Escaping orbits are rare in the quasi-periodic Fermi-Ulam ping-pong

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Abstract

We consider the quasi-periodic Fermi-Ulam ping-pong model with no diophantine condition on the frequencies and show that typically the set of initial data which lead to escaping orbits has Lebesgue measure zero.

1 Introduction

The Fermi-Ulam ping-pong, basically introduced by Fermi [10] in 1949, serves as a simplified model for charged particles, like a proton or an electron, which bounce off an interstellar magnetic field at high energies. One of the central issues was to determine whether, upon repeated bouncing, a particle can gain so much energy that its speed would come close to the speed of light.

On a mathematical level, a common formulation of the model is as follows: consider two vertical rackets, the left one being fixed at $x = 0$, whereas the right one moves according to some law $x = p(t)$ for a prescribed function $p = p(t)$. The two rackets alternately hit the particle, which impacts completely elastically and experiences no gravity. Furthermore, the particle is assumed to travel without being accelerated between the impacts. The successor map is $f : (t_0, v_0) \mapsto (t_1, v_1)$ and sends a time $t_0 \in \mathbb{R}$ of impact to the left racket $x = 0$ and the corresponding velocity $v_0 > 0$ immediately after the impact to their successors t_1 and v_1, describing in the same way the subsequent impact to $x = 0$. Thus defining the forward orbits $(t_n, v_n) = f^n(t_0, v_0)$ for $n \geq 0$, the problem is to study the 'escape set'

$$E = \{(t_0, v_0) : \lim_{n\to\infty} v_n = \infty\} \tag{1.1}$$

of initial conditions along whose orbit the particle will become infinitely fast. Ulam was interested in this question and conjectured [16, p. 318] an increase in velocity on the average, i.e. he believed that escape would be the typical behavior of the trajectories. Although the computing power at this time was low as compared to today, he however realized that numerically such a pattern could not be seen and large fluctuations seemed more likely to be typical. This turned out to be true in a rigorous sense and was first shown for periodic forcing functions $p(t)$ which are sufficiently regular, see [12, Thm. 2]; also related is [15, Part 2, Chapter 1]. The proof relies on the invariant curve theorem [13] and yields that the velocity is bounded along every orbit. In particular, $E = \emptyset$ in this case. It was furthermore shown in [17] that escaping orbits can exist, if the periodic function $p(t)$ is not smooth enough; also see [11, Thm. 3.4] for C^∞-examples in the case where $p(t)$ is not periodic. In [18] the boundedness result was generalized to quasiperiodic forcing functions $p(t)$ such that the frequencies satisfy a diophantine condition, and once again the proof uses an appropriate invariant curve theorem. In the last years a new approach has been developed in the works of Dolgopyat and De Simoi [5, 2, 8, 3]. They consider periodic forcings and some maps which can be seen as approximations of the successor map f. Several results concerning the Lebesgue measure of E (and in some cases its Hausdorff dimension) are obtained in these papers.

The result of [18], together with [6, Problem 4], can be viewed as a starting point for the present paper, since it leaves open the question of what would be the typical behavior in the quasi-periodic case, if no diophantine condition was assumed. Next we state our main result. The reader will have noticed a certain lack of precision in the definition of the escape set. This is related to the definition of the ping-pong map $\mathcal{D} \ni (t_0, v_0) \mapsto (t_1, v_1)$, whose domain $\mathcal{D}$ is sometimes not all of $\mathbb{R}\times]0,\infty[$, but a proper subset. It can be shown that this map is well-defined on $v > \nu_*$, where $\nu_* = 2\max\{\sup_{t\in\mathbb{R}} \dot{p}(t), 0\}$; see Section 5 for more details. From now on we take $\mathcal{D} = \mathbb{R}\times]\nu_*,\infty[$. Then the escape set is comprised by those points $(t_0, v_0) \in \mathcal{D}$ such that

(a) the forward orbit $(t_n, v_n)_{n\geq 0}$ is complete, i.e., $(t_n, v_n) \in \mathcal{D}$ for each $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} v_n = \infty$.

Theorem 1.1 *Assume $0 < a < b$ and $P \in C^2(\mathbb{T}^N)$ is such that*

$$0 < a \leq P(\bar{\Theta}) \leq b, \quad \bar{\Theta} \in \mathbb{T}^N. \tag{1.2}$$

Let $\omega_1, \ldots, \omega_N > 0$ be rationally independent and consider the family of quasi-periodic forcing functions

$$p_{\bar{\Theta}}(t) = P(\overline{\theta_1 + t\omega_1}, \ldots, \overline{\theta_N + t\omega_N}), \quad \bar{\Theta} \in \mathbb{T}^N, \quad t \in \mathbb{R}. \tag{1.3}$$

Let $E_{\bar{\Theta}}$ denote the escape set for the ping-pong map with forcing function $p(t) = p_{\bar{\Theta}}(t)$. Then, for almost all $\bar{\Theta} \in \mathbb{T}^N$, the set $E_{\bar{\Theta}} \subset \mathbb{R}^2$ has Lebesgue measure zero.

Here $\mathbb{T}^N$ denotes the N-torus and $C^2(\mathbb{T}^N)$ consists of those functions $P = P(\theta_1, \ldots, \theta_N)$ which are 1-periodic in each variable θ_i and of class C^2. Furthermore, $\bar{\theta} = \theta + \mathbb{Z}$.

Remarks 1.2 (a) We emphasize again that besides being rationally independent there is no further assumption on the frequencies. In particular, it is not needed that $\omega = (\omega_1, \dots, \omega_N)$ satisfies a diophantine condition.

(b) In the periodic case $N = 1$ the theorem applies to yield the conclusion for $p(t) = P(t\omega_1)$. Even this gives something new as compared to the results in e.g. [12, Thm. 2], since only $P \in C^2$ has to be assumed. On the other hand, the conclusion that the escape set has Lebesgue measure zero is weaker than the statement that the velocity is bounded along every orbit.

(c) It is natural to ask whether Theorem 1.1 could be generalized to an almost periodic set-up, or even further, to arbitrary skew products, in the sense that $p_y(t) = P(g_t(y))$ for the forcing functions, where $g_t : Y \to Y$ is a flow on a space Y which preserves a measure μ; in the quasi-periodic case $Y = \mathbb{T}^N$, $g_t(\bar{\Theta}) = (\overline{\theta_1 + t\omega_1}, \dots, \overline{\theta_N + t\omega_N})$ and μ is the Haar measure. We have considered the almost periodic case, but don't have a definite answer at the moment. We are grateful to the referee for pointing out this possible generalization.

Before we are going to outline the proof of Theorem 1.1 we include a simple example in order to illustrate its application.

Example 1.3 *Let $z \in \mathbb{R}$ be a Liouville number, for instance $z = \sum_{j=1}^{\infty} 10^{-j!}$. For $a \in \mathbb{R}$ introduce*

$$p_a(t) = 3 + \sin(2\pi(a + t)) + \sin(2\pi z t) \tag{1.4}$$

and denote by E_a the escape set (1.1) for the ping-pong map with the forcing function $p(t) = p_a(t)$. Then, for almost all $a \in \mathbb{R}$, the escape set $E_a \subset \mathbb{R}^2$ has Lebesgue measure zero.

To see this, it suffices to define $P(\theta_1, \theta_2) = 3 + \sin(2\pi\theta_1) + \sin(2\pi\theta_2)$ and $\omega = (1, z)$ as well as

$$\tilde{p}_{a,b}(t) = 3 + \sin(2\pi(a + t)) + \sin(2\pi(b + zt))$$

for $a, b \in \mathbb{R}$, where we introduced one more parameter b as compared to (1.4). Then $p_a(t) = \tilde{p}_{a,1}(t)$, $P \in C^\infty(\mathbb{T}^2)$, and ω is rationally independent. Hence Theorem 1.1 applies and shows that for almost all $(a, b) \in \mathbb{R}^2$ the escape set $\tilde{E}_{a,b} \subset \mathbb{R}^2$ for the ping-pong map with the forcing function $p(t) = \tilde{p}_{a,b}(t)$ has Lebesgue measure zero. This yields the claim on the $E_a = \tilde{E}_{a,1}$ by Fubini's theorem, using that $\tilde{E}_{a,b}$ and $\tilde{E}_{a+\tau,b+z\tau}$ have the same Lebesgue measure for all $\tau \in \mathbb{R}$. Note that here ω does not satisfy a diophantine condition.

To prove Theorem 1.1, all the techniques related to invariant curve theorems will no longer be applicable and one has to come up with new methods. The basic strategy will be to show that most orbits are recurrent and so they are not contained in the escape set. This starting point suggests the use of Poincaré's recurrence theorem since the map f preserves the measure $v\,dtdv$. However we are dealing with a dynamical system on a space of infinite measure and thus an extension of Poincaré's theorem will be needed. Consequently an important role in the proof will be played by the following lemma, which is basically due to Dolgopyat [7, Lemma 4.1].

