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A forced pendulum equation without stable periodic solutions of a fixed period

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Abstract. The pendulum equation is forced by a continuous and T-periodic function with zero average. In these conditions it is well known that there exist at least two T-periodic solutions. We construct some examples where there are exactly two T-periodic solutions and both are unstable.

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1. Introduction

Consider the differential equation

$$\ddot{x} + \beta \sin x = f(t) \tag{1}$$

where $\beta > 0$ is a real parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function satisfying

$$\int_0^T f(t)dt = 0. \tag{2}$$

Many authors have discussed different properties of this equation and an extensive list of references can be found in the survey paper [3]. In a recent paper [5] I discussed the existence of T-periodic solutions that are stable in the Lyapunov sense. It was proved in [5] that if

$$\beta \leq (\frac{\pi}{T})^2$$

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then there exists a stable T-periodic solution for a large class of forcings satisfying (2). Two examples of non-existence of stable periodic solutions will be constructed in the present paper. They show that, in some aspects, the result obtained in [5] is sharp.

Theorem 1.1. *Assume that*

$$\beta > (\frac{2\pi}{3T})^2.$$

Then there exists a real analytic function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(t+T) = f(t), \quad f(-t) = -f(t), \quad t \in \mathbb{R} \tag{3}$$

and such that the equation (1) has no stable T-periodic solution.

The function $f(t)$ in the above result is odd and, in particular, the condition (2) holds. This shows that the conclusion of the theorem in [5] cannot hold for every forcing. Therefore, the phrase *for almost every forcing* employed in [5] is essential, at least for parameters lying in the interval

$$(\frac{2\pi}{3T})^2 < \beta \leq (\frac{\pi}{T})^2.$$

Let us recall the precise meaning in our context of a property that holds almost everywhere. The space X of continuous and T-periodic functions with mean value zero has infinite dimensions. It becomes a Banach space with the norm

$$||f|| = \max_{t \in \mathbb{R}} |f(t)|.$$

We say that a property holds for almost every forcing if it holds for every $f \in \mathcal{P}$, where $\mathcal{P}$ is a prevalent subset of X. The definition of prevalence is analyzed in [6]. We just recall this definition: a subset $\mathcal{P}$ of X is prevalent if there exist a Borel measure μ on X, a Borel subset $\mathcal{N} \subset X$ and a compact set $K \subset X$ such that $X \setminus \mathcal{P} \subset \mathcal{N}$, $\mu(K) > 0$ and $\mu(\mathcal{N} + g) = 0$ for every $g \in X$. Prevalent sets are dense in X and so the next result shows that the number $(\frac{\pi}{T})^2$ is optimal for the theorem in [5].

Theorem 1.2. *Assume that*

$$\beta > (\frac{\pi}{T})^2.$$

Then there exist a number $\epsilon > 0$ and a real analytic function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions in (3) and such that the equation

$$\ddot{x} + \beta \sin x = g(t)$$

has no stable T-periodic solution if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function satisfying

$$\int_0^T g(t)dt = 0, \quad ||f - g|| < \epsilon.$$

The rest of the paper is dedicated to the proof of these results. The proof of Theorem 1.1 is based on the phenomenon of strong resonance at the third root of unity, as described in Section 2. This explains the restriction on β imposed in Theorem 1.1. Perhaps the analysis of resonances at higher roots of unity could help to improve this result. I do not know if the conclusion of Theorem 1.1 is valid for arbitrary $\beta > 0$. In my experience with the pendulum equation, I have found that an useful strategy to produce examples with special properties is to assume first that (1) is an autonomous equation with impulses, that is $f = \sum_j c_j D\delta_j$, where $D\delta_j$ is the derivative of a Dirac measure. Once a preliminary example of this type has been constructed, one can try to change f by an authentic function via perturbation arguments. In this process weak topologies play an important role. This explains why two sections of the paper, 3 and 6, are devoted to study the effect of the weak* topology on equations of pendulum type. The examples proving the two theorems are presented in sections 4 and 5. The period $T > 0$ is arbitrary in both theorems but, after a re-scaling of time, it is not restrictive to assume that T has a fixed value. To simplify computations we will choose $T = \frac{2\pi}{3}$ in the first theorem and $T = 2\pi$ in the second one.

With respect to the notations employed in the paper, I think they are more or less standard. In particular $L^p(\mathbb{R}/T\mathbb{Z})$ and $C^p(\mathbb{R}/T\mathbb{Z})$ are spaces of T-periodic functions.

2. Resonance at the third root of unity

Let $\mathcal{R}_3$ be the class of 2×2 matrices $L \in \mathbb{R}^{2\times 2}$ satisfying

$$L^3 = I, \quad L \neq I.$$

These matrices are conjugate to the rotation of angle 120° and they play a singular role in stability theory. Assume that L is in $\mathcal{R}_3$ and consider the difference equations

$$x_{n+1} = Lx_n \tag{4}$$

and

$$x_{n+1} = Lx_n + N(x_n), \tag{5}$$

where N is a nonlinear map satisfying $N(0) = 0$, $N'(0) = 0$. For the linear equation (4) all non-trivial solutions are 3-cycles and the origin $x = 0$ is stable. However, for a typical nonlinear perturbation (5) the origin is unstable. The purpose of this section is to describe this phenomenon in precise terms. The exposition will be inspired by the book [7].

Let $\mathcal{U}$ be any open subset of the plane containing the origin and assume that

$$F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$$

is a C^2 map with $F(0) = 0$. The origin $x = 0$ is stable with respect to F if given any neighborhood $\mathcal{V}$ there exists another neighborhood $\mathcal{W}$ such that the iterates $F^n(\mathcal{W})$ are well defined for the future and

$$F^n(\mathcal{W}) \subset \mathcal{V} \text{ if } n \geq 0.$$

The fixed point $x = 0$ is unstable when the previous condition does not hold. The notions of stability and instability are topological, meaning that they are invariant under conjugacy by homeomorphisms.

The map F is C^2 and so it can be expanded as

$$F(x) = Lx + Q(x) + o(|x|^2) \text{ as } |x| \to 0$$

where $L \in \mathbb{R}^{2\times 2}$ and Q is a quadratic polynomial of the type

$$Q(x_1, x_2) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2, \quad \alpha, \beta, \gamma \in \mathbb{R}^2.$$

Let us assume that $L \in \mathcal{R}_3$ and define

$$Q^\sharp(x) = L^2 Q(x) + LQ(Lx) + Q(L^2 x).$$

This is a new quadratic polynomial. Sometimes we will write Q_F and $Q_F^\sharp$ to emphasize the dependence on F.

The 2-jet of F at the origin will be described as

$$J_0^2 F = (L; \alpha, \beta, \gamma).$$

To measure the distance between jets we fix some norm $|\cdot|$ in the space $\mathbb{R}^{2\times 2} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$.

