Stable periodic solutions in the forced pendulum equation

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Dedicated to Alain Chenciner, on the occasion of his seventieth birthday

1 Introduction

Consider the differential equation

$$\ddot{x} + \beta \sin x = f(t) \tag{1}$$

where $\beta > 0$ is a real parameter and the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. In addition it is assumed that f has zero average, that is

$$\int_0^T f(t)dt = 0.$$

Given a solution $x(t)$, then $x(t) + 2\pi N$ is also a solution for each $N = \pm 1, \pm 2, \ldots$ Two solutions are called geometrically different if they are not related in this way. Mawhin and Willem proved in [12] that there exist at least two T-periodic solutions $x_1(t)$ and $x_2(t)$ that are geometrically different. More information on this result and on the history of this equation can be found in the survey paper [11]. These periodic solutions were found as critical points of the action functional

$$\mathcal{A}[x] = \int_0^T \{\frac{1}{2}\dot{x}^2(t) + \beta \cos x(t) + f(t)x(t)\}dt, \quad x(t+T) = x(t).$$

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In particular one of them, say $x_1(t)$, is a global minimizer of the action. It was already observed by Poincaré that minimizers are unstable in the Lyapunov sense and a proof for the non-degenerate case can be seen in the classical book by Carathéodory [3]. The general case is treated in [18]. In consequence the equation (1) has always one unstable T-periodic solution. The problem of the existence of stable T-periodic solutions is more delicate and the answer depend on the forcing f. If we define the resonance set

$$\mathcal{R} = \{(\frac{2\pi p}{Tq})^2 : \ q = 1, 2, 3, 4; \ p = 1, 2, \ldots\}$$

then there exists a stable T-periodic solution when $\beta \notin \mathcal{R}$ and f is small enough. This follows from a perturbation argument that uses Birkhoff Normal Form and KAM theory. To obtain non-local results it seems that these tools must be combined with ideas coming from Nonlinear Analysis. The first results in this direction were obtained in [15] by Núñez. He used the method of upper and lower solutions (in the reversed order) to locate the periodic solution. More recently Lei, Li, Yan and Zhang have obtained in [5] a non-local extension of the perturbation result mentioned above. They can find explicitly a non-negative function $P(\beta)$ that only vanishes on $\mathcal{R}$ and such that there exists a stable T-periodic solution if

$$\int_0^T |f(t)|dt < P(\beta).$$

In [6] this result was improved to allow some resonances. In the present paper I obtain a stability result without imposing restrictions on the size of the forcing. More precisely, it will be proved that if

$$0 < \beta \leq (\frac{\pi}{T})^2 \tag{2}$$

then there exists a stable T-periodic solution for almost every forcing f with zero average.

The space of continuous and T-periodic functions with zero average will be denoted by

$$X = \{f \in C(\mathbb{R}/T\mathbb{Z}) : \ \int_0^T f(t)dt = 0\}.$$

It has infinite dimensions and so the phrase *for almost every forcing f in X* needs further clarifications. To this end we interpret X as a separable Banach space endowed with the uniform norm

$$||f||_\infty = \max_{t \in \mathbb{R}} |f(t)|$$

and use the notion of prevalence introduced by Ott and Yorke in [20]. That paper explains the relationship with a related notion introduced previously by Kolmogorov. The main result of the paper can be stated now in precise terms.

Theorem 1 *Assume that the condition (2) holds. Then the set*

$$\mathcal{S} = \{f \in X : \text{ (1) } \textit{has a stable } T-\textit{periodic solution}\}$$

is prevalent in X*.*

In a paper in preparation I will construct two examples of forcings $f(t)$ such that the equation (1) has exactly two T-periodic solutions and both of them are unstable. The first example will be constructed for $\beta > (\frac{2\pi}{3T})^2$ and this implies that $\mathcal{S} \neq X$, at least when $\beta > (\frac{2\pi}{3T})^2$. For the second example it will be assumed that $\beta > (\frac{\pi}{T})^2$ but the new feature will be the existence of $\epsilon > 0$ such that the equation $\ddot{x} + \beta \sin x = f^*(t)$ has exactly two T-periodic solutions, both of them unstable, if $||f - f^*||_\infty < \epsilon$. This shows that $\mathcal{S}$ is not prevalent when $\beta > (\frac{\pi}{T})^2$. I do not know if the set $\mathcal{S}$ is also large in the sense of category when (2) holds. Perhaps some of the techniques of the paper by Markus and Meyer [9] could be applied but it does not seem an easy task.

The rest of the paper is dedicated to the proof of theorem 1. In section 2 we recall some well known properties of linear periodic equations related to the Floquet multipliers μ_1, μ_2 and the discriminant $\Delta = \mu_1 + \mu_2$. Given a T-periodic solution $\varphi(t)$ of (1), we say that it is elliptic if the multipliers of the linearized equation satisfy $\mu_1 = \overline{\mu}_2$, $\mu_1 = e^{i\theta}$, $\theta \neq k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. In section 3 we assume that (2) holds and prove that there exists an elliptic solution for almost every $f \in X$. The main tool in this section is degree theory. Elliptic solutions can be unstable in the Lyapunov sense, however this cannot occur if the number $\frac{\theta}{2\pi}$ satisfies certain arithmetic conditions. This is explained in section 4, where we invoke a result due to Russmann [22]. In particular the solution $\varphi(t)$ is stable if the number $\frac{\theta}{2\pi}$ is Diophantine. The set of Diophantine real numbers is small in the sense of category but it has full measure. The Floquet exponent θ is easily computed in terms of the discriminant Δ of the linearized equation. In section 5 we interpret the discriminant as a functional depending on the elliptic solution φ, $\Delta = \Delta[\varphi]$, and prove that all numbers in the elliptic region $-2 < \Delta < 2$ are regular values of this functional. This section is inspired by ideas of Moser in [14]. In section 6 a result of more abstract nature is presented. We describe it in a simplified form. Consider a separable Banach space $\mathbb{E}$ and a C^1 functional

$d : \mathbb{E} \to \mathbb{R}$ without singular values. Given a set $F \subset \mathbb{R}$ of full measure, we can expect that $d^{-1}(F)$ is prevalent in $\mathbb{E}$. In finite dimension this is a more or less direct consequence of Fubini's theorem. In infinite dimensions some work is needed, especially because, as shown in [7], prevalence is not preserved by C^1 diffeomorphisms. In section 7 this abstract result is applied and the proof of the theorem is completed. Roughly speaking, $\mathbb{E} = X$, $d = \Delta$ and F is in correspondence with Diophantine numbers. It follows that, for almost every $f \in X$, the elliptic solution φ is in the conditions of Russmann's theorem.

