Prevalence of non-degenerate periodic solutions in the forced pendulum equation

Rafael Ortega*

Departamento de Matemática Aplicada
Facultad de Ciencias
Universidad de Granada, 18071 Granada, Spain
rortega@ugr.es

Dedicated to Professor Klaus Schmitt

1 Introduction

Consider the forced pendulum equation

$$\ddot{x} + a \sin x = f(t) \tag{1}$$

where $a > 0$ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function with

$$\int_0^T f(t)dt = 0. \tag{2}$$

The paper by Mawhin [7] contains a very complete survey on this problem. As can be seen in that survey, many of the known properties for (1)-(2) are of generic nature. This means that they do not need to hold for all forcings f but just for a sub-class which is large in the sense of category. Among the known generic properties we mention:

(a) finite number of T-periodic solutions
(b) existence of infinitely many sub-harmonic solutions
(c) chaotic dynamics.

*Supported by D.G.I. MTM2011-23652, Spain

At this moment it is convenient to mention that the periodicity of the equation implies that if $x(t)$ is a solution then $x(t) + 2\pi$ is also a solution. For this reason the properties (a) and (b) refer to solutions satisfying $x(0) \in [0, 2\pi]$ and the dynamics in (c) is analyzed in the cylinder $(x(0), \dot{x}(0))$ with $x(0) \equiv x(0) + 2\pi$. The property (a) does not hold for $f \equiv 0$ and $a > (\frac{2\pi}{T})^2$. This shows that the nature of (a) is indeed generic. Also (c) fails if $f \equiv 0$. As far as I know, it has not been decided if the property (b) holds for all forcings. Concerning the proofs, the genericity of the property (a) is a direct consequence of the results in [6]. The genericity of (b) follows from a result by Fonda and Willem [3] together with [6]. The genericity of (c) is proved by Bosetto, Serra and Terracini in [2]. All these proofs have used another generic property obtained in [6],

(d) every T-periodic solution is non-degenerate.

Let us recall that a T-periodic solution $x(t)$ is non-degenerate if $y \equiv 0$ is the unique T-periodic solution of the variational equation

$$\ddot{y} + (a \cos x(t))y = 0. \tag{3}$$

The property (d) looks rather technical but it seems an useful tool for the obtention of other generic results.

Up to now I have not been precise on the meaning of genericity. To remedy this, let us consider the set $\mathcal{F}$ of T-periodic and continuous functions satisfying (2). It becomes a Banach space with the norm

$$||f||_\infty = \max_{t \in \mathbb{R}} |f(t)|.$$

A set $\mathcal{G} \subset \mathcal{F}$ is generic if there exists a sequence $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ of open and dense subsets of $\mathcal{F}$ such that $\bigcap_{n \in \mathbb{N}} \mathcal{G}_n \subset \mathcal{G}$. This is the standard notion of large set in the sense of category. In spaces of finite dimension large sets can be understood also in the sense of measure. The duality between measure and category is nicely described in the book by Oxtoby [9]. For infinite dimensional spaces like $\mathcal{F}$, a prevalent set can be seen as the analogue of a set of full measure in finite dimension. The papers by Ott and Yorke [8] and by Fraysse, Jaffard and Kahane [4] discuss this point in detail. Once the notion of prevalence is available, it seems natural to ask whether the known generic results for the pendulum equation are also prevalent. In this paper we prove that the property (d) is prevalent.

Theorem 1 *There exists an open and prevalent set $\mathcal{R} \subset \mathcal{F}$ such that (d) holds for each $f \in \mathcal{R}$.*

Prevalent sets are dense in the ambient space and so $\mathcal{R}$ is large in both senses, category and prevalence.

The rest of the paper is divided in two sections. First we present a prevalent version of the theorem of parametric transversality. There are related results by Kaloshin [5] and Shannon [10] but our assumptions are slightly different. In the last section of the paper the abstract transversality theorem is applied to the pendulum equation. The proof does not use many properties of the sine function and the pendulum has been replaced by an equation of the type

$$\ddot{x} + s(x) = f(t)$$

where $s(x)$ is a rather general periodic function.

2 A transversality theorem

Let $\mathbb{E}$ be a separable Banach space of infinite dimension. A subset $\mathcal{N}$ of $\mathbb{E}$ is Haar-null if there exist a Borel set $\mathcal{B}$ and a Borel measure μ on $\mathbb{E}$ such that

- $\mathcal{N} \subset \mathcal{B}$
- $0 < \mu(C) < \infty$ for some compact subset C of $\mathbb{E}$
- $\mu(e + \mathcal{B}) = 0$ for each $e \in \mathbb{E}$.

Compact sets are always Haar-null. Also, a countable union of Haar-null sets is Haar-null. We refer to [8] for a proof of these results as well as for additional information on the notion of Haar-null sets, also called shy sets. A subset of $\mathbb{E}$ is prevalent if its complement is Haar-null. A prevalent set can be small in the sense of category but it is always dense in $\mathbb{E}$.