Proposition 2.1. *Given F in the previous conditions, assume that $Q_F^\sharp$ is not identically zero. Then there exists $\epsilon_F > 0$ such that the origin $x = 0$ is unstable with respect to any C^2 map $G : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

$$G(0) = 0, \; G'(0) \in \mathcal{R}_3, \;\; |J_0^2 F - J_0^2 G| < \epsilon_F.$$

Later we will present some examples on how to apply this result. By now we just concentrate ourselves on the proof. As a first step we mention two straightforward properties of the polynomial $Q^\sharp$.

- Given F in the conditions of the above proposition, there exists $\epsilon_F > 0$ such that $Q_G^\sharp \neq 0$ if $G : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a C^2 map with $G(0) = 0$, $G'(0) \in \mathcal{R}_3$ and $|J_0^2 F - J_0^2 G| < \epsilon_F$.
- Assume that $L = PRP^{-1}$ where P is a non-singular matrix and R is one of the two rotations of angle $\frac{2\pi}{3}$ (with positive or negative orientation). If we define $F_*(x) = P^{-1} F(Px)$ then $F_*'(0) = R$ and $Q_{F_*}^\sharp(x) = P^{-1} Q_F^\sharp(Px)$.

After these remarks the above proposition becomes equivalent to the following simplified result: *Given a C^2 map $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

$$F(0) = 0, \quad F'(0) = R, \quad Q^\sharp_F \neq 0,$$

the origin is unstable with respect to F.

The use of complex notation is very convenient for the proof of this result. We identify $\mathbb{C}$ and $\mathbb{R}^2$ with $z = x_1 + ix_2$. The map $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ can be expressed as

$$F(z, \overline{z}) = \omega z + Q(z, \overline{z}) + o(|z|^2) \ \text{ as } |z| \to 0$$

with $\omega^2 + \omega + 1 = 0$ and $Q(z, \overline{z}) = Az^2 + Bz\overline{z} + C\overline{z}^2$, $A, B, C \in \mathbb{C}$. The polynomial $Q^\sharp$ becomes

$$Q^\sharp(z, \overline{z}) = 3\overline{\omega}C\overline{z}^2,$$

and the assumption $Q^\sharp \neq 0$ is equivalent to $C \neq 0$. For simplicity we assume that $3\overline{\omega}C = 1$. This is always the case after a change $Z = \lambda z$ for appropriate λ. The third iterate of F has the expansion

$$z_3 = z + Q^\sharp(z, \overline{z}) + o(|z|^2) = z + \overline{z}^2 + o(|z|^2),$$

and a direct computation leads to

$$Re(z_3^3) = Re(z^3) + 3|z|^4 + o(|z|^4).$$

Let us fix $r > 0$ such that if $|z| \leq r$ then

$$Re(z_3^3) > Re(z^3) + 2|z|^4. \tag{6}$$

We are going to prove that no forward orbit with $Re(z_0^3) > 0$ can remain in the disk $D = \{z \in \mathbb{C} : \ |z| \leq r\}$. This proves the instability of $z = 0$.

We proceed by contradiction and assume that the positive orbit $\{z_n\}_{n \geq 0}$ is well defined and satisfies

$$|z_n| \leq r, \quad n \geq 0$$

for some z_0 in the above conditions. From the inequality (6) we deduce that

$$2 \sum_{n=0}^{N} |z_{3n}|^4 < Re(z_{3N+3}^3) - Re(z_0^3) \leq 2r^3.$$

This implies that the series $\sum |z_{3n}|^4$ converges and, in particular, $|z_{3n}| \to 0$. This fact is not compatible with another consequence of (6), namely

$$\cdots > Re(z_{3n}^3) > Re(z_{3n-3}^3) > \cdots > Re(z_3^3) > Re(z_0^3) > 0.$$

Once we have completed the proof of Proposition 2.1 we apply it to an example that will be important later. Consider the differential equation

$$\ddot{y} + y + \gamma(t)y^2 = 0$$

where $\gamma \in L^1(\mathbb{R}/T\mathbb{Z})$. Further assume that the period is

$$T = \frac{2\pi}{3}.$$

We will prove that the trivial solution $y = 0$ is unstable if

$$\int_0^{\frac{2\pi}{3}} \gamma(t)e^{3it}dt \neq 0. \tag{7}$$

The equation is understood in the Carathéodory sense. Solutions are well defined on the interval $[0, T]$ for small initial conditions and the Poincaré map can be defined as

$$F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad F(y_0, v_0) = (y(T; y_0, v_0), \dot{y}(T; y_0, v_0))$$

where $\mathcal{U}$ is a neighborhood of the origin and $y(T; y_0, v_0)$ is the solution with initial conditions $y(0) = y_0$, $\dot{y}(0) = v_0$. This is a C^∞ map and the origin $y_0 = 0$, $v_0 = 0$ is a fixed point corresponding to the trivial solution $y = 0$. Note that the two notions of stability, for the map and the differential equation, are equivalent. The variational equation at $y = 0$ is

$$\ddot{y} + y = 0$$

and so

$$F'(0, 0) = \begin{pmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{pmatrix}.$$

This matrix is precisely the rotation of angle $\frac{2\pi}{3}$ in the clockwise sense. To compute the quadratic polynomial Q it is useful to employ complex notation. Letting $w = y + i\dot{y}$ and $z = y_0 + iv_0$, the equation is written as

$$\dot{w} = -iw - i\gamma(t)(\frac{w + \overline{w}}{2})^2, \quad w(0) = z.$$

Using the formula of variation of constants we transform the initial value problem in the integral equation

$$w(t) = e^{-it}z - i\int_0^t e^{-i(t-s)}\gamma(s)(\frac{w(s) + \overline{w}(s)}{2})^2 ds.$$

The variational equation leads to the first order approximation

$$w(t) = e^{-it}z + O(|z|^2) \ \text{ as } |z| \to 0,$$

uniformly in $t \in [0, T]$. Combining the previous identities we obtain the expansion $F(z, \overline{z}) = \overline{\omega}z + Q(z, \overline{z}) + \cdots$ with Q given by

$$-\frac{i\overline{\omega}}{4}(\int_0^{\frac{2\pi}{3}} e^{-is}\gamma(s)ds)z^2 - \frac{i\overline{\omega}}{2}(\int_0^{\frac{2\pi}{3}} e^{is}\gamma(s)ds)z\overline{z} - \frac{i\overline{\omega}}{4}(\int_0^{\frac{2\pi}{3}} e^{3is}\gamma(s)ds)\overline{z}^2$$

and $\omega = e^{\frac{2\pi}{3}i}$. From (7) we deduce that $Q^\sharp \neq 0$ and the conclusion follows.