2 Remarks on Hill's equation

Given a linear equation

$$\ddot{y} + a(t)y = 0 \tag{3}$$

with $a \in C(\mathbb{R}/T\mathbb{Z})$, we consider the solutions $\phi_1(t)$ and $\phi_2(t)$ satisfying

$$\phi_1(0) = \dot{\phi}_2(0) = 1, \ \ \dot{\phi}_1(0) = \phi_2(0) = 0. \tag{4}$$

The Floquet multipliers μ_1, μ_2 are the eigenvalues of the monodromy matrix

$$M = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \dot{\phi}_1(T) & \dot{\phi}_2(T) \end{pmatrix}.$$

They satisfy $\mu_1 \cdot \mu_2 = 1$. The equation is called

- elliptic if $\mu_1 = \overline{\mu_2} \in \mathbb{S}^1 \setminus \{-1, 1\}$
- hyperbolic if $\mu_1, \mu_2 \in \mathbb{R} \setminus \{-1, 1\}$
- parabolic if $\mu_1 = \mu_2 = \pm 1$.

Here $\mathbb{S}^1 = \{z \in \mathbb{C} : \ |z| = 1\}$. When the multipliers are real they have the same sign, under some additional conditions they must be positive.

Lemma 2 *Assume that the equation (3) is hyperbolic or parabolic and*

$$a(t) \leq (\frac{\pi}{T})^2, \ \ t \in \mathbb{R}$$

with strict inequality somewhere. Then $\mu_i > 0$, $i = 1, 2$.

This lemma can be seen as a particular case of lemma 1.4 in [16]. In that paper it was assumed that there was a friction term but the proof also works without friction.

The discriminant of (3) is defined as

$$\Delta := \phi_1(T) + \dot{\phi}_2(T) = \mathrm{trace}(M) = \mu_1 + \mu_2.$$

This is a very significant number in the theory of Hill's equation (see [8]). It is easy to verify that the equation is elliptic if $|\Delta| < 2$, hyperbolic if $|\Delta| > 2$ and parabolic if $|\Delta| = 2$. The existence of non-trivial T-periodic solutions for (3) is equivalent to $\mu_1 = \mu_2 = 1$ or $\Delta = 2$. In such a case we say that the equation is degenerate. In the elliptic case the monodromy matrix M is similar to a rotation. More precisely, there exist a number $\theta \in]0, 2\pi[$, $\theta \neq \pi$, and a 2×2 matrix P with $\det P = 1$ such that $M = PR[\theta]P^{-1}$ with

$$R[\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In this case the Floquet multipliers are $\mu_1 = e^{i\theta}$, $\mu_2 = e^{-i\theta}$ and the discriminant is

$$\Delta = 2\cos\theta.$$

3 Existence of an elliptic periodic solution

Let $x(t)$ be a T-periodic solution of the pendulum equation (1). We say that $x(t)$ is elliptic if the variational equation

$$\ddot{y} + (\beta \cos x(t))y = 0 \tag{5}$$

is elliptic. Similar definitions are given for the hyperbolic and parabolic case. The main result of this section is the following.

Proposition 3 *Assume that $\beta \leq (\frac{\pi}{T})^2$. Then there exists an open and prevalent set $\mathcal{E} \subset X$ such that the equation (1) has at least one elliptic solution if $f \in \mathcal{E}$.*

The main tools for the proof of this result will be some connections between degree and stability (see [16, 17]) and a result on non-degeneracy proved in [19]. We start with a more elementary result on a priori bounds.

Lemma 4 *Let $x(t)$ be a T-periodic solution of (1). Then*

$$|\dot{x}(0)| \leq (\beta + ||f||_\infty)\frac{T}{2}.$$

Proof. The periodicity of $x(t)$ implies that the derivative vanishes at some instant, say $\dot{x}(\tau) = 0$ for some $\tau \in [-\frac{T}{2}, \frac{T}{2}]$. The mean value theorem and the equation lead to the estimate

$$|\dot{x}(0)| = |\dot{x}(0) - \dot{x}(\tau)| = |\ddot{x}(\xi)||\tau| \leq (\beta|\sin x(\xi)| + |f(\xi)|)|\tau|,$$

where ξ is some number between 0 and τ.$\blacksquare$

Next we recall the result in [19]. We say that a T-periodic solution $x(t)$ is non-degenerate if 1 is not a multiplier of the equation (5). Let $\mathcal{R}$ be the class of functions $f \in X$ such that all T-periodic solutions of (1) are non-degenerate. It is proved in [19] that $\mathcal{R}$ is open and prevalent.

Lemma 5 *Assume that $f \in \mathcal{R}$. Then there is only a finite number of T-periodic solutions of (1) satisfying $x(0) \in [0, 2\pi]$.*

Proof. The periodic differential equation (1) has an associated Poincaré map defined as

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^2, \;\; \Pi(\xi) = (x(T;\xi), \dot{x}(T;\xi))$$

where $x(t;\xi)$ is the solution of (1) satisfying $x(0) = \xi_1, \dot{x}(0) = \xi_2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. The fixed points of Π are the initial conditions of the periodic solutions with period T. Given a T-periodic solution $x(t)$ and the fixed point $\xi_* = (x(0), \dot{x}(0))$, the theorem on differentiability with respect to initial conditions implies that the derivative of Π at the fixed point is precisely the monodromy matrix of (5); that is,

$$\Pi'(\xi_*) = M.$$

In consequence, if $f \in \mathcal{R}$, all fixed points of Π will satisfy $\det(I - \Pi'(\xi_*)) \neq 0$. The implicit function theorem can be applied to deduce that all these fixed points are isolated. If we combine this fact with the bound in Lemma 4 we can conclude that the set of fixed points

$$F = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \; \Pi(\xi) = \xi, \; \xi_1 \in [0, 2\pi]\}$$

is finite (=compact and discrete).$\blacksquare$

Let us fix $f \in \mathcal{R}$ and a number $\sigma \in \mathbb{R}$ such that $x(0) \neq \sigma$ for any T-periodic solution. Let $x_1(t), \ldots, x_N(t)$ be the family of T-periodic solutions satisfying

$$x_i(0) \in]\sigma, \sigma + 2\pi[, \;\;\; i = 1, \ldots, N.$$

Each of these solutions has an associated index that can be computed according to the formula

$$\gamma_T(x_i) = \text{sign}\{(1-\mu_1)(1-\mu_2)\},$$

where μ_1 and μ_2 are the Floquet multipliers associated to the variational equation (5) with $x \equiv x_i$. An important global property of these indexes is that

$$\sum_{i=1}^{N} \gamma_T(x_i) = 0.$$

This identity can be derived in many ways but we refer to [17] for an elementary proof. Each index $\gamma_T(x_i)$ can only take the values 1 or -1 and so at least one of the solutions, say x_1, must satisfy $\gamma_T(x_1) = 1$. We define $\mathcal{E} = \mathcal{R}\setminus\{0\}$ and claim that if $f \in \mathcal{E}$ then $x_1(t)$ is elliptic. Since $\mathcal{E}$ is open and prevalent this will complete the proof of proposition 3. To prove the claim we discuss some properties of the Floquet multipliers. From the asumption on β we know that $\beta \cos x(t) \leq (\frac{\pi}{T})^2$ and so lemma 2 applies unless we are in the exceptional case $\beta \cos x(t) \equiv (\frac{\pi}{T})^2$. This situation is excluded because it would lead to $x(t) \equiv 2\pi n$, $n \in \mathbb{Z}$ and $f \equiv 0$, but the forcing $f \equiv 0$ is not in $\mathcal{E}$. Once we know that lemma 2 is applicable, we observe that no parabolic solution can exist if $f \in \mathcal{E}$. We compute the index. In the hyperbolic case $0 < \mu_1 < 1 < \mu_2$ and so $\gamma_T(x) = \text{sign}\{(1-\mu_1)(1-\mu_2)\} = -1$, while in the elliptic case $\overline{\mu}_1 = \mu_2$ and $\gamma_T(x) = \text{sign}\{|1-\mu_1|^2\} = 1$. Summing up, when $f \in \mathcal{E}$, a periodic solution is elliptic if and only if $\gamma_T(x) = 1$.

