RECURRENT EQUATIONS WITH SIGN AND FREDHOLM ALTERNATIVE

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To George R. Sell, in memoriam

Abstract. We prove that a Fredholm–type Alternative holds for recurrent equations with sign, extending a previous result by Cieutat and Haraux in [3]. Moreover, we show that this can be seen a particular case of [1] and we provide a solutions to an interesting problem posed by Hale in [6]. Finally we characterize the existence of exponential dichotomies also in the nonrecurrent case.

1. Introduction

Consider a linear differential equation in $\mathbb{R}^N$:

$$\dot{x} = A(t)x + f(t) \tag{1.1}$$

where the vector f is bounded continuous on time and the matrix A is *recurrent*. That is, A is bounded and uniformly continuous and its hull:

$$H(A, f) \;=\; \mathrm{cls}\big\{(A\tau, f\tau) : \tau \in \mathbb{R}\big\}$$

is compact minimal with respect to the compact–open topology and the flow induced by translations $(A\tau)(t) = A(t + \tau)$. A necessary condition in order (1.1) to admit bounded solutions can be easily obtained integrating by parts, namely:

$$\big\langle f, v\big\rangle \in BP(\mathbb{R}) \tag{1.2}$$

for every bounded solution v of the adjoint equation:

$$\dot{v} = -A(t)^T v\ . \tag{1.3}$$

Here BP stands for having bounded primitive. In [1] we discussed whether or not this condition gives rise to a Fredholm Alternative in the recurrent framework, as implicitly suggested in [14]. To be more precise the question in [1] is to decide if, under the assumption that f is jointly recurrent with A, condition (1.2) is also sufficient for equation (1.1) to admit a solution x which is not only bounded, but even recurrent and with the same recurrence properties of A and f. This can be expressed by saying that a continuous flow epimorphism exits:

$$H(A, f) \to H(x) \qquad\qquad (A, f) \mapsto x\ . \tag{1.4}$$

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It is worth to stress that recurrent solutions to (1.1) always exist when bounded solutions do, but having solutions satisfying (1.4) is much more restrictive and, in general, is not for free. If for instance A and f are almost periodic, then this restriction implies that x is almost periodic too.
The main result of [1] is that (1.2) is sufficient to get such x satisfying (1.4), as soon as A and f are jointly recurrent and A fulfills the additional condition:

$$d_F(A) \;=\; d_S(A)\;. \tag{1.5}$$

The above quantities are respectively called the *Favard dimension* and the *Sacker–Sell dimension* of A. To introduce them one has to look at the hull of the homogeneous equations:

$$\dot u = A(t)u \tag{1.6}$$

that is the class of equations:

$$\dot u = B(t)u \qquad\qquad B \in H(A)\;. \tag{1.7}$$

The number d_B of independent bounded solutions may vary with B, and the Favard dimension of A is the smallest possible value:

$$d_F(A) \;=\; \min_{B\in H(A)} d_B\;.$$

The Sacker–Sell spectrum $\sigma(A)$ of A is defined as the set of real λ's such that the equation:

$$\dot u = \big[A(t) - \lambda\big]u$$

does not admit an exponential dichotomy. The spectrum is the union of a finite number of closed intervals and, roughly speaking, the Sacker–Sell dimension $d_S(A)$ is the number of independent solutions of (1.6) having Lyapunov exponents in the spectral interval containing zero. See Section 2 for a precise definition and a description of the consequences of (1.5). Here we just recall that the same condition has been already used by Sacker and Sell in [12], to obtain a special case of their successive Spectral Theorem [13].
The aim of the present paper is to test the result of [1] in some concrete situations, showing that it gives new insights even in cases already studied in the literature. The first concrete result concerns a matrix A which has a sign, say for instance it is nonnegative:

$$A(t) \;\geq\; 0 \qquad\quad \forall t\;. \tag{1.8}$$

Here A is possibly asymmetric and the sign is that of its symmetric part:

$$S_A(t) \;=\; \frac{A(t) + A(t)^T}{2}\;.$$

This situation has been already considered by Cieutat and Haraux in [3], when $A(t)$ and $f(t)$ are both almost periodic and the antisymmetric part of $A(t)$, that is:

$$K_A(t) \;=\; \frac{A(t) - A(t)^T}{2}$$

is purely periodic. Under these assumptions, the authors obtain a Fredholm–type Alternative which looks quite different from that of [1]. Precisely, they prove that (1.1) has an almost periodic solution if and only if:

$$\big\langle f, v\big\rangle \in AP(\mathbb{R};\mathbb{R})$$

for every solution v of (1.3) which is almost periodic and satisfies the following pair of conditions:

$$\dot{v} = K_A(t)v \qquad\qquad S_A(t)\,v(t) = 0$$

the second of which is not even differential. The point is that, contrarily to the appearance, the result of [3] can be seen as a particular case of that proved in [1]. In Section 3 we show indeed that condition (1.5) is satisfied regardless of K_A, obtaining the following improvement.

Theorem 1.1. *Let A be recurrent and f bounded continuous, and assume the sign condition (1.8) is satisfied. Then condition (1.5) is satisfied too and equation (1.1) has bounded solutions if and only if condition (1.2) holds for every bounded solution to (1.3).*

In Section 4 we present a scalar example showing that the recurrence of A is optimal for the result, in spite of the fact that the conclusion refers to bounded solutions with any prescribed recurrence property. To grant such properties, we need to strengthen a little bit the assumptions.

Corollary 1.2. *Assume that A and f are jointly recurrent and (1.8) is satisfied. Then equation (1.1) admits solutions satisfying (1.4) if and only if condition (1.2) holds for every bounded solution to (1.3).*

Because condition (1.5) is satisfied, the Corollary follows from the mentioned result of [1]. In Section 3 we give however and independent proof of Theorem 1.1 and Corollary 1.2 for the following reasons: the paper [1] is not yet published and, moreover, the way to prove (1.5) gives the full result in few additional steps.
The comparison of Corollary 1.2 with the aforementioned result of [3] open the doors to a natural question. Imagine indeed that A and f are both almost periodic with A satisfying the sign condition (1.8), and we are interested in almost periodic solutions to equation (1.1): *is it enough testing condition (1.2) on those v which are almost periodic too?* In Section 4 we show that the answer is negative, by using a well know example by Lillo [7] and a special case of Fredholm Alternative already published in [14].
The last contribution of the present work concerns the real extent of condition (1.5). Beside showing that it is optimal for the validity of the Fredholm Alternative in the recurrent framework, in [1] we proved that is is also *necessary* for small Sacker–Sell dimensions:

$$d_S(A) \;\leq\; 2\,.$$

In Section 4 we show that such restriction can be dropped in the periodic case, does not matter A has a sign or not.