3. Weak convergence and number of periodic solutions

Let $\varphi(t)$ be a T-periodic solution of the forced pendulum equation (1). The change of variables $x = y + \varphi(t)$ transforms the original equation in

$$\ddot{y} + \beta \sin(y + \varphi(t)) = \beta \sin \varphi(t). \tag{8}$$

When φ is non-smooth or even discontinuous this equation still makes sense, even if it does not come from an equation of the type (1). Due to the properties of the sine function, this equation can be rewritten as

$$\ddot{y} + \beta \cos \varphi(t) \sin y + \beta \sin \varphi(t) \cos y = \beta \sin \varphi(t).$$

We will work with the more general class of equations

$$\ddot{y} + a(t) \sin y + b(t) \cos y = c(t) \tag{9}$$

with $a, b, c \in L^\infty(\mathbb{R}/T\mathbb{Z})$. This equation is understood in the Carathéodory sense. Given a T-periodic solution $\psi(t)$, the linearized equation is

$$\ddot{\xi} + (a(t) \cos \psi(t) - b(t) \sin \psi(t))\xi = 0. \tag{10}$$

We say that $\psi(t)$ is simple when 1 is not a Floquet multiplier of (10). The equation (9) will be called simple if all T-periodic solutions are simple.

As an example consider the pendulum equation

$$\ddot{y} + \lambda \sin y = 0$$

corresponding to $a(t) = \lambda > 0$, $b(t) = 0$, $c(t) = 0$. This equation is simple whenever

$$\lambda < (\frac{2\pi}{T})^2.$$

To prove this we first recall that the closed orbits of this autonomous equation have minimal period $\tau > \frac{2\pi}{\sqrt{\lambda}}$. Hence they do not produce T-periodic solutions if $\lambda \leq (\frac{2\pi}{T})^2$. Under this condition the only T-periodic solutions are $y = 0$ and $y = \pi$. A direct computation shows that $y = \pi$ is always simple and $y = 0$ is also simple excepting for $\lambda = (\frac{2\pi n}{T})^2$, $n = 1, 2, \ldots$

Note that, given a solution $y(t)$ of (9), new solutions can be produced by adding 2π. The family $y(t) + 2\pi n$, $n \in \mathbb{Z}$, is interpreted as a single solution. This identification, already employed in the previous example, will apply in the rest of the paper. Next we present an important property of simple equations.

Lemma 3.1. *Assume that the equation (9) is simple. Then the number of T-periodic solutions is finite.*

From now on this number will be indicated by $N(a, b, c)$. In particular $N(\lambda, 0, 0) = 2$ if $\lambda \leq (\frac{2\pi}{T})^2$.

Proof. Given a T-periodic solution $\psi(t)$, there exists some $\tau \in [-\frac{T}{2}, \frac{T}{2}]$ such that $\dot{\psi}(\tau) = 0$. Thus

$$|\dot{\psi}(0)| = |\int_{\tau}^{0} \ddot{\psi}(s)ds| \leq (||a||_{L^\infty(\mathbb{R}/T\mathbb{Z})} + ||b||_{L^\infty(\mathbb{R}/T\mathbb{Z})} + ||c||_{L^\infty(\mathbb{R}/T\mathbb{Z})})\frac{T}{2} =: C.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré map associated to (9),

$$F(y_0, v_0) = (y(T; y_0, v_0), \dot{y}(T; y_0, v_0)).$$

This is a real analytic diffeomorphism whose fixed points are in correspondence with T-periodic solutions. Moreover, if $(\psi(0), \dot{\psi}(0)) = (y_0, v_0)$ then the eigenvalues of $F'(y_0, v_0)$ are precisely the Floquet multipliers of (10). In particular 1 is not an eigenvalue if $\psi(t)$ is simple. The inverse function theorem applied to $id - F$ implies that $(\psi(0), \dot{\psi}(0))$ is an isolated fixed point of F. Summing up, we can say that the set of fixed points

$$\mathcal{F} = \{(y_0, v_0) \in [0, 2\pi] \times \mathbb{R} : \ F(y_0, v_0) = (y_0, v_0)\}$$

is closed, bounded ($|v_0| \leq C$) and discrete, and so it is finite.$\blacksquare$.

Assume now that (9) is simple and consider sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $L^\infty(\mathbb{R}/T\mathbb{Z})$ with $a_n \to a$, $b_n \to b$ and $c_n \to c$ in an appropriate topology. We will prove that for large n the number of T-periodic solutions of

$$\ddot{y} + a_n(t) \sin y + b_n(t) \cos y = c_n(t) \tag{11}$$

is independent of n. This is a more or less standard result if the convergence of the sequences a_n, b_n, c_n is strong, the main point of this section is that it also holds when the convergence is understood in a weak sense. To be precise we recall the notion of weak* convergence in L^∞.

Given a sequence $\{f_n\}$ in $L^\infty(\mathbb{R}/T\mathbb{Z})$, we say that it converges to $f \in L^\infty(\mathbb{R}/T\mathbb{Z})$ in the weak* sense if

$$\int_0^T f_n \phi \to \int_0^T f \phi$$

for every $\phi \in L^1(\mathbb{R}/T\mathbb{Z})$. Sometimes we will employ the notation $f \rightharpoonup f$ to indicate this convergence.

Weak* convergence in $L^\infty(\mathbb{R}/T\mathbb{Z})$ can be characterized by the two properties below,

- $\sup_n ||f_n||_{L^\infty(\mathbb{R}/T\mathbb{Z})} < \infty$
- $\int_0^T f_n \chi_{[a,b]} \to \int_0^T f \chi_{[a,b]}$ for every compact interval $[a, b] \subset [0, T]$.

Here $\chi_{[a,b]}$ denotes the characteristic function of the set $[a, b]$. The proof of this characterization is based on Banach-Steinhaus theorem and the density of simple functions in $L^1(0, T)$. Note that $L^1(\mathbb{R}/T\mathbb{Z})$ and $L^1(0, T)$ can be identified.

We are ready to present the main result of the section.

Proposition 3.2. *Assume that the equation (9) is simple and the sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *in* $L^\infty(\mathbb{R}/T\mathbb{Z})$ *satisfy*

$$a_n \rightharpoonup a, \quad b_n \rightharpoonup b, \quad c_n \rightharpoonup c.$$

Then, for large n, *the equation (11) is simple and* $N(a_n, b_n, c_n) = N(a, b, c)$.

To illustrate this proposition we present some concrete examples of sequences in the above conditions.