Given a vector $e \in \mathbb{E}$ with norm $||e||$, the open ball of radius r centered at e is denoted by

$$B(e, r) = \{f \in \mathbb{E} : \ ||f - e|| < r\}.$$

The norm of a vector ξ in the space of finite dimension $\mathbb{R}^d$ will be denoted by $|\xi|$. We will work with a map

$$h : \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d, \ (\xi, e) \mapsto h(\xi, e)$$

and we are interested in the set of zeros

$$\mathcal{Z} = \{(\xi, e) \in \mathbb{R}^d \times \mathbb{E} : \ h(\xi, e) = 0\}.$$

We impose three conditions on the map h, the first of them is concerned with the differentiability (in Fréchet sense),

$(C1)$ $\quad h \in C^1(\mathbb{R}^d \times \mathbb{E}, \mathbb{R}^d)$.

The partial derivatives will be denoted by $\partial_1 h(t,\xi)$ and $\partial_2 h(t,\xi)$ respectively. At each point (ξ, e) the derivative $\partial_1 h$ can be interpreted as an endomorphism of $\mathbb{R}^d$ or as a matrix of dimension $d \times d$. The derivative $\partial_2 h$ is a bounded linear operator from $\mathbb{E}$ to $\mathbb{R}^d$. The second condition on h is concerned with this derivative.

$(C2)$ *There exists a compact set $K \subset \mathbb{E}$ such that the linear operator $\partial_2 h(t,\xi) : \mathbb{E} \to \mathbb{R}^d$ is onto if $(\xi, e) \in \mathcal{Z}$ and $e \notin K$.*

The last condition is concerned with the existence of a priori bounds for the zeros of h.

$(C3)$ *Given $b > 0$ there exists $B > 0$ such that if $(\xi, e) \in \mathcal{Z}$ and $||e|| \le b$ then $|\xi| \le B$.*

Theorem 2 *Assume that the conditions $(C1)$, $(C2)$ and $(C3)$ hold. Then the set*

$$\mathcal{R} = \{e \in \mathbb{E} : \text{ 0 is a regular value of } h(\cdot, e)\}$$

is open and prevalent.

Remarks. 1. The map $h(\cdot, e)$ goes from $\mathbb{R}^d$ into $\mathbb{R}^d$. In this setting, to say that 0 is a regular value is equivalent to the non-degeneracy of the zeros of $h(\cdot, e)$; that is, $\det[\partial_1 h(\xi, e)] \neq 0$ for each $\xi \in \mathbb{R}^d$ such that $h(\xi, e) = 0$.
2. The condition $(C3)$ can be weakened if h satisfies a periodicity condition. More precisely, assume that

$$h(T(\xi), e) = h(\xi, e), \tag{4}$$

where $T(\xi_1, \xi_2, ..., \xi_d) = (\xi_1 + 2\pi, \xi_2, ..., \xi_d)$. Then the condition $(C3)$ in the previous theorem can be replaced by

$(C3)_{per}$ *Given $b > 0$ there exists $B > 0$ such that if $(\xi, e) \in \mathcal{Z}$ and $||e|| \le b$ then $|\hat{\xi}| \le B$, where $\hat{\xi} = (\xi_2, ..., \xi_d)$*

Notice that, in contrast with $(C3)$, no bound is required on the first coordinate ξ_1.

Two preliminary results are needed before the proof of theorem 2.

Lemma 3 *A subset $\mathcal{N}$ of $\mathbb{E}$ is Haar-null if there exist a Borel set $\mathcal{B}^*$ and a compact set K such that $\mathcal{N} \subset \mathcal{B}^*$ and the property below holds for each $e \in \mathbb{E} \setminus K$,*

(P_e) *There exist $\epsilon > 0$, an integer $N \geq 1$ and linearly independent vectors $e_1^*, \ldots, e_N^*$ in $\mathbb{E}$ such that if $||e - f|| < \epsilon$ then*

$$\text{meas}(\{\lambda \in \mathbb{H}_N : \ f + \sum_{i=1}^{N} \lambda_i e_i^* \in \mathcal{B}^*\}) = 0.$$

Here $\mathbb{H}_N$ is the N-dimensional hypercube $]-1,1[\times \cdots \times]-1,1[$ and meas *stands for the Lebesgue measure in $\mathbb{R}^N$.*