Proposition 1.3. *Let A be continuous and periodic. Assume moreover that, for every recurrent f, equation (1.1) has a solution satisfying (1.4) if and only if condition (1.2) holds for every bounded solution to (1.3). Then condition (1.5) is satisfied.*

We believe that this fact and the results of [1] provide a complete answer to an exercise proposed by Hale in his textbook [6], asking for the appropriate Fredholm Alternative when A is purely periodic and f is almost periodic: see Exercise 1.1 at page 147.

Notations.
The symbols $|x|$ and $\langle x, y\rangle$ stand for Euclidean norm and inner product in $\mathbb{R}^N$. With $\mathcal{L}(N)$ we denote the $N{\times}N$ matrices with real entries, while $\mathcal{GL}(N)$ and $\mathcal{O}(N)$ stand for the subsets of the invertible and the orthogonal matrices respectively.
The symbol $BP(\mathbb{R})$, $AP(\mathbb{R})$ and $AP^1(\mathbb{R})$ stand respectively for the space of the continuous functions having bounded primitive, that of the almost periodic function and that the almost periodic functions having almost periodic derivative. When $f \in AP(\mathbb{R})$ we denote by $\overline{f}$ its mean value.

2. Preliminaries

We start assuming that the function A and f are just *continuous* and we denote by ϕ_A and ϕ_A^* the Cauchy operators of the homogeneous equations (1.6) and (1.3) respectively. Moreover we set:

$$\mathcal{B}(A) \;=\; \big\{\xi \in \mathbb{R}^N : \sup_t |\phi_A(t)\xi| < +\infty\big\}$$

for the initial data giving rise to bounded solutions of (1.6). We use $\mathcal{B}^*(A)$ for the same set when referred to the adjoint equation (1.3).
A straightforward integration by parts gives:

$$\begin{aligned}\int_0^t \big\langle f(s), v(s)\big\rangle ds \;&= \big[\big\langle x(s), v(s)\big\rangle\big]_0^t - \int_0^t \big\langle \dot v(s) + A(s)^T v(s), x(s)\big\rangle ds \\ &= \big\langle x(t), v(t)\big\rangle - \big\langle x(0), v(0)\big\rangle\end{aligned} \tag{2.1}$$

for every x solving (1.1) and every v solving the adjoint homogeneous equation (1.3). Thus the following condition must be satisfied:

$$\big\langle f, \phi_A^*(\cdot)\zeta\big\rangle \in BP(\mathbb{R}) \qquad\qquad \forall \zeta \in \mathcal{B}^*(A) \tag{2.2}$$

in order (1.1) may admit a bounded solution.
This necessary condition becomes trivially sufficient when the equation (1.6) admits an exponential dichotomy. This means that there exist a time independent projector P in $\mathbb{R}^N$ and constants $K, \alpha > 0$ such that:

$$\begin{aligned}\big\|\phi_A(t)P\,\phi_A(s)^{-1}\big\| &\le Ke^{-\alpha(t-s)} &\quad \forall t \ge s \\ \big\|\phi_A(t)(I-P)\phi_A(s)^{-1}\big\| &\le Ke^{-\alpha(s-t)} &\quad \forall s \ge t\end{aligned} \tag{2.3}$$

Sometimes we say that A has an exponential dichotomy too. When this is true, also the adjoint equation admits an exponential dichotomy and hence $\mathcal{B}^*(A) = \{0\}$ and condition (2.2) is empty. On the other hand it is easy to check that, for every bounded and continuous f, the function:

$$x(t) \;=\; \int_{-\infty}^t \phi_A(t)P\phi_A(s)^{-1} f(s)\,ds \;-\int_t^{+\infty} \phi_A(t)(I-P)\phi_A(s)^{-1} f(s)\,ds\;.$$

is a bounded solution to (1.6), actually the only one.
A kind of opposite situation has been considered in [14]. There the author assumes

that all the solutions to (1.6) are not only bounded but also separated from zero, namely:

$$(2.4) \qquad 0 \ < \ \inf_t |\phi_A(t)\xi| \ \leq \ \sup_t |\phi_A(t)\xi| \ < \ +\infty \qquad\qquad \forall \xi \neq 0$$

and proves the following result.

Proposition 2.1. *Assume A and f are continuous and condition (2.4) is satisfied. Then equation (1.1) admits bounded solutions if and only if condition (2.2) is satisfied.*

Actually, the boundedness of A and f is explicitly requested in [14] but not used in the proof. The proof depends on the classical variation of constants formula, once one has checked that ϕ^*_A also satisfies condition (2.4).
The above proposition is at the core of the result of [1] for the recurrent framework, which we now introduce. We assume from now on that A is *bounded and uniformly continuous*: the hull $H(A)$ is then compact metrizable and connected, and both spectral theory and Favard theory behave well. The Sacker–Sell spectrum $\sigma(A)$ has been introduced in [13] as the set of the real λ's such that:

$$\dot{u} = \big[A(t) - \lambda\big] u$$

does not admit an exponential dichotomy. Since having an exponential dichotomy transfers from a matrix to its hull, when A is recurrent the previous condition is independent of $B \in H(A)$ in the sense that $\sigma(B) = \sigma(A)$ for every $B \in H(A)$. When on the contrary A is not recurrent, it survives only the inclusion $\sigma(B) \subset \sigma(A)$ for every $B \in H(A)$. The spectrum is always nonempty and is the union of at most N compact intervals. Each spectral interval s canonically associated to a spectral vector subbundle of $H(A) \times \mathbb{R}^N$. Taken any interval (λ, μ) which contains the desired spectral interval and whose closure avoids all the others, the fiber over $B \in H(A)$ is given by the initial data ξ such that:

$$\lim_{t \to -\infty} e^{-\mu t} \phi_B(t)\xi \ = \ 0 \ = \ \lim_{t \to +\infty} e^{-\lambda t} \phi_B(t)\xi \ .$$

In [13] it is also proved that the all these subbundles are linearly independent and their sum is the whole $H(A) \times \mathbb{R}^N$. We are mainly interested in the spectral boundle associated to the interval containing 0. We denote its fibres by $\mathcal{V}(B)$ and we agree that $\mathcal{V}(B) = \{0\}$ when $0 \notin \sigma(A)$. The dimension being independent of $B \in H(A)$ due to the connectedness of $H(A)$, we set:

$$d_S(A) \ = \ \dim \mathcal{V}(B) \qquad\qquad \forall B \in H(A)$$

and we call it the *Sacker–Sell dimension* of A.
From the definition is clear that $\mathcal{V}(B) \supset \mathcal{B}(B)$ for every $B \in H(A)$. Contrarily to $\mathcal{V}(B)$ however, the dimension of $\mathcal{B}(B)$ may vary with B. We set:

$$d_F(A) \ = \ \min_{B \in H(A)} \dim \mathcal{B}(B) \qquad\qquad \forall B \in H(A)$$

and we call it the *Favard dimension* of A. Automatically the following inequality:

$$(2.5) \qquad d_F(A) \ \leq \ \dim \mathcal{B}(B) \ \leq \ \dim \mathcal{V}(B) \ = \ d_S(A)$$

holds for all $B \in H(A)$. The event that this inequality becomes an equality is particularly relevant for Fredholm Alternative: indeed in [1] the following statement is proved for the recurrent framework.

