whereas Corollary 2.6 is a result widely used in the method of upper and lower solutions. An extended version with a L^α-condition was proved in [14] for the case of non-negative q and later extended to the case of indefinite sign in [2]. In this last paper, the authors gives a result for the general operator $\mathcal{L}[p,q]$ with both p, q changing sign, assuming that $\int_0^T p(s)ds = 0$ (see [2, Th. 5.1]). It can be checked that our Corollary 2.4 is just this result for $\alpha = 1$. The case $\int_0^T p(s)ds \ne 0$ was considered in [13, 17], but in these papers both p, q are assumed positive. As to the knowledge of the authors, a first result for the general operator $\mathcal{L}[p,q]$ with both p, q changing sign can be found in [6]. The papers [4, 6, 7] give also effective criteria for nonresonance of a general linear n-th order operator. Corollary 2.1 and Theorems 2.2 and 2.3 provide essentially new information which complement or generalize the previously mentioned results.

We finish the paper with some comments concerning the applications to nonlinear problems. With a nonresonance criterion for the operator $\mathcal{L}$, one can deduce existence and uniqueness results for the nonlinear equation

$$\mathcal{L}u = g(t,u) \quad (5.3)$$

by using the arguments of [5, 15, 16]. On the other hand, $\mathcal{L} \in V^-$ provides a method of well-ordered upper and lower solutions for eq. (5.3), whereas $\mathcal{L} \in V_S^+$ gives a method of upper and lower solutions on the reversed order (see [1, 10, 13, 14] and their references). Besides, an anti-maximum principle enables the applications of classical fixed point theorems like Schauder's or

Krasnoselskii's to regular or singular problems like in [3, 9, 12, 13] and many others. The inclusion of delays [17] does not suppose major changes. We consider that the previously mentioned techniques are sufficiently developed in the literature.

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