Index and persistence of stable Cantor sets*

Rafael Ortega and Alfonso Ruiz-Herrera

Departamento de Matemática Aplicada

Universidad de Granada, 18071 Granada, Spain

e-mail: rortega@ugr.es, alfonsoruiz@ugr.es

Dedicated to Professor Fabio Zanolin on the occasion of his sixtieth birthday

1 Introduction

Cantor sets often appear as invariant sets of planar homeomorphisms. Well known examples are the Bernoulli shift in Smale's horseshoe, Aubry-Mather sets in non-integrable twist maps or adding machines obtained as sections of a solenoid. Some concrete constructions can be found in [1, 3, 6]. In general we will consider a homeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and a Cantor set $\Lambda \subset \mathbb{R}^2$ with

$$h(\Lambda) = \Lambda.$$

In this paper homeomorphisms are understood as surjective maps, so that $h(\mathbb{R}^2) = \mathbb{R}^2$. Also, to avoid trivialities, it will be assumed that Λ is transitive. This means that for some $p \in \Lambda$,

$$L_\omega(p, h) = \Lambda,$$

where $L_\omega(p, h)$ is the corresponding ω-limit set. A Cantor set is a compact, perfect and totally disconnected metric space. All Cantor sets are homeomorphic but they can support many different transitive dynamics. In the examples mentioned above one can find chaos, Denjoy dynamics or almost-periodicity.

An invariant set $\Lambda \subset \mathbb{R}^2$ is stable (in the sense of Lyapunov) if each neighborhood U of Λ contains another neighborhood V such that

$$h^n(V) \subset U \text{ for every } n \geq 1.$$

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In [2], Bell and Meyer obtained a remarkable result: in the plane, stable Cantor sets are never isolated, in fact they can be approximated by periodic points lying outside Λ. The purpose of our paper is to prove that these periodic points have non-zero index. Here we refer to the fixed point index that can be expressed in terms of Brouwer's degree. As a consequence we will prove that stable Cantor sets are persistent as invariant sets. An invariant compact set Λ is persistent if, given any positive $\varepsilon > 0$, there exists $\delta > 0$ such that for any homeomorphism $\widetilde{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$\|h(x) - \widetilde{h}(x)\| \leq \delta$$

for each $x \in \mathbb{R}^2$, there exists a compact set $\widetilde{\Lambda} \subset \mathbb{R}^2$ such that

$$\widetilde{h}(\widetilde{\Lambda}) = \widetilde{\Lambda} \quad \text{and} \quad D_H(\Lambda, \widetilde{\Lambda}) \leq \epsilon.$$

The symbol D_H refers to the Hausdorff distance between compact subsets of the plane. In our result, $\widetilde{\Lambda}$ will be composed by periodic points derived from the properties of degree. Summing up we can say that stable Cantor sets in the plane are simultaneously non-isolated and persistent. This is in contrast with the properties enjoyed by stable finite sets. At the end of the paper we will present an example of a fixed point that is stable and non-persistent. The structure of the paper is as follows. The main theorem on index and a corollary on persistence are stated in Section 2. The proofs of both results are presented in Section 3. Finally, in Section 4 we discuss some connections with the literature. To finish this introduction we notice that an example constructed in [2] shows that our results do not admit a direct extension to higher dimensions.

2 Main results

Given a Jordan curve $\Gamma \subset \mathbb{R}^2$, the bounded component of $\mathbb{R}^2 \backslash \Gamma$ will be indicated by $\widehat{\Gamma}$. Brouwer's degree in the plane will be denoted by $d[f, G, 0]$ where $G \subset \mathbb{R}^2$ is a bounded and open set and $f : cl(G) \longrightarrow \mathbb{R}^2$ is a continuous function defined on the closure of G. We must also assume that f does not vanish on ∂G, the boundary of G. We recall two properties of the degree that will be employed later,

i) existence of zeros: the function f has at least one zero on G if $d[f, G, 0] \neq 0$,

ii) continuity of the degree: there exists $\eta > 0$, depending on f, such that if $g : cl(G) \longrightarrow \mathbb{R}^2$ is a continuous function with

$$\|f(x) - g(x)\| \leq \eta$$

for each $x \in \partial G$, then g does not vanish on ∂G and $d[g, G, 0] = d[f, G, 0]$.