Lemma 1.4 *Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that the map $T : X \to X$ is one-to-one and such that the following holds:*

(a) T is measurable, in the sense that $T(B), T^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{F}$,

(b) T is measure-preserving, in the sense that $\mu(T(B)) = \mu(B)$ for $B \in \mathcal{F}$, and

(c) there is a set $A \in \mathcal{F}$ such that $\mu(A) < \infty$ with the property that almost all points from X visit A in the future.

Then for every measurable set $B \subset X$ almost all points of B visit B infinitely many times in the future (i.e., T is infinitely recurrent).

A main insight is that in many situations it can be beneficial to apply this result with $X = U$, the set of unbounded orbits of a given dynamical system. The advantage derived from this choice is that the property of future visit has to be only checked for unbounded orbits. The time/velocity coordinates (t, v) will be replaced by the angle/energy coordinates $(\bar{\Theta}, E)$, where $\bar{\Theta} = (\bar{\theta}_1, \dots, \bar{\theta}_N)$, i.e., the new phase space will be $\mathbb{T}^N \times]0, \infty[$. Our main abstract result, Theorem 3.1 below, deals with maps which are defined on $\mathbb{T}^N \times]0, \infty[$ and provides a quite general method for constructing a suitable 'section' A of finite measure as is required in (c) of Lemma 1.4. The construction of A is done in Lemma 4.1, and the set's geometry has to be carefully adapted according to a function, called W, which is supposed to satisfy an estimate of the type

$$W(f(\bar{\Theta}, E)) \leq W(\bar{\Theta}, E) + c(E),$$

where $c :]0, \infty[\to \mathbb{R}$ is a decreasing and bounded function such that $\lim_{E \to \infty} c(E) = 0$.

At first sight this function is reminiscent of a discrete Lyapunov function, but it will be more accurate to interpret it as a generalized adiabatic invariant. For large energy E the quantity W can decrease freely, while any growth has to be very slow. Thus the construction of A can be reduced to finding the function W. Since $W(\bar{\Theta}, E) = P(\bar{\Theta})^2 E$ is such an adiabatic invariant for the ping-pong map [17, 11], the argument can finally be closed. We expect that Theorem 3.1 will have several further applications in other quasi-periodic problems.

2 Measure-preserving embeddings

First we need to fix some notation. Let $\mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$ denote the standard N-torus, which is an additive quotient group. Vectors in $\mathbb{R}^N$ are denoted by $\Theta = (\theta_1, \dots, \theta_N)$ and the corresponding class in $\mathbb{T}^N$ is $\bar{\Theta} = (\bar{\theta}_1, \dots, \bar{\theta}_N)$. The invariant (Haar) measure $\mu_{\mathbb{T}^N}$ is unique after the normalization $\mu_{\mathbb{T}^N}(\mathbb{T}^N) = 1$. On the contrary, there are many invariant metrics on $\mathbb{T}^N$. We will use the quotient metric $\|\bar{\Theta}\| = \min\{|\Theta + z| : z \in \mathbb{Z}^N\}$ for $\bar{\Theta} \in \mathbb{T}^N$, where $|\cdot|$ is a fixed norm on $\mathbb{R}^N$. The distance between $\bar{\Theta}_1, \bar{\Theta}_2 \in \mathbb{T}^N$ is given by $d(\bar{\Theta}_1, \bar{\Theta}_2) = \|\bar{\Theta}_1 - \bar{\Theta}_2\|$. With the above definitions the map $\bar{\Theta} \mapsto \bar{\Theta} + \bar{\varphi}$ preserves the measure and the distance; here $\bar{\varphi}$ is a fixed element of $\mathbb{T}^N$.

Let rationally independent $\omega_1, \dots, \omega_N > 0$ be chosen. Then (for $N > 1$) the map

$$\iota : \mathbb{R} \to \mathbb{T}^N, \quad \iota(t) = (\overline{t\omega_1}, \dots, \overline{t\omega_N}),$$

is a monomorphism of topological groups and the image $\iota(\mathbb{R}) \subset \mathbb{T}^N$ is dense. In the case $N = 1$ the map i is an epimorphism of groups. The flow

$$\Phi : \mathbb{R} \times \mathbb{T}^N \to \mathbb{T}^N, \quad \Phi_t(\bar{\Theta}) = \bar{\Theta} + \iota(t),$$

preserves the measure $\mu_{\mathbb{T}^N}$ and is ergodic. For $\bar{\Theta} \in \mathbb{T}^N$ we will also need the maps

$$\iota_{\bar{\Theta}} : \mathbb{R} \to \mathbb{T}^N, \quad \iota_{\bar{\Theta}}(t) = \Phi_t(\bar{\Theta}) = \bar{\Theta} + \iota(t). \tag{2.1}$$

In what follows we will work on the phase space $\mathbb{T}^N \times]0, \infty[$ with coordinates $(\bar{\Theta}, r)$. The product map given by $(\iota_{\bar{\Theta}} \times \mathrm{id})(t, r) = (\iota_{\bar{\Theta}}(t), r)$ will appear in many places. As the measure on the phase space we will take the product measure $\mu_{\mathbb{T}^N} \otimes \lambda$, where λ denotes the Lebesgue measure on the real line.

Now we consider maps

$$f : \mathcal{D} \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[, \quad (\bar{\Theta}_1, r_1) = f(\bar{\Theta}, r),$$

which are defined on some open set $\mathcal{D}$. It will always be assumed that f is continuous and one-to-one. The proof of the following lemma is a well-known consequence of the theorem on the invariance of the domain; see [4, IV.7 and VIII.1].

Lemma 2.1 *The range $\tilde{\mathcal{D}} = f(\mathcal{D})$ is open and the inverse map*

$$f^{-1} : \tilde{\mathcal{D}} \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[, \quad (\bar{\Theta}, r) = f^{-1}(\bar{\Theta}_1, r_1),$$

is continuous as well.

In fact $f : \mathcal{D} \to \tilde{\mathcal{D}}$ is a homeomorphism and, in particular, both f and f^{-1} are Borel measurable. We will say that f is a measure-preserving embedding, if in addition

$$(\mu_{\mathbb{T}^N} \otimes \lambda)(f(\mathcal{B})) = (\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{B}) \tag{2.2}$$

holds for all Borel sets $\mathcal{B} \subset \mathcal{D}$.

Remarks 2.2 (a) It should be noted that the relation $(\mu_{\mathbb{T}^N} \otimes \lambda)(f^{-1}(\mathcal{B})) = (\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{B})$ may fail, if $\mathcal{B}$ is not contained in $\tilde{\mathcal{D}}$; here $f^{-1}(\mathcal{B})$ is understood to be the preimage of $\mathcal{B}$ under f. Already the simple example $\mathcal{D} = \mathbb{T}^N \times]0, \infty[$, $f(\bar{\Theta}, r) = (\bar{\Theta}, r + 1)$, $\mathcal{B} = \mathbb{T}^N \times]\frac{1}{2}, 2[$, $f^{-1}(\mathcal{B}) = \mathbb{T}^N \times]0, 1[$ illustrates this fact.
(b) Every measure-preserving embedding is Lebesgue measurable. This can be proved by adapting arguments from the case of homeomorphisms; see [14, Section 13].

Next we have to introduce the forward iterates of f. Some care must be taken since the iterates will be defined on smaller and smaller domains. To this end, let

$$\mathcal{D}_1 = \mathcal{D}, \quad f^1 = f, \quad \mathcal{D}_{n+1} = f^{-1}(\mathcal{D}_n), \quad f^{n+1} = f^n \circ f.$$

Then f^n is well-defined on $\mathcal{D}_n$ and $\mathcal{D}_{n+1} \subset \mathcal{D}_n \subset \mathcal{D}$ for all $n \in \mathbb{N}$, as induction shows that $\mathcal{D}_{n+1} = \{(\bar{\Theta}, r) \in \mathcal{D} : f(\bar{\Theta}, r), \ldots, f^n(\bar{\Theta}, r) \in \mathcal{D}\}$. Similarly, using (2.2), the following result is obtained.

Lemma 2.3 *Let f be a measure-preserving embedding. Then also*

$$f^n : \mathcal{D}_n \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[$$

is measure-preserving for each $n \geq 2$.

The set of initial conditions

$$\mathcal{D}_\infty = \bigcap_{n=1}^{\infty} \mathcal{D}_n \subset \mathbb{T}^N \times]0, \infty[$$

will give rise to complete forward orbits. In fact, if $(\bar{\Theta}_0, r_0) \in \mathcal{D}_\infty$, then

$$(\bar{\Theta}_n, r_n) = f^n(\bar{\Theta}_0, r_0), \quad n \in \mathbb{N}_0,$$

is defined. It should be noted that it is possible that $\mathcal{D}_\infty = \emptyset$ or even $\mathcal{D}_n = \emptyset$ for some $n \geq 2$. A complete orbit is called unbounded if $\limsup_{n\to\infty} r_n = \infty$. The corresponding set of initial data is denoted by

$$\mathcal{U} = \{(\bar{\Theta}_0, r_0) \in \mathcal{D}_\infty : \limsup_{n\to\infty} r_n = \infty\}. \tag{2.3}$$

It is Borel measurable, owing to $\mathcal{U} = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} f^{-k}(\mathbb{T}^N \times [m, \infty[)$. The set of initial data $\mathcal{E} \subset \mathcal{U}$ which lead to escaping orbits is defined as

$$\mathcal{E} = \{(\bar{\Theta}_0, r_0) \in \mathcal{D}_\infty : \lim_{n\to\infty} r_n = \infty\}.$$

Also this set is Borel measurable, since $\mathcal{E} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} f^{-k}(\mathbb{T}^N \times [m, \infty[)$.