Example 1. For each $n > 2$ consider the uniform partition of $[0, T]$

$$t_0 = 0 < t_1 = \frac{T}{n} < t_2 = \frac{2T}{n} < \cdots < t_n = T$$

and fix a number θ. Define

$$\varphi_n(t) = (-1)^k \theta \;\text{ if } t \in]t_k, t_{k+1}[.$$

We consider the equation

$$\ddot{y} + \beta \sin(y + \varphi_n(t)) = \beta \sin \varphi_n(t)$$

and try to apply the above proposition. The sequence $a_n = \beta \cos \varphi_n$ is constant and so it converges to $\lambda = \beta \cos \theta$ in any topology. Next we prove that $b_n = c_n = \beta \sin \varphi_n$ converges to 0 in the weak* sense. Note that $||\sin \varphi_n||_{L^\infty} \leq 1$ and so we can assume $\phi = \chi_{[a,b]}$. Then

$$\int_0^T \sin \varphi_n \chi_{[a,b]} = \int_a^{t_r} \sin \varphi_n + \sum_{k=r}^{s-1} \int_{t_k}^{t_{k+1}} \sin \varphi_n + \int_{t_s}^{b} \sin \varphi_n$$

where r and s are the indexes such that

$$t_{r-1} < a \leq t_r \leq t_s \leq b < t_{s+1}.$$

All the integrals inside the sum have the same value up to an alternating sign. Hence they cancel in pairs and the sum is either zero or it reduces to one single integral, depending on the parity of $s - 1 - r$. In both cases we obtain the estimate

$$\left| \int_0^T \sin \varphi_n \chi_{[a,b]} \right| \leq \frac{3T}{n}.$$

The limit equation is $\ddot{y} + \lambda \sin y = 0$ and so, for large n, $N(a_n, b_n, c_n) = 2$ if $\lambda < (\frac{2\pi}{T})^2$.

Example 2. Consider a function $f(t)$ satisfying

$$f \in L^1(\mathbb{R}/T\mathbb{Z}), \quad \int_0^T f(t)dt = 0$$

and let $h(t)$ be the unique solution of

$$\ddot{h} = f(t), \quad h \text{ is } T-\text{periodic}, \quad \int_0^T h(t)dt = 0.$$

The change of variables $x = y + h(t)$ transforms the pendulum equation (1) into

$$\ddot{y} + \beta \sin(y + h(t)) = 0.$$

When this equation is simple and has N periodic solutions of period T, the above proposition can be applied. In particular there exists $\epsilon > 0$ such that for any function g satisfying

$$g \in L^1(\mathbb{R}/T\mathbb{Z}), \quad \int_0^T g(t)dt = 0, \quad ||f - g||_{L^1(\mathbb{R}/T\mathbb{Z})} < \epsilon$$

the equation

$$\ddot{x} + \beta \sin x = g(t)$$

has exactly N periodic solutions of period T.

Example 3. Consider the equation of a pendulum of variable length

$$\ddot{y} + a(t) \sin y = 0.$$

We fix two positive numbers M and $\lambda < (\frac{2\pi}{T})^2$. The above result can be applied to deduce that there exists $\epsilon > 0$ such that the equation has exactly two T-periodic solutions if $a \in L^\infty(\mathbb{R}/T\mathbb{Z})$ satisfies

$$||a||_{L^\infty(\mathbb{R}/T\mathbb{Z})} < M, \quad ||a - \lambda||_{L^1(\mathbb{R}/T\mathbb{Z})} < \epsilon.$$

The key for the proof of Proposition 3.2 is the following result on continuous dependence with respect to weak topologies.

Lemma 3.3. *Assume that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in the conditions of Proposition* 3.2 *and let $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré map associated to (*11*). Then, for each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$,*

$$\partial^\alpha F_n(y_0, v_0) \to \partial^\alpha F(y_0, v_0),$$

and the convergence is uniform on $(y_0, v_0) \in K$ for each compact set $K \subset \mathbb{R}^2$.

We are using the multi-index notation $\partial^\alpha = \frac{\partial^{\alpha_1+\alpha_2} F}{\partial^{\alpha_1} y_0 \partial^{\alpha_2} v_0}$. The proof of this lemma is unrelated to the rest of the paper and we postpone it to Section 6.

Proof of Proposition 3.2. Consider the sets of fixed points

$$\mathcal{F}_n = \{(y_0, v_0) \in [0, 2\pi] \times \mathbb{R} : \; F_n(y_0, v_0) = (y_0, v_0)\}.$$

From the proof of Lemma 3.1 we know that the sets $\mathcal{F}_n$ are compact and contained in a common rectangle $\mathcal{R} = [0, 2\pi] \times [-C, C]$. Assume that the set $\mathcal{F}$ associated to

(9) has N points, that is $\sharp\mathcal{F} = N$. By assumption these fixed points are simple. Let us fix closed disks $D_1, \dots, D_N$ centered in these fixed points and pairwise disjoint. The maps F_n converge to F in $C^1(D_i)$ and so $\mathcal{F}_n \cap D_i$ is a singleton for large n. Moreover, the points in these intersections are also simple as fixed points of F_n. We are lead to the inequality

$$N(a_n, b_n, c_n) = \sharp\mathcal{F}_n \geq \sharp\mathcal{F} = N(a, b, c) = N.$$

To prove the reversed inequality we proceed by contradiction. If $N(a_k, b_k, c_k)$ were greater than N for some subsequence, then $\mathcal{F}_k$ should have a point lying in $\mathcal{R} \setminus \cup_{i=1}^{N} D_i$. By a passage to the limit we conclude that the same should occur to the set $\mathcal{F}$ and this is absurd.■

Remark. The previous proof shows that the set $\mathcal{F}_n$ converges to $\mathcal{F}$ in the Hausdorff topology on the space of compact subsets of $\mathbb{R}^2$. Later this fact will be combined with a consequence of the proof of Lemma 3.3. The general solution of (11), denoted by $y_n(t; y_0, v_0)$, satisfies that $(y_n(t; y_0, v_0), \dot{y}_n(t; y_0, v_0))$ converges to $(y(t; y_0, v_0), \dot{y}(t; y_0, v_0))$ uniformly in $t \in [0, T]$, $(y_0, v_0) \in K$, where K is any compact subset of $\mathbb{R}^2$.