We refer to [10] for more information on degree theory. Given a continuous function $\phi : cl(G) \longrightarrow \mathbb{R}^2$, the fixed point index is defined as the degree of $f = id - \phi$. The zeros of f are precisely the fixed points of ϕ.

We will prove that the existence of a stable Cantor set has strong consequences on the fixed point index of the map $h^N = h \circ \cdot \cdot^{(N} \cdot \circ h$. Notice that the fixed points of h^N are the periodic points of h whose minimal period is a divisor of N.

Theorem 2.1 *Assume that $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a homeomorphism and Λ is an invariant Cantor set that is stable and has a transitive point. Then for every $\delta > 0$ and $p \in \Lambda$ there exist a Jordan curve $\Gamma = \Gamma(\delta, p)$ and an integer $N = N(\delta, p) \geq 1$ such that the following properties hold,*

$$D_H(\Gamma, \{p\}) \leq \delta, \quad h^N(x) \neq x \text{ if } x \in \Gamma, \quad d[id - h^N, \widehat{\Gamma}, 0] = 1.$$

The existence property of the degree implies that each region $\widehat{\Gamma}(\delta, p)$ contains a periodic point. This implies that Λ can be obtained as a limit of periodic points.

Theorem 2.2 *(Bell and Meyer) In the assumptions of Theorem 2.1 and given $p \in \Lambda$, there exist a sequence of points $\{x_n\}$ in $\mathbb{R}^2$ and integers $\sigma(n) \geq 1$ such that*

$$x_n \longrightarrow p \quad \text{and} \quad h^{\sigma(n)}(x_n) = x_n.$$

The persistence of Λ will be deduced from the continuity of the degree.

Corollary 2.1 *In the assumptions of Theorem 2.1, the set Λ is persistent.*

3 Proofs

The proof by Bell and Meyer in [2] is based on a well known fixed point theorem due to Cartwright and Littlewood. This theorem deals with orientation preserving homeomorphisms and it has been extended to the orientation reversing case by Bell. We will employ a strategy similar to that in [2] but without making use of this fixed point theorem. Instead we will use the following result which is a consequence of Brouwer's theory on translations arcs.

Lemma 3.1 *Assume that $\Omega \subset \mathbb{R}^2$ is an open and simply connected set and let $H : \Omega \longrightarrow \Omega$ be an orientation preserving embedding. In addition, assume that H has a recurrent point that is not fixed. Then there exists a Jordan curve $\Gamma \subset \Omega$ such that $H(x) \neq x$ if $x \in \Gamma$ and*

$$d[id - H, \widehat{\Gamma}, 0] = 1.$$

Let us recall that an embedding is a continuous and one-to-one map. In contrast to homeomorphisms, embeddings are not necessarily onto, that is $H(\Omega) \subset \Omega$. For this reason, orbits are well defined for the future but not necessarily for the past. The embedding is orientation-preserving if

$$d[H, B, y] = 1,$$

where y is any point in $H(\Omega)$ and B is an open ball centered at $H^{-1}(y)$. Given any embedding H, the second power $H^2 = H \circ H$ is always orientation-preserving. This is well known and follows from the properties of the degree of a composition