Proof. This result is essentially contained in [8] but a complete proof will be presented. The numbers ϵ and N in the property (P_e) depend on e and we write $\epsilon = \epsilon_e$ and $N = N_e$ to emphasize this dependence. Define also $\delta_e = \frac{1}{2}\epsilon_e$. The family of open balls $\{B(e, \delta_e)\}_{e \in \mathbb{E} \setminus K}$ covers $\mathbb{E} \setminus K$. The space $\mathbb{E}$ is separable and so Lindelöf theorem applies (see [11]).Therefore it is possible to find a family $\{e_n : \ n \in A\}$ with $e_n \in \mathbb{E} \setminus K$ and $A \subset \mathbb{N}$ and such that

$$\mathbb{E} \setminus K \subset \bigcup_{n \in A} B(e_n, \delta_n),$$

where $\delta_n = \delta_{e_n}$. Let us consider the Borel set $\mathcal{B}_n^* = \mathcal{B}^* \cap B(e_n, \delta_n)$. We prove that $\mathcal{B}_n^*$ is Haar-null. To this end we use that the property (P_e) holds for $e = e_n$ and consider the space F spanned by $e_1^*, \ldots, e_N^*$. Notice that also F depends on n but we do not make explicit this dependence. The map $\lambda = (\lambda_1, \ldots, \lambda_N) \mapsto \sum_{i=1}^{N} \lambda_i e_i^*$ defines an isomorphism between $\mathbb{R}^N$ and F. The Lebesgue measure transported to F will be denoted by meas_F and $\mathbb{H}_F$ will be the image of the hypercube in F. We define the Borel measure on $\mathbb{E}$,

$$\mu(\mathcal{B}) = \text{meas}_F[(\mathcal{B} \cap B(e_n, \delta_n) \cap (e_n + \mathbb{H}_F)) - e_n].$$

This is a measure supported on $B(e_n, \delta_n) \cap (e_n + \mathbb{H}_F)$ and the compact set $e_n + \overline{\mathbb{H}_F}$ satisfies $0 < \mu(\mathbb{H}_F) \leq 2^N$. To prove that

$$\mu(\varphi + \mathcal{B}_n^*) = 0 \ \text{ for each } \varphi \in \mathbb{E}$$

we distinguish two cases. Assume first that $||\varphi|| \geq \epsilon_n$ with $\epsilon_n = 2\delta_n$. In this case the set $\varphi + \mathcal{B}_n^*$ is outside the support of μ. In fact the balls $\varphi + B(e_n, \delta_n)$ and $B(e_n, \delta_n)$ are disjoint and so

$$(\varphi + \mathcal{B}_n^*) \cap B(e_n, \delta_n) \subset (\varphi + B(e_n, \delta_n)) \cap B(e_n, \delta_n) = \emptyset.$$

To discuss the second case we assume that $||\varphi|| < \epsilon_n$. Now it is more convenient to express μ directly in terms of the Lebesgue measure in $\mathbb{R}^N$.

More precisely,

$$\mu(\varphi + \mathcal{B}_n^*) = \text{meas}(\{\lambda \in \mathbb{H}_N : e_n + \sum_{i=1}^{N} \lambda_i e_i^* \in (\varphi + \mathcal{B}_n^*) \cap B(e_n, \delta_n)\}).$$

Since $\mathcal{B}_n^*$ is contained in $\mathcal{B}^*$,

$$\mu(\varphi + \mathcal{B}_n^*) \leq \text{meas}(\{\lambda \in \mathbb{H}_N : e_n - \varphi + \sum_{i=1}^{N} \lambda_i e_i^* \in \mathcal{B}^*\}),$$

and we can apply (P_e) with $e = e_n$ and $f = e_n - \varphi$.

Once we know that each $\mathcal{B}_n^*$ is Haar-null, we arrive easily to the conclusion of the lemma because $\mathcal{N}$ is contained in $K \cup [\bigcup_{n \in A} \mathcal{B}_n^*]$.

The second preliminary result is an analogue of theorem 2 in finite dimensions. It is inspired by lemma 1 in [5] and lemma 3.2 in [10].

Lemma 4 *Let $H : \mathbb{R}^d \times \mathbb{H}_N \to \mathbb{R}^d$, $(\xi, \lambda) \mapsto H(\xi, \lambda)$ be a C^1 function and let $Z = H^{-1}(0)$ be the set of zeros. Assume that for each $(\xi, \lambda) \in Z$ the second partial derivative is onto. This condition can be expressed in terms of the rank of matrices by*

$$\text{rank}[\frac{\partial H}{\partial \lambda}(\xi, \lambda)] = d \quad \text{if } (\xi, \lambda) \in Z.$$

Then

$$R = \{\lambda \in \mathbb{H}_N : \text{ 0 is a regular value of } H(\cdot, \lambda)\}$$

is of full measure in $\mathbb{H}_N$; that is, $\text{meas}(R) = 2^N$.