4. The first construction

We fix the period $T = \frac{2\pi}{3}$ and assume $\beta > 1$. For each integer $n \geq 3$ consider a partition of the interval $[0, \frac{T}{2}]$ of the type

$$\tau_0 = 0 < \tau_1 < \tau_2 = \frac{2T}{2n} < \tau_3 = \frac{3T}{2n} < \cdots < \tau_n = \frac{T}{2}.$$

There are two differences with respect to the partition considered in Example 1 after Proposition 3.2, now the interval has length $\frac{T}{2}$ and one of the knots is not fixed. Indeed τ_1 can be interpreted as a parameter lying in the interval $]0, \frac{T}{n}[$. Let θ be the number in $]0, \frac{\pi}{2}[$ satisfying

$$\cos\theta = \frac{1}{\beta}.$$

We define the function $p_n \in L^\infty(\mathbb{R}/T\mathbb{Z})$ by

$$p_n(-t) = -p_n(t), \quad p_n(t + T) = p_n(t), \quad a.e.\ t \in \mathbb{R},$$

$$p_n(t) = (-1)^k \theta \ \text{ if } t \in]\tau_k, \tau_{k+1}[, \quad 0 \leq k < n.$$

Note that the function p_n depends upon the choice of τ_1. We will use this freedom to select a value of τ_1 such that the condition

$$\int_0^T e^{3it} \sin p_n(t) dt \neq 0 \tag{12}$$

holds. To prove that this choice is possible we observe that e^{3it} is T-periodic and so the integral can be translated to the interval $[-\frac{T}{2}, \frac{T}{2}]$,

$$\int_0^T e^{3it} \sin p_n(t) dt = \int_0^{\frac{T}{2}} + \int_{-\frac{T}{2}}^0 = \int_0^{\frac{T}{2}} (e^{3it} - e^{-3it}) \sin p_n(t) dt =$$

$$2i(\sin\theta) \sum_{k=0}^{n-1} (-1)^k \int_{\tau_k}^{\tau_{k+1}} \sin 3t dt = -\frac{4}{3} i \sin\theta \cos 3\tau_1 + \gamma$$

where γ is a complex number depending on n and θ but independent of τ_1. The condition (12) will hold for all numbers $\tau_1 \in]0, \frac{T}{n}[$ excepting perhaps for a finite number of choices.

Once τ_1 has been fixed we observe that

$$\cos p_n(t) = \frac{1}{\beta} \quad a.e.\ t \in \mathbb{R}$$

and

$$\sin p_n \rightharpoonup 0$$

in the weak* sense. The proof of this fact is almost the same as the proof given in Example 1 after Proposition 3.2. This proposition can be applied to conclude that the equation

$$\ddot{y} + \beta \sin(y + p_n(t)) = \beta \sin p_n(t) \tag{13}$$

has exactly two T-periodic solutions, $y = 0$ and $z_n(t)$. Moreover $z_n(t)$ converges uniformly to π. At this point the remark after the proof of Proposition 3.2 is useful. We will prove that both solutions are unstable if $n \geq n_0$ for some n_0. Let us start with $y = 0$ and expand the equation to obtain

$$\ddot{y} + \beta(\cos p_n(t)) y - \frac{\beta}{2} (\sin p_n(t)) y^2 + \cdots = 0,$$

where the remainder is of order y^3. In consequence this equation has a contact of order two with

$$\ddot{y} + y + \gamma_n(t) y^2 = 0$$

and

$$\gamma_n(t) = -\frac{\beta}{2} \sin p_n(t).$$

Note that the condition (12) implies that (7) holds. Let $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré map associated to (13). The expansion of F_n around the origin coincides, up to second order, with the expansion computed in Section 2. Then Proposition 2.1 is applicable to F_n and $y = 0$ is unstable as a solution of (13). To prove the instability of $z_n(t)$ set $z = y - z_n(t)$ in (13) to obtain

$$\ddot{z} + \beta \sin(z + z_n(t) + p_n(t)) - \beta \sin(z_n(t) + p_n(t)) = 0.$$

The linearization at $z = 0$ is

$$\ddot{\xi} + \beta \cos(z_n(t) + p_n(t))\xi = 0. \tag{14}$$

Since $z_n(t) \to \pi$ uniformly, this equation converges to

$$\ddot{\xi} - \xi = 0.$$

The Floquet multipliers of the limit equation are $0 < \mu_1 = e^{-T} < 1 < \mu_2 = e^T$. By continuous dependence, the Floquet multipliers of (14) also satisfy $0 < \mu_1 < 1 < \mu_2$ for large n. The first Lyapunov method says that $z_n(t)$ is unstable as a solution of (13).

If the function $p_n(t)$ were real analytic, the proof of Theorem 1.1 would be complete, since it would be sufficient to define $x = y + p_n(t)$ and $f(t) = \ddot{p}_n(t) + \beta \sin p_n(t)$. Of course this is not right and our strategy will be to perform this change of variable after approximating p_n by an appropriate analytic function.

Lemma 4.1. *For each $n \geq n_0$ and $\delta > 0$ there exists $q_n \in C^\omega(\mathbb{R}/T\mathbb{Z})$ satisfying*

(i) q_n *is odd*

(ii) $||p_n - q_n||_{L^2(\mathbb{R}/T\mathbb{Z})} < \delta$

(iii) the linear equation

$$\ddot{y} + \beta \cos q_n(t) y = 0 \tag{15}$$

has Floquet multipliers $\mu_1 = \omega$, $\mu_2 = \overline{\omega}$ with $\omega^2 + \omega + 1 = 0$.

Before proving this result let us discuss how to complete the proof of Theorem 1.1. We claim that there exists a sequence $\delta_n \downarrow 0$ such that the equation

$$\ddot{y} + \beta \sin(y + q_n(t)) = \beta \sin q_n(t) \tag{16}$$

has exactly two T-periodic solutions and both of them are unstable if n is large enough. Here q_n is the function given by the previous lemma when $\delta = \delta_n$. First note that

$$\sin q_n \rightharpoonup 0 \text{ weak}^* \text{ in } L^\infty(\mathbb{R}/T\mathbb{Z}), \quad \cos q_n \to \frac{1}{\beta} \text{ in } L^2(\mathbb{R}/T\mathbb{Z})$$

for any sequence $\delta_n \downarrow 0$. Indeed, to justify the first convergence, we take a characteristic function $\chi_{[a,b]}$ and observe that

$$|\int_0^T (\sin q_n)\chi_{[a,b]}| \leq |\int_0^T (\sin p_n)\chi_{[a,b]}| + |\int_0^T (\sin q_n - \sin p_n)\chi_{[a,b]}| \leq$$

$$|\int_0^T (\sin p_n)\chi_{[a,b]}| + \sqrt{T}||p_n - q_n||_{L^2(\mathbb{R}/T\mathbb{Z})} \to 0.$$

The second convergence is obtained in a similar way. The next step is to select δ_n in an appropriate way. The Poincaré maps associated to (13) and (16) are denoted by F_n and $\tilde{F}_n$ and, by Lemma 3.3, they are close in the C^2 topology on compact sets. According to Proposition 2.1 we adjust δ_n so that

$$|J_0^2 F_n - J_0^2 \tilde{F}_n| < \epsilon_{F_n}.$$

The eigenvalues of the matrix $\tilde{F}_n'(0)$ are the Floquet multipliers of (15); that is, ω and $\overline{\omega}$. This implies that $\tilde{F}_n'(0) \in \mathcal{R}_3$. Now Proposition 2.1 also says that $y = 0$ is unstable as a solution of (16). Moreover, from Proposition 3.2 we know that the equation (16) has exactly two T-periodic solutions $y = 0$ and $\tilde{z}_n \to \pi$. As before we observe that the equation

$$\ddot{\xi} + \beta \cos(\tilde{z}_n(t) + q_n(t))\xi = 0$$

converges, in the sense of $L^2(\mathbb{R}/T\mathbb{Z})$, to $\ddot{\xi} - \xi = 0$. This is sufficient to guarantee that also $\tilde{z}_n(t)$ is unstable for large n. The proof of Theorem 1.1 is complete, we just define $f(t) = \ddot{q}_n(t) + \beta \sin q_n(t)$.