Theorem 2.2. *Let A be recurrent and f bounded and continuous, and assume moreover that:*

$$d_F(A) \;=\; d_S(A)\ . \tag{2.6}$$

Then equation (1.1) admits bounded solutions if and only if condition (2.2) is satisfied. If in addition f is jointly recurrent with A, then one of these solutions satisfies condition (1.4).

Actually, only the second claim is stated in[1] but the first one can be easily deduced from the proof. The bridge between the two claims in the statement is of course Favard theory, introduced by Favard in [5] for the almost periodic framework and extended to the recurrent framework by Palmer in [9]. The central assumption in this theory is a separation condition on the bounded solutions of the equations (1.7), namely:

$$\inf_t |\phi_B(t)\xi| \;>\; 0 \qquad\qquad \forall \xi \in \mathcal{B}(B)\setminus\{0\}\ . \tag{2.7}$$

When this is true for every $B \in H(A)$, one says that the Favard condition (F_A) is satisfied. If in addition $H(A, f)$ is minimal, then the following result is proved: *equation (1.1) admits a recurrent solution satisfying (1.4) if and only if it admits a bounded solution.*

The relationship between Favard dimension and Favard condition has been investigated in [15] and [2]. In the latter the authors prove that a given $B \in H(A)$ satisfies the separation condition (2.7) if and only if:

$$\dim \mathcal{B}(B) \;=\; d_F(A)$$

which moreover defines a residual subset of $H(A)$ itself. Thus Favard condition (F_A) holds if and only if the previous dimensional equality is satisfied in the whole $H(A)$. This conclusion was already proved in [15]. By taking into account the inequality (2.5), it is now manifest that (2.6) implies the validity of (F_A) and hence that the second part of Theorem 2.2 is a consequence of the first one. It is also worth noticing that condition (2.6) actually implies a much stronger equality, that is:

$$d_F(A) \;=\; d_S(A) \;=\; d_S^*(A) \;=\; d_F^*(A) \tag{2.8}$$

and that moreover the Favard condition (F_A^*) for the adjoint equation is satisfied too. This is proved in [1], where it is also shown that the same conclusions are false in general: the two Favard conditions (F_A) and (F_A^*) may indeed be not equivalent, and both can be true with different Favard dimensions. The symmetric role of the direct and the adjoint equations is at the basis of a partial inverse of Theorem 2.2, showing that condition (2.5) is much more than optimal for the Fredholm Alternative. Indeed in [1] the following result is proved.

Theorem 2.3. *Let A be recurrent such that $d_S(A) \le 2$ and (F_A) and (F_A^*) are satisfied. Assume moreover that, for every f jointly recurrent with A, condition (2.2) is sufficient to get a solution of (1.1) satisfying (1.4). Then $d_F(A) = d_S(A)$.*

We will show in Section 4 that, as already anticipated in the Introduction, the restriction $d_S(A) \le 2$ can be removed in the class of purely periodic A's.

In the next section we will show that all the results stated in the Introduction can be seen as particular instances of the above theorem. The proof makes use of some specialized normal forms for the equation, obtained via some suitable change of variables. Actually, these normal forms allow to give an independent and short proof of these results. We end the present section by commenting what happens when we do a change of variables. By that we precisely mean a differentiable function $Q : \mathbb{R} \to \mathcal{GL}(N)$ such that Q, Q^{-1} and $\dot{Q}$ are bounded and uniformly continuous on time. It is not difficult to check that, in this case, all functions in $H(Q)$ preserve all these properties. Setting $x = Q(t)\mathfrak{x}$ transforms the equation (1.1) into:

$$\dot{\mathfrak{x}} = \mathfrak{A}(t)\mathfrak{x} + \mathfrak{f}(t) \tag{2.9}$$

where:

$$\mathfrak{A} = Q^{-1}\big\{AQ - \dot{Q}\big\} \qquad\qquad \mathfrak{f} = Q^{-1}f\ .$$

This defines an equivalence relation $A \sim \mathfrak{A}$ which is usually called kinematic similarity. Since $\phi_A(t) = Q(t)\phi_{\mathfrak{A}}(t)Q(0)^{-1}$ for every t, it is clear that (1.6) has an exponential dichotomy if and only if equation:

$$\dot{\mathfrak{x}} = \mathfrak{A}(t)\mathfrak{x} \tag{2.10}$$

has. This implies $\sigma(\mathfrak{A}) = \sigma(A)$ and also $d_S(\mathfrak{A}) = d_S(A)$. Concerning hulls, an obvious flow epimorphism exists $H(A, Q) \to H(\mathfrak{A})$, showing that for every element of $\mathfrak{B} \in H(\mathfrak{A})$ there exists a $B \in H(A)$ such that $B \sim \mathfrak{B}$. The same result also holds when the roles of A and $\mathfrak{A}$ are swapped, due to the symmetry of kinematic similarity. Since kinematic similarity does not affect neither boundedness of solutions nor their separation from zero, we can conclude that the Favard conditions (F_A) and $(F_{\mathfrak{A}})$ are equivalent and $d_F(A) = d_F(\mathfrak{A})$.
Passing to the adjoint equations, it is easy to check $(Q^{-1})^T$ is again a change of variables, transforming the equation (1.3) into:

$$\dot{\mathfrak{v}} = -\mathfrak{A}^T(t)\mathfrak{v} \tag{2.11}$$

so that the conclusions we obtained for the direct equations extend to their adjoint equations. Finally, notice that:

$$\big\langle \mathfrak{f}, \mathfrak{v} \big\rangle \ = \ \big\langle Q^{-1}f, \mathfrak{v} \big\rangle \ = \ \big\langle f, v \big\rangle$$

where $v = (Q^{-1})^T\mathfrak{v}$ is a bounded solution of equation (2.11) if and only if v is for (1.3). Thus condition (2.2) is satisfied for the equation (1.1) if and only if the analogous condition is satisfied for equation (2.9).
Summing up, the whole problem of Fredholm Alternative is totally unaffected by changes of variables. Nevertheless, there is a point that deserves some more attention, which concerns recurrence properties. Indeed, in general the recurrence of A does not implies that of $\mathfrak{A}$, not even if Q is recurrent: it can be easily checked that this is however true when A and Q are jointly recurrent, in which case $\mathfrak{A}$ is jointly recurrent with them too.