Proof. The assumptions imply that 0 is a regular value of H and so Z is a manifold of class C^1 and dimension N. Given a point (ξ, λ) in Z, vectors lying in the tangent space $T_{(\xi,\lambda)}(Z)$ will be denoted by $(\dot{\xi}, \dot{\lambda})$. These are the vectors $(\dot{\xi}, \dot{\lambda}) \in \mathbb{R}^d \times \mathbb{R}^N$ satisfying

$$\frac{\partial H}{\partial \xi}(\xi, \lambda)\dot{\xi} + \frac{\partial H}{\partial \lambda}(\xi, \lambda)\dot{\lambda} = 0. \tag{5}$$

The projections $\pi_1(\xi, \lambda) = \xi$ and $\pi_2(\xi, \lambda) = \lambda$ can be restricted to the manifold Z. These restrictions, denoted again by π_i, are functions of class C^1 whose differentials $(d\pi_i)_{(\xi,\lambda)}$ are the restrictions of π_i to the tangent space. From the identity (5) we deduce that

$$\frac{\partial H}{\partial \xi}(\xi, \lambda) \circ (d\pi_1)_{(\xi,\lambda)} = -\frac{\partial H}{\partial \lambda}(\xi, \lambda) \circ (d\pi_2)_{(\xi,\lambda)}. \tag{6}$$

We intend to apply Sard's lemma to $\pi_2 : Z \to \mathbb{R}^N$. This map has the required smoothness but the classical version of the lemma only deals with maps between Euclidean spaces. To overcome this difficulty one can use a version of Sard's lemma on manifolds (see [5] and [1]) but there is a simpler approach in our case. After invoking again Lindelöf theorem, this time on $\mathbb{R}^d \times \mathbb{R}^N$, we cover Z with a countable family of charts, then the classical version of the lemma is applied to the composition of π_2 with each chart. In any of the two ways we conclude that the set $\tilde{R}$ of regular values of $\pi_2 : Z \to \mathbb{R}^N$ is of full measure in $\mathbb{R}^N$. Given $\lambda \in \tilde{R} \cap \mathbb{H}_N$, for each $(\xi, \lambda) \in Z$ we know that the differential $(d\pi_2)_{(\xi,\lambda)} : T_{(\xi,\lambda)}(Z) \to \mathbb{R}^N, (\dot{\lambda}, \dot{\xi}) \mapsto \dot{\lambda}$ is an isomorphism. From the assumption we conclude that $-\frac{\partial H}{\partial \lambda}(\xi, \lambda) \circ (d\pi_2)_{(\xi,\lambda)}$ is onto. The identity (6) implies that $\frac{\partial H}{\partial \xi}(\xi, \lambda)$ is also onto. Since this map is an endomorphism of $\mathbb{R}^d$ we conclude that $\det[\frac{\partial H}{\partial \xi}(\xi, \lambda)] \neq 0$. Summing up, we have proved that 0 is a regular value of $H(\cdot, \lambda)$ if $\lambda \in \tilde{R} \cap \mathbb{H}_N$. In consequence $\tilde{R} \cap \mathbb{H}_N$ is contained in R and the proof is complete.

Proof of theorem 2. We divide the proof in three steps.
Step 1: $\mathcal{R}$ is open.
We prove that the complement $\mathbb{E} \setminus \mathcal{R}$ is closed. Given a sequence $e_n \notin \mathcal{R}$ converging to e_∞, we must prove that e_∞ is not in $\mathcal{R}$. From the definition of $\mathcal{R}$ we find a sequence of degenerate zeros $(\xi_n, e_n) \in \mathcal{Z}$. They satisfy

$$h(\xi_n, e_n) = 0, \;\; \det[\partial_1 h(\xi_n, e_n)] = 0. \tag{7}$$

From $(C3)$ we deduce that the sequence ξ_n is bounded and so we can extract a convergent subsequence ξ_k. Let ξ_∞ be the limit. Letting $k \to \infty$ in (7), we deduce that ξ_∞ is a degenerate zero of $h(\cdot, e_\infty)$ and so $e_\infty \notin \mathcal{R}$.

When h is periodic and $(C3)$ is replaced by $(C3)_{per}$, the previous argument needs some modifications. By periodicity we find another sequence (ζ_n, e_n) with $\xi_n - \zeta_n \in 2\pi\mathbb{Z}$ and $\zeta_n \in [0, 2\pi]$. Moreover, the differential of h coincides at the points (ξ_n, e_n) and (ζ_n, e_n). By $(C3)_{per}$ we know that $\hat{\zeta}_n$ is bounded and the rest of the proof is the same.

Given a subspace F of $\mathbb{E}$ we consider the restriction of $\partial_2 h(\xi, e)$ to F. This is a linear operator that will be denoted by $\partial_{2,F} h(\xi, e) : F \to \mathbb{R}^d$. Also, for each $e \in \mathbb{E}$, we employ the notation

$$\mathcal{Z}_e = \{\xi \in \mathbb{R}^d : \; (\xi, e) \in \mathcal{Z}\}.$$

Step 2: Given $e \in \mathbb{E} \setminus K$ there exists a subspace F of $\mathbb{E}$ of finite dimension and such that $\partial_{2,F} h(\xi, e)$ is onto for every $\xi \in \mathcal{Z}_e$.

