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THE NUMBER OF LIMIT CYCLES OF A GENERALIZED ABEL EQUATION

TESIS DOCTORAL

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La presente memoria, ***The Number of Limit Cycles of a Generalized Abel Equation***, ha sido realizada por Naeem M. H. Alkoumi bajo la dirección de los Drs. Rafael Ortega Ríos y Pedro José Torres Villarroya, del Departamento de Matemática Aplicada de la Universidad de Granada, para la obtención del título de Doctor en Ciencias Matemáticas por la Universidad de Granada.

V. B. Los Directores **El Doctorando**

♡♡♡♡♡♡

To my parents, and to my wife Samar for her unwavering care and patience rooted in love.

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Contents

Chapter 1

Introduction

The general purpose of this work is to study the number of isolated periodic solutions (limit cycles) of the polynomial differential equation

$$u' = a_n(t)u^n + a_{n-1}(t)u^{n-1} + \ldots + a_0(t), \tag{1.1}$$

where the coefficients $a_i(t)$, $i = 0, 1, \ldots, n$, are continuous and T-periodic functions for some $T > 0$. Classically, the interest on this problem comes from the study of the number of isolated closed orbits of a planar polynomial vector field, which is a part of Hibert's sixteen problem [20], also listed as Problem 7 by S. Smale in [41]. This line of research has attracted the attention of a large number of mathematicians, giving as a result a huge number of papers concerning this topic, which makes very difficult if not impossible any attempt of a systematic review. Therefore, we will limit ourselves to give a general overview of some aspects of particular interest for our purposes.

For $n = 1$, equation (1.1) is a linear equation, consequently having at most one isolated periodic solution, whereas for $n = 2$ it is a Riccati equation with at most two isolated periodic solutions. When $n = 3$, equation (1.1) is known as the Abel differential equation. This is the first non-trivial case. It was shown in [37] that the Abel differential equation has at most three periodic solutions when $a_3 > 0$. However, for a leading coefficient a_3 with indefinite sign, Lins Neto [31] gave examples with an arbitrary number of limit cycles. Such a case is important because local questions related to Hilbert's sixteenth problem (bifurcation of small-amplitude isolated periodic solution and center conditions) are reduced to polynomial equations in which the leading coefficient changes sign. The examples constructed by Lins-Neto can be extended to higher-order polynomial equations, even with a constant leading coefficient a_n [31, 13]. At this point, to get a more accurate information on the number of limit cycles, most of the papers in the literature require that only some the polynomial coefficients $a_k(t)$ do not vanish, in such a way that the polynomial nonlinearity has only three or four terms (see for instance [9, 4, 6, 14, 13, 26, 36, 40, 42, 43]).

More concretely, the equation with three terms

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3}, \tag{1.2}$$

where $n_1 > n_2 > n_3$, has been considered on some related works. From now on, a continuous function $f : [0, T] \to \mathbb{R}$ is said to have a definite sign if it is not null and either $f(t) \geq 0$ or $f(t) \leq 0$, and we write $f \succ 0$ in the first case and $f \prec 0$ in the second case. Gasull and Guillamon [13] proved that if $n_3 = 1$ and $a_{n_2}(t)$ or $a_{n_3}(t)$ have a definite sign, then equation (1.2) has at most two positive limit cycles. This gives a total maximum number of five limit cycles by the change $y = -u$, since $u = 0$ is always a solution. In addition, if $n_3 > 1$ and only one of the coefficients has a definite sign, examples are given with an arbitrary number of limit cycles.

Therefore, for equations with 3 or more monomials, in order to obtain bounds on the number of limit cycles, it is natural in some sense to assume that at least two coefficients have a definite sign. A result following this idea was proved by Alwash in [9], where it is proved that if $n \geq 3$ and $a_{n-2}(t) \leq 0$, the equation

$$u' = u^n + a_{n-1}(t)u^{n-1} + a_{n-2}(t)u^{n-2}, \tag{1.3}$$

has at most one positive limit cycle.