3 Quasi-periodic maps

Assume now that $f : \mathcal{D} \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[$ is as in Section 2 and has the special form

$$f(\bar{\Theta}, r) = (\bar{\Theta} + \iota(F(\bar{\Theta}, r)), r + G(\bar{\Theta}, r)), \tag{3.1}$$

where $F, G : \mathcal{D} \to \mathbb{R}$ are continuous. For $\bar{\Theta} \in \mathbb{T}^N$ let

$$D_{\bar{\Theta}} = (\iota_{\bar{\Theta}} \times \mathrm{id})^{-1}(\mathcal{D}) \subset \mathbb{R} \times]0, \infty[\tag{3.2}$$

and consider the family of planar maps $\{f_{\bar{\Theta}}\}_{\bar{\Theta} \in \mathbb{T}^N}$ given by

$$\begin{aligned} &f_{\bar{\Theta}} : D_{\bar{\Theta}} \subset \mathbb{R} \times]0, \infty[\to \mathbb{R} \times]0, \infty[, \\ &f_{\bar{\Theta}}(t, r) = (t + F(\bar{\Theta} + \iota(t), r), r + G(\bar{\Theta} + \iota(t), r)). \end{aligned} \tag{3.3}$$

Then $D_{\bar{\Theta}}$ is open and $f_{\bar{\Theta}}$ is continuous. Moreover, the identity

$$f \circ (\iota_{\bar{\Theta}} \times \mathrm{id}) = (\iota_{\bar{\Theta}} \times \mathrm{id}) \circ f_{\bar{\Theta}} \text{ on } D_{\bar{\Theta}} \tag{3.4}$$

implies that $f_{\bar{\Theta}}$ is one-to-one. All forward iterates of $f_{\bar{\Theta}}$ are defined on the set

$$D_{\bar{\Theta},\infty} = \bigcap_{n=1}^{\infty} D_{\bar{\Theta},n} \subset \mathbb{R} \times]0, \infty[, \tag{3.5}$$

where $D_{\bar{\Theta},1} = D_{\bar{\Theta}}$ and $D_{\bar{\Theta},n+1} = f_{\bar{\Theta}}^{-1}(D_{\bar{\Theta},n})$. Defining $(t_n, r_n) = f_{\bar{\Theta}}^n(t_0, r_0)$ for $n \in \mathbb{N}_0$, the unbounded orbits are generated by the initial conditions in the set

$$U_{\bar{\Theta}} = \{(t_0, r_0) \in D_{\bar{\Theta},\infty} : \limsup_{n\to\infty} r_n = \infty\},$$

whereas the ones in

$$E_{\bar\Theta} = \{(t_0, r_0) \in D_{\bar\Theta,\infty} : \lim_{n\to\infty} r_n = \infty\} \tag{3.6}$$

will lead to escaping complete orbits. We also note the relations

$$D_{\bar\Theta,n} = (\iota_{\bar\Theta} \times \mathrm{id})^{-1}(\mathcal{D}_n),\ n = 1, 2, \ldots, \infty, \quad U_{\bar\Theta} = (\iota_{\bar\Theta} \times \mathrm{id})^{-1}(\mathcal{U}), \quad E_{\bar\Theta} = (\iota_{\bar\Theta} \times \mathrm{id})^{-1}(\mathcal{E}).$$

The following theorem is our main abstract result on escaping orbits. Its proof will be given in Section 4, whereas the application to the ping-pong problem and the proof of Theorem 1.1 is the content of Section 5.

Theorem 3.1 *Let $f : \mathcal{D} \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[$ be a measure-preserving embedding of the form (3.1) and suppose that there is a function $W = W(\bar\Theta, r)$ satisfying $W \in C^1(\mathbb{T}^N \times]0, \infty[)$,*

$$0 < \beta \le \frac{\partial W}{\partial r}(\bar\Theta, r) \le \gamma \quad \textit{for} \quad \bar\Theta \in \mathbb{T}^N, \quad r \in]0, \infty[, \tag{3.7}$$

with some constants $\beta, \gamma > 0$, and furthermore

$$W(f(\bar\Theta, r)) \le W(\bar\Theta, r) + c(r) \quad \textit{for} \quad (\bar\Theta, r) \in \mathcal{D}, \tag{3.8}$$

where $c :]0, \infty[\to \mathbb{R}$ is a decreasing and bounded function such that $\lim_{r\to\infty} c(r) = 0$. Then, for almost all $\bar\Theta \in \mathbb{T}^N$, the set $E_{\bar\Theta} \subset \mathbb{R} \times]0, \infty[$ has Lebesgue measure zero.

The following example illustrates the usefulness of Theorem 3.1.

Example 3.2 (A toy model) For $N = 1$ (where we write $\bar\Theta = \bar\theta$), we consider the generating function

$$h(\theta, \theta_1) = -g(\theta_1)(\theta_1 - \theta)^{1/2}, \quad \theta_1, \theta \in \mathbb{R}, \quad \theta_1 \ge \theta,$$

where g is 1-periodic, more precisely we assume $g \in C^1(\mathbb{T})$. Using the relation $r = \partial_1 h$ and $r_1 = -\partial_2 h$, it induces the map with lift

$$\theta_1 = \theta + \frac{g(\theta_1)^2}{4r^2}, \quad r_1 = r + \frac{g'(\theta_1)g(\theta_1)}{2r}. \tag{3.9}$$

which is well-defined by the implicit function theorem on a half-plane $\mathcal{D} = \mathbb{T} \times]R, \infty[$ for $R > 0$ sufficiently large (depending upon $\|g\|_{C^1}$). It is not difficult to prove that then $f : \mathcal{D} \to \mathbb{T} \times]0, \infty[$ is continuous, one-to-one, and measure-preserving. Furthermore, $D := D_{\bar\theta} = \mathbb{R} \times]R, \infty[$ is independent of $\bar\theta$, where we take $\omega = 1$ for $\iota_{\bar\theta}$. Also f has the required form (3.1). In fact, if $\theta_1 = \theta_1(\theta, r)$ denotes the solution to the scalar equation $\theta_1 = \theta + \frac{g(\theta_1)^2}{4r^2}$ for $r > R$, then $\theta_1(\theta + k, r) = \theta_1(k, r) + k$ for $k \in \mathbb{Z}$ and we can set

$$F(\bar\theta, r) = \frac{g(\overline{\theta_1(\theta, r)})^2}{4r^2} \quad \text{and} \quad G(\bar\theta, r) = \frac{g'(\overline{\theta_1(\theta, r)})g(\overline{\theta_1(\theta, r)})}{2r}.$$

To satisfy (3.7) and (3.8), we can use $W(\bar\theta, r) = r$, $\beta = \gamma = 1$, and $c(r) = \frac{1}{2r}\|g\|^2_{C^1}$ for $r \in]R, \infty[$. Therefore Theorem 3.1 applies to this example, and consequently for almost all

$\bar{\theta} \in \mathbb{T}$ the escape set $E_{\bar{\theta}} \subset \mathbb{R}\times]0,\infty[$ has Lebesgue measure zero. Since $N = 1$ the map $\iota_{\bar{\theta}} \times \mathrm{id} : \mathbb{R}\times]0,\infty[\to \mathbb{T}\times]0,\infty[$ is onto and $(\iota_{\bar{\theta}} \times \mathrm{id})(E_{\bar{\theta}}) = \mathcal{E}$ for each $\bar{\theta} \in \mathbb{T}$. Moreover, $\iota_{\bar{\theta}}$ is Lipschitz continuous, and so also the escape set $\mathcal{E} \subset \mathbb{T}\times]0,\infty[$ has measure zero. Note that already this is a nontrivial piece of information, since the invariant curve theorem is not applicable: Firstly the map is only continuous and the theorems in [13] require more regularity. Even if we assume that g is smooth and apply some change of variable, the results in [13] cannot be used in some cases. Namely, if we suppose that g has a zero on $\{\theta = \theta_*\}$, then all the points (θ_*, r) will be fixed under the map and there are no invariant curves with irrational rotation number.

We add one further technical result that will be needed later on in the application of Theorem 3.1 to the ping-pong map. The proof is a straightforward calculation and hence omitted.