We apply $(C2)$ to each $\xi \in \mathcal{Z}_e$ and find vectors $f_1^{(\xi)}, \dots, f_d^{(\xi)} \in \mathbb{E}$ with

$$\partial_2 h(\xi, e) f_i^{(\xi)} = c_i, \ \ i = 1, \dots, d,$$

where $c_1, \dots, c_d$ is the canonical basis of $\mathbb{R}^d$. The continuity of $\partial_2 h$ implies the existence of a number $\delta_\xi > 0$ such that the family $\{\partial_2 h(\xi^*, e) f_i^{(\xi)}\}_{1 \leq i \leq d}$ is a basis of $\mathbb{R}^d$ if $|\xi - \xi^*| \leq \delta_\xi$. The condition $(C3)$ and the continuity of h imply in particular that $\mathcal{Z}_e$ is compact. Then we can find a finite set $\xi_1, \dots, \xi_r \in \mathcal{Z}_e$ such that the balls

$$B_1 = B(\xi_1, \delta_{\xi_1}), \dots, B_r = B(\xi_r, \delta_{\xi_r})$$

is a covering of $\mathcal{Z}_e$. We put all the vectors $f_i^{(\xi_j)}$ together, with $i = 1, \dots, d$ and $j = 1, \dots, r$. They span a vector space F having at most dimension $r \cdot d$. We claim that F satisfies the conditions of Step 2. Indeed, given $\xi \in \mathcal{Z}_e$ we find some $j \in \{1, \dots, r\}$ with $|\xi - \xi_j| < \delta_{\xi_j}$ and so the family of vectors $\partial_2 h(\xi, e) f_i^{(\xi_j)}$, $i = 1, \dots, d$, is a basis of $\mathbb{R}^d$.

When h is periodic and $(C3)_{per}$ holds, the set $\mathcal{Z}_e$ is not compact but the proof still works if one replaces $\mathcal{Z}_e$ by

$$\tilde{\mathcal{Z}}_e = \mathcal{Z}_e \cap \{\xi_1 \in [0, 2\pi]\}.$$

Given a vector $e \in \mathbb{E}$ and N linearly independent vectors $\varphi_1, \dots, \varphi_N \in \mathbb{E}$ we define the function

$$H : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d, \ \ H(\xi, \lambda) = h(\xi, e + \sum_{i=1}^{N} \lambda_i \varphi_i).$$

At some moment it will be convenient to emphasize the dependence of H with respect to e and we will write $H = H(\xi, \lambda; e)$. The function H is of class C^1 and if we select $e \in \mathbb{E} \setminus K$ and a basis $\varphi_1, \dots, \varphi_N$ of the subspace F given by step 2, then

$$\text{rank}[\frac{\partial H}{\partial \lambda}(\xi, 0)] = d \ \text{ for each } \xi \in \mathcal{Z}_e. \tag{8}$$

Step 3: Given $e \in \mathbb{E} \setminus K$ there exist $\epsilon > 0$ and a basis $\varphi_1, \dots, \varphi_N$ of F such that if $f \in \mathbb{E}$ and $||f - e|| < \epsilon$ then

$$\text{rank}[\frac{\partial H}{\partial \lambda}(\xi, \lambda; f)] = d \ \ \textit{if } H(\xi, \lambda, f) = 0 \textit{ and } \lambda \in \mathbb{H}_N.$$

Let us fix a basis $\hat{\varphi}_1, \dots, \hat{\varphi}_N$ of F and construct the associated function H. We claim that there exists $\eta > 0$ such that

$$\operatorname{rank}[\frac{\partial H}{\partial \lambda}(\xi, \lambda; f)] = d \ \ if \ H(\xi, \lambda, f) = 0 \ and \ |\lambda| \leq \eta, \ ||f - e|| \leq \eta.$$

This claim is proved by contradiction. Assume the existence of a sequence $(\xi_n, \lambda_n; f_n)$ with $\operatorname{rank}[\frac{\partial H}{\partial \lambda}(\xi_n, \lambda_n; f_n)] < d$ and $H(\xi_n, \lambda_n; f_n) = 0$, $|\lambda_n| \leq \frac{1}{n}$, $||f_n - e|| \leq \frac{1}{n}$. From $(C3)$ we deduce that ξ_n is bounded. After extracting a subsequence $\xi_k \to \xi_\infty$ we notice that $H(\xi_\infty, 0; e) = 0$ and $\operatorname{rank}[\frac{\partial H}{\partial \lambda}(\xi_\infty, 0; e)] < d$. This is a contradiction with step 2 since ξ_∞ belongs to $\mathcal{Z}_e$. The proof of step 3 is complete if we consider the basis $\varphi_i = \frac{1}{\eta}\hat{\varphi}_i$.

The modifications for the periodic case are now rather obvious. We replace Z_η by $\tilde{Z}_\eta = Z_\eta \cap \{\xi_1 \in [0, 2\pi]\}$ and proceed as before.