3. Sign and normal forms

In this section we enter the sign condition (1.8) and discuss his consequences: the devices we use are essentially the same of [3] but we are able to strengthen the conclusions there. Start supposing that u is a solution of the equation:

$$\dot{u} = [A(t) - \lambda]u$$

and notice that:

$$\frac{d}{dt}\left(|u(t)|^2\right) \;=\; 2\left\{\left\langle A(t)u(t), u(t)\right\rangle - \lambda|u(t)|^2\right\} \;\geq\; -2\lambda\,|u(t)|^2 \;. \tag{3.1}$$

By integrating we get the following inequality:

$$|u(s)| \;\leq\; |u(t)|\,e^{\lambda(s-t)} \qquad\qquad \forall t \geq s \;.$$

As a consequence, if $\lambda = 0$ then $|u(t)|$ is not decreasing in time. When on the contrary $\lambda < 0$ we have $u(-\infty) = 0$ and the above equation admits an exponential dichotomy, which yields:

$$\sigma(A) \;\subset\; [0, +\infty) \;.$$

If now $0 \notin \sigma(A)$ then equation (1.6) has an exponential dichotomy with $P \equiv 0$ as projector, namely it has the whole $\mathbb{R}^N$ as unstable subspace and trivial stable subspace. Next proposition uses this fact to characterize the existence of an exponential dichotomy.

Theorem 3.1. *Assume A is bounded uniformly continuous, symmetric and satisfying (1.8). Then $0 \notin \sigma(A)$ if and only if for every recurrent $B_0 \in H(A)$ one has:*

$$\bigcap_t \ker B_0(t) \;=\; \{0\} \;. \tag{3.2}$$

Notice that a recurrent B_0 exists in any closed invariant subset of $H(A)$ due to the compactness of $H(A)$ itself.

Proof. Since exponential dichotomy passes from A to $H(A)$, the only if part is trivial. To prove the if part we use Theorem 2 in [11], saying that $0 \notin \sigma(A)$ if we have two ingredients. The first one is that, for every $B \in H(A)$, the equation:

$$\dot{u} = B(t)u \tag{3.3}$$

does not admit nontrivial bounded solutions. To show that this is true, suppose by contradiction that for some B we have a nontrivial bounded solution $u(t)$ and define:

$$\lim_{t\to+\infty} |u(t)| = \beta > 0 \;.$$

Let now B_0 be a recurrent element in the ω–limit of B and choose a sequence $\tau_n \to +\infty$ such that $B\tau_n \to B_0$. Standard compactness arguments show that $u\tau_n \to u_0$ up to subsequences, where $\dot{u}_0 = B_0(t)u_0$. Finally:

$$|u_0(t)| = \beta \qquad\qquad \forall t$$

holds by the very construction and hence:

$$0 = \frac{d}{dt}\left(|u_0(t)|^2\right) \;=\; 2\left\langle B_0(t)u_0(t), u_0(t)\right\rangle \;.$$

Since $B_0 \geq 0$ and is symmetric, this implies:

$$\dot{u}_0(t) = B_0(t)u_0(t) = 0 \ .$$

In other words $u_0(t) = \xi$ for some suitable $\xi \in \mathbb{R}^N$ in the kernel of each $B_0(t)$. Since $|\xi| = \beta > 0$, we conclude that the intersection of these kernels is nontrivial, contradicting (3.2).

The second ingredient for Theorem 2 in [11] is that the stable and unstable subspaces of (3.3) have complementary dimensions, which moreover are independent of $B \in H(A)$. We claim that the stable subspace of (3.3) is always trivial while the unstable subspace is the whole $\mathbb{R}^N$. Due indeed to $B \geq 0$ we know that $|u(t)|$ in nondecreasing in time for every solution to (3.3). The triviality of the stable subspace is then manifest. Concerning the unstable subspace, suppose by contradiction that, for some $B \in H(A)$, the equation (3.3 has a solution u with:

$$\lim_{t \to +\infty} |u(t)| = \alpha > 0 \ .$$

Then the failure of (3.2) follows again from the very same arguments already used in the first part of the proof. □

Condition (3.2) means that the corresponding equation (1.7) has no nontrivial *constant* solutions, an occurrence which is very easy to check and then gives rise to an useful test for exponential dichotomy. Notice also that, due to the symmetry of A and hence of every $B \in H(A)$, these solutions coincide with the constant solutions of the adjoint equation:

$$\dot{v} = -B(t)^T v = -B(t)v \ .$$

Later on we will see that a related property holds even in the asymmetric case: see Corollary 3.7.

Next results say what happens when recurrence is added.

Corollary 3.2. *Let A be recurrent, symmetric and satisfying (1.8). Then $0 \notin \sigma(A)$ if and only if:*

$$\bigcap_t \ker A(t) \ = \ \{0\} \ . \tag{3.4}$$

Proof. Just use recurrence to prove that actually $\bigcap_t \ker B(t) = \bigcap_t \ker A(t)$ holds for every $B \in H(A)$. □

Remark 3.3. The corollary extends to the recurrent case an analogous result by [3] for the case of an almost periodic A. In fact, there the authors state that exponential dichotomy is equivalent to:

$$\overline{A} \ := \ \lim_{T \to +\infty} \frac{1}{T} \int_0^T A(t) \, dt \ > \ 0$$

This result can be also extended to the recurrent case: we don't provide here such extension since it is not relevant for our aims.

We already know that Fredholm Alternative holds, though trivially, when $0 \notin \sigma(A)$. To understand what happens in the opposite case, we change the variables and put the equation in a convenient normal form. Set:

$$V \ = \ \bigcap_t \ker A(t)$$

and choose a $P\in\mathcal{O}(N)$ straightening the orthogonal decomposition:

$$\mathbb{R}^N \;=\; V^{\perp} \oplus V$$

namely realizing an isomorphism of the first m coordinates of $\mathbb{R}^N$ onto the first factor in the decomposition, and the second $N-m$ coordinates onto the second factor. Here m is the codimension of V. The time–independent change of variables $u = P\mathfrak{u}$ transforms equation (1.6) into:

$$\dot{\mathfrak{u}} \;=\; P^T A(t) P\, \mathfrak{u} \tag{3.5}$$

and next proposition shows why the new equation is more convenient to deal with.