of maps, see for instance [10].
By a recurrent point $x_* \in \Omega$ we mean a point such that $H^{\sigma_n}(x_*) \to x_*$ for some increasing sequence of positive integers $\{\sigma_n\}$. Notice that the sequence $\{H^n(x_*)\}_{n\geq 0}$ could be unbounded.
Proof of Lemma 3.1. This is a well known result and we refer to [4, 9, 8] for the case of homeomorphisms. The proof for the case of embeddings is similar. We sketch it. Since Ω is homeomorphic to $\mathbb{R}^2$ we can restrict to the case $\Omega = \mathbb{R}^2$. For this reduction we are using the invariance of the fixed point index under topological conjugation. This is again a consequence of the properties of the degree of a composition.
Let C be a connected component of $\mathbb{R}^2 \setminus Fix(H)$ containing the recurrent point x_*. We can find a small and closed disk D centered at x_* and such that $D \subset C$ and $D \cap H(D) = \emptyset$. This is possible because x_* is not fixed. From Proposition 20 in chapter 3 of [15] we know that $H(D)$ is contained in C. The recurrence of x_* allows us to obtain an integer $\sigma \geq 2$ such that $y_* = H^\sigma(x_*)$ belongs to the interior of D. The points x_* and y_* lie on D and so it is possible to apply Proposition 17 in chapter 3 of [15] to deduce the existence of a translation arc α containing x_* and y_*. In consequence, y_* belongs to $\alpha \cap H^\sigma(\alpha)$ and Brouwer's Arc Translation Lemma is applicable. An adaptation to embeddings of the proof by Brown of this lemma can be found in [15].

We will also use the following result on minimal homeomorphisms.

Lemma 3.2 *Assume that K is a compact metric space and $\phi : K \longrightarrow K$ is a minimal homeomorphism. Then, for each integer $N \geq 1$, the set*

$$\mathcal{R}_N = \{k \in K : k \in L_\omega(k, \phi^N)\}$$

is dense in K.

We recall that ϕ is minimal if every point is transitive; that is, $L_\omega(k, \phi) = K$ for each $k \in K$.

Proof. First of all we prove that $\mathcal{R}_N$ is non-empty. The existence of minimal sets for general homeomorphisms implies that there exists a non-empty compact set $M \subset K$ that is minimal for ϕ^N. This means that $\phi^N(M) = M$ and if N is a compact subset of M with $\phi^N(N) = N$ then either $N = \emptyset$ or $N = M$. In particular, the set $L_\omega(m, \phi^N)$ has to coincide with M for each $m \in M$. This implies that M is contained in $\mathcal{R}_N$. The second observation is that $\mathcal{R}_N$ is invariant under ϕ. This is easily checked and leads to the identity $\phi(cl(\mathcal{R}_N)) = cl(\mathcal{R}_N)$. The minimality of ϕ implies that $cl(\mathcal{R}_N) = K$.

We need two more lemmas. The setting and the assumptions correspond to those of the main theorem.

Lemma 3.3 *The restricted homeomorphism $h_\Lambda : \Lambda \longrightarrow \Lambda$ is minimal.*

Proof. This is a particular case of Lemma 2 in [5] but we present the proof for completeness. Assume by contradiction that h is not minimal on Λ. Then there exists a point $p \in \Lambda$ such that the limit set $L_\omega(p,h)$ is a proper subset of Λ. Let us fix another point $q \in \Lambda \setminus L_\omega(p,h)$. The compact sets $L_\omega(p,h)$ and $\{q\}$ can be separated by two open sets U and V of $\mathbb{R}^2$. Since Λ is totally disconnected they can be chosen so that

- $\Lambda \subset U \cup V$,
- $cl(V) \cap cl(U) = \emptyset$,
- $L_\omega(p,h) \subset U$,
- $q \in V$.

Let V_* be the connected component of V containing q. Notice that this is also a component of the larger set $U \cup V$. The stability of Λ implies the existence of an open set $W \subset \mathbb{R}^2$ satisfying that

$$\Lambda \subset W \subset U \cup V, \quad h^n(W) \subset U \cup V$$

for each $n \geq 2$. Let W_* be the connected component of W containing p. By assumption we know that Λ contains a transitive point. All the points in the orbit will be transitive and therefore we know that transitive points are dense in Λ. Let $r \in \Lambda$ be a transitive point close enough to p in order to guarantee that $r \in W_*$. Let (σ_n) be an increasing sequence of positive integers with $h^{\sigma_n}(r) \longrightarrow q$. This implies that $h^{\sigma_n}(r)$ belongs to V_* for large n and so $h^{\sigma_n}(W_*) \cap V_* \neq \emptyset$. Since $h^{\sigma_n}(W_*)$ is a connected subset of $U \cup V$ we conclude that it must be contained in one component. Hence $h^{\sigma_n}(W_*) \subset V_*$. Finally, we observe that the iterates $h^{\sigma_n}(p)$ belong to $h^{\sigma_n}(W_*) \subset V_*$ and therefore $L_\omega(p,h)$ has to contain a point in $cl(V_*)$. This is a contradiction with the conditions imposed on U and V.