In Chapter 2, new results are proved on the maximum number of isolated T-periodic (limit cycles) of a first order polynomial differential equation with four terms

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m. \tag{1.4}$$

This case has been considered very recently in [4]. Our method of proof is different and it is based on the known result that a sign on derivative up to order three of the nonlinearity on a given region gives a bound on the number of limit cycles (see for instance [13, 24, 39]), but we exploit the fact that this sign is not invariant under changes of variables. We have found that a similar idea has been used very recently in [19] to obtain different results. We will pay some attention to compare our results with the available literature (specially [9, 4]). It should be emphasized the particularity that with our results it is possible to consider the case of negative powers, a situation which has been scarcely explored in the related literature. It is worthwhile to consider this case for applications to the study of the number of periodic solution in polynomial planar systems on the cylinder, as we will show in more detail with examples inspired by [7]. The results of this chapter are contained in the paper [3], recently accepted for publication.

Comparatively, the number of papers providing an explicit bound on the number of periodic solution in the general case where all the coefficients of the polynomial nonlinearity are present is small. The most known result is due to Ilyashenko [23], who proved that the number of real periodic solutions does not exceed the bound

$$8\exp\left\{(3C+2)\exp\left[\frac{3}{2}(2C+3)^n\right]\right\},$$

where $C > 1$ is a uniform upper bound for $|a_k(t)|$. The importance of this result relies in the fact that it states the finiteness of the number of periodic solutions more that in the explicit estimate, which is rather conservative. Later, Calanchi and Ruf [11], proved that there are at most n periodic solutions if n is odd, the leading term is fixed and the remaining terms are small enough. Finally, in a very recent work [10], Alwash proved related results giving more precise information about the number of limit cycles. On the other hand, Panov [36] demonstrated that there is no upper bound for the number of periodic solutions by proving that the set of return maps of these equations are dense in the space of orientation preserving homeomorphisms. It follows from these examples that there is no upper bound, in terms of n only, for the number of periodic solutions of equation (1.1).

In Chapter 3, we prove new results about the number of isolated periodic solutions of a first order differential equation with a polynomial nonlinearity when all the coefficients are present. To this aim, we will write (1.1) in the more general form

$$x' = x^m \sum_{k=0}^{n} a_k(t)x^k, \tag{1.5}$$

where $m \in \mathbb{Z}$. Thus, once again we consider a polynomial nonlinearity with possible negative powers. We will justify the interest of this case with the study of a new family of planar vector fields

$$x' = -y^{2p-1} + \frac{x}{q}P(x,y) \quad , \quad y' = x^{2q-1} + \frac{y}{p}P(x,y) \tag{1.6}$$

where $P(x, y)$ is a polynomial and p, q are natural numbers. When $p = q = 1$, it is a rigid system (see for instance [15, 16, 17]). Up to our knowledge, such a generalized family has not been considered in previous works. The results of this paper correspond to the paper [2], which has been submitted for publication.

Finally in Chapter 4, we show that the method of Ilyashenko [23] can also be applied to more general equations. As mentioned before, assuming that all the coefficients $a_j(t)$ are dominated by a common bound $C > 1$, Ilyahenko [23] obtained an explicit estimate (depending on n and C). The purpose of this Chapter is to show that this method can also be applied to more general equations. As a model we will consider the equation

$$x' = x^3 + 10\sin x + p(t), \quad p(t+1) = p(t) \tag{1.7}$$

and we will obtain an estimate on the number of periodic solutions depending only on

$$\|p\|_\infty = \max_{t\in[0,1]} |p(t)| \le C. \tag{1.8}$$

The basic tool employed in [23] is the so called Jensen's Lemma. This is a result in Complex Analysis that allows to estimate the number of zeros of a holomorphic function in a domain D in terms of the behavior on the boundary ∂D. The standard version of this Lemma assumes that D is a disc but the result can be transported to other domains via Riemann's Theorem on conformal mappings. The approach in [23] was to consider certain domains with the shape of a stadium and to employ ideas taken from hyperbolic geometry to estimate some quantities related to the Riemann's mapping for these domains. Our approach will be more straightforward, we will consider the explicit Christoffel-Schwarz formula mapping the unit disc onto a rectangle. This will allow us to state a version of Jensen's Lemma for the rectangle where all the quantities involved are explicit and expressed in terms of elliptic functions. Once we have obtained this result in Complex Analysis we will present a modified version of Ilyashenko's technique applicable to equation (1.7). It seems to us that the results are presented so that the method is flexible and can be applied to many other equations. The original results contained in this paper correspond to the preprint [1].