Proposition 3.4. *Assume A is recurrent and symmetric and (1.8) holds. Then:*

$$P^T A(t) P \;=\; \operatorname{diag}\big\{A_*(t),\, 0\big\} \qquad \forall t \tag{3.6}$$

with blocks of dimension m and $N-m$ respectively. Moreover A_* is recurrent, symmetric and satisfies (1.8) together with $0 \notin \sigma(A_*)$.

Proof. Since P is orthogonal, the matrix funtion $P^T A\, P$ is again symmetric and satisfying (1.8), while recurrence follows trivially from the fact that P is time–independent. Consider now the the block–decomposition driven by P that is:

$$P^T A(t)\, P = \begin{pmatrix} A_*(t) & B_*(t) \\ C_*(t) & D_*(t) \end{pmatrix}$$

The definition of V and the construction of P say that $B_*(t)$ and $D_*(t)$ are identically zero. The same is actually true also for $C_*(t)$, since the symmetry of $P^T A(t) P$ implies $C_*(t) = B_*(t)^T$. The remaining block $A_*(t)$ is then the only nontrivial one, and hence is recurrent, symmetric and fulfills (1.8).
It remain to prove that $0 \notin \sigma(A_*)$. To this aim, suppose $\xi \in \mathbb{R}^m$ is such that:

$$A_*(t)\xi = 0 \qquad \forall t\ .$$

This means $P(\xi, 0) \in V$ which, taking into account that $P(\xi, 0) \in V^{\perp}$ by construction of P, implies $\xi = 0$. Corollary 3.2 then implies $0 \notin \sigma(A_*)$. □

As a trivial consequence of Proposition 3.4 we have a first partial result about Fredholm Alternative.

Corollary 3.5. *Assume A is recurrent and symmetric and (1.8) holds. Then the equality:*

$$d_F(A) \;=\; d_S(A)$$

is satisfied and (1.6) and (1.3) have exactly the same bounded solutions, which moreover are the constants belonging to V. If in addition f is bounded and continuous, then equation (1.1) admits bounded solutions if and only if:

$$\big\langle f, \xi \big\rangle \in BP(\mathbb{R}) \qquad \forall \xi \in V\ . \tag{3.7}$$

Notice that (3.7) is just a specialized form of the more general condition (2.2). That is, Fredholm Alternative has not changed.

Proof. Because of the normal form (3.6) the new equation (3.5) splits into the two independent equations:

$$\dot{\mathfrak{u}}_1 = A_*(t)\mathfrak{u}_1 \qquad\qquad \dot{\mathfrak{u}}_2 = 0$$

where $\mathfrak{u}_1 \in \mathbb{R}^m$ and $\mathfrak{u}_2 \in \mathbb{R}^{N-m}$. From Proposition 3.4 we know that $0 \notin \sigma(A_*)$. This implies that:

$$\mathcal{B}(P^T AP) = \mathcal{V}(P^T AP) = \{0\} \times \mathbb{R}^k = P^T V$$

and that these constant are indeed the only bounded solutions of (3.5). Exactly the same conclusions hold for the adjoint equation, which in the new variables writes:

$$\dot{\mathfrak{v}}_1 = -A_*(t)\mathfrak{v}_1 \qquad\qquad \dot{\mathfrak{v}}_2 = 0$$

where by symmetry $\sigma(-A_*) = \sigma(-A_*^T) = -\sigma(A_*)$. The first part of the Corollary the follows from the final part of Section 2.
To prove the second part, define $\mathfrak{f} = P^T f$ and write $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)$. Then consider the equations:

$$\dot{\mathfrak{x}}_1 = A_*(t)\mathfrak{x}_1 + \mathfrak{f}_1 \qquad\qquad \dot{\mathfrak{x}}_2 = \mathfrak{f}_2\ .$$

Again due to $0 \notin \sigma(A_*)$, the first equation has one and only one bounded solution, whatever the bounded and continuous f is. On the other hand, the second equation has bounded solutions if and only if $\mathfrak{f}_2 \in BP(\mathbb{R})$. But for every $\zeta_2 \in \mathbb{R}^{N-m}$ we have:

$$\big\langle \mathfrak{f}_2, \zeta_2 \big\rangle = \big\langle (\mathfrak{f}_1, \mathfrak{f}_2), (0, \zeta_2) \big\rangle = \big\langle f, P(0, \zeta_2) \big\rangle$$

so that condition (3.7) is obtained by construction of P. □

We now see what happens when symmetry is removed from the assumptions. Next result shows that symmetry can can be recovered by means of a suitable change of variables. In the proof, we use the decomposition:

$$A(t) \ = \ S_A(t) + K_A(t)$$

into the symmetric and the antisymmetric parts of $A(t)$.

Proposition 3.6. *Assume A is recurrent and satisfying (1.8). Then there exists a change of variables $Q : \mathbb{R} \to \mathcal{O}(N)$ which is jointly recurrent with A and such that:*

$$\mathfrak{A} = Q^{-1}\big\{AQ - \dot{Q}\big\} \tag{3.8}$$

is again recurrent and satisfying (1.8) but in addition is symmetric.

Proof. The rule:

$$(B, R)\tau \ = \ \big(B\tau,\, \phi_{K_B}(\tau)R\big)\ .$$

defines a continuous flow on the compact space $H(A) \times \mathcal{O}(N)$. Let M be a minimal subset. The canonical projection $p : M \to H(A)$. is a continuous flow morphism which, due to the minimality of $H(A)$, is automatically surjective. Thus there exists a matrix $R_A \in \mathcal{O}(N)$ such that $(A, R_A) \in M$.
Consider now the other canonical projection $q : M \to \mathcal{O}(N)$. It is a continuous map whose multiplicative inverse:

$$q(B, R)^{-1} = R^{-1} = R^T$$

is also continuous. Moreover it admits a derivative along the flow:

$$\frac{d}{d\tau}\, q\big((B, R)\tau\big)\big|_{\tau=0} \ = \ \dot{\phi}_{K_B}(0)R \ = \ K_B(0)\phi_{K_B}(0)R \ = \ K_B(0)R$$

which is also continuous on the whole M.
By taking into account the compactness of M, all the mentioned continuities are indeed uniform continuities. Thus it can be easily checked that the map:

$$Q(t) \ = \ q\big((A, R_A)t\big) \ = \ \phi_{K_A}(t)R_A$$

is a change of variables in the sense of Section 2. Since K_A is antisymmetric we know that $\phi_{K_A}(t) \in \mathcal{O}(N)$ and hence the same is true for $Q(t)$ for every t. Due to the minimality of M, such Q is moreover jointly recurrent with the flow line $t \mapsto p\big((A, R_A)t\big) = At$ in $H(A)$ and hence with A itself.
It remains to show that the matrix $\mathfrak{A}$ defined by (3.8) has the properties claimed in the statement. This is standard fact, also proved in [3]. Indeed $\dot{Q} = K_A Q$ and hence:

$$\mathfrak{A} = Q^{-1}\big\{AQ - \dot{Q}\big\} = Q^T\big\{AQ - K_A Q\big\} = Q^T S_A\, Q$$

so that symmetry and sign of $\mathfrak{A}$ are the same of A, due to the orthogonality of Q. That $\mathfrak{A}$ is recurrent follows trivially from the joint recurrence of Q and A. □

We are finally ready to describe Fredholm Alternative in the general, possibly asymmetric case.