Proof of Lemma 4.1. Let us recall some well known facts on the discriminant of Hill's equation. Given $a \in L^2(\mathbb{R}/T\mathbb{Z})$ consider the linear equation

$$\ddot{y} + a(t)y = 0 \tag{17}$$

with Floquet multipliers μ_1 and μ_2 and $\mu_1 \cdot \mu_2 = 1$. The discriminant is defined as

$$\Delta = \mu_1 + \mu_2$$

and so the condition (iii) on the multipliers is equivalent to $\Delta = \omega + \overline{\omega} = -1$. The discriminant can be thought as a functional

$$\Delta : L^2(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}, \quad a \mapsto \Delta[a].$$

It is well known that Δ is continuous. We will be interested in the discriminant of the equation (17) with $a(t) = \beta \cos q(t)$. For this reason we define the new functional

$$\mathcal{D} : L^2(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}, \quad \mathcal{D}[q] = \Delta[\beta \cos q].$$

It is also continuous because it can be expressed as the composition $\mathcal{D} = \Delta \circ N$ where N is the Lipschitz-continuous operator

$$N : L^2(\mathbb{R}/T\mathbb{Z}) \to L^2(\mathbb{R}/T\mathbb{Z}), \quad N(q) = \beta \cos q.$$

To work with odd functions we introduce the subspace of $L^2(\mathbb{R}/T\mathbb{Z})$,

$$L^2_\natural = \{q \in L^2(\mathbb{R}/T\mathbb{Z}) : \ q(-t) = -q(t) \ \ a.e.\ t\}.$$

It is a Hilbert space with the dense subspace of odd analytic functions

$$V = C^\omega(\mathbb{R}/T\mathbb{Z}) \cap L^2_\natural.$$

To prove the density of V we expand every function $q \in L^2_\natural$ in a Fourier series of sines,

$$q(t) \sim \sum_{n=1}^{\infty} q_n \sin \frac{2\pi n t}{T}$$

converging to q in the L^2 sense. The partial sums $\sum_{n=1}^{N} q_n \sin \frac{2\pi n t}{T}$ belong to V and q is the L^2-limit.

Define

$$\mathcal{C}^\omega = \{q \in V : \ ||q - p_n||_{L^2(\mathbb{R}/T\mathbb{Z})} < \delta\}.$$

This is a non-empty convex set whose closure in $L^2_\natural$ is the ball

$$\mathcal{C} = \{q \in L^2_\natural : \ ||q - p_n||_{L^2(\mathbb{R}/T\mathbb{Z})} \leq \delta\}.$$

To prove the lemma we must show that the functional $\mathcal{D}$ takes the value -1 at some function in $\mathcal{C}^\omega$. Note that $\mathcal{D}[p_n] = -1$. We proceed by contradiction and assume that

$$\mathcal{D}[q] \neq -1 \quad \text{if } q \in \mathcal{C}^\omega.$$

Since $\mathcal{C}^\omega$ is convex, we can assume that that $\mathcal{D} + 1$ does not change sign on this set. The two possible cases are $\mathcal{D} + 1 > 0$ and $\mathcal{D} + 1 < 0$ and both can be handled similarly. From now on we assume $\mathcal{D} + 1 > 0$. By a density argument

$$\mathcal{D}[q] \geq -1 \quad \text{if } q \in \mathcal{C}. \tag{18}$$

Next we compute the variations of $\mathcal{D}$. Given $f \in L^2_\natural$ and $\lambda \in \mathbb{R}$, define

$$D(\lambda) = \mathcal{D}[p_n + \lambda f].$$

This is a smooth function and

$$D'(0) = -\beta \int_0^T \chi(t)(\sin p_n(t)) f(t) dt$$

with

$$\chi(t) = -\frac{\sqrt{3}}{2} \sin^2 t - \frac{\sqrt{3}}{2} \cos^2 t = -\frac{\sqrt{3}}{2}.$$

For this computation we refer to [5]. In particular, for the choice $f(t) = \sin p_n(t)$,

$$D'(0) = \beta \frac{\sqrt{3}}{2} \int_0^T \sin^2 p_n(t) dt > 0.$$

Hence $D(\lambda) < -1$ for λ negative and small. This is incompatible with (18) because $p_n + \lambda \sin p_n$ belongs to $\mathcal{C}$ when $|\lambda|$ is small. ■

5. The second construction

In this section we assume that $T = 2\pi$ and $\beta > \frac{1}{4}$. The key for this construction will be the notion of hyperbolicity. Let us first consider the linear equation (17). This equation is called hyperbolic if the Floquet multipliers satisfy $0 < |\mu_1| < 1 < |\mu_2|$. In terms of the discriminant $\Delta = \Delta[a]$ this is equivalent to $|\Delta| > 2$. Given a nonlinear equation and a periodic solution $\psi(t)$, we say that ψ is hyperbolic if the linearized equation satisfies the above condition. We recall that hyperbolic solutions are unstable in the Lyapunov sense. Next we present examples for the linear and nonlinear cases.