4 Lyapunov stability and linearization

Typically, the stability of a periodic solution of the forced pendulum equation is decided using some information on the higher order tems in the Taylor expansion of the equation (see [2, 13]). It is also possible to prove stability using information coming exclusively from the linearized equation (5) but then some restrictions on the Floquet multipliers must be imposed. A general result in this line is proved by Russmann in [22] and we will apply it to the pendulum equation. First we need some basic facts on the arithmetic of irrational numbers. We refer to [21] for more details. An irrational number θ is a Liouville number if for each integer $n > 1$ there exist integers p and $q > 1$ such that

$$|\theta - \frac{p}{q}| < \frac{1}{q^n}.$$

The set of Liouville numbers $L \subset \mathbb{R}$ has zero measure but it is large in the sense of category. Irrational numbers which are not of Liouville type satisfy a Diophantine condition. The set $D = \mathbb{R} \setminus (L \cup \mathbb{Q})$ has full measure but it is small in the sense of category. The function $\varphi(x) = 2\cos(2\pi x)$ is Lipschitz-continuous and so it maps null sets into null sets; in particular

$$C = \varphi(L \cup \mathbb{Q}) \tag{6}$$

has zero measure. This set can also be expressed as

$$C = \varphi((L \cup \mathbb{Q}) \cap [0, \frac{1}{2}])$$

with

$$L \cup \mathbb{Q} = \bigcap_{n>1} \bigcup_{p,q \in \mathbb{Z}, q>1}]\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}[.$$

From here it is easy to prove that C is a Borel set. Note that φ defines a homeomorphism between $[0, \frac{1}{2}]$ and $[-2, 2]$.

Proposition 6 *Assume that $x(t)$ is a periodic solution of (1) and the discriminant of the linearized equation (5) satisfies $|\Delta| < 2$ and $\Delta \notin C$. Then $x(t)$ is stable.*

Proof. The Poincaré map Π associated to the differential equation (1) is an area-preserving analytic diffeomorphism. Moreover, the stability of the solution $x(t)$ is equivalent to the stability of the corresponding fixed point of Π. The discriminant satisfies $|\Delta| < 2$ and so the solution is elliptic and M is similar to a rotation $R[\theta]$. By assumption we know that $\Delta = 2\cos\theta$ is not in C and so $\frac{\theta}{2\pi} \in D$. Theorem 1.2 in [22] can be applied and therefore the fixed point is stable under Π. Notice that the arithmetic condition imposed in [22] is of Bruno type. This is less restrictive than the diophantine condition. ■

Remark. The previous proposition is not particularly linked to the pendulum equation. The same proof is valid for a general equation

$$\ddot{x} + g(x) = f(t)$$

when g is real analytic. The results is also valid when g is C^∞ but now Russmann's theorem is not applicable. Instead one can use Herman's theorem (see [4] for more details).

5 Regular values of the discriminant

The discriminant of the linear equation (3) can be interpreted as a functional depending on the coefficient $a(t)$. More precisely, we consider the functional

$$\Delta : C(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}, \quad a \mapsto \Delta[a],$$

where $\Delta[a]$ is the discriminant of (3). This approach was taken by Moser in [14]. The following result was essentially proved in [14] but a complete proof will be presented, since it seems a good opportunity to introduce some notation.

Lemma 7 *The functional Δ is of class C^1 and it has exactly two critical values, $\Delta = \pm 2$.*

Proof. Given $\delta \in C(\mathbb{R}/T\mathbb{Z})$, the theorem on differentiability with respect to parameters implies that the directional derivative along δ is given by

$$\Delta'[a]\delta := \frac{d}{ds}\Delta[a + s\delta]_{|s=0} = z_1(T) + \dot{z}_2(T)$$

where $z_i(t)$ is the solution of

$$\ddot{z} + a(t)z + \delta(t)\phi_i(t) = 0, \quad z(0) = \dot{z}(0) = 0.$$

From the formula of variation of constants we deduce that

$$z_i(t) = \int_0^t G(t,s)\phi_i(s)\delta(s)ds$$

with

$$G(t,s) = \phi_1(t)\phi_2(s) - \phi_1(s)\phi_2(t).$$

Then

$$\Delta'[a]\delta = \int_0^T \chi(s;a)\delta(s)ds \tag{7}$$

with $\chi(s;a) = G(T,s)\phi_1(s) + \frac{\partial G}{\partial t}(T,s)\phi_2(s) =$

$$-\phi_2(T)\phi_1(s)^2 + (\phi_1(T) - \dot{\phi}_2(T))\phi_1(s)\phi_2(s) + \dot{\phi}_1(T)\phi_2(s)^2.$$

This shows that Δ is Gateaux differentiable. The function $\chi(\cdot;a)$ is continuous and so the the derivative $\Delta'[a]$ can be interpreted as an element of the dual space $C(\mathbb{R}/T\mathbb{Z})^*$. Next we prove that the differential map

$$a \in C(\mathbb{R}/T\mathbb{Z}) \mapsto \Delta'[a] \in C(\mathbb{R}/T\mathbb{Z})^*$$

is continuous. Consider a sequence $a_n \in C(\mathbb{R}/T\mathbb{Z})$ converging to a, $||a_n - a||_\infty \to 0$. By continuous dependence we know that $\chi(s; a_n) \to \chi(s; a)$, uniformly in $s \in [0, T]$. For each $\delta \in C(\mathbb{R}/T\mathbb{Z})$ with $||\delta||_\infty \leq 1$,

$$|(\Delta'[a_n] - \Delta'[a])\delta| \leq \int_0^T |\chi(s; a_n) - \chi(s; a)| ds \to 0.$$

Moreover this convergence is uniform in the unit ball of $C(\mathbb{R}/T\mathbb{Z})$, $||\delta||_\infty \leq 1$. The previous discussions imply that Δ is Fréchet differentiable and of class C^1.

