Periodic oscillations of the relativistic pendulum with friction

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Abstract: It is proved that the forced pendulum equation with friction and a singular ϕ-laplacian operator of relativistic type has periodic solutions, in contrast to what happen in the newtonian case.
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1 Introduction and main result.

In this paper we consider the existence of periodic solutions for the forced pendulum equation with relativistic effects

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}} \right)' + kx' + a \sin x = p(t) \tag{1.1}$$

where $c > 0$ is the speed of light in the vacuum, $k \geq 0$ is a possible viscous friction coefficient and p is a continuous and T-periodic forcing term with mean value $\overline{p} = \frac{1}{T} \int_0^T p(t)dt = 0$. This equation can be derived from an appropriate lagrangian formulation [3]. Physically, we are assuming a basic principle of Special Relativity: the mass of a moving object is not constant but depends on its velocity. From a more mathematical perspective, the equation can be seen as a singular ϕ-laplacian oscillator. The recent publication of [2] has renewed the interest in the study of equations with singular ϕ-laplacian operators. If compared with the classical or newtonian case, the relativistic pendulum has been scarcely studied, therefore at this stage it is important to point out the dynamical differences between both models. The aim of this note is to reveal a new dynamical response when relativistic effects are considered.

In the non-relativistic regime, that is, if c is assumed to be $+\infty$, we have the classical forced pendulum equation

$$x'' + kx' + a\sin x = p(t), \tag{1.2}$$

which is a paradigm in Classical Mechanics and Dynamical Systems. The long story around this equation can be found in the reviews [6, 7]. Concerning the existence of periodic solutions, the first result was proved by Hamel [4] in 1922 for the conservative case $k = 0$.

Theorem 1 (Hamel's theorem) *If $k = 0$, then equation (1.2) has at least one T-periodic solution.*

Let us note that the original result by Hamel was proved for $p(t) = \sin t$ but the idea is easily extended to the general case. The proof is of variational type and hence the conservative nature of the problem plays the fundamental role. However, J. Mawhin [5] conjectured that a topological approach may be useful to prove the existence of periodic solutions in the presence of friction. The first counterexample was presented by R. Ortega [8]. Later, J.M. Alonso [1] provided a different counterexample, but the more general non-existence result is given in [9]. From now on, let us denote by C_T the Banach space of the continuous and T-periodic functions and by $\tilde{C}_T$ the space of the functions of C_T with zero mean value.

Theorem 2 ([9]) *Given positive constants a, k and T, there exists $p \in \tilde{C}_T$ such that the equation (1.2) has no T-periodic solutions.*

Our main aim is to prove that Mawhin's conjecture is partially true in the relativistic framework. Our main result is as follows.

Theorem 3 *Let us assume that $2cT \leq 1$. For any values a, k and for any $p \in \tilde{C}_T$, equation (1.2) has at least one T-periodic solution.*

The proof is an elementary application of the Schauder's Fixed Point Theorem and will be given in the next section. Of course, now an interesting open question arises: are there proper counterexamples for higher values of the period?

2 Proof of the main result.

The equation (1.1) can be written as

$$\phi(x')' + kx' + a\sin x = p(t)$$

where $\phi :]-c, c[\to I\!R$ is given by

$$\phi(u) = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Of course, the inverse ϕ^{-1} is a bounded operator. The first step of the proof is to make the change of variables $x = \arcsin y$. Then the equation (1.1) is equivalent to

$$\phi\left(\frac{y'}{\sqrt{1-y^2}}\right)' + k\frac{y'}{\sqrt{1-y^2}} + ay = p(t). \tag{2.3}$$

The second step is to write the problem of finding a T-periodic solution of (2.3) as a fixed point problem for a suitable operator. An integration of the equation gives

$$\phi\left(\frac{y'}{\sqrt{1-y^2}}\right)' + k\arcsin y = \int_0^t (p(s) - ay(s))ds + C,$$

where C is a constant to be fixed later. For convenience, let us define the operator

$$F[y](t) = \int_0^t (p(s) - ay(s))ds - k\arcsin y.$$

Then, we get

$$y' = \sqrt{1-y^2}\phi^{-1}\left(F[y](t) + C\right).$$

Finally, a new integration gives

$$y(t) = \int_0^t \sqrt{1-y^2}\phi^{-1}\left(F[y](t) + C\right)ds + D.$$

Lemma 1 *For any $y \in \tilde{C}_T$, there exists a unique choice of C_y, D_y such that*

$$\mathcal{T}[y](t) \equiv \int_0^t \sqrt{1-y^2}\phi^{-1}\left(F[y](t) + C_y\right)ds + D_y \in \tilde{C}_T. \tag{2.4}$$

Proof. . Periodicity is equivalent to

$$\int_0^T \sqrt{1-y^2}\phi^{-1}\left(F[y](t) + C_y\right)ds = 0.$$

As a function of C_y, the left-hand side of this equation is continuous and increasing, so the existence of a unique solution C_y for such equation follows from a basic application of the Mean Value Theorem. Once C_y is fixed, D_y is given by

$$D_y = -\frac{1}{T}\int_0^T \int_0^t \sqrt{1-y(s)^2}\phi^{-1}\left(F[y](s) + C_y\right)dsdt.$$

□

Therefore, we have a well-defined functional $\mathcal{T} : \tilde{C}_T \to \tilde{C}_T$. It is easy to prove that it is continuous and compact. Let us define the closed and convex set

$$K = \{y \in \tilde{C}_T : \|y\|_\infty \leq 2cT\}.$$

Take $y \in K$. Note that $\left\|\phi^{-1}[y]\right\|_\infty < c$ for every $h \in C_T$. Therefore,

$$|\mathcal{T}[y](t)| \leq 2\left|\int_0^t \sqrt{1-y(s)^2}\phi^{-1}\left(F[y](s)+C_y\right)ds\right| < 2cT.$$

for all t. In consequence, by the Schauder's fixed point Theorem there exists a T-periodic solution y of (2.3). The hypothesis $2cT \leq 1$ enables to invert the change and hence $x = \arcsin y$ is a T-periodic solution of the original equation (1.1).

As a final note, let us remark that the same proof works if the linear friction term kx' is replaced by $h(x)x'$ without further restrictions on the continuous function h.

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