The last lemma needs some preliminary remarks on the topology of $\mathbb{R}^2$. Given an open set G in $\mathbb{R}^2$, the set $\widehat{G} \subset \mathbb{R}^2$ is the smallest open and simply connected set containing G. We refer to [14] for an elementary construction of this set. In [2], this set $\widehat{G}$ is called the topological hull of G. In fact its construction is purely topological and this explains the property $h(\widehat{G}) = \widehat{h(G)}$.

Lemma 3.4 *Given a point $p \in \Lambda$ and a disk D centered at p, there exists an integer $N \geq 1$ and an open and simply connected domain $\Omega \subset \mathbb{R}^2$ satisfying that*

$$p \in \Omega \subset D, \quad h^N(\Omega) \subset \Omega.$$

Proof. Since Λ is totally disconnected it is possible to find open sets A and B in $\mathbb{R}^2$ satisfying that

$$p \in A \subset int(D),$$

$$\Lambda \subset A \cup B,$$

$$cl(A) \cap cl(B) = \emptyset.$$

The open set $A \cup B$ is a neighborhood of Λ and the stability of this set implies the existence of another open set $V \subset \mathbb{R}^2$ with $\Lambda \subset V \subset A \cup B$ and $h^n(V) \subset A \cup B$ if $n \geq 1$. Define $W = \bigcup_{n \geq 0} h^n(V)$. This is also a neighborhood of Λ satisfying

$$\Lambda \subset W \subset A \cup B \text{ and } h^n(W) \subset W \text{ if } n \geq 1.$$

Let G be the connected component of W containing p. This component has to be contained in A, and hence in D. In consequence $\widehat{G}$ is also contained in D. We know by Lemma 3.3 that the limit set $L_\omega(p, h)$ is the whole Cantor set Λ. From here we deduce that $p \in L_\omega(p, h)$ and there exists an integer $N \geq 1$ such that $h^N(p)$ belongs to G. This implies that $G \cap h^N(G) \neq \emptyset$. But $h^N(G)$ is a connected set inside W and so it must be contained in one component of W. This component is obviously G. From $h^N(G) \subset G$ we obtain that $h^N(\widehat{G}) = \widehat{h^N(G)} \subset \widehat{G}$ and the set $\widehat{G}$ is the searched domain Ω.

Proof of Theorem 2.1. We fix $p \in \Lambda$ and a disk D of radius $\delta > 0$. From Lemma 3.4 we obtain a simply connected domain $\Omega \subset \mathbb{R}^2$ and an integer $N \geq 1$ with

$$p \in \Omega \subset D, \ \ h^N(\Omega) \subset \Omega.$$

Consider the orientation preserving embedding $H = h^{2N} : \Omega \longrightarrow \Omega$. We know from Lemmas 3.3 and 3.2 that the set

$$\mathcal{R}_{2N} = \{q \in \Lambda : q \in L_\omega(q, h^{2N})\}$$

is dense in Λ. In consequence we can find a point lying in $\Omega \cap \mathcal{R}_{2N}$. This point is recurrent for H and lemma 3.1 applies.

Proof of Corollary 2.1. We fix $\varepsilon > 0$. The stability of Λ as an invariant set of h guarantees the existence of $\delta_* > 0$ such that

$$dist(x, \Lambda) \leq \delta_* \Longrightarrow dist(h^i(x), \Lambda) \leq \frac{\varepsilon}{2}$$