Lemma 3.3 *A quasi-periodic map f of class C^1 of the form (3.1) is orientation and measure-preserving if and only if $(1+\partial_\omega F)(1+\partial_r G) - (\partial_r F)(\partial_\omega G) = 1$ in $\mathcal{D}$, where $\partial_\omega = \sum_{j=1}^{N} \omega_j \frac{\partial}{\partial \theta_j}$.*

It should be noted that the condition from Lemma 3.3 holds as soon as one of the planar maps $f_{\bar{\Theta}}$ is orientation-preserving and area-preserving; this observation is used below for the case of the ping-pong map.

4 Proof of Theorem 3.1

We continue to use the general setup from the previous sections and start with an auxiliary result.

Lemma 4.1 *Let $f : \mathcal{D} \subset \mathbb{T}^N\times]0,\infty[\to \mathbb{T}^N\times]0,\infty[$ be a measure-preserving embedding and suppose that there is a function $W = W(\bar{\Theta}, r)$ satisfying $W \in C^1(\mathbb{T}^N\times]0,\infty[)$, (3.7) and (3.8). Let $(\varepsilon_j)_{j\in\mathbb{N}}$ and $(W_j)_{j\in\mathbb{N}}$ be sequences of positive numbers with the properties $\sum_{j=1}^{\infty} \varepsilon_j < \infty$, $\lim_{j\to\infty} W_j = \infty$ and $\lim_{j\to\infty} \varepsilon_j^{-1} c(\frac{1}{4\gamma} W_j) = 0$. Denote*

$$\mathcal{A} = \bigcup_{j\in\mathbb{N}} \mathcal{A}_j, \quad \mathcal{A}_j = \{(\bar{\Theta}, r) \in \mathbb{T}^N\times]0,\infty[: |W(\bar{\Theta}, r) - W_j| \le \varepsilon_j\}. \tag{4.1}$$

Then $\mathcal{A}$ has finite measure and every unbounded orbit of f enters $\mathcal{A}$. More precisely, if $(\bar{\Theta}_0, r_0) \in \mathcal{U}$, where $\mathcal{U}$ is from (2.3), and if $(\bar{\Theta}_n, r_n)_{n\in\mathbb{N}}$ denotes the forward orbit under f, then there is $K \in \mathbb{N}$ so that $(\bar{\Theta}_K, r_K) \in \mathcal{A}$.

Proof: First we shall show that $\mathcal{A}$ has finite measure. By Fubini's theorem,

$$(\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{A}_j) = \int_{\mathbb{T}^N} \lambda(\mathcal{A}_{j,\bar{\Theta}})\, d\mu_{\mathbb{T}^N}(\bar{\Theta}) \tag{4.2}$$

for the sections $\mathcal{A}_{j,\bar{\Theta}} = \{r \in]0,\infty[: (\bar{\Theta}, r) \in \mathcal{A}_j\}$. We are going to prove that $\lambda(\mathcal{A}_{j,\bar{\Theta}}) \le 2\beta^{-1}\varepsilon_j$. To establish this assertion, we note that every function $w_{\bar{\Theta}} : r \mapsto W(\bar{\Theta}, r)$ provides a diffeomorphism between the intervals $]0,\infty[$ and $]W(\bar{\Theta}, 0),\infty[$. Its inverse function $w_{\bar{\Theta}}^{-1}$ is

Lipschitz continuous with constant β^{-1}, by (3.7), and moreover $\mathcal{A}_{j,\bar{\Theta}} = w_{\bar{\Theta}}^{-1}(]W_j - \varepsilon_j, W_j + \varepsilon_j[)$. Thus $\mathcal{A}_{j,\bar{\Theta}}$ is an interval of length at most $2\beta^{-1}\varepsilon_j$, and therefore

$$(\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{A}) \le \sum_{j=1}^{\infty} (\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{A}_j) \le \sum_{j=1}^{\infty} \frac{2\varepsilon_j}{\beta} < \infty.$$

To prove the recurrence property, we fix $(\bar{\Theta}_0, r_0) \in \mathcal{U}$ and we denote by $(\bar{\Theta}_n, r_n)$ the forward orbit under f. We need to start with some preliminaries. According to (3.7) there is $r_* > 0$ such that

$$\frac{\beta}{2} \le \frac{W(\bar{\Theta}, r)}{r} \le 2\gamma, \quad r \in [r_*, \infty[. \tag{4.3}$$

Furthermore, owing to the properties of the sequences (ε_j) and (W_j), we find an integer $j_0 \ge 2$ such that

$$W_{j_0} > \max\{W(\bar{\Theta}_1, r_1), \|c\|_\infty + \max_{\bar{\Theta} \in \mathbb{T}^N} W(\bar{\Theta}, r_*), 2\|c\|_\infty\} \quad \text{and} \quad c\Big(\frac{1}{4\gamma} W_{j_0}\Big) \le \varepsilon_{j_0}.$$

Then $\limsup_{n\to\infty} W(\bar{\Theta}_n, r_n) = \infty$. To verify this assertion, observe that for $r_n \ge r_1$ by (3.7):

$$W(\bar{\Theta}_n, r_n) \ge \beta(r_n - r_1) + W(\bar{\Theta}_n, r_1),$$

and $\limsup_{n\to\infty} W(\bar{\Theta}_n, r_n) = \infty$ follows from $\limsup_{n\to\infty} r_n = \infty$ and the compactness of $\mathbb{T}^N$. Due to $\limsup_{n\to\infty} W(\bar{\Theta}_n, r_n) = \infty$ and $W(\bar{\Theta}_1, r_1) < W_{j_0}$ we can select a first index $K \ge 2$ so that $W(\bar{\Theta}_K, r_K) > W_{j_0}$. In particular, this implies that $W(\bar{\Theta}_{K-1}, r_{K-1}) \le W_{j_0}$. Now from (3.8) we obtain the bound $W(\bar{\Theta}_K, r_K) \le W(\bar{\Theta}_{K-1}, r_{K-1}) + c(r_{K-1})$, and this in turn leads to

$$W(\bar{\Theta}_{K-1}, r_{K-1}) \ge W(\bar{\Theta}_K, r_K) - \|c\|_\infty > W_{j_0} - \|c\|_\infty \ge \max_{\bar{\Theta} \in \mathbb{T}^N} W(\bar{\Theta}, r_*) \ge W(\bar{\Theta}_{K-1}, r_*).$$

The monotonicity of $W(\bar{\Theta}_{K-1}, \cdot)$ thus shows that $r_{K-1} > r_*$. Combining (4.3) and the previous estimate, it thus follows that

$$r_{K-1} \ge \frac{1}{2\gamma} W(\bar{\Theta}_{K-1}, r_{K-1}) \ge \frac{1}{2\gamma} (W_{j_0} - \|c\|_\infty) \ge \frac{1}{4\gamma} W_{j_0}.$$

Finally, using $W(\bar{\Theta}_K, r_K) > W_{j_0} \ge W(\bar{\Theta}_{K-1}, r_{K-1})$ and that $c(r)$ is decreasing, we get

$$|W(\bar{\Theta}_K, r_K) - W_{j_0}| \le W(\bar{\Theta}_K, r_K) - W(\bar{\Theta}_{K-1}, r_{K-1}) \le c(r_{K-1}) \le c\Big(\frac{1}{4\gamma} W_{j_0}\Big) \le \varepsilon_{j_0},$$

which means that $(\bar{\Theta}_K, r_K) \in \mathcal{A}_{j_0}$. □

In order to prove recurrence for a measure-preserving transformation $T : X \to X$, the Poincaré recurrence theorem can be applied if X has finite measure. We will use it in the following version.

Lemma 4.2 *Let $(X, \mathcal{F}, \mu)$ be a measure space such that $\mu(X) < \infty$. Suppose that there exist a measurable set $\Gamma \subset X$ of measure zero and a map $T : X \setminus \Gamma \to X$ which is one-to-one so that the following holds:*

(a) T is measurable, in the sense that $T(B), T^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{F}$, and

(b) T is measure-preserving, in the sense that $\mu(T(B)) = \mu(B)$ for $B \in \mathcal{F}$.

Then for every measurable set $B \subset X$ almost all points of B visit B infinitely many times in the future (i.e., T is infinitely recurrent).

Note that in particular $\mu(T(X \setminus \Gamma)) = \mu(X \setminus \Gamma) = \mu(X)$ by (b) and so T is both almost onto and almost one-to-one. In fact a stronger property holds: there exists a set $X_\infty \in \mathcal{F}$ of full measure such that $X_\infty \subset X \setminus \Gamma$ and $T(X_\infty) = X_\infty$. It can be constructed recursively by setting

$$X_1 = (X \setminus \Gamma) \cap T(X \setminus \Gamma), \quad X_{n+1} = T^{-1}(X_n) \cap T(X_n) \cap X_n, \quad X_\infty = \bigcap_{n=1}^{\infty} X_n.$$

In the opposite case that $\mu(X) = \infty$, Lemma 1.4 by Dolgopyat will be important. We include a proof, following [7, Lemma 4.1], to make the paper self-contained.