To show that ± 2 are critical values, we find functions $a_\pm \in C(\mathbb{R}/T\mathbb{Z})$ such that the monodromy matrices associated to the equations $\ddot{y} + a_\pm(t) y = 0$ are $M_\pm = \pm I$. Then $\Delta[a_\pm] = \pm 2$ and $\chi(\cdot; a_\pm) \equiv 0$. In view of (7) we conclude that $\Delta'[a_\pm] = 0$. The simplest choices for a_+ and a_- are the constant functions $a_+ \equiv (\frac{2\pi}{T})^2$ and $a_- \equiv (\frac{\pi}{T})^2$.

To complete the proof we must check that all values in $\mathbb{R} \setminus \{-2, 2\}$ are regular. Assume that $a \in C(\mathbb{R}/T\mathbb{Z})$ is such that $\Delta[a] \neq \pm 2$. Going back to the formula (7) we notice that we must prove that $\chi(\cdot; a)$ is not identically zero. Assume by contradiction that $\chi(t; a) = 0$ for each $t \in \mathbb{R}$. In particular, $\chi(0; a) = 0$ and $\dot{\chi}(0; a) = 0$. This implies that $\phi_2(T) = 0$ and $\phi_1(T) - \dot{\phi}_2(T) = 0$ and so the function χ takes the simplified form $\chi(s; a) = \dot{\phi}_1(T)\phi_2(s)^2$. Now we conclude that $\dot{\phi}_1(T) = 0$ since χ vanishes everywhere. Then the monodromy matrix M takes the form αI with $\alpha = \phi_1(T) = \dot{\phi}_2(T)$. Since M has determinant one, we deduce that $M = \pm I$. This is impossible if $\Delta[a] \neq \pm 2$.■

We will be interested in similar properties for another functional: the discriminant of the linearized forced pendulum equation. Given $f \in X$ and a T-periodic solution $x(t)$ of the equation (1) we define $\mathcal{D} = \mathcal{D}[x]$ as the discriminant of the linearized equation (5). To be precise on the domain of the functional we introduce the set

$$\mathcal{M} = \{x \in C^2(\mathbb{R}/T\mathbb{Z}) : \int_0^T \sin x(t) dt = 0\}.$$

The definition of $\mathcal{M}$ is motivated by the condition $f \in X$ and the identity

$$\beta \int_0^T \sin x(t) dt = \int_0^T f(t) dt, \tag{8}$$

valid for any T-periodic solution $x(t)$ of (1). As noticed in [10], $\mathcal{M}$ is a Banach manifold of codimension one in the space $C^2(\mathbb{R}/T\mathbb{Z})$. The tangent space at $x \in \mathcal{M}$ is

$$T_x(\mathcal{M}) = \{y \in C^2(\mathbb{R}/T\mathbb{Z}) : \int_0^T (\cos x(t))y(t)dt = 0\}.$$

The rigorous definition of the functional is

$$\mathcal{D} : \mathcal{M} \to \mathbb{R}, \;\; \mathcal{D}[x] = \Delta[\beta \cos x].$$

It can be seen as the composition $\mathcal{D} = \Delta \circ N \circ i$,

$$N : C^2(\mathbb{R}/T\mathbb{Z}) \to C(\mathbb{R}/T\mathbb{Z}), \;\; i : \mathcal{M} \to C^2(\mathbb{R}/T\mathbb{Z}),$$

where N is the substitution operator $Nx(t) = \cos x(t)$ and i is the inclusion of $\mathcal{M}$ in the ambient space. From the chain rule we deduce that $\mathcal{D}$ is C^1 and, for each $y \in T_x(\mathcal{M})$,

$$\mathcal{D}'[x]y = -\Delta'[\beta \cos x](\beta \sin x)y = -\beta \int_0^T \chi_x(t)(\sin x(t))y(t)dt,$$

where $\chi_x = \chi(\cdot, a)$ with $a(t) = \beta \cos x(t)$. In contrast to the previous example, $\mathcal{D}$ can have critical values lying in the elliptic region $|\mathcal{D}| < 2$. The constant functions $x_n(t) \equiv n\pi$, $n \in \mathbb{Z}$ satisfy $\mathcal{D}'[x_n] = 0$ and $\mathcal{D}[x_{2n}] = \Delta[\beta] = 2\cos\sqrt{\beta}T$, $\mathcal{D}[x_{2n+1}] = \Delta[-\beta] = 2\cosh\sqrt{\beta}T$. Then $2\cos\sqrt{\beta}T$ and $2\cosh\sqrt{\beta}T$ are critical values. To avoid this unwanted fact we eliminate the functions x_n and consider the new domain for the functional

$$\mathcal{M}_* = \mathcal{M} \setminus \{x_n : \; n \in \mathbb{Z}\}.$$

This is an open subset of $\mathcal{M}$ and the restriction of the functional will be denoted by $\mathcal{D}_* : \mathcal{M}_* \to \mathbb{R}$.

Proposition 8 *All real numbers different from ± 2 are regular values of $\mathcal{D}_*$.*

We need a preliminary result on linear equations, whose proof is postponed to the end of the section.

Lemma 9 *Assume that $\phi_1(t)$ and $\phi_2(t)$ are the solutions of the equation (3) satisfying the initial conditions (4). Then the three functions $\phi_1(t)^2$, $\phi_1(t)\phi_2(t)$ and $\phi_2(t)^2$ are linearly independent on any open interval $I \subset \mathbb{R}$.*

Proof of Proposition 8. Assume that $x \in \mathcal{M}_*$ is a critical point of $\mathcal{D}_*$. Then $\mathcal{D}'[x] = 0$ and

$$\int_0^T \chi_x(t)(\sin x(t))y(t)dt = 0$$

for each $y \in T_x(\mathcal{M})$. This integral can be interpreted as an inner product in the Hilbert space $L^2(\mathbb{R}/T\mathbb{Z})$,

$$\langle \chi_x \sin x, y \rangle_{L^2} = 0.$$

This is equivalent to say that the function $\chi_x \sin x$ is orthogonal to the tangent space,

$$\chi_x \sin x \in T_x(\mathcal{M})^\perp.$$

From the general theory of Hilbert spaces we know that

$$T_x(\mathcal{M})^\perp = V^\perp,$$

where V is the closure of $T_x(\mathcal{M})$ in $L^2(\mathbb{R}/T\mathbb{Z})$. The space V can also be described as the hyperplane orthogonal to the line spanned by $\cos x(t)$,

$$V = L^\perp, \quad L = \{\lambda \cos x(\cdot) : \ \lambda \in \mathbb{R}\}.$$

Hence, the function x is a critical point of $\mathcal{D}$ if and only if

$$\chi_x \sin x \in V^\perp = (L^\perp)^\perp = L.$$

This means that

$$\chi_x(t) \sin x(t) = \lambda \cos x(t) \tag{9}$$

for some $\lambda \in \mathbb{R}$. In principle this identity should be understood in the L^2-sense but, since we are dealing with continuous functions, it holds everywhere. In particular it must hold at an instant τ where $\sin x(\tau) = 0$. Notice that such an instant exists because x is in the manifold $\mathcal{M}$. Evaluating (9) at $t = \tau$ we obtain $\lambda = 0$. At this moment we use that x is in $\mathcal{M}_*$ and so the function $\sin x(t)$ must be positive on some open interval I. The identity (9) now leads to $\chi_x(t) = 0$ for each $t \in I$. The function χ_x can be thought as a linear combination of ϕ_1^2, $\phi_1\phi_2$ and ϕ_2^2 and using Lemma 9 we conclude that

$$\dot{\phi}_1(T) = \phi_2(T) = \phi_1(T) - \dot{\phi}_2(T) = 0.$$

As before, these identities imply that the monodromy matrix is $\pm I$ and therefore $\mathcal{D}[x] = \pm 2$.■