for each $i \geq 0$. In particular, $\delta_* \leq \frac{\varepsilon}{2}$. Since Λ is compact it can be covered by a finite number of open balls $B_1, ..., B_k$ of radius δ_* and centered at points $p_1, ..., p_k$ lying in Λ. Next we apply Theorem 2.1 at each p_i to find Jordan curves Γ_1,...,Γ_k and integers $N_1, ..., N_k \geq 1$ such that $\Gamma_j \subset B_j$ and $d[id - h^{N_j}, \widehat{\Gamma}_j, 0] = 1$, $j = 1, ..., k$. Define $K = \bigcup_{j=1}^k (\Gamma_j \cup \widehat{\Gamma}_j)$ and $N = \max\{N_1, ..., N_k\}$.
We consider the family $\mathcal{F}_1$ composed by homeomorphisms $\widetilde{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying

$$\|h - \widetilde{h}\|_\infty := \sup_{x \in \mathbb{R}^2} \|h(x) - \widetilde{h}(x)\| \leq 1.$$

We need some properties of the iterates of $\widetilde{h}$ which are common to the whole family $\mathcal{F}_1$.
Claim 1: There exists a compact set $K_* \subset \mathbb{R}^2$ such that

$$\widetilde{h}^i(K) \subseteq K_*$$

for all $i = 0, 1, ..., N$ and for each $\widetilde{h} \in \mathcal{F}_1$.
Let $C_0 > 0$ be a large number so that K is contained in the ball of radius C_0 centered at the origin. By induction, we define

$$C_{i+1} = 1 + \max_{\|x\| \le C_i} \|h(x)\|, \qquad i \ge 0.$$

We claim that

$$\|\widetilde{h}^i(x)\| \le C_i \quad \text{if} \ \ x \in K.$$

Indeed, using the induction method,

$$\|\widetilde{h}^{i+1}(x)\| \le \|\widetilde{h}(\widetilde{h}^i(x)) - h(\widetilde{h}^i(x))\| + \|h(\widetilde{h}^i(x))\| \le$$

$$\|\widetilde{h} - h\|_\infty + \max_{\|x\| \le C_i} \|h(x)\|.$$

Claim 2: Given $\Delta > 0$ there exists $\delta_2 > 0$ such that $\widetilde{h} \in \mathcal{F}_1$ and $\|h - \widetilde{h}\|_\infty \le \delta_2$ implies that $\|h^i(x) - \widetilde{h}^i(x)\| \le \Delta$ if $x \in K$, $i = 1, ..., N$.
In view of Claim 1 we can find a modulus of continuity for h on K_*. This means a function $\omega : [0, \infty[\longrightarrow \mathbb{R}$ with $\lim_{r \to 0^+} \omega(r) = 0$ and

$$\|h(x) - h(y)\| \le \omega(\|x - y\|) \quad \text{if } x, y \in K_*.$$

Define $D_i = \max_{x \in K} \|\widetilde{h}^i(x) - h^i(x)\|$. Then, by induction, we prove that

$$D_{i+1} \le \|\widetilde{h} - h\|_\infty + \omega(D_i), \ \ i = 1, ..., N-1$$

and the claim follows easily. Notice that

$$\|\widetilde{h}^{i+1}(x) - h^{i+1}(x)\| \le \|\widetilde{h}(\widetilde{h}^i(x)) - h(\widetilde{h}^i(x))\| + \|h(\widetilde{h}^i(x)) - h(h^i(x))\|.$$

After these claims we are ready to prove the existence of $\widetilde{\Lambda}$. First we apply the continuity of the degree to find positive numbers $\eta_1, ..., \eta_k$ such that if

$$\|h^{N_j}(x) - \widetilde{h}^{N_j}(x)\| \le \eta_j, \ \ x \in \Gamma_j,$$

then

$$d[id - \widetilde{h}^{N_j}, \widehat{\Gamma}_j, 0] = d[id - h^{N_j}, \widehat{\Gamma}_j, 0] = 1.$$