Corollary 3.7. *Assume A is recurrent and (1.8) holds. Then the equality:*

$$d_F(A) \;=\; d_S(A)$$

is satisfied and (1.6) and (1.3) have exactly the same bounded solutions, which moreover have constant norm. If in addition f is bounded and continuous, then equation (1.1) admits bounded solutions if and only if condition (2.2) is satisfied.

Since Favard separation condition (F_A) is also satisfied, when f is jointly recurrent with A Favard theory applies to show that equation (1.1) has a solution satisfying (1.4). This proves Corollary 1.2 in the Introduction.

Proof. Do the change of variables $u = Q(t)\mathfrak{u}$ with Q as in Proposition 3.6 obtaining:

$$\dot{\mathfrak{u}} = \mathfrak{A}(t)\mathfrak{u} \tag{3.9}$$

as new equation, where $\mathfrak{A}$ is given by (3.8). Then apply Corollary 3.5 to equation (3.9), so proving for it that Favard dimension is equal to Sacker–Sell dimension and that Fredholm Alternative works fine. Since these conclusions are not affected by change of variables (see the final part of Section 2) and then holds for the original equation.
It remain to prove the claim about the bounded solutions. Again due to Corollary 3.5, we know that the bounded solutions of (3.9) and its adjoint equation coincide, and are exactly the constants belonging to:

$$\mathfrak{V} \;=\; \bigcap_t \ker \mathfrak{A}(t)\ .$$

Thus the bounded solutions to (1.6) are:

$$u = Q(t)\xi \qquad\qquad \xi \in \mathfrak{V}\ .$$

On the other hand, we explained in Section 2 that the adjoint equations to (1.6) and to (3.9) are connceted by the change of variables $(Q^T)^{-1}$. Thus, by using the orthogonality of Q, we can conclude that the bounded solutions to the adjoint equation to (3.9) are:

$$v = \big(Q(t)^T\big)^{-1}\xi = Q(t)\xi \qquad\qquad \xi \in \mathfrak{V}$$

that is, they are exactly the same of (3.9). That these solutions have constant norm, it follows once again from the orthogonality of Q. □

The rest of the section is devoted to compare the previous results with the existing literature. Comparison with [1] is rather trivial: Corollaries 3.5 and 3.7 are particular cases of Theorem 2.2, since condition (2.6) is obtained in both cases.
To do the comparison with [3], we need to strengthen recurrence by requiring that *A and f are both almost periodic*. Notice that joint almost periodicity is never a problem. The analogous of our Corollary 3.5 and Corollary 3.7 in the paper [3] are Proposition 3.3 and Theorem 3.6 respectively. We start comparing conclusions which, apparently, are stronger in [3] than here: the authors in [3] grant indeed the existence of almost periodic solutions of (1.1) instead of bounded ones only. However, Favard theory applies to pass from bounded solutions to solutions satisfying (1.4), which are almost periodic since A and f are.
Concerning now the assumptions of the symmetric case, Proposition 3.3 in [3] differs from our Corollary 3.5 for the class of functions involved in condition (3.7). In fact, since $\langle f, \xi \rangle$ is almost periodic, its primitive is almost if and only if it is bounded.
In the possibly asymmetric case, the comparison between the assumptions of Theorem 3.6 in [3] and our Corollary 3.7 is more delicate. Indeed the assumptions in [3] are expressed in a completely different form, which does not involve the adjoint equation. More precisely, in [3] the authors assume that:

$$K_A \text{ is purely periodic} \tag{3.10}$$

and the test condition for the Fredholm Alternative is that:

$$\int_0^t \langle f(s), v(s) \rangle ds \ \in \ AP(\mathbb{R}) \tag{3.11}$$

for every v solving the following system of conditions:

$$\begin{cases} \dot{v} = K_A(t)v \\ S_A(t)v(t) = 0 \end{cases} \tag{3.12}$$

the second one not even being differential. Notice however that such v is bounded from the first equation, since K_A is antisymmetric, and hence also almost periodic due to (3.10) and classical Floquet theory: see also the final part of Section 4. Thus again the primitive of $\langle f, v \rangle$ in (3.11) is almost periodic if and only if it is bounded. The following lemma then close the comparison.

Lemma 3.8. *Assume A is recurrent and (1.8) holds. Then the solutions of (3.12) are exactly the bounded solutions of the adjoint equation (1.3).*

Proof. It is trivial to check that (3.12) is equivalent to the system of differential equations:

$$\begin{cases} \dot{v} = A(t)v \\ \dot{v} = -A(t)^T v \end{cases} \tag{3.13}$$

where the adjoint equation plays now a role. The first part of Corollary 3.7 says that the bounded solutions of the two equations in the system (3.13) are exactly the same. $\square$

4. Counterexamples and other results

We start showing that the recurrence of A is optimal for the validity of Theorem 1.1 in the Introduction or, which is equivalent, Corollary 3.7 in the previous section. Let indeed $a > 0$ be a bounded and uniformly continuous function satisfying:

$$a(\pm\infty) = 0 \qquad \widehat{a}(-\infty) = -\infty \qquad \widehat{a}(+\infty) < +\infty$$

where we set:

$$\widehat{a}(t) = \int_0^t a(s)\,ds \; .$$

Since 0 is a minimal subset of $H(a)$, the function a is not recurrent. Moreover it is clear that $0 \in \sigma(a)$. Consider now the scalar equation:

$$\dot{x} = a(t)x + f(t) \; . \tag{4.1}$$

The adjoint equation $\dot{v} = -a(t)v$ has no nontrivial bounded solutions, so that each continuous function f passes the test (1.2). We claim that, on the contrary, equation (4.1) does not admit bounded solutions as soon as f is such that:

$$f(-\infty) = c \neq 0 \; .$$

Indeed the general solution of (4.1) is:

$$x(t) \;=\; e^{\widehat{a}(t)} \left\{ x_0 + \int_0^t e^{-\widehat{a}(s)} f(s)\,ds \right\} \; .$$

Thus applying Hôpital's rule we have:

$$\lim_{t\to-\infty} x(t) \;=\; \lim_{t\to-\infty} \frac{x_0 + \displaystyle\int_0^t e^{-\widehat{a}(s)} f(s)\,ds}{e^{-\widehat{a}(t)}} \;=\; -\lim_{t\to-\infty} \frac{f(t)}{a(t)}$$

which explodes, showing that x is always unbounded. As a final comment, notice that:

$$\mathcal{B}(b) = \mathbb{R} = \mathcal{V}(b) \qquad \forall b \in H(a)$$

so that the example works also for the same type of optimality in Theorem 2.2 and Proposition 2.1.