Given $\epsilon \in [0, \frac{1}{2}[$, we consider the even and 2π-periodic function $a_\epsilon \in L^\infty(\mathbb{R}/T\mathbb{Z})$ defined by

$$a_\epsilon(t) = \begin{cases} (\frac{1}{2}+\epsilon)^2, & |t| < \frac{\pi}{2} \\ (\frac{1}{2}-\epsilon)^2, & \frac{\pi}{2} < |t| < \pi. \end{cases}$$

This function is piecewise constant and the associated Hill's equation $\ddot{y}+a_\epsilon(t)y = 0$ can be integrated explicitly. After some computations that can be found in chapter 5 of the book [1], it can be checked that the discriminant satisfies

$$\Delta[a_\epsilon] < -2$$

if $\epsilon > 0$ is small enough. The equation $\ddot{y}-a_\epsilon(t)y = 0$ is hyperbolic for any $\epsilon \in [0, \frac{1}{2}[$. This can also be checked by direct integration. In this case

$$\Delta[-a_\epsilon] > 2.$$

Next we consider the nonlinear equation

$$\ddot{y} + a_\epsilon(t)\sin y = 0. \tag{19}$$

The sequence a_ϵ converges to the constant $\frac{1}{4}$ in a strong sense,

$$\lim_{\epsilon\downarrow 0} ||a_\epsilon - \frac{1}{4}||_{L^\infty(\mathbb{R}/T\mathbb{Z})} = 0.$$

We are in the conditions of the third example after Proposition 3.2 and so, for small ϵ, the equation (19) has exactly two periodic solutions with period $T = 2\pi$. These solutions are $y = 0$ and $y = \pi$. The above discussion on linear equations implies that these two solutions are hyperbolic. Up to this point the construction is essentially the same as in the last section of [4]. From now on we fix a number ϵ positive and small. In addition to the previous conditions we also assume that it satisfies

$$\beta > (\frac{1}{2}+\epsilon)^2.$$

This new condition is employed to find numbers $\theta_\pm$ in $]0, \frac{\pi}{2}[$ with

$$\beta\cos\theta_\pm = (\frac{1}{2}\pm\epsilon)^2.$$

Next we take uniform partitions of the intervals $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$,

$$t_0 = 0 < t_1 = \frac{\pi}{2n} < t_2 = \frac{2\pi}{2n} < \cdots < t_n = \frac{\pi}{2},$$

$$t_0^* = \frac{\pi}{2} < t_1^* = \frac{\pi}{2} + \frac{\pi}{2n} < t_2^* = \frac{\pi}{2} + \frac{2\pi}{2n} < \cdots < t_n^* = \pi.$$

Let H_n be the odd periodic function in $L^\infty(\mathbb{R}/T\mathbb{Z})$ defined by

$$H_n(t) = \begin{cases} (-1)^k \theta_+, & \text{if } \ t \in]t_k, t_{k+1}[\\ (-1)^k \theta_-, & \text{if } \ t \in]t_k^*, t_{k+1}^*[. \end{cases}$$

We observe that $\beta \cos H_n(t) = a_\epsilon(t)$ almost everywhere and $\beta \sin H_n \rightharpoonup 0$ in the weak* sense. The equation

$$\ddot{y} + \beta \sin(y + H_n(t)) = \beta \sin H_n(t) \tag{20}$$

converges to (19) in the sense of Proposition 3.2. Since the equation (19) is simple we deduce that for large n the equation (20) has exactly two periodic solutions with period $T = 2\pi$. We label them as $y_1 = 0$ and $y_{2,n}(t)$. In view of the remark at the end of Section 3 we know that $y_{2,n}(t)$ converges to π uniformly. The linearization at $y_1 = 0$ is precisely (19) and we know that this equation is hyperbolic. The linearization at $y_{2,n}(t)$ is

$$\ddot{\xi} + \beta \cos(y_{2,n}(t) + H_n(t))\xi = 0.$$

It is easily checked that $\beta \cos(y_{2,n}(t) + H_n(t)) + a_\epsilon(t)$ converges to 0 uniformly. The properties of continuity of the discriminant functional imply that the discriminant of the above equation converges to $\Delta[-a_\epsilon] > 2$. From now on we fix n large enough so that (20) has exactly two periodic solutions of period 2π. Moreover both of them are hyperbolic. Once again we have used Proposition 3.2.

Our next step will be to expand $H_n(t)$ in Fourier series

$$H_n(t) \sim \sum_{k=1}^{\infty} \gamma_k \sin kt.$$

For large N the trigonometric polynomial

$$K_N(t) = \sum_{k=1}^{N} \gamma_k \sin kt$$

is such that the equation

$$\ddot{y} + \beta \sin(y + K_N(t)) = \beta \sin K_N(t)$$

has exactly two T-periodic solutions, both of them hyperbolic. The change of variables $x = y + K_N(t)$ leads to an equation of the type (1). In view of the second example after Proposition 3.2 we can conclude that this is the searched equation for the proof of Theorem 1.2.

6. Proof of Lemma 3.3

We divide the proof in five steps.

6.1. A remark on uniform convergence. Let Λ be a compact metric space. Convergent sequences in Λ will be denoted by $\lambda_n \to \lambda_\infty$, where λ_∞ is the limit. Given a sequence of functions

$$X_n : [0,T] \times \Lambda \to \mathbb{R}^2, \;\; (t,\lambda) \mapsto X_n(t,\lambda), \; n = 1,2,\dots,\infty,$$

we say that there is c-convergence if the following property holds: for each $\lambda_n \to \lambda_\infty$,

$$X_n(t,\lambda_n) \to X_\infty(t,\lambda_\infty)$$

uniformly in $t \in [0,T]$.
This notion is related to the so-called continuous convergence (see [2]). We will employ the notation $X_n \rightrightarrows X_\infty$.

Lemma 6.1. *In the previous setting assume that $X_n \rightrightarrows X_\infty$ and X_∞ is continuous. Then X_n converges to X_∞ uniformly in $[0,T] \times \Lambda$.*

Proof. By contradiction assume the existence of a number $\delta > 0$ and sequences t_n and λ_n such that

$$|X_{\sigma(n)}(t_n,\lambda_n) - X_\infty(t_n,\lambda_n)| \geq \delta$$

for some integer $\sigma(n) \geq 1$ with $\sigma(n) \to \infty$. After extracting a subsequence we can assume that $\lambda_n \to \lambda_\infty$. Then

$$|X_{\sigma(n)}(t_n,\lambda_n) - X_\infty(t_n,\lambda_n)| \leq$$

$$|X_{\sigma(n)}(t_n,\lambda_n) - X_\infty(t_n,\lambda_\infty)| + |X_\infty(t_n,\lambda_\infty) - X_\infty(t_n,\lambda_n)|.$$

The assumption on c-convergence and the uniform continuity of X_∞ imply that the two terms in the sum tend to zero. This is incompatible with the existence of the number δ. ∎

6.2. Convergence of the solution. We prove that the solution of equation (11) converges to the solution of (9) in the following sense,

$$y_n(t;y_0,v_0) \to y(t;y_0,v_0), \;\; \dot{y}_n(t;y_0,v_0) \to \dot{y}(t;y_0,v_0)$$

uniformly in $t \in [0,T]$, $(y_0,v_0) \in K$, where K is a compact subset of $\mathbb{R}^2$.