Next we apply Claim 2 with $\Delta = \min\{\frac{\epsilon}{2}, \eta_1, ..., \eta_k\}$ and find $\delta_2 \in]0, \delta_*[$ such that the conclusion of the claim holds if $\|h - \widetilde{h}\|_\infty \le \delta_2$. The existence property of the degree allows us to select points $\widetilde{x}_j \in \widehat{\Gamma}_j$ such that $\widetilde{h}^{N_j}(\widetilde{x}_j) = \widetilde{x}_j$. The set

$$\widetilde{\Lambda} = \{\widetilde{h}^i(\widetilde{x}_j) : \ j = 1, ..., k, \ 0 \le i < N_j\}$$

is finite and invariant under $\widetilde{h}$. It remains to prove that $D_H[\Lambda, \widetilde{\Lambda}] \le \epsilon$. Assume first that p is a point in Λ. Since Λ is covered by $B_1, ..., B_k$ we find an index j such that $p \in B_j$. The ball B_j also contains the point $\widetilde{x}_j$. In consequence,

$$dist(p, \widetilde{\Lambda}) \le \|p - \widetilde{x}_j\| \le 2\delta_* \le \epsilon.$$

Consider now a point in $\widetilde{\Lambda}$, say $\widetilde{h}^i(\widetilde{x}_j)$. From

$$dist(\widetilde{x}_j, \Lambda) \leq \|\widetilde{x}_j - p_j\| \leq \delta_*,$$

we deduce that

$$dist(h^i(\widetilde{x}_j), \Lambda) \leq \frac{\epsilon}{2}.$$

Hence, using Claim 2 and this estimate, if $\|h - \widetilde{h}\|_\infty \leq \delta_2$,

$$\begin{aligned} dist(\widetilde{h}^i(\widetilde{x}_j), \Lambda) &\leq \|\widetilde{h}^i(\widetilde{x}_j) - h^i(\widetilde{x}_j)\| + dist(h^i(\widetilde{x}_j), \Lambda) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

4 Miscelaneous remarks

4.1 Invariant finite sets can be stable and non-persistent

A finite and invariant set Λ has to be composed by periodic points. We consider the simple case of a singleton $\Lambda = \{p\}$ and present an example of a stable fixed point that is not persistent as invariant set.
Consider the map

$$h : \mathbb{C} \longrightarrow \mathbb{C}$$
$$h(z) = z \exp(\frac{iy}{1 + |z|^2})$$

with $z = x + iy$. We have expressed it in complex notation but for many purposes it is more convenient the use of polar coordinates,

$$h : \begin{cases} \theta_1 = \theta + \frac{r}{1+r^2} \sin\theta, \\ r_1 = r. \end{cases}$$

It is not hard to prove that h is a real analytic diffeomorphism of the plane. We also observe that every disk of the type $|z| \leq$ *constant* is invariant under h and so the fixed point $z = 0$ is stable. An useful property of h is that $V(z) = \Re e\ z = x$ is a Lyapunov function. This means that

$$V(h(z)) \leq V(z)$$

for each $z \in \mathbb{C}$. Let us now consider the perturbed map $h_\varepsilon = T_\varepsilon \circ h$ where $T_\varepsilon(z) = z - \varepsilon$ is a horizontal translation with $\varepsilon > 0$. Again V is a Lyapunov function with

$$V(h_\varepsilon(z)) = V(h(z)) - \varepsilon \leq V(z) - \varepsilon.$$

More generally, if $n \geq 1$,

$$V(h_\varepsilon^n(z)) \leq V(z) - n\varepsilon$$

and so all the orbits for h_ε are unbounded. This shows that h_ε has no compact invariant sets. Since $\|h - h_\varepsilon\|_\infty = \varepsilon$, the maps h and h_ε are close and $\Lambda = \{0\}$ is

not persistent.
Incidentally, we notice that the set of fixed points $Fix(h)$ is the real axis and so $z = 0$ is not an isolated fixed point. This is no surprise because stable fixed points are persistent as soon as they are isolated in $Fix(h)$. This is a consequence of the main result in [7]: if $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orientation-preserving homeomorphism and $p = h(p)$ is a stable fixed point which is isolated in $Fix(h)$, then

$$d[id - h, \widehat{\Gamma}, 0] = 1$$

for each Jordan curve $\Gamma \subset \mathbb{R}^2$ with $\widehat{\Gamma} \cap Fix(h) = \{p\}$, $\Gamma \cap Fix(h) = \emptyset$. The case of orientation-reversing homeomorphisms was treated by Ruiz del Portal in [16].