After these steps we are ready for the proof of the theorem. We want to apply lemma 3 with $\mathcal{N} = \mathcal{B}^* = \mathbb{E} \setminus \mathcal{R}$. Given $e \in \mathbb{E} \setminus K$ we must check the property (P_e). We know from step 3 that the assumption of lemma 4 holds for $H(\cdot, \cdot; f)$ if $||f - e|| < \epsilon$. Then 0 is a regular value of $H(\cdot, \lambda; f)$ for almost every λ in $\mathbb{H}_N$. Since $h(\cdot, f + \sum_{i=1}^N \lambda_i \varphi_i) = H(\cdot, \lambda; f)$, we conclude that

$$\operatorname{meas}(\{\lambda \in \mathbb{H}_N : f + \sum_{i=1}^{N} \lambda_i \varphi_i \notin \mathcal{R}\}) = 0.$$

3 General periodic non-linearities

In this section we work with a function $s : \mathbb{R} \to \mathbb{R}$ of class C^1 satisfying

$(\sigma 1)$ $\quad$ *$s(x + 2\pi) = s(x)$ for each $x \in \mathbb{R}$,*

$(\sigma 2)$ $\quad$ *s is not locally trivial.*

The condition $(\sigma 2)$ means that for every open and non-empty interval $I \subset \mathbb{R}$ there exists some $x \in I$ such that $s(x) \neq 0$. The function $s(x) = a \sin x + b$ satisfies both conditions if $(a, b) \neq (0, 0)$.

Consider the differential equation

$$\ddot{x} + s(x) = f(t) \tag{9}$$

with $f \in \mathcal{F}$. Given a T-periodic solution $x(t)$ of (9), the variational equation is

$$\ddot{y} + s'(x(t))y = 0. \tag{10}$$

We say that $x(t)$ is non-degenerate if $y \equiv 0$ is the unique T-periodic solution of (10).

Theorem 5 *Assume that* $(\sigma 1)$ *and* $(\sigma 2)$ *hold. Then the set*

$$\mathcal{R} = \{f \in \mathcal{F} : \text{ every } T-\text{periodic solution of (9) is non}-\text{degenerate}\}$$

is open and prevalent.

The condition $(\sigma 1)$ is not sufficient to prove the theorem because $\mathcal{R} = \emptyset$ if $s \equiv 0$. At the end of the paper it will be shown that $(\sigma 2)$ is optimal.

The proof of the theorem will consist in an application of theorem 2 with $\mathbb{E} = \mathcal{F}$ and $d = 2$. Before the proof we need some preliminary remarks. We work with column vectors $\xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2$ and the norm $|\xi| = |\xi_1| + |\xi_2|$. In the space of 2×2 matrices $\mathbb{R}^{2\times 2}$ we consider the associated norm

$$|A| = \max_{|\xi|\leq 1} |A\xi| = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\},$$

where $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$. The space of bounded linear operators from $\mathcal{F}$ to $\mathbb{R}^2$ will be denoted by $\mathcal{L}(\mathcal{F}, \mathbb{R}^2)$ with norm

$$||L|| = \sup\{|Lf| : \ f \in \mathcal{F}, ||f||_\infty \leq 1\}$$

if $L \in \mathcal{L}(\mathcal{F}, \mathbb{R}^2)$. The integral operator

$$L : \mathcal{F} \to \mathbb{R}^2, \quad L(f) = \int_0^T f(t)(\psi_1(t), \psi_2(t))^* dt$$

belongs to $\mathcal{L}(\mathcal{F}, \mathbb{R}^2)$ if $\psi_1, \psi_2 : [0, T] \to \mathbb{R}$ are two given integrable functions. We present a preliminary result on this class of operators.

Lemma 6 *Assume that* ψ_1 *and* ψ_2 *are functions in* $C^1[0, T]$*. The following statements are equivalent:*
(i) *The derivatives* $\dot{\psi}_1$ *and* $\dot{\psi}_2$ *are linearly independent in* $C[0, T]$*,*
(ii) *The map* $L \in \mathcal{L}(\mathcal{F}, \mathbb{R}^2)$ *is onto.*

Proof. The condition (ii) is equivalent to
$(ii)^*$ *The functionals* $\ell_i : \mathcal{F} \to \mathbb{R}$, $\ell_i(f) = \int_0^T f(t)\psi_i(t)dt$, $i = 1, 2$, *are linearly independent in the dual space* $\mathcal{F}^*$.
This is a consequence of general arguments in abstract Linear Algebra. We also recall that if $\psi \in C^1[0, T]$ then the condition

$$\int_0^T f(t)\psi(t)dt = 0 \text{ for each } f \in \mathcal{F}$$

is equivalent to $\dot{\psi} \equiv 0$. From this last statement we notice that if $k_1, k_2 \in \mathbb{R}$, the identity in $\mathcal{F}^*$

$$k_1\ell_1 + k_2\ell_2 = 0$$

is equivalent to $k_1\dot{\psi}_1 + k_2\dot{\psi}_2 \equiv 0$. The equivalence of (i) and $(ii)^*$ follows.