Proof of Lemma 9. It is sufficient to prove that the Wronskian $W = W(t)$ of these three functions never vanishes. A simple computation shows that

$$W = \det \begin{pmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_2^2 \\ 2\phi_1\dot{\phi}_1 & \phi_2\dot{\phi}_1 + \phi_1\dot{\phi}_2 & 2\phi_2\dot{\phi}_2 \\ 2\dot{\phi}_1^2 - 2a\phi_1^2 & 2\dot{\phi}_1\dot{\phi}_2 - 2a\phi_1\phi_2 & 2\dot{\phi}_2^2 - 2a\phi_2^2 \end{pmatrix}.$$

In particular,

$$W(0) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2a(0) & 0 & 2 \end{pmatrix} = 2.$$

Assume for the moment that the function $a(t)$ is of class C^1. The three functions $z_{ij} = \phi_i \phi_j$, $1 \leq i \leq j \leq 2$, are solutions of the third order linear equation

$$\frac{d^3 z}{dt^3} + 4a(t)\frac{dz}{dt} + 2\dot{a}(t)z = 0.$$

This fact is observed in the book [8] and can be checked by direct substitution. Liouville's formula can be applied to this equation and it follows that the Wronskian is constant ($W = 2$) everywhere. The same conclusion also holds for the general case, when we can only assume that $a(t)$ is continuous. The function $a(t)$ can be approximated in the uniform sense by C^1 functions $a_n(t)$. By continuous dependence, for a fixed initial condition, the solutions of the second order equations $\ddot{y} + a_n(t)y = 0$ converge to the solutions of (3) in $C^1[0, T]$. This fact implies that the corresponding Wronskians $W_n = 2$ converge to W.$\blacksquare$

6 Regular values and prevalent sets

Let $\mathbb{E}$ be a separable Banach space. Given a vector $e \in \mathbb{E}$, the ball centered at e of radius $r > 0$ will be denoted by

$$\mathbb{B}(e, r) = \{f \in \mathbb{E} : \ ||f - e|| < r\}.$$

The closed ball will be $\overline{\mathbb{B}}(e, r)$. The σ-algebra of Borel sets on $\mathbb{E}$ will be denoted by $\mathcal{B}_{\mathbb{E}}$.

A set $N \in \mathcal{B}_{\mathbb{E}}$ is Haar-null if there exists a Borel measure μ on $\mathbb{E}$ such that $\mu(K) > 0$ for some compact subset K of $\mathbb{E}$ and

$$\mu(N + e) = 0 \quad \text{for each } e \in \mathbb{E}.$$

More generally, a subset of $\mathbb{E}$ is Haar-null if it is contained in some Borel set that is Haar-null.

To illustrate this notion we present a simple class of Haar-null sets. Assume that $\mathbb{E}$ is split as $\mathbb{E} = \mathbb{E}_1 \bigoplus \mathbb{E}_2$, where $\mathbb{E}_1$ is a closed subspace and $\mathbb{E}_2$ has finite dimension. The vector space $\mathbb{E}_2$ is locally compact and has an associated Haar measure, denoted by $\lambda_{\mathbb{E}_2}$. To normalize we can assume that the unit ball has measure one. Next we fix a set $S \subset \mathbb{E}_2$ such that $S \in \mathcal{B}_{\mathbb{E}_2}$ and $\lambda_{\mathbb{E}_2}(S) = 0$. Then we define

$$N_S = \{e_1 + e_2 : \ e_1 \in \mathbb{E}_1, e_2 \in S\} \tag{10}$$

and claim that N_S is Haar-null in $\mathbb{E}$. To prove that N_S belongs to $\mathcal{B}_{\mathbb{E}}$ we observe that it can be expressed as $N_S = \pi_2^{-1}(S)$, where $\pi_2 : \mathbb{E} \to \mathbb{E}$ is the projection onto $\mathbb{E}_2$ associated to the splitting. Now it is enough to recall that the inverse image of a Borel set under a continuous function is also a Borel set. The subspace $\mathbb{E}_2$ is closed in $\mathbb{E}$ and so the σ-algebra of Borel sets of $\mathbb{E}_2$ can be expressed as $\mathcal{B}_{\mathbb{E}_2} = \{B \cap \mathbb{E}_2 : \ B \in \mathcal{B}_{\mathbb{E}}\}$. This fact allows us to induce a measure on $\mathbb{E}$ from the measure on $\mathbb{E}_2$,

$$\mu(B) := \lambda_{\mathbb{E}_2}(B \cap \mathbb{E}_2) \ \text{ if } B \in \mathcal{B}_{\mathbb{E}}.$$

It is easy to prove that μ is indeed a Borel measure. Moreover, the compact set $K = \overline{\mathbb{B}}(0,1) \cap \mathbb{E}_2$ satisfies $\mu(K) = 1$. Finally, if we denote $e_2 = \pi_2(e)$,

$$\mu(N_S + e) = \lambda_{\mathbb{E}_2}(S + e_2) = \lambda_{\mathbb{E}_2}(S) = 0.$$

Going back to the general theory, we recall that the countable union of Haar-null sets is also a Haar-null set. Also we recall that Haar-null sets were called shy sets in [20]. The complement of a shy set is a prevalent set. Next we present the main result of the section.

Proposition 10 *Let G be an open and prevalent subset of $\mathbb{E}$. Assume that there exist a family $\{U_\alpha\}_{\alpha \in A}$ of open subsets of $\mathbb{E}$ and functions $d_\alpha \in C^1(U_\alpha, \mathbb{R})$ such that*

$$G \subset \bigcup_{\alpha \in A} U_\alpha$$

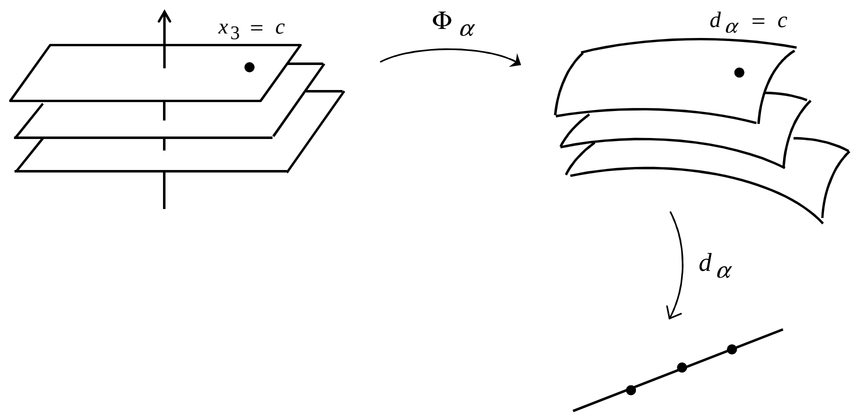
$$d'_\alpha(e) \neq 0 \ \text{ for each } e \in U_\alpha, \ \alpha \in A.$$

Finally assume that C is a Borel subset of $\mathbb{R}$ with zero measure. Then the set

$$\tilde{G} = \bigcup_{\alpha \in A} d_\alpha^{-1}(\mathbb{R} \setminus C)$$

is prevalent in $\mathbb{E}$.