Proof of Lemma 1.4: Let $\Gamma \subset X$ be measurable so that $\mu(\Gamma) = 0$ and all points from $X \setminus \Gamma$ visit A in the future. Then the first return time $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in A\}$ is well-defined for $x \in X \setminus \Gamma$; here $\mathbb{N} = \{1, 2, \ldots\}$. It induces a map $S : X \setminus \Gamma \to A$ given by $S(x) = T^{r(x)}(x)$. The restriction of this map to $A \setminus \Gamma$, i.e., $S : A \setminus \Gamma \to A$, is one-to-one and measure-preserving; see [9, Lemma 2.43] for a similar statement. Now let $B \subset X$ be measurable and define $B_j = \{y \in B \setminus \Gamma : r(y) \le j\}$ as well as

$$A_j = S(B_j) = \bigcup_{k=1}^{j} (T^k(B) \cap A) \subset A \quad \text{for} \quad j \in \mathbb{N}.$$

Since $\mu(A) < \infty$ by hypothesis, the Poincaré recurrence theorem applies to A_j. Hence there are measurable sets $\Gamma_j \subset A_j$ such that $\mu(\Gamma_j) = 0$ and every point $x \in A_j \setminus \Gamma_j$ returns, via S, to A_j infinitely often. Now consider the null set $F \subset B$ given by

$$F = B \cap \Big(\Gamma \cup \bigcup_{j \in \mathbb{N}} S^{-1}(\Gamma_j) \Big).$$

If $y \in B \setminus F$, then y will return to B infinitely many times in the future. In fact, select $j \in \mathbb{N}$ such that $r(y) \le j$, i.e., $y \in B_j$. Then $x = S(y) \in A_j \setminus \Gamma_j$. Thus, by construction, there exist infinitely many $k \in \mathbb{N}$ with the property that $k \ge j$ and $S^k(x) \in A_j$. Let us fix one of these k. Then $S^k(x) = S(z)$ for some $z \in B_j$. Writing out this relation, we arrive at

$$T^{r(z)}(z) = S(z) = S^k(x) = S^{k+1}(y) = T^{\sum_{j=0}^{k} r(S^j(y))}(y).$$

Noting that $\sum_{j=0}^{k} r(S^j(y)) \ge k + 1 \ge j \ge r(z)$, the fact that T is one-to-one leads to $T^m(y) = z \in B_j \subset B$, here $m = \sum_{j=0}^{k} r(S^j(y)) - r(z) \in \mathbb{N}$. $\square$

Now we can give the

Proof of Theorem 3.1 : Recall the definition of

$$\mathcal{U} = \{(\bar{\Theta}_0, r_0) \in \mathcal{D}_\infty : \limsup_{n\to\infty} r_n = \infty\}$$

from (2.3), where $(\bar{\Theta}_n, r_n) = f^n(\bar{\Theta}_0, r_0)$ gives the orbit of $(\bar{\Theta}_0, r_0)$ under the map f. Since the assertion is immediate in the case where $\mathcal{U} = \emptyset$, we will henceforth assume that $\mathcal{U} \neq \emptyset$.

Step 1: Almost all unbounded orbits of f are oscillatory. We are going to show that there is a set $\mathcal{Z} \subset \mathcal{U}$ of measure zero such that if $(\bar{\Theta}_0, r_0) \in \mathcal{U} \setminus \mathcal{Z}$, then

$$\liminf_{n\to\infty} r_n < \infty. \tag{4.4}$$

Hence if $(\bar{\Theta}_0, r_0) \in \mathcal{U} \setminus \mathcal{Z}$, then $\limsup_{n\to\infty} r_n = \infty$, but $\liminf_{n\to\infty} r_n < \infty$. To prove the assertion about the existence of $\mathcal{Z}$ we first note that the restriction $T = f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ is well-defined and one-to-one. Furthermore, since f is assumed to be measure-preserving, so is T. Now we are going to distinguish three cases:

(i) $(\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{U}) = 0$,

(ii) $0 < (\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{U}) < \infty$, and

(iii) $(\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{U}) = \infty$.

In case (i) we can simply take $\mathcal{Z} = \mathcal{U}$. In the second case (ii) we apply the Poincaré recurrence theorem (Lemma 4.2) to $T : \mathcal{U} \to \mathcal{U}$. It follows that for every Borel set $\mathcal{B} \subset \mathcal{U}$ almost every point $(\bar{\Theta}_0, r_0) \in \mathcal{B}$ returns to $\mathcal{B}$ infinitely often under the iteration of f. In case (iii) we arrive at exactly the same conclusion, owing to Lemma 1.4 with $X = \mathcal{U}$, $\mu = \mu_{\mathbb{T}^N} \otimes \lambda$, and the set $A = \mathcal{A} \cap \mathcal{U}$ from (4.1) in Lemma 4.1. To exploit this fact we introduce the distance function $d((\bar{\Theta}_1, r_1), (\bar{\Theta}_2, r_2)) = \|\bar{\Theta}_1 - \bar{\Theta}_2\| + |r_1 - r_2|$ on $\mathbb{T}^N \times]0, \infty[$ and cover $\mathbb{T}^N \times]0, \infty[$ by the sets $\mathfrak{B}_j = \{(\Theta, r) : |r - j| \le 1\}$ for $j \in \mathbb{N}$. Then for $\mathcal{B}_j = \mathfrak{B}_j \cap \mathcal{U} \subset \mathcal{U}$ we apply the recurrence property to find sets $\mathcal{Z}_j \subset \mathcal{B}_j$ of measure zero such that every $(\bar{\Theta}_0, r_0) \in \mathcal{B}_j \setminus \mathcal{Z}_j$ returns to $\mathcal{B}_j$ infinitely often. By definition of $\mathfrak{B}_j$ this means that $|r_n - r_0| < 2$ for infinitely many n, and hence $\liminf_{n\to\infty} r_n \le r_0 + 2$. To summarize, if $j \in \mathbb{N}$ and $(\bar{\Theta}_0, r_0) \in \mathcal{B}_j \setminus \mathcal{Z}_j$, then $\liminf_{n\to\infty} r_n < \infty$. Therefore we can take $\mathcal{Z} = \bigcup_{j\in\mathbb{N}} \mathcal{Z}_j \subset \mathcal{U}$ which has the desired properties, owing to $\bigcup_{j\in\mathbb{N}} \mathcal{B}_j = \mathcal{U}$.

Step 2: The conclusion of the theorem is valid on a subgroup of $\mathbb{T}^N$. Let $\Sigma = \{0\} \times \mathbb{T}^{N-1} \subset \mathbb{T}^N$ and denote the points in Σ by $\bar{\varphi} = (\bar{0}, \bar{\varphi}_2, \ldots, \bar{\varphi}_N)$. The normalized Haar measure on Σ is called μ_Σ. We are going to show that for μ_Σ–almost all $\bar{\varphi} \in \Sigma$ the set $E_{\bar{\varphi}} \subset \mathbb{R} \times]0, \infty[$ has Lebesgue measure zero. This will be a consequence of the previous step and Fubini's theorem. First we assert that

$$(\iota_{\bar{\Theta}} \times \mathrm{id})(E_{\bar{\Theta}}) \subset \mathcal{Z}, \quad \bar{\Theta} \in \mathbb{T}^N, \tag{4.5}$$

where $\iota_{\bar{\Theta}}$ is from (2.1) and $\mathcal{Z}$ denotes the set of measure zero from Step 1. To establish (4.5), let $(t_0, r_0) \in E_{\bar{\Theta}}$. If $(t_n, r_n) = f^n_{\bar{\Theta}}(t_0, r_0)$ denotes the orbit of (t_0, r_0) under $f_{\bar{\Theta}}$, then $\lim_{n\to\infty} r_n = \infty$ by definition. For $(\bar{\Theta}_0, r_0) = (\iota_{\bar{\Theta}} \times \mathrm{id})(t_0, r_0)$ we obtain, using (3.4) iteratively,

$$f^n(\bar{\Theta}_0, r_0) = (\iota_{\bar{\Theta}} \times \mathrm{id}) f^n_{\bar{\Theta}}(t_0, r_0) = (\iota_{\bar{\Theta}} \times \mathrm{id})(t_n, r_n).$$

Thus $(\bar{\Theta}_n, r_n) = f^n(\bar{\Theta}_0, r_0) = (\iota_{\bar{\Theta}}(t_n), r_n)$ is the corresponding orbit of $(\bar{\Theta}_0, r_0)$ under f and it has the property $\lim_{n\to\infty} r_n = \infty$. In particular, $(\bar{\Theta}_0, r_0) \in \mathcal{U}$ which proves (4.5) in the case (i). For (ii) and (iii) we have $\liminf_{n\to\infty} r_n < \infty$ for $(\bar{\Theta}_0, r_0) \in \mathcal{U} \setminus \mathcal{Z}$, see (4.4). Hence we must have $(\bar{\Theta}_0, r_0) \in \mathcal{Z}$, which completes the proof of (4.5).