Next we focus on condition (1.2), discussing whether or not the class of the test function v is appropriate to obtain for the Fredholm Alternative in some special recurrent frameworks. When for instance A is purely periodic, Floquet theory says that the bounded solutions of the adjoint equation (1.3) are almost periodic, actually even quasi periodic: testing condition (1.2) only on the almost periodic v's is then enough, in this case. The results of [3] seem to suggest that the same restriction may work when *A and f are both almost periodic* and the sign condition (1.8) is satisfied. Indeed, we already proved that this is true when, in addition, A is symmetric: see the final part of the previous section. To show that, on the contrary, the conclusion may be false in the asymmetric case, consider a scalar almost periodic function a and introduce the antisymmetric matrix:

$$A \;=\; \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \; . \tag{4.2}$$

The sign condition (1.8) holds trivially, even if the sign is reversed. Moreover the direct equation (1.6) and the adjoint equation (1.3) coincide, and their solutions all have constant norm. In particular, all of them satisfy condition (2.4) and then Proposition 2.1 provides a working Fredholm–type Alternative, does not matter the almost periodic a we take. Assume now a satisfies the following additional conditions:

$$\bar{a} = 0 \qquad\qquad a \notin BP(\mathbb{R};\mathbb{R}) \,. \tag{4.3}$$

Lillo proved in [7] that, in this case, none of the solutions to (1.6) or, which is equivalent, to (1.3) is almost periodic. Thus the weak form of condition (1.2) is trivially satisfied whatever f is. Next result closes the question, showing that there exists almost periodic f's for which things go wrong.

Proposition 4.1. *Asssume $a \in AP(\mathbb{R})$ satisfies (4.3) and define A as in (4.2). Then there exists an almost periodic f such that equation (1.1) does not admit bounded solutions.*

Proof. Arguing by contradiction suppose that, for every $f \in AP(\mathbb{R})$, equation (1.1) admits bounded solutions. Classical Favard theory applies to show it admits also an almost periodic solution: see for instance [14]. Such almost periodic solution is unique, since (1.1) has no almost periodic solutions. Summing up, the map:

$$L_a x \;=\; \dot{x} - A(t)x$$

is a bijection and then a Banach space isomorphism $AP^1(\mathbb{R}) \cong AP(\mathbb{R})$. Consider now a sequence a_n of trigonometric polynomials such that:

$$\bar{a}_n = 0 \qquad\qquad \|a_n - a\|_\infty \to 0 \,.$$

Since the class of isomorphisms is open, eventually the map L_{a_n} must be also an isomorphism. On the other hand, the solutions of the equation:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & -a_n \\ a_n & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

are all almost periodic, inasmuch they write:

$$\begin{pmatrix} \cos \widehat{a}_n & -\sin \widehat{a}_n \\ \sin \widehat{a}_n & \cos \widehat{a}_n \end{pmatrix} \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}$$

where u_{10}, u_{20} are the initial data and:

$$\widehat{a}_n(t) = \int_0^t a_n(s)\, ds$$

is almost periodic. Thus the kernel of L_{a_n} is nontrivial, contradicting the previous conclusion. $\square$

The last part of the section is devoted to the case where A is *purely periodic*, does not matter it has a sign or not, but once more with the aim of discussing the solvability of equation (1.1) in some suitable class of recurrent functions. We stress that here we are not interested in the most classical case, namely that where the nonhomogeneous term f and the solution x are also periodic of the same period of A: this is well known (see for instance Hale's textbook [6]) and our condition (1.5) is not relevant to it. Our concern is instead the solvability when f and x are required to be almost periodic, or even bounded. The former case has been also

considered in [6] but left to the reader, see Exercise 1.1 at page 147: out aim here is to give a complete solution to the exercise.
Theorem 2.2 says that condition (1.5) is relevant to our aim: if it is satisfied, then the Fredholm–type Alternative we proposed works fine. The point is then to decide wether or not, beside being sufficient, condition (1.5) is also necessary to our aim. To prove necessity, we start construction a suitable normal form: we suspect that the result is known, but we sketch the proof since we are unable to provide an explicit reference.

Proposition 4.2. *Let A be continuous and periodic. Then there exists a quasi–periodic change of variables Q such that:*

$$D \;=\; Q^{-1}\big\{AQ - \dot{Q}\big\} \;=\; \operatorname{diag}\{D_1, \ldots, D_n\}$$

in independent of time and each diagonal block D_k is a real Jordan cell:

$$D_k \;=\; \begin{pmatrix} \lambda_k & 1 & 0 & \ldots & 0 \\ 0 & \lambda_k & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_k \end{pmatrix}$$

for some suitable real eigenvalue λ_k.

It is not difficult to check that the λ_k are exactly the Lyapunov exponents of the solutions of the homogeneous equation (1.6). Notice however that to each exponent can correspond many different blocks.

Proof. Real Floquet theory applies to show that A is kinematically similar to a constant matrix C, by means of a periodic change of variables whose period is twice that of A. We can now such C into its real Jordan canonical form, by means of a time independent change of variable: see for instance [4]. To complete the proof, we must ave the better of the real Jordan cells corresponding to complex eigenvalues of C. Since different cells give rise to uncoupled equations, this can be done cell by cell. Consider then the equation associated to the real Jordan cell, corresponding to a complex eigenvalue $a + ib$ repeated m times. That is:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \vdots \\ \dot{u}_{2m-3} \\ \dot{u}_{2m-2} \\ \dot{u}_{2m-1} \\ \dot{u}_{2m} \end{pmatrix} = \begin{pmatrix} a & -b & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ b & a & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & -b & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & a & -b & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \vdots \\ u_{2m-3} \\ u_{2m-2} \\ u_{2m-1} \\ u_{2m} \end{pmatrix}$$