Before the proof we present an informal discussion for the case of a single index $A = \{\alpha\}$ and $\mathbb{E} = \mathbb{R}^3$. In this case a Haar-null set is just a set of zero measure. Since the function d_α has no critical values, the open set U_α is foliated by the smooth surfaces $d_\alpha(x_1, x_2, x_3) = c$, where c is constant.



At least locally this family of surfaces has an explicit equation, say $x_3 = \phi_\alpha(x_1, x_2, c)$. The diffeomorphism $\Phi_\alpha : (x_1, x_2, c) \mapsto (x_1, x_2, \phi_\alpha(x_1, x_2, c))$ is defined locally and maps a subset Z contained in $\mathbb{R}^2 \times C$ into $d_\alpha^{-1}(C)$. It can be proved that $d_\alpha^{-1}(C)$ can be covered by the image of a countable family $\{Z_\lambda\}_{\lambda\in\Lambda}$ of sets this type. The set $\mathbb{R}^2 \times C$ is Haar-null in $\mathbb{R}^3$. This is a consequence of Fubini theorem but it can also be deduced from previous discussions. Indeed $\mathbb{R}^2 \times C$ can be interpreted as the set N_C corresponding to the splitting of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ along the coordinate axes. The function ϕ_α is C^1 and so Φ_α preserves sets of zero measure. This implies that $\Phi_\alpha(Z_\lambda)$ has zero measure. Then $d_\alpha^{-1}(C)$ can be covered by a countable union of zero measure sets.

As will be seen later, the proof in the general case follows along these lines excepting at one point. In contrast to the case of $\mathbb{R}^n$, in infinite dimensions Haar-null sets are not preserved by diffeomorphisms. An example of this phenomenon is presented in [7]. To overcome this difficulty we will show that the image under a diffeomorphism of a set of the type N_S is a Haar-null set.

The proof of Proposition 10 will be obtained after a sequence of lemmas. We start with a characterization of Haar-null sets.

Lemma 11 *Assume that $N \in \mathcal{B}_{\mathbb{E}}$. The following statements are equivalent:*
(i) N is Haar-null
(ii) For each $n \in N$ there exists a neighborhood V_n in $\mathbb{E}$ such that $N \cap V_n$ is Haar-null.

Proof. This result is mentioned in [20]. The implication $(i) \Rightarrow (ii)$ is obvious. To prove $(ii) \Rightarrow (i)$ we notice that it is not restrictive to assume that V_n is open. Since $\mathbb{E}$ is a separable metric space, it has the Lindelöf property (see [23]) and so we can extract a countable sub-covering of $\{V_n\}_{n\in N}$. That is, $\{V_\lambda\}_{\lambda\in\Lambda}$ with $\Lambda \subset \mathbb{N}$ and $\bigcup_\lambda V_\lambda \supset N$. Then N is the countable union of the sets $N \cap V_\lambda$, that are Haar-null sets.■

The next result will be very useful to simplify proofs. Instead of using a single measure as in the definition of Haar-null set, we play with many measures. The advantage is that the condition $\mu(N+e) = 0$ is only needed when $||e||$ is small.

Lemma 12 *Assume that $N \in \mathcal{B}_\mathbb{E}$ and for each $n \in N$ there exists a number $\delta_n > 0$, a Borel measure μ_n on $\mathbb{E}$ and a compact set $K_n \subset \mathbb{E}$ such that $\mu_n(K_n \cap \mathbb{B}(n, r)) > 0$ for each $r > 0$ and*

$$\mu_n(N + e) = 0 \text{ if } ||e|| < \delta_n.$$

Then N is Haar-null.

Proof. In view of Lemma 11 it is sufficient to prove that $N \cap \mathbb{B}(n, \frac{1}{3}\delta_n)$ is Haar-null for each $n \in N$. To this end we define a new measure

$$\hat{\mu}_n(B) := \mu_n(B \cap \mathbb{B}(n, \frac{1}{3}\delta_n)).$$

It is easy to check that $\hat{\mu}_n$ is a Borel measure and, by assumption, $\hat{\mu}_n(K_n) > 0$. It remains to prove that

$$\hat{\mu}_n[(N \cap \mathbb{B}(n, \frac{1}{3}\delta_n)) + e] = 0$$

for each $e \in \mathbb{E}$. This is a consequence of the assumptions if $||e|| < \delta_n$. Assume now that $||e|| \geq \delta_n$, then the balls $\mathbb{B}(n, \frac{1}{3}\delta_n)$ and $\mathbb{B}(n, \frac{1}{3}\delta_n) + e$ are disjoint and so

$$\hat{\mu}_n[(N \cap \mathbb{B}(n, \frac{1}{3}\delta_n)) + e] = \mu_n(\emptyset) = 0.■$$

In the next result we assume that there is a splitting $\mathbb{E} = \mathbb{E}_1 \bigoplus \mathbb{E}_2$ as before and N_S is defined by (10).

Lemma 13 *Let $\varphi : U \to V$ be a C^1-diffeomorphism between two open sets contained in $\mathbb{E}$. In addition assume that φ^{-1} is Lipschitz continuous. Then $N = \varphi(N_S \cap U)$ is Haar-null.*

Proof. Given $n \in N$ we find $m \in N_S \cap U$ with $n = \varphi(m)$. We split m as $m = m_1 + m_2$ with $m_1 \in \mathbb{E}_1$, $m_2 \in \mathbb{E}_2$. It is possible to find sets W_1, W_2 and W satisfying that W_i is open in $\mathbb{E}_i$, $m_i \in W_i$, $i = 1, 2$, W is open in $\mathbb{E}$, $0 \in W$, and such that the map $\psi : (W_2 \times W) \times W_1 \subset (\mathbb{E}_2 \times \mathbb{E}) \times \mathbb{E}_1 \to \mathbb{E}_1$,

$$\psi(f_2, e; f_1) = \pi_1 \varphi^{-1}(\varphi(f_1 + f_2) + e) - m_1$$

is well defined. Here π_1 is the projection associated to the splitting. The map ψ is of class C^1 and satisfies

$$\psi(m_2, 0; m_1) = 0, \quad \partial_3 \psi(m_2, 0; m_1) = id_{\mathbb{E}_1}.$$