Now we turn to the subgroup Σ and observe that given any interval $I \subset \mathbb{R}$ of length $\frac{1}{\omega_1}$ the map

$$\psi : I \times \Sigma \to \mathbb{T}^N, \quad \psi(t, \bar{\varphi}) = \iota_{\bar{\varphi}}(t) = \Phi_t(\bar{\varphi}) = \bar{\varphi} + \iota(t), \tag{4.6}$$

is an isomorphism of measure spaces; see [1, Prop. 6.3] for a related construction in a more general framework. This means the following: (a) up to sets of measure zero ψ is bijective; (b) ψ and ψ^{-1} are measurable; (c) we have

$$\frac{1}{\omega_1}\,\mu_{\mathbb{T}^N}(\psi(\mathcal{B})) = (\lambda \otimes \mu_\Sigma)(\mathcal{B}) \tag{4.7}$$

for all Borel sets $\mathcal{B} \subset I \times \Sigma$. In particular, if we consider $\psi \times \mathrm{id} : I \times \Sigma \times]0,\infty[\to \mathbb{T}^N \times]0,\infty[$, then

$$\frac{1}{\omega_1}\,(\mu_{\mathbb{T}^N} \otimes \lambda)(\psi \times \mathrm{id})(B) = (\lambda \otimes \mu_\Sigma \otimes \lambda)(B) \tag{4.8}$$

for all Borel sets $B \subset I \times \Sigma \times]0,\infty[$. Let

$$C_I = \{(t, \bar{\varphi}, r) \in I \times \Sigma \times]0,\infty[: (\psi(t,\bar{\varphi}), r) \in \mathcal{Z}\}.$$

Since $(\mu_{\mathbb{T}^N} \otimes \lambda)(\mathcal{Z}) = 0$ we deduce from (4.8) that also $(\lambda \otimes \mu_\Sigma \otimes \lambda)(C_I) = 0$. Next define the sections

$$C_{I,\bar{\varphi}} = \{(t, r) \in I \times]0,\infty[: (t, \bar{\varphi}, r) \in C_I\}.$$

Then it follows from Fubini's theorem that $C_{I,\bar{\varphi}}$ has Lebesgue measure zero for μ_Σ–almost all $\bar{\varphi} \in \Sigma$. Let $(I_j)_{j\in\mathbb{Z}}$ be a countable family of intervals $I_j \subset \mathbb{R}$ of length $\frac{1}{w_1}$ which cover $\mathbb{R}$. For each $j \in \mathbb{Z}$ select a set $S_j \subset \Sigma$ such that $\mu_\Sigma(S_j) = 0$ and moreover $\lambda^2(C_{I_j,\bar{\varphi}}) = 0$ for $\bar{\varphi} \in \Sigma \setminus S_j$. Hence the set $S = \bigcup_{j\in\mathbb{Z}} S_j \subset \Sigma$ has measure zero and

$$\lambda^2\Big(\bigcup_{j\in\mathbb{Z}} C_{I_j,\bar{\varphi}}\Big) = 0, \quad \bar{\varphi} \in \Sigma \setminus S.$$

But

$$\bigcup_{j\in\mathbb{Z}} C_{I_j,\bar{\varphi}} = \{(t, r) \in \mathbb{R} \times]0,\infty[: (\iota_{\bar{\varphi}}(t), r) \in \mathcal{Z}\} = (\iota_{\bar{\varphi}} \times \mathrm{id})^{-1}(\mathcal{Z})$$

by construction and therefore, recalling that $E_{\bar{\varphi}} \subset (\iota_{\bar{\varphi}} \times \mathrm{id})^{-1}(\mathcal{Z})$ from (4.5), we finally obtain $\lambda^2(E_{\bar{\varphi}}) = 0$ for $\bar{\varphi} \in \Sigma \setminus S$.

Step 3: From Σ to $\mathbb{T}^N$. First we remark that for all $\bar{\Theta} \in \mathbb{T}^N$ and $s \in \mathbb{R}$:

$$f_{\iota_{\bar{\Theta}}(s)} = \tau_{-s} \circ f_{\bar{\Theta}} \circ \tau_s \ \text{ on } \ D_{\iota_{\bar{\Theta}}(s)},$$

where $\tau_s(t, r) = (t + s, r)$. For the escape sets it directly follows that $\tau_s(E_{\iota_{\bar{\Theta}}(s)}) = E_{\bar{\Theta}}$ for $s \in \mathbb{R}$, since $f^n_{\iota_{\bar{\Theta}}(s)} = \tau_{-s} \circ f^n_{\bar{\Theta}} \circ \tau_s$ on $D_{\iota_{\bar{\Theta}}(s),\infty}$ for all $n \in \mathbb{N}$. In particular,

$$\lambda^2(E_{\iota_{\bar{\Theta}}(s)}) = \lambda^2(E_{\bar{\Theta}}), \quad \bar{\Theta} \in \mathbb{T}^N, \quad s \in \mathbb{R}. \tag{4.9}$$

Define $I = [0, \frac{1}{\omega_1}]$ and consider the map $\psi : I \times \Sigma \to \mathbb{T}^N$ given by (4.6). Take $S \subset \Sigma$ from Step 2 and put $Z_* = \psi(I \times S) \subset \mathbb{T}^N$. Then (4.7) and $\mu_\Sigma(S) = 0$ imply that $\mu_{\mathbb{T}^N}(Z_*) = 0$. Let $\bar\Theta \in \mathbb{T}^N \setminus Z_*$ be fixed and introduce $s = \frac{\Theta_1}{\omega_1} \in I$, where $\bar\Theta = (\bar\Theta_1, \dots, \bar\Theta_N)$ and $\Theta_1 \in [0,1]$. With $\bar\varphi = \iota_{\bar\Theta}(-s)$ we then have $\bar\varphi = \bar\Theta - \iota(s) = \bar\Theta - \overline{\frac{\Theta_1}{\omega_1}\omega} \in \Sigma$ and moreover $\bar\varphi \notin S$, since otherwise $\psi(s, \bar\varphi) = \bar\varphi + \iota(s) = \bar\Theta \in Z_*$. Hence, using (4.9) and Step 2, $\lambda^2(E_{\bar\Theta}) = \lambda^2(E_{\bar\varphi}) = 0$, which completes the proof of Theorem 3.1. □

5 Proof of the main result

First we are going to introduce the ping-pong map in some detail. Let p be a forcing function such that

$$p \in C^2(\mathbb{R}), \quad 0 < a \le p(t) \le b \quad (t \in \mathbb{R}), \quad \|p\|_{C_b^2} = \|p\|_\infty + \|\dot p\|_\infty + \|\ddot p\|_\infty < \infty. \tag{5.1}$$

We study the successor map, which sends a time $t_0 \in \mathbb{R}$ of impact to the left racket $x = 0$ and the corresponding velocity $v_0 > 0$ immediately after the impact to their successors t_1 and v_1 describing in the same way the subsequent impact to $x = 0$. As is derived in [11], this map is given by

$$(t_0, v_0) \mapsto (t_1, v_1), \quad t_1 = \tilde t + \frac{p(\tilde t)}{v_1}, \quad v_1 = v_0 - 2\dot p(\tilde t), \tag{5.2}$$

where $\tilde t = \tilde t(t_0, v_0)$ denotes the time at which the right racket $x = p(t)$ is hit. It is implicitly defined by the relation

$$(\tilde t - t_0)v_0 = p(\tilde t). \tag{5.3}$$

In order to have the map well-defined, it will be assumed that $\tilde t$ is the first root of equation (5.3) in $]t_0, \infty[$. From a mechanical point of view it must be ensured that there is no further impact to the moving plate before t_1, i.e.,

$$x(t) = p(\tilde t) + (t - \tilde t)\dot x(\tilde t+) < p(t), \quad t \in]\tilde t, t_1[,$$

is needed. Then, by [11, Remark 3.1], for that it is sufficient to take $v_0 > 3\|\dot p\|_\infty$. Ignoring the physical side and looking at the model more formally, the map will be well-defined as soon as

$$v_0 > \nu_* := 2 \max \{\sup_{t \in \mathbb{R}} \dot p(t), 0\}, \tag{5.4}$$

since this condition will guarantee that v_1 is positive. It also implies that $\tilde t = \tilde t(t_0, v_0)$ is a (unique) smooth function, at least of class C^1. From now on we understand that the map (5.2), (5.3) is defined on the half-plane $\mathbb{R}\times]\nu_*, \infty[$.