Proof of theorem 5. Given $\xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2$ and $f \in \mathcal{F}$, the solution of the initial value problem

$$\ddot{x} + s(x) = f(t), \;\; x(0) = \xi_1, \; \dot{x}(0) = \xi_2 \tag{11}$$

will be denoted by $x(t; \xi, f)$. Since s is bounded, this solution is globally defined. The notations $\Phi(t)$ and $\Phi(t; \xi, f)$ will be employed for the matrix solution of

$$\dot{Y} = A(t)Y, \; Y(0) = I_2, \;\; \text{with } A(t) = \begin{pmatrix} 0 & 1 \\ -s'(x(t;\xi,f)) & 0 \end{pmatrix}. \tag{12}$$

Notice that this first order system is associated to the variational equation (10). The theorem on continuous dependence can be applied to the Cauchy problems (11) and (12). It implies that the map

$$(t; \xi, f) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{F} \mapsto \Phi(t; \xi, f) \in \mathbb{R}^{2\times 2}$$

is continuous. In particular it is uniformly continuous on compact sets. This implies that if $\xi_n \to \xi$ and $||f_n - f||_\infty \to 0$ then

$$\Phi(t; \xi_n, f_n) \to \Phi(t; \xi, f) \;\; \text{uniformly in } t \in [0, T].$$

We also consider the map

$$h : \mathbb{R}^2 \times \mathcal{F} \to \mathbb{R}^2, \;\; h(\xi, f) = (x(T; \xi, f) - \xi_1, \dot{x}(T; \xi, f) - \xi_2)^*$$

and observe that the zeros of $h(\cdot, f)$ are the initial conditions producing T-periodic solutions. This map is continuous and the theorem on differentiability with respect to initial conditions and parameters implies that it is Gâteaux differentiable with partial derivatives $\partial_1 h(\xi, f) \in \mathbb{R}^{2\times 2}$ and $\partial_2 h(\xi, f) \in \mathcal{L}(\mathcal{F}, \mathbb{R}^2)$ given by

$$\partial_1 h(\xi, f) = \Phi(T; \xi, f) - I_2, \;\; \partial_2 h(\xi, f)g = (y(T), \dot{y}(T))^*$$

where g is an arbitrary function in $\mathcal{F}$ and $y(t)$ is the solution of

$$\ddot{y} + s'(x(t; \xi, f))y = g(t), \;\; y(0) = \dot{y}(0) = 0.$$

A more explicit expression for $y(t)$ can be obtained via the formula of variation of constants and the identity $\det \Phi(t;\xi,f)=1$,

$$y(t)=\int_0^T G(t,s;\xi,f)g(s)ds \tag{13}$$

and $G(t,s;\xi,f)=\phi_2(t)\phi_1(s)-\phi_2(s)\phi_1(t)$, $\phi_i=\phi_i(\cdot;\xi,f)$. Similarly,

$$\dot{y}(t)=\int_0^T \frac{\partial G}{\partial t}(t,s;\xi,f)g(s)ds. \tag{14}$$

The continuity of Φ and the formulas (13) and (14) can be employed to prove the continuity of the partial derivatives of h. In particular the continuity of $(\xi,f)\in\mathbb{R}^2\times\mathcal{F}\mapsto\partial_2 h(\xi,f)\in\mathcal{L}(\mathcal{F},\mathbb{R}^2)$ is a consequence of the estimate

$$||\partial_2 h(\xi,f)-\partial_2 h(\hat{\xi},\hat{f})||\leq$$

$$\int_0^T\{|G(T,s;\xi,f)-G(T,s;\hat{\xi},\hat{f})|+|\frac{\partial G}{\partial t}(T,s;\xi,f)-\frac{\partial G}{\partial t}(T,s;\hat{\xi},\hat{f})|\}ds.$$

The previous discussions show that h is Fréchet differentiable and $(C1)$ holds. The condition $(\sigma 1)$ implies that $x(t;T(\xi),f)=x(t;\xi,f)+2\pi$. From here we deduce that h satisfies the periodicity condition (4) and we check $(C3)_{per}$. Given $(\xi,f)\in\mathcal{Z}$ we know that $x(t;\xi,f)$ is a T-periodic solution of (9). Hence

$$||\ddot{x}(\cdot;\xi,f)||_\infty\leq||s||_\infty+||f||_\infty.$$

The periodicity of $x(\cdot;\xi,f)$ implies that the derivative vanishes somewhere, say $\dot{x}(\tau;\xi,f)=0$ for some $\tau\in[0,T]$. Then

$$|\xi_2|=|\dot{x}(0;\xi,f)|=|\int_0^\tau \ddot{x}(t;\xi,f)dt|\leq(||s||_\infty+||f||_\infty)T.$$

The condition $(C3)_{per}$ holds with $B=(||s||_\infty+b)T$.