Chapter 2

On the Number of Limit Cycles of a Generalized Abel Equation

This chapter is motivated by some recent results on the number of limit cycles of the first order differential equation with polynomial nonlinearity

$$x' = \sum_{i=0}^{n} a_i(t)x^i, \tag{2.1}$$

where the coefficients a_i are continuous and T-periodic functions for some $T > 0$. This chapter is divided into four Sections. In Section 2.1 we will present some known results. Section 2.2 we will prove a new result with 4 monomials and discuss some consequences. The method of proof is based on the known result that a sign on the derivative up to order three of the nonlinearity on a given region gives a bound on the number of limit cycles (see for instance [13, 24, 39]), but we exploit the fact that this sign is not invariant under changes of variables. In Section 2.3, we combine this technique with upper and lower solutions in order to get multiplicity results for the fourth-order differential equation. Finally, in the last section the main results are applied to some specific examples of polynomial planar systems in order to get information on the maximum number of limit cycles.
To conclude this introduction, we fix some notation. Following [13], we say that equation (2.1) is of $(d_1, d_2, ..., d_r)$ type if $a_j \equiv 0$ for

all $j \neq d_1, ..., d_r$. The set of continuous and T-periodic function is denoted as $\mathbb{C}_T$. Given $a \in \mathbb{C}_T$, we write $a \succ 0$ if $a(t) \geq 0$ for all $t \in [0, T]$ and $a(t_0) > 0$ for some t_0. We write $a \prec 0$ if $-a \succ 0$. D^+ denotes the subset of functions $x \in \mathbb{C}_T$ such that $x(t) > 0$ for all t, whereas D^- denotes the subset of functions $x \in \mathbb{C}_T$ such that $x(t) < 0$ for all t.

2.1 Preliminary Results

Let us consider a first order differential equation

$$x' = \sum_{k=0}^{n} a_k(t)x^k, \tag{2.2}$$

where a_k are continuous and T-periodic functions for some $T > 0$. In this section we collect some elementary results employed on the proofs of many theorems.

We are interested in the problem of the existence and multiplicity of limit cycles of equation (2.2). Let us consider a general first order equation

$$x' = g(t, x). \tag{2.3}$$

with g continuous and T-periodic in t and with continuous derivatives in x up to order 3.

Definition 1 *A periodic solution $x = x(t)$ is said to be isolated if there exists an $\epsilon > 0$ such that there are no periodic solutions other than $x = x(t)$ in the region*

$$\{(t, x) : x(t) - \epsilon \leq x \leq x(t) + \epsilon\}$$

Definition 2 *A T-periodic solution of equation* (2.3) *which is isolated in the set of all the periodic solution is called a limit cycle.*

Let $x(t, c)$ be the unique solution of (2.3) with initial condition $x(0, c) = c$. Then $x(t, c)$ is a limit cycle of (2.3) if and only if c is an isolated zero of the displacement function $q(c) := x(T, c) - c$.

Definition 3 *A given limit cycle $x(t, c)$ is said* hyperbolic *if the characteristic exponent*

$$\delta(T, c) = \int_0^T \frac{\partial g(t, x(t, c))}{\partial x} dt$$

is different from zero. As a consequence, a limit cycle is asymptotically stable if $\delta < 0$ and unstable if $\delta > 0$.

Definition 4 *A T-periodic function α is called a strict lower (resp. upper) solution of equation* (2.3) *if*

$$\alpha'(t) < g(t, \alpha(t)) \quad (\textit{resp. } \alpha'(t) > g(t, \alpha(t))),$$

for all t.

Lemma 1 *A T-periodic solution does not intersect any eventual strict upper or lower solution.*

Proof. Suppose that $\alpha(t)$ is a lower solution of equation (2.3). By contradiction, assume that for a given T-periodic solution $x(t)$ there exists t_0 such that $x(t_0) = \alpha(t_0)$. Define the T-periodic function $d(t) = x(t) - \alpha(t)$, then

$$d'(t_0) = x'(t_0) - \alpha'(t_0) > g(t_0, x(t_0)) - g(t_0, \alpha(t_0)) = 0.$$

But then d should be strictly increasing in every zero, which it is impossible since d is T-periodic and continuous. Therefore d has no zeroes. The proof for the upper solution is analogous.