The last identity concerning the partial derivative along the subspace $\mathbb{E}_1$ is a consequence of the chain rule because

$$\partial_3 \psi(f_2, e; f_1) = \pi_1 \circ (\varphi^{-1})'(\varphi(f_1 + f_2) + e) \circ \varphi'(f_1 + f_2) \circ i_1,$$

where $i_1 : \mathbb{E}_1 \to \mathbb{E}$ is the inclusion. It follows from the implicit function theorem that there exist sets W_1^*, W_2^* and W^* and a function $F : W_2^* \times W^* \to W_1^*$ such that W_i^* is open in $\mathbb{E}_i$, $m_i \in W_i^* \subset W_i$, $i = 1, 2$, W^* is open in $\mathbb{E}$, $0 \in W^* \subset W$, F is of class C^1, $F(m_2, 0) = m_1$ and the equation

$$\psi(f_2, e; f_1) = 0, \quad (f_2, e; f_1) \in W_2^* \times W^* \times W_1^*$$

is equivalent to $f_1 = F(f_2, e)$.

Given $e \in W^*$ we observe that the function

$$M_e : \; f_2 \in W_2^* \mapsto \varphi^{-1}(\varphi(F(f_2, e) + f_2) + e)$$

takes values into the affine space $\mathbb{E}_2 + m_1 = \mathbb{E}_2 + m$. Then M_e can be interpreted as a C^1 map between two spaces of the same finite dimension. In consequence the set $Z_e = M_e(S \cap W_2^*)$ satisfies $\lambda_{\mathbb{E}_2+m}(Z_e) = 0$. Here we are using a new but obvious notation. The Haar measure on $\mathbb{E}_2$ is transported to $\mathbb{E}_2 + m$ via the formula $\lambda_{\mathbb{E}_2+m}(B) := \lambda_{\mathbb{E}_2}(B - m)$. We intend to apply Lemma 12 to the Borel set $N = \varphi(N_S \cap U)$. For each $n = \varphi(m)$ we take a ball β centered at m. The radius of this ball will be chosen later. Consider the Borel measure in $\mathbb{E}$,

$$\mu_n(B) := \lambda_{\mathbb{E}_2+m}(\varphi^{-1}(B \cap V) \cap \beta \cap (\mathbb{E}_2 + m))$$

for each $B \in \mathcal{B}_{\mathbb{E}}$. Define

$$K_n = \varphi(\overline{\mathbb{B}}(m, r_*) \cap (\mathbb{E}_2 + m))$$

where $r_* > 0$ is so small that $\overline{\mathbb{B}}(m, r_*) \subset U$. This is a compact set contained in V and

$$\varphi^{-1}(K_n \cap \mathbb{B}(n,r)) = \overline{\mathbb{B}}(m, r_*) \cap (\mathbb{E}_2 + m) \cap \varphi^{-1}(\mathbb{B}(n,r) \cap V)$$

contains a ball (relative to $\mathbb{E}_2 + m$) centered at m. Hence, $\mu_n(K_n \cap \mathbb{B}(n,r)) > 0$ for each $r > 0$. It remains to prove that $\mu_n(N+e) = 0$ if $||e||$ is small. Assume that x is a point in $\Sigma = \varphi^{-1}((\varphi(N_S \cap U) + e) \cap V) \cap \beta \cap (\mathbb{E} + m)$. Then $x = \varphi^{-1}(\varphi(f_1 + f_2) + e)$ with $f = f_1 + f_2 \in N_S \cap U$ and

$$||f - m|| \leq ||\varphi^{-1}(\varphi(f)) - \varphi^{-1}(\varphi(f) + e)|| + ||\varphi^{-1}(\varphi(f) + e) - m||$$

$$\leq [\varphi^{-1}]_{Lip} ||e|| + ||x - m|| \leq [\varphi^{-1}]_{Lip} ||e|| + radius\ of\ \beta.$$

Here $[\varphi^{-1}]_{Lip}$ is the best Lipschitz constant of φ^{-1} and we are using that x is in the ball β. Adjusting the radius and choosing $||e||$ small enough we conclude that $f_1 \in W_1^*$, $f_2 \in W_2^*$ and $e \in W^*$. Since x belongs to $\mathbb{E}_2 + m$, $m_1 = \pi_1(x) = \pi_1 \varphi^{-1}(\varphi(f_1 + f_2) + e)$, implying that $\psi(f_2, e; f_1) = 0$. This is equivalent to $f_1 = F(f_2, e)$ and therefore $x = M_e(f_2)$ with $f_2 \in S \cap W_2^*$. We conclude that the set Σ is contained in $M_e(S \cap W_2^*) = Z_e$ and so

$$\mu_n(N + e) = \lambda_{\mathbb{E}_2 + m}(\Sigma) \leq \lambda_{\mathbb{E}_2 + m}(Z_e) = 0. \blacksquare$$

Next we show that the Lipschitz condition on φ^{-1} is not essential.

Proposition 14 *Let $\varphi : U \to V$ be a C^1-diffeomorphism between two open subsets of $\mathbb{E}$. Then $N = \varphi(N_S \cap U)$ is a Haar null set in $\mathbb{E}$.*

Proof. For each $n \in N$, $n = \varphi(m)$, we find an open neighborhood U_n of m and a number M_n such that $U_n \subset U$ and

$$||(\varphi^{-1})'(x)||_{\mathcal{L}(\mathbb{E},\mathbb{E})} \leq M_n$$

for each $x \in V_n = \varphi(U_n)$. Here $|| \cdot ||_{\mathcal{L}(\mathbb{E},\mathbb{E})}$ is the norm in the space of bounded endomorphisms of $\mathbb{E}$. The restriction of φ, $\varphi_n : U_n \to V_n$ is a C^1-diffeomorphism and φ_n^{-1} is Lipschitz-continuous with $[\varphi_n^{-1}]_{Lip} \leq M_n$. Then Lemma 13 can be applied and the set $\varphi(N_S \cap U_n)$ is Haar-null. Since

$$\varphi(N_S \cap U_n) = \varphi(N_S \cap U) \cap V_n = N \cap V_n,$$

we can apply Lemma 11 and deduce that N is a Haar-null set. $\blacksquare$

We are ready to prove a simplified version of Proposition 10.

Lemma 15 *Assume that U is an open subset of $\mathbb{E}$ and there exists a function $d \in C^1(U, \mathbb{R})$ with*

$$d'(e) \neq 0 \;\text{ for each } e \in U.$$

In addition C is a Borel subset of $\mathbb{R}$ with zero measure. Then $Z = d^{-1}(C)$ is Haar-null in $\mathbb{E}$.