As a preparation for the proof of Theorem 1.1 we first consider the map $(t_0, v_0) \mapsto (t_1, v_1)$ from (5.2) and (5.3) for a fixed forcing function $p(t)$ such that (5.1) holds. Since this map is not symplectic, we need to reformulate the model in terms of time t and energy $E = \frac{1}{2}v^2$. Defining E_0 and E_1 by $v_0 = \sqrt{2E_0}$ and $v_1 = \sqrt{2E_1}$, respectively, (5.2) reads as

$$\Psi : (t_0, E_0) \mapsto (t_1, E_1),$$

$$t_1 = \tilde t + \frac{p(\tilde t)}{\sqrt{2E_1}}, \quad E_1 = E_0 - 2\sqrt{2E_0}\,\dot p(\tilde t) + 2\dot p(\tilde t)^2 = (\sqrt{E}_0 - \sqrt{2}\,\dot p(\tilde t))^2, \tag{5.5}$$

where $\tilde{t} = \tilde{t}(t_0, E_0)$ is implicit and to be determined from the relation $\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}$. Then the map Ψ from (5.5) is defined for $(t_0, E_0) \in \mathbb{R}\times]\frac{1}{2}\nu_*^2, \infty[$ and moreover it is area-preserving. The latter may be derived by a direct calculation or from the fact that it has a generating function; see [11, Section A.6]. In addition, the inverse function theorem implies that it is locally one-to-one. At the end of the paper we will present an example (cf. Remark 5.3) showing that in general Ψ will fail to be one-to-one globally.

The bound (5.6) in the following lemma expresses the crucial fact that $W = p(t)^2 E$ is an adiabatic invariant [11]. Therefore its increase can be conveniently controlled by a 'modulus of continuity' Δ.

Lemma 5.1 *There is a constant $C > 0$, depending only upon $\|p\|_{C_b^2}$ and $a, b > 0$ from (5.1), such that*

$$|p(t_1)^2 E_1 - p(t_0)^2 E_0| \le C\Delta(t_0, E_0) \quad \textit{for} \quad (t_0, E_0) \in \mathbb{R}\times]\nu_*^2/2, \infty[, \tag{5.6}$$

where $(t_1, E_1) = \Psi(t_0, E_0)$ denotes the ping-pong map for the forcing function p, and $\Delta(t_0, E_0) = E_0^{-1/2} + \sup\{|\ddot{p}(t) - \ddot{p}(s)| : t, s \in [t_0 - C, t_0 + C], |t - s| \le CE_0^{-1/2}\}$.

Proof: Since $E_1 \le (\sqrt{E_0} + \sqrt{2}\,\|\dot{p}\|_\infty)^2$, it is sufficient to prove (5.6) for $E_0 \ge 1$, as we will henceforth assume. Defining $\varphi(t) = p(t)^2$, we obtain from [11, Lemma A.5] the key relation

$$p(t_1)^2 E_1 - p(t_0)^2 E_0 = \frac{1}{2}\,\varphi(\tilde{t}) \int_0^1 (1-\lambda)\Big[\ddot{\varphi}((1-\lambda)\tilde{t} + \lambda t_0) - \ddot{\varphi}((1-\lambda)\tilde{t} + \lambda t_1)\Big]\, d\lambda\,; \tag{5.7}$$

we also note that for large values of the energy this formula is a consequence of [11, Lemma 3.7]. Furthermore, $t_1 - t_0 \le C_2\, E_0^{-1/2}$, by [11, p. 1494] or directly from (5.5) and the definition of $\tilde{t}$, for a constant $C_2 > 0$ which depends upon b and $\|\dot{p}\|_\infty$. Since $\tilde{t} \in [t_0, t_1]$, also the convex combinations $(1-\lambda)\tilde{t} + \lambda t_0$ and $(1-\lambda)\tilde{t} + \lambda t_1$ belong to this interval, so that their distance to t_0 is bounded by $C_2\, E_0^{-1/2} \le C_2$. This yields the claim, observing that $\ddot{\varphi} = 2\dot{p}^2 + 2p\ddot{p}$. □

Thus far we have presented the general setup for the ping-pong map. Now we start to investigate its properties w.r. to quasi-periodicity, in the sense that we fix $\bar{\Theta} \in \mathbb{T}^N$ and replace $p(t)$ by $p_{\bar{\Theta}}(t)$ from (1.3). Since $P \in C^2(\mathbb{T}^N)$ by hypothesis, we have the bound $\|\dot{p}_{\bar{\Theta}}\|_\infty + \|\ddot{p}_{\bar{\Theta}}\|_\infty \le C$ uniformly in $\bar{\Theta} \in \mathbb{T}^N$. Furthermore, $0 < a \le p_{\bar{\Theta}}(t) \le b$ for all $\bar{\Theta} \in \mathbb{T}^N$ and $t \in \mathbb{R}$ by (1.2), which means that the above considerations apply with uniform constants; also note that ν_* from (5.4) becomes uniform in $\bar{\Theta} \in \mathbb{T}^N$ if it is replaced by $\nu_{**} = 2\max\{\max_{\bar{\Psi}\in\mathbb{T}^N} \partial_\omega P(\bar{\Psi}), 0\}$, where $\partial_\omega = \sum_{j=1}^N \omega_j\, \frac{\partial}{\partial\theta_j}$. Furthermore, from $\ddot{p}_{\bar{\Theta}}(t) = \sum_{i,j=1}^N \omega_i\omega_j \frac{\partial^2 P}{\partial\theta_i\partial\theta_j}(\Phi_t(\bar{\Theta}))$ we observe that the functions $\Delta(t_0, E_0)$ for $p_{\bar{\Theta}}$ can be uniformly bounded by

$$\Delta(E_0) = E_0^{-1/2} + \sup\Big\{\sum_{i,j=1}^N \omega_i\omega_j\,\Big|\frac{\partial^2 P}{\partial\theta_i\partial\theta_j}(\bar{\Theta}) - \frac{\partial^2 P}{\partial\theta_i\partial\theta_j}(\bar{\Psi})\Big| : \bar{\Theta}, \bar{\Psi} \in \mathbb{T}^N, \|\bar{\Theta} - \bar{\Psi}\| \le CE_0^{-1/2}\Big\}.$$

Therefore Lemma 5.1 leads to the following result.

Lemma 5.2 *There is a constant $C > 0$, uniform in $\bar{\Theta}$, such that*

$$|p_{\bar{\Theta}}(t_1)^2 E_1 - p_{\bar{\Theta}}(t_0)^2 E_0| \le C\Delta(E_0) \quad \textit{for} \quad (t_0, E_0) \in \mathbb{R}\times]\nu_{**}^2/2, \infty[,$$

where $(t_1, E_1) = f_{\bar{\Theta}}(t_0, E_0)$ denotes the ping-pong map for the forcing function $p_{\bar{\Theta}}$.

Since in particular $P \in C^1(\mathbb{T}^N)$, the equation

$$\tau = \frac{1}{\sqrt{2E_0}}\, P(\bar{\Theta}_0 + \iota(\tau)) \tag{5.8}$$

can be solved for $\tau = \tau(\bar{\Theta}_0, E_0)$, if $\bar{\Theta}_0 \in \mathbb{T}^N$ and $E_0 > E_*$ are fixed; here it suffices to choose E_* so large that $\sqrt{2E_*} > \nu_{**}$, with ν_{**} from above. Furthermore, owing to $P \in C^2(\mathbb{T}^N)$, in particular τ will be a C^1-function of its arguments.

To match the family of ping-pong maps to the general framework, we take $\mathcal{D} = \mathbb{T}^N \times]E^*, \infty[$ for $E^* > \max\{E_*, E_{**}\}$ fixed, where E_{**} will be determined below. Consider $f : \mathcal{D} \subset \mathbb{T}^N \times]0, \infty[\to \mathbb{T}^N \times]0, \infty[$, $(\bar{\Theta}_1, E_1) = f(\bar{\Theta}_0, E_0)$, given by

$$f: \quad \bar{\Theta}_1 = \bar{\Theta}_0 + \iota(F(\bar{\Theta}_0, E_0)), \quad E_1 = E_0 + G(\bar{\Theta}_0, E_0), \tag{5.9}$$

where

$$\begin{aligned} F(\bar{\Theta}_0, E_0) &= \Big(\frac{1}{\sqrt{2E_0}} + \frac{1}{\sqrt{2E_1}}\Big) P(\bar{\Theta}_0 + \iota(\tau)), \\ G(\bar{\Theta}_0, E_0) &= -2\sqrt{2E_0}\, \partial_\omega P(\bar{\Theta}_0 + \iota(\tau)) + 2\partial_\omega P(\bar{\Theta}_0 + \iota(\tau))^2, \end{aligned}$$

for $\tau = \tau(\bar{\Theta}_0, E_0)$. Then f has the special form (3.1) and the associated family of planar maps $\{f_{\bar{\Theta}}\}_{\bar{\Theta} \in \mathbb{T}^N}$ is defined by (3.3). For a fixed $\bar{\Theta} \in \mathbb{T}^N$ the function $\tilde{t} = \tilde{t}(t_0, E_0)$ has to be determined as the solution to $\tilde{t} = t_0 + \frac{p_{\bar{\Theta}}(\tilde{t})}{\sqrt{2E_0}} = t_0 + \frac{P(\bar{\Theta} + \iota(\tilde{t}))}{\sqrt{2E_0}}$. Comparing this to (5.8), it is found that $\tilde{t}(t_0, E_0) = t_0 + \tau(\bar{\Theta} + \iota(t_0), E_0)$. As a consequence, it turns out that $f_{\bar{\Theta}}$ is just the ping-pong map (5.5) with forcing function $p_{\bar{\Theta}}(t)$. By (3.2) and the definition of $\mathcal{D} = \mathbb{T}^N \times]E^*, \infty[$, it is defined on $D_{\bar{\Theta}} = (\iota_{\bar{\Theta}} \times \mathrm{id})^{-1}(\mathcal{D}) = \mathbb{R} \times]E^*, \infty[$, independently of $\bar{\Theta} \in \mathbb{T}^N$.