4.2 Unstable Cantor sets can be isolated and non-persistent

With the help of a Denjoy homeomorphism on $\mathbb{S}^1$, it is possible to construct homeomorphisms $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ having an unique fixed point p_* and an invariant Cantor set Λ. In addition, the limit set of any point $x \in \mathbb{R}^2$ is either the fixed point, $L_\omega(x, h) = \{p_*\}$, or the Cantor set, $L_\omega(x, h) = \Lambda$. In particular, Λ is minimal. The details of the construction can be found in [11]. The map h has not periodic points and this implies that

$$d[id - h^N, \widehat{\Gamma}, 0] = 0$$

for any $N \geq 1$ and any Jordan curve $\Gamma \subset \mathbb{R}^2$ such that p_* lies in the exterior, that is, $p_* \notin \Gamma \cup \widehat{\Gamma}$. This example shows that the conclusion of Theorem 2.1 does not hold if we drop the stability assumption. In the example constructed in [11], the fixed point was placed at the origin, $p_* = 0$, and the Cantor set was inside the unit circumference, $\Lambda \subset \mathbb{S}^1$. Moreover the Euclidean norm $V(x) = \|x\|$ was a Lyapunov function satisfying

$$V(h(x)) < V(x)$$

if $x \in \mathbb{R}^2 \backslash (\Lambda \cup \{0\})$. Consider the perturbed homeomorphism $h_\varepsilon = D_\varepsilon \circ h$, with $\varepsilon > 0$ and

$$D_\varepsilon(x) = \begin{cases} (1-\varepsilon)x, & \text{if } \|x\| \leq 2; \\ (1 - 3\varepsilon + \varepsilon\|x\|)x, & \text{if } 2 \leq \|x\| \leq 3; \\ x, & \text{if } \|x\| \geq 3. \end{cases}$$

Then $\|h_\varepsilon - h\|_\infty = 2\varepsilon$ and

$$V(h_\varepsilon(x)) < V(x)$$

if $x \in \mathbb{R}^2 \backslash \{0\}$. La Salle's invariance principle implies that the origin is a global attractor for h_ε. This shows that Λ is not persistent.
The dynamics of h_Λ in the preceding example is of Denjoy type, a case that can be excluded if Λ is stable. The reason for this exclusion lies in a result by Buescu and Stewart [5] implying that stable Cantor sets are conjugate to adding machines. The family of adding machines is composed by certain explicit maps describing all possible almost periodic dynamics on a Cantor set. Denjoy dynamics is presented in [13] as the prototype of minimal dynamics that is not almost periodic and so it is not conjugate to an adding machine.

4.3 Adding machines cannot be isolated

In [17], Thomas obtained a result on the dynamics of solenoids in 3D flows that can be adapted to a 2D discrete setting for adding machines. Assume now that $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a C^1 diffeomorphism that is orientation-preserving and has an invariant Cantor set Λ such that h_Λ is almost periodic. Then it is possible to construct a T-periodic differential equation in the plane such that h is the Poincaré map. See [12] for an explicit construction. In this way, we obtain a $\mathcal{C}^1$ flow on the manifold $M = (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^2$ and the results in [17] are applicable. The closure of the orbit starting at any point of Λ ia a solenoid $S \subset M$ and Theorem 3 in [17] implies that S is not isolated as an invariant set of the flow. The invariant sets accumulating on S must intersect the global section $M_0 = \{0\} \times \mathbb{R}^2$ and so Λ cannot be isolated as an invariant set of h. Notice that the result by Bell and Meyer does not follow from [17] and [5] because in principle one could find invariant sets without periodic points. The smoothness of h was needed in [17] to work with a smooth isolating block. At the end of that paper it is mentioned that the smoothness hypotheses can be weakened. It seems reasonable to expect that the previous discussion can be extended to homeomorphisms. We do not know if the conclusion of Bell and Meyer is also valid when the assumption of stability for Λ is replaced by almost periodicity.

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