Let us now consider the rotation:

$$R(t) \;=\; \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$$

and the periodic change of variables $u = P(t)v$ where:

$$P(t) \;=\; \operatorname{diag}\big\{R(t), \ldots, R(t)\big\}\;.$$

With some lengthy but straightforward computations one proves that the new equation is:

$$\begin{pmatrix} \dot v_1 \\ \dot v_2 \\ \dot v_3 \\ \dot v_4 \\ \vdots \\ \dot v_{2m-3} \\ \dot v_{2m-2} \\ \dot v_{2m-1} \\ \dot v_{2m} \end{pmatrix} = \begin{pmatrix} a & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2m-3} \\ v_{2m-2} \\ v_{2m-1} \\ v_{2m} \end{pmatrix}$$

To conclude is now sufficient to reorder the components of v, separating those with odd index from those with even index:

$$z_1 = v_1 \quad z_2 = v_3 \quad \dots \quad z_m = v_{2m-1} \quad z_{m+1} = v_2 \quad z_{m+2} = v_4 \quad \dots \quad z_m = v_{2m}$$

obtaining:

$$\dot z = \begin{pmatrix} D_* & 0 \\ 0 & D_* \end{pmatrix} z$$

where of course $z = (z_1, \dots, z_{2m})$ and D_* is the $m \times m$ Jordan cell:

$$D_* = \begin{pmatrix} a & 1 & 0 & \dots & 0 \\ 0 & a & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & a \end{pmatrix}$$

Summing up, we replaced the initial $2m \times 2m$ complex block, with two $m \times m$ real blocks. To obtain such result we have composed four changes of variables: two which are time independent and two which are periodic, but with possibly unrelated periods. The composition is then quasi–periodic. □

Next step is to discuss condition (1.5). Since A is kinematically similar to D as in Proposition 4.2, the arguments of the final part of Section 2 apply to say that this codition can be equivalently discussed for D itself.

Lemma 4.3. *Let D be as in Proposition Proposition 4.2 and assume it has the zero eigenvalue. Then $d_F(D) = d_S(D)$ if and only if all the cells corresponding to the zero eigenvalue are one–dimensional.*

From the proof it will be clear that $d_F(D)$ is the geometric multiplicity of zero and $d_S(D)$ the algebraic multiplicity.

Proof. It is clear that nonzero eigenvalue of D gives rise to solutions of:

$$\dot u = Du$$

which behave exponentially at infinity, and then do not contribute to the Sacker–Sell dimension of D. Consider then only the cells corresponding to the zero eigenvalue. Different cells gives rise to uncoupled equations, so that the Favard and the Sacker–Sell dimensions of D can be obtained by summing the contributions of the single cells. In other words $d_F(D) = d_S(D)$ if and only if this is true for every Jordan cell

corresponding to the zero eigenvalue.
Fix then the attention on the equation corresponding to a single cell:

$$\begin{pmatrix} \dot u_1 \\ \dot u_2 \\ \vdots \\ \dot u_{m-1} \\ \dot u_m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix}$$

It is not difficult to check that its Favard dimension is 1 while the Scaker–Sell dimension is m. Thus the two coincide if and only if $m = 1$. □

We are finally ready for the main result, which close our solution to the above mentioned exercise of [6].

Proposition 4.4. *Let A be continuous and periodic. If:*

$$d_F(A) < d_S(A) \tag{4.4}$$

the there exists a quasi–periodic f such that condition (1.2) is satisfied for every bounded solution to (1.3), but nevertheless equation (1.1) does not admit any bounded solution.

Since it is well known that the Favard separation conditions (F_A) and (F_A^*) are automatically satisfied in view of the periodicity of A, we obtain that the conclusions of Theorem 2.3 in Section 2 hold, independently of the assumption $d_S(A) \le 2$.

Proof. Let D be the matrix considered in Proposition 4.2. Since condition (4.4) is equivalent to $d_F(D) < d_S(D)$, in particular we have $0 \in \sigma(D)$ and hence 0 is an eigenvalue of D. Moreover we can use Lemma 4.3 to find a Jordan cell for D with zero eigenvalue and dimension $m > 1$. We fix now the attention on such cell and consider the equation:

$$\begin{cases} \dot{\mathfrak{z}}_1 & = \mathfrak{z}_2 + \quad \mathfrak{g}_1(t) \\ & \vdots \\ \dot{\mathfrak{z}}_{m-1} & = \mathfrak{z}_m + \quad \mathfrak{g}_{m-1}(t) \\ \dot{\mathfrak{z}}_m & = \qquad\quad \mathfrak{g}_m(t) \end{cases} \tag{4.5}$$

where $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ are quasi–periodic functions. The bounded solutions of the corresponding adjoint equation:

$$\begin{cases} \dot{\mathfrak{w}}_1 & = 0 \\ \vdots & \\ \dot{\mathfrak{w}}_2 & = -\mathfrak{w}_1 \\ \dot{\mathfrak{w}}_m & = -\mathfrak{w}_{m-1} \end{cases}$$

are the one–dimensional space $\mathfrak{w}_1 = 0, \dots, \mathfrak{w}_{m-1} = 0$, $\mathfrak{w}_m = \mathfrak{w}_{m0}$. Thus the test condition (1.2) for equation (4.5) only involves the component $\mathfrak{g}_m$ and writes:

$$\mathfrak{g}_m \in BP(\mathbb{R}) \; . \tag{4.6}$$

Assume that this condition is satisfied. This implies that the primitives of $\mathfrak{g}_m$ are quasi–periodic: we denote by $\mathfrak{G}_m$ the primitive satisfying $\overline{\mathfrak{G}_m} = 0$.
The point is that the existence of bounded solutions for equation (4.5) requires much stronger conditions, inasmuch it involves other $m - 1$ conditions which can

be described recursively. The first one is that determining the boundedness of $\mathfrak{z}_{m-1}$ and writes:

$$\mathfrak{G}_m + \mathfrak{g}_{m-1} - \overline{\mathfrak{g}_{m-1}} \in BP(\mathbb{R}) . \tag{4.7}$$

If we now take for $\mathfrak{g}_{m-1}$ any quasi–periodic function with mean value zero but unbounded primitive, the previous condition fails and hence equation (4.5) does not admit any bounded solutions.

Define now $\mathfrak{f}$ to be the quasi–periodic function obtained by completing to zero outside the block $(\mathfrak{g}_1, \ldots, \mathfrak{g}_m)$. It is clear that:

$$\dot{\mathfrak{x}} = D\mathfrak{x} + \mathfrak{f}(t)$$

does not admit bounded solutions but fulfills the test condition (1.2). To conclude it enough to transfer the counterexample to the original equation (1.1) by setting:

$$x = Q(t)\mathfrak{x} \qquad\qquad f(t) = Q(t)\,\mathfrak{f}(t)$$

where Q is the change of variables involved in Proposition 4.2. Since such Q is quasi–periodic, the same is true for f. □

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