Proof. First of all we notice that Z is the inverse image under d of a Borel set. This implies that $Z \in \mathcal{B}_{\mathbb{E}}$. Next we apply the local submersion theorem as stated in [1]. Given $e_* \in U$, we must check that the functional $d'(e_*) : \mathbb{E} \to \mathbb{R}$ is surjective and has split kernel $\mathbb{E}_1$ with closed complement $\mathbb{E}_2$, $\mathbb{E} = \mathbb{E}_1 \bigoplus \mathbb{E}_2$. This is immediate because $d'(e_*)$ is a non-zero element of the dual space $\mathbb{E}^*$ and so $\mathbb{E}_1 = \text{Ker } d'(e_*)$ is a closed hyperplane. For the subspace $\mathbb{E}_2$ we can take any line $\mathbb{E}_2 = \{t\bar{e} : \ t \in \mathbb{R}\}$ with $d'(e_*)\bar{e} \neq 0$. The submersion theorem implies the existence of open sets V and W in $\mathbb{E}$ and a C^1-diffeomorphism $\varphi : V \to W$ such that

$$V = \{e_1 + t\bar{e} : \ e_1 \in \mathbb{E}_1, \ ||e_1|| < r_1, \ |t| < r_2\} \;\text{ for appropriate } \mathrm{r}_1 \text{ and } \mathrm{r}_2,$$

$e_* \in W \subset U$, $\varphi(0) = e_*$ and $d(\varphi(e_1 + t\bar{e})) = t$. Then $Z \cap W$ can be described as the image under φ of the set

$$\{e_1 + t\bar{e} : \ e_1 \in \mathbb{E}_1, \ ||e_1|| < r_1, \ |t| < r_2, \ t \in C\}.$$

That is, $Z \cap W = \varphi(N_C \cap V)$, and Proposition 14 implies that $Z \cap W$ is Haar-null. Once again we invoke Lemma 11 to finish the proof.$\blacksquare$

Proof of Proposition 10. The Lindelöf property allows us to extract a countable sub-covering $\{U_\lambda\}_{\lambda \in \Lambda}$, $\Lambda \subset \mathbb{N}$, $\bigcup_\lambda U_\lambda \supset G$. From Lemma 15 we deduce that each set $d_\lambda^{-1}(C)$ is Haar-null. In consequence also $\bigcup_{\lambda \in \Lambda} d_\lambda^{-1}(C)$ is Haar-null. From

$$\mathbb{E} \setminus \tilde{G} \subset (\mathbb{E} \setminus G) \cup \bigcup_{\lambda \in \Lambda} d_\lambda^{-1}(C)$$

we conclude that $\tilde{G}$ is prevalent.$\blacksquare$

7 Proof of Theorem 1

Let us consider the operator

$$\mathcal{F} : \mathcal{M} \to X, \;\; \mathcal{F}[x] = \ddot{x} + \beta \sin x,$$

where $\mathcal{M}$ is the manifold introduced in Section 5. In view of the identity (8), every T-periodic solution of (1) belongs to $\mathcal{M}$ if $f \in X$. Hence the periodic problem for (1) with $f \in X$ is equivalent to the equation

$$\mathcal{F}[x] = f.$$

The map $\mathcal{F}$ is smooth with derivative

$$\mathcal{F}'[x] : T_x(\mathcal{M}) \to X, \;\; \mathcal{F}'[x]y = \ddot{y} + \beta(\cos x)y.$$

The next result was already obtained in [10] in a more general context. We present a simple proof.

Lemma 16 *Given $x \in \mathcal{M}$, the derivative $\mathcal{F}'[x]$ is an isomorphism if and only if $x(t)$ is a non-degenerate T-periodic solution of (1).*

Proof. Assume first that $\mathcal{F}'[x]$ is an isomorphism and let $y(t)$ be a T-periodic solution of the linearized equation

$$\ddot{y} + \beta(\cos x(t))y = 0.$$

Integrating over a period we obtain $\beta \int_0^T (\cos x(t))y(t)dt = 0$ and so $y \in T_x(\mathcal{M})$. From the linearized equation, $\mathcal{F}'[x]y = 0$, and this implies $y = 0$ because Ker $\mathcal{F}'[x] = \{0\}$.

Assume now that $x(t)$ is non-degenerate. Then it is obvious that the kernel of $\mathcal{F}'[x]$ is trivial. To prove that $\mathcal{F}'[x]$ is onto let p be a given function in X. By Fredholm alternative we know that the non-homogeneous equation

$$\ddot{y} + \beta(\cos x(t))y = p(t)$$

has a unique T-periodic solution. This solution satisfies

$$\beta \int_0^T (\cos x(t))y(t)dt = \int_0^T p(t)dt = 0.$$

Then $y \in T_x(\mathcal{M})$ and $\mathcal{F}'[x]y = p$. ∎

The set $\mathcal{E}$ defined by Proposition 3 is open and prevalent in X. We recall that it was defined as $\mathcal{R} \setminus \{0\}$ where $\mathcal{R}$ was the set constructed in [19]. Given $f \in \mathcal{E}$, we can select $x \in \mathcal{M}$ such that $\mathcal{F}[x] = f$ and the linearized equation at $x(t)$ is elliptic. Indeed x belongs to the sub-manifold $\mathcal{M}_* = \mathcal{M} \setminus \{x_n : \; n \in \mathbb{Z}\}$ because $f \neq 0$ and $\mathcal{F}[x_n] = 0$. In the notations of Section 5

$$|\mathcal{D}[x]| < 2.$$

In particular the solution $x(t)$ is non-degenerate and so $\mathcal{F}'[x]$ is an isomorphism. We apply the inverse function theorem and find open sets $V_f \subset \mathcal{M}_*$, $U_f \subset \mathcal{E}$ with $x \in V_f$, $f \in U_f$, and such that the restriction $\mathcal{F} : V_f \to U_f$ is a diffeomorphism. After restricting the size of U_f we can assume that the functional

$$d_f : U_f \to \mathbb{R}, \ \ d_f(g) = \mathcal{D}[\mathcal{F}^{-1}(g)]$$

is smooth and takes values in the interval $]-2, 2[$. From Proposition 8 we deduce that $d_f'(g) \neq 0$ for each $g \in U_f$. We are ready to apply Proposition 10. Define $\mathbb{E} = X$, $G = \mathcal{E}$ and consider the open covering $\{U_f\}_{f\in\mathcal{E}}$. Finally, let C be the set defined in Section 4 by the formula (6). Then

$$\tilde{G} = \bigcup_{f\in\mathcal{E}} d_f^{-1}(\mathbb{R} \setminus C)$$

is prevalent in X.

We claim that the equation

$$\ddot{x} + \beta \sin x = g(t)$$

has a stable T-periodic solution for each $g \in \tilde{G}$. According to the definition of $\tilde{G}$ there exists a T-periodic solution $x(t)$ such that the discriminant of the linearized equation is given by $\Delta = d_f(g)$, for some $f \in \mathcal{E}$ with $g \in U_f$. Moreover, $|\Delta| < 2$ and $\Delta \notin C$. Proposition 6 is applicable and we conclude that $\tilde{G} \subset \mathcal{S}$.

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