To check $(C2)$ we define $K=\{0\}$ and prove that $\partial_2 h(\xi,f):\mathcal{F}\to\mathbb{R}^2$ is onto if $(\xi,f)\in\mathcal{Z}$ and $f\not\equiv 0$. After some manipulations with the formulas (13) and (14) we obtain

$$\partial_2 h(\xi,f)g=\Phi(T;\xi,f)J\int_0^T g(t)(\phi_1(t),\phi_2(t))^*dt$$

with $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Since Φ and J have an inverse, it is enough to prove that

$$L:g\in\mathcal{F}\mapsto\int_0^T g(t)(\phi_1(t),\phi_2(t))^*dt\in\mathbb{R}^2$$

is onto. In view of lemma 6 we must prove that $\dot\phi_1$ and $\dot\phi_2$ are linearly independent. Actually we will prove that the Wronskian of these functions is not identically zero. From the equation (10) and the identity $W(\phi_1,\phi_2)\equiv 1$,

$$W(\dot\phi_1,\dot\phi_2)=\dot\phi_1\ddot\phi_2-\dot\phi_2\ddot\phi_1=s'(x(t;\xi,f))(\phi_1\dot\phi_2-\phi_2\dot\phi_1)=s'(x(t;\xi,f)).$$

Assume by contradiction that $W(\dot\phi_1,\dot\phi_2)\equiv 0$. Then $s'(x(t;\xi,f))$ vanishes identically and so $s(x(t;\xi,f))$ is a constant k. From the equation (9), $f(t)=\ddot x(t;\xi,f)+k$. The solution $x(t;\xi,f)$ is T-periodic and f has zero average and hence $k=0$. In consequence also $s(x(t;\xi,f))$ vanishes identically and the assumption $(\sigma 2)$ implies that $x(t;\xi,f)$ must be constant. Then $f(t)=k=0$ but this forcing has been excluded by the definition of K. The proof of the theorem is complete because the sets $\mathcal{R}$ appearing in theorems 2 and 5 are the same. Notice that 0 is a regular value of $h(\cdot,f)$ if and only if $\det[\Phi(T;\xi,f)-I_2]\neq 0$ for each T-periodic solution $x(t;\xi,f)$.

We finish the paper with a result on the sharpness of the condition $(\sigma 2)$. The set $\mathcal{R}$ is the same as in theorem 5.

Proposition 7 *Assume that $s\in C^1(\mathbb{R})$ vanishes on some open and non-empty interval $I\subset\mathbb{R}$. Then there exists an open and non-empty subset $\mathcal{D}\subset\mathcal{F}$ with $\mathcal{D}\cap\mathcal{R}=\emptyset$.*

Proof. Given $f\in\mathcal{F}$ let F be the unique solution of

$$\ddot F=f(t),\quad F\in\mathcal{F}.$$

The linear operator $f\in\mathcal{F}\mapsto F\in\mathcal{F}$ is bounded. We pick a point $x_*\in I$ and define

$$\mathcal{D}=\{f\in\mathcal{F}:\ x_*+F(\mathbb{R})\subset I\}.$$

This set contains $f\equiv 0$ and is open. Moreover, if $f\in\mathcal{D}$ then $x(t)=x_*+F(t)$ is a T-periodic solution of (9) and the variational equation (10) becomes $\ddot y=0$. In consequence $x(t)$ is degenerate and $f\notin\mathcal{R}$.

References

[1] V. Arnold, Chapitres supplémentaires de la théorie des équations différentielles ordinaires, Mir 1984.

[2] E. Bosetto, E. Serra, and S. Terracini, Generic-type results for chaotic dynamics in equations with periodic forcing term, J. Diff. Equs., 180 (2002) 99-124.

[3] A. Fonda, M. Willem, Subharmonic oscillations of forced pendulum-type equations, J. Diff. Equs. 81 (1989) 215-220

[4] A. Fraysse, S. Jaffard, J.P. Kahane, Quelques propriétés génériques en analyse, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 65-651.

[5] V. Kaloshin, Prevalence in the space of finitely smooth maps, Functional Analysis and its Applications 31 (1997) 95-99.

[6] P. Martínez-Amores, J. Mawhin, R. Ortega, and M. Willem, Generic results for the existence of nondegenerate periodic solutions of some differential systems with periodic nonlinearities, J. Diff. Equs. 91 (1991) 138-148.

[7] J. Mawhin, Global results for the forced pendulum equation, Handbook of differential equations, 533-589. Elsevier/North Holland 2004.

[8] W. Ott, J.A. Yorke, Prevalence, Bull. Am. Mat. Soc. 42 (2005) 263-290.

[9] J.C. Oxtoby, Measure and category, Springer-Verlag 1980.

[10] C. Shannon, A prevalent transversality theorem for Lipschitz functions, Proc. Am. Math. Soc. 134 (2006) 2755-2765.

[11] W. Sierpiński, General topology, Dover Pub. 2000.