To prove this assertion we take $\Lambda = K$ with $\lambda = (y_0,v_0)$ and prove that there is c-convergence of $X_n = (y_n,\dot{y}_n)$ towards $X_\infty = (y,\dot{y})$. By continuous dependence with respect to initial conditions we know that X_∞ is continuous and so Lemma 6.1 will imply that X_n converges to X_∞ in the uniform sense. Let us take a sequence

$\lambda_n = (y_{0n}, v_{0n}) \in K$ converging to $\lambda_\infty = (y_{0\infty}, v_{0\infty})$. We abbreviate the notation to

$$y_n(t) = y(t; y_{0n}, v_{0n}).$$

We must prove that

$$y_n(t) \to y_\infty(t), \quad \dot{y}_n(t) \to \dot{y}_\infty(t)$$

uniformly in $t \in [0,T]$. The sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are bounded in $L^\infty(\mathbb{R}/T\mathbb{Z})$ and so the same can be said for the sequence $\{\ddot{y}_n\}$ in $L^\infty(0,T)$. From here it is easy to deduce that the sequences $\{y_n\}$ and $\{\dot{y}_n\}$ are uniformly bounded and equi-continuous so that Ascoli theorem can be applied. Let $\{y_k\}$ be a subsequence of $\{y_n\}$ converging to some y_* in $C^1[0,T]$, we will prove that $y_\infty = y_*$. This is sufficient to guarantee the convergence of the whole sequence $\{y_n\}$ to y_∞.

Given a test function $\varphi \in C^\infty[0,T]$, $\varphi(0) = \varphi(T) = 0$, we deduce from (11) that

$$-\int_0^T \dot{y}_k \dot{\varphi} + \int_0^T \{a_k \sin y_k + b_k \cos y_k - c_k\}\varphi = 0.$$

The sequence $\sin y_k$ converges uniformly to $\sin y_*$, hence $a_k \sin y_k \rightharpoonup a \sin y_*$. This type of argument allows a passage to the limit when $k \to \infty$, showing that y_* is a solution of (9). This solution must be understood in a weak sense but, since we are dealing with ordinary equations, it is also a solution in the Carathéodory sense. The initial conditions satisfied by y_∞ and y_* coincide at $t = 0$ and so $y_* = y_\infty$.

6.3. Linear equations depending on parameters. Consider the Cauchy problem

$$\ddot{z} + G(t,\lambda)z = g(t,\lambda), \quad z(0) = w_0, \ \dot{z}(0) = w_1 \tag{21}$$

where $G, g : [0,T] \times \Lambda \to \mathbb{R}$ satisfy the following conditions:

- For each $\lambda \in \Lambda$, $G(\cdot,\lambda), g(\cdot,\lambda) \in L^\infty(0,T)$
- If $\lambda_n \to \lambda_\infty$ then $G(\cdot,\lambda_n) \rightharpoonup G(\cdot,\lambda_\infty)$, $g(\cdot,\lambda_n) \rightharpoonup g(\cdot,\lambda_\infty)$ in the weak* sense.

The solution of (21) will be denoted by $z(t,\lambda)$ and we claim that it depends continuously on λ. More precisely,

$$(t,\lambda) \in [0,T] \times \Lambda \mapsto (z(t,\lambda), \dot{z}(t,\lambda))$$

is continuous. The proof of this fact uses again Ascoli theorem and an argument of uniqueness. We omit the details.

6.4. Successive derivatives of the solution. The derivatives $\partial^\alpha y_n$ with $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} y_0 \partial^{\alpha_2} v_0}$, $|\alpha| = \alpha_1 + \alpha_2$, can be computed by differentiating the equation (11) with respect to initial conditions. They satisfy a linear Cauchy problem of the type

$$\ddot{z} + \Gamma_n(t; y_0, v_0)z = \gamma_n(t; y_0, v_0), \quad z(0) = w_0, \ \dot{z}(0) = w_1 \tag{22}$$

where

$$\Gamma_n = a_n \cos y_n - b_n \sin y_n$$

and γ_n depends upon a_n, b_n and the derivatives $\partial^\beta y_n$ with $|\beta| < |\alpha|$. For instance, $\gamma_n = 0$, $w_0 = 0$, $w_1 = 1$ if $\alpha = (0,1)$ or $\gamma_n = (a_n \sin y_n + b_n \cos y_n)\partial^{e_1} y_n \partial^{e_2} y_n$, $w_0 = w_1 = 0$ if $\alpha = (1,1)$ and $e_1 = (1,0)$, $e_2 = (0,1)$.

The same can be said about the solution of (9). To unify the discussion we employ the notation y_∞ for the solutions of (9). We claim that for each $n = 1, 2, \ldots, \infty$ the map

$$(t; y_0, v_0) \in [0,T] \times \mathbb{R}^2 \mapsto (\partial^\alpha y_n(t; y_0, v_0), \partial^\alpha \dot{y}_n(t; y_0, v_0)) \in \mathbb{R}^2$$

is continuous. This is proved by induction on $N = |\alpha|$. We assume that the map is continuous for each β with $|\beta| < N$ and prove that the same holds for α. We can apply the previous remark on the linear problem (21) where Λ is a closed ball in $\mathbb{R}^2$, $G = \Gamma_n$ and $g = \gamma_n$. Given a convergent sequence $\lambda_m = (y_{0m}, v_{0m}) \to \lambda_\infty = (y_{0\infty}, v_{0\infty})$, the continuity of y_n implies that $\Gamma_n(\cdot, \lambda_m)$ converges, as $m \to \infty$, to $\Gamma_n(\cdot, \lambda_\infty)$ in the weak* sense. The same can be said about $\gamma_n(\cdot, \lambda_m)$ and $\gamma_n(\cdot, \lambda_\infty)$ but now the inductive assumption has to be invoked.

6.5. Convergence of the derivatives of the solution. To complete the proof we show that the solution of equation (11) converges to the solution of (9) in the C^N topology on compact sets, that is

$$\partial^\alpha y_n(t; y_0, v_0) \to \partial^\alpha y(t; y_0, v_0), \quad \partial^\alpha \dot{y}_n(t; y_0, v_0) \to \partial^\alpha \dot{y}(t; y_0, v_0)$$

uniformly in $t \in [0,T]$, $(y_0, v_0) \in K$ for each $|\alpha| \leq N$.

Again the proof is by induction on the order of α. We consider the solution of (22), $Z_n(t; y_0, v_0) = (z_n(t; y_0, v_0), \dot{z}_n(t; y_0, v_0))$ and prove that it converges to $Z_\infty(t; y_0, v_0)$ uniformly on $[0,T] \times K$. To this end we apply again Lemma 6.1 with $\Lambda = K$. We already know that Z_∞ is continuous and so it remains to prove that $Z_n \rightrightarrows Z_\infty$. This is more or less a repetition of previous arguments based on Ascoli theorem and the uniqueness of the initial value problem.

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