Next we are going to argue that f is measure-preserving, using Lemma 3.3. This amounts to deriving the identity $(1 + \partial_\omega F)(1 + \partial_{E_0} G) - (\partial_{E_0} F)(\partial_\omega G) = 1$. However, since $f_{\bar{0}}(t_0, E_0) = (t_0 + F(\iota(t_0), E_0), E_0 + G(\iota(t_0), E_0))$ by (3.3), it turns out that the desired relation is equivalent to the condition $\det Df_{\bar{0}} = 1$ on the Jacobian determinant of $f_{\bar{0}}$. Recalling that $f_{\bar{0}}$ is even exact symplectic, it follows that f is measure-preserving.

We also need to find a C^1-function $W = W(\bar{\Theta}_0, E_0)$ such that (3.7) and (3.8) are verified. For this let

$$W(\bar{\Theta}_0, E_0) = P(\bar{\Theta}_0)^2 E_0.$$

Regarding (3.7) we have $\frac{\partial W}{\partial E_0}(\bar{\Theta}_0, E_0) = P(\bar{\Theta}_0)^2$, so we can take $\beta = a^2$ and $\gamma = b^2$ by (1.2). For (3.8) we recall the definition of f from (5.9), and we get

$$W(f(\bar{\Theta}_0, E_0)) - W(\bar{\Theta}_0, E_0) = p_{\bar{\Theta}_0}(F(\bar{\Theta}_0, E_0))^2 [E_0 + G(\bar{\Theta}_0, E_0)] - p_{\bar{\Theta}_0}(0)^2 E_0. \tag{5.10}$$

Writing $(t_1, E_1) = f_{\bar{\Theta}_0}(t_0, E_0)$ with $t_1 = t_1(t_0, E_0)$ and $E_1 = E_1(t_0, E_0)$, we have $F(\bar{\Theta}_0, E_0) = t_1(0, E_0)$ and $E_0 + G(\bar{\Theta}_0, E_0) = E_1(0, E_0)$, cf. (3.3). Therefore (5.10) and Lemma 5.2 for $(t_0, E_0) = (0, E_0)$ yield

$$W(f(\bar{\Theta}_0, E_0)) - W(\bar{\Theta}_0, E_0) = p_{\bar{\Theta}_0}(t_1(0, E_0))^2 E_1(0, E_0) - p_{\bar{\Theta}_0}(0)^2 E_0,$$

which finishes the proof of (3.8) upon taking $c(E_0) = C\Delta(E_0)$. Note that then $\lim_{E_0 \to \infty} c(E_0) = 0$, due to the compactness of $\mathbb{T}^N$.

Next we need to show that f is one-to-one on $\mathbb{T}^N\times]E_{**},\infty[$, if E_{**} is fixed sufficiently large. To establish this claim note that $E_1=\mathcal{O}(E_0)$ leads to

$$Df=\frac{\partial(\bar{\Theta}_1,E_1)}{\partial(\bar{\Theta}_0,E_0)}=\left(\begin{array}{cc} I_N+\mathcal{O}(E_0^{-1/2}) & \mathcal{O}(E_0^{-3/2}) \\ \mathcal{O}(E_0^{1/2}) & 1+\mathcal{O}(E_0^{-1/2}) \end{array}\right)$$

for the Jacobian matrix of f (which is a square matrix of size $N+1$) and I_N denoting the $N\times N$-identity matrix. Using the mean value theorem for both coordinates, from this it is straightforward to check that f is indeed one-to-one for E_{**} large enough.

To summarize the argument thus far, we are in a position to apply Theorem 3.1 to $f:\mathcal{D}\subset\mathbb{T}^N\times]0,\infty[\to\mathbb{T}^N\times]0,\infty[$. This leads to the following conclusion: If we define $D_{\bar{\Theta},\infty}$ as in (3.5), then $\lambda^2(\hat{E}_{\bar{\Theta}})=0$ for almost all $\bar{\Theta}\in\mathbb{T}^N$, where $\hat{E}_{\bar{\Theta}}=\{(t_0,E_0)\in D_{\bar{\Theta},\infty}:\lim_{n\to\infty}E_n=\infty\}$ by (3.6). Translated back to the original variables $(t,v)=(t,\sqrt{2E})$, this yields the following. Let $g_{\bar{\Theta}}$ be the ping-pong map $(t_0,v_0)\mapsto(t_1,v_1)$ from (5.2) for $p(t)=p_{\bar{\Theta}}(t)$ and let

$$\tilde{D}_{\bar{\Theta}}=\mathbb{R}\times]\sqrt{2E^*},\infty[,\quad \tilde{D}_{\bar{\Theta},1}=\tilde{D}_{\bar{\Theta}},\quad \tilde{D}_{\bar{\Theta},n+1}=g_{\bar{\Theta}}^{-1}(\tilde{D}_{\bar{\Theta},n}),\quad \tilde{D}_{\bar{\Theta},\infty}=\bigcap_{n=1}^{\infty}\tilde{D}_{\bar{\Theta},n}.$$

Then $\lambda^2(\tilde{E}_{\bar{\Theta}})=0$ for almost all $\bar{\Theta}\in\mathbb{T}^N$, where

$$\tilde{E}_{\bar{\Theta}}=\{(t_0,v_0)\in\tilde{D}_{\bar{\Theta},\infty}:\lim_{n\to\infty}v_n=\infty\}. \tag{5.11}$$

It remains to relate $\tilde{E}_{\bar{\Theta}}$ from (5.11) to $E_{\bar{\Theta}}$, as introduced in Theorem 1.1. The ping-pong map $g_{\bar{\Theta}}$ is in fact defined and in particular C^1 on $D^*=\{(t_0,v_0):v_0>2\max_{\bar{\Psi}\in\mathbb{T}^N}\partial_\omega P(\bar{\Psi})\}$, since, provided that $v_0>2\max_{\bar{\Psi}\in\mathbb{T}^N}\partial_\omega P(\bar{\Psi})$, the equation (5.3) can be uniquely solved for $\tilde{t}$. Thus we find

$$E_{\bar{\Theta}}=\{(t_0,v_0)\in D^*:(t_n,v_n)_{n\in\mathbb{N}}\text{ is well-defined and }\lim_{n\to\infty}v_n=\infty\}.$$

Hence if $(t_0,v_0)\in E_{\bar{\Theta}}$, then there is $n_0\in\mathbb{N}$ such that $(t_n,v_n)\in\tilde{E}_{\bar{\Theta}}$ for all $n\geq n_0$. In particular, we have $E_{\bar{\Theta}}\subset\bigcup_{n=1}^{\infty}g_{\bar{\Theta}}^{-n}(\tilde{E}_{\bar{\Theta}})$. Noting that $g_{\bar{\Theta}}$ is a local diffeomorphism, it follows that $\lambda^2(E_{\bar{\Theta}})=0$ whenever we have $\lambda^2(\tilde{E}_{\bar{\Theta}})=0$. This completes the proof of Theorem 1.1. $\Box$

Remark 5.3 The map Ψ from (5.5) will in general fail to be one-to-one globally. To see this consider a smooth forcing function $p(t)$ such that the derivative $\dot{p}(t)$ reaches its maximum at two instants $\tilde{t}_1$ and $\tilde{t}_2$ satisfying $\tilde{t}_1<\tilde{t}_2$ and $p(\tilde{t}_1)>p(\tilde{t}_2)$. For simplicity we will use the original coordinates (t,v). Let $v_1>0$ be the unique number so that $\tilde{t}_1+\frac{p(\tilde{t}_1)}{v_1}=\tilde{t}_2+\frac{p(\tilde{t}_2)}{v_1}$. Next define $v_0=v_1+2\dot{p}(\tilde{t}_1)=v_1+2\dot{p}(\tilde{t}_2)$ as well as $t_{0i}=\tilde{t}_i-\frac{p(\tilde{t}_i)}{v_0}$ for $i=1,2$. Since $v_0=v_1+2\sup_{t\in\mathbb{R}}\dot{p}(t)>\nu_*$ by (5.4) and $t_{01}<t_{02}$, the latter due to $p(\tilde{t}_2)<p(\tilde{t}_1)$, the points (t_{0i},v_0) are in the domain of Ψ and furthermore $\Psi(t_{0i},v_0)=(t_1,v_1)$, where $t_1=\tilde{t}_1+\frac{p(\tilde{t}_1)}{v_1}$.

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