Lemma 2 *Let $\alpha(t)$ be a strict lower (resp. upper) solution of* (2.3)*. Then $q(\alpha(0)) > 0$ (resp. $q(\alpha(0)) < 0$).*

Proof. Let us assume that $\alpha(t)$ is a strict lower solution and define $d(t) = x(t, \alpha(0)) - \alpha(t)$. Then, $d(0) = 0$ and $d'(0) = g(0, \alpha(0)) - \alpha'(0) > 0$. Let us prove that $d(t) > 0$ for all $t > 0$. By contradiction, let us suppose that there exists $t_1 > 0$ such that $d(t_1) = 0$ and $d(t) > 0$ for all $0 < t < t_1$. Then $x(t_1, \alpha(0)) = \alpha(t_1)$ and $d'(t_1) = g(t_1, \alpha(t_1)) - \alpha'(t_1) > 0$, which is a contradiction with the fact that t_1 is the first positive zero. Therefore, $d(T) = x(T, \alpha(0)) - \alpha(T) > 0$, but because $\alpha(t)$ is T-periodic, this means that $q(\alpha(0)) = x(T, \alpha(0)) - \alpha(0) > 0$. The proof for the upper solution is analogous.

Proposition 1 *Let us assume that $\alpha(t), \beta(t)$ are strict lower and upper solutions such that $\alpha(t) < \beta(t)$ (resp $\alpha(t) > \beta(t)$). Then, there exists a T-periodic solution $x(t)$ such that $\alpha(t) < x(t) < \beta(t)$ (resp. $\beta(t) < x(t) < \alpha(t)$).*

Proof. Since the displacement function is continuous, it is a direct consequence of Bolzano's theorem and Lemmas 2 and 1.

Lemma 3 ([10, 30]) *Consider the real differential equation*

$$x' = g(t,x) = x^n + a_{n-1}(t)x^{n-1} + \ldots + a_1(t)x + a_0. \tag{2.4}$$

The derivative of the displacement function are given by

$$q'(c) = \exp(\delta(T,c)) - 1, \tag{2.5}$$

$$q''(c) = \exp(\delta(T,c)) \int_0^T D(t,c)dt, \tag{2.6}$$

$$q'''(c) = \exp(\delta(T,c)) \left[\frac{3}{2}(G(T,c))^2 + \int_0^T (\exp(\delta(T,c)))^2 \frac{\partial^3 g(x(t,c),t)}{\partial x^3} dt\right] \tag{2.7}$$

where

$$D(t,c) = \exp(\delta(t,c)) \frac{\partial^2 g(x(t,c),t)}{\partial x^2}$$

and

$$G(t,c) = \int_0^t D(\tau,c)d\tau.$$

These formulae imply that for $k \leq 3$, if $\frac{\partial^k g(t,x)}{\partial x^k}$ does not change sign on an interval then the kth derivative of q does not change sign on that interval.

Proof. $q(c) = x(T,c) - c = \int_0^T dx = \int_0^T g(t,x(t,c))dt$, then

$$q'(c) = \frac{\partial x(T,c)}{\partial c} - 1 = \int_0^T \frac{\partial g(t,x(t,c))}{\partial x} . \frac{\partial x(t,c)}{\partial c} dt.$$

By assuming that $U(t) = \frac{\partial x(t,c)}{\partial c}$, we get a first order linear differential equation

$$U'(t) - \frac{\partial g(t,x(t,c))}{\partial x} U(t) = 0, \quad U(0) = \frac{\partial x(0,c)}{\partial c} = 1,$$

so the solution is

$$U(T) = \exp\left(\int_0^T \frac{\partial g(t, x(t, c))}{\partial x} dt\right).$$

Therefore

$$q'(c) = \exp(\delta(T, c)) - 1.$$

To show (2.6), differentiating (2.5), we have

$$q''(c) = \exp(\delta(T, c)) \int_0^T \frac{\partial^2 g(x(t, c), t)}{\partial x^2} . \frac{\partial x}{\partial c} dt,$$

but $\frac{\partial x(T,c)}{\partial c} = q'(c) + 1 = \exp(\delta(T, c))$. Therefore

$$q''(c) = \exp(\delta(T, c)) \int_0^T D(t, c) dt.$$

Further differentiation leads to

$$\begin{aligned}
q'''(c) = \exp(\delta(T, c)) & \left[\int_0^T \left(\exp(\delta(t, c)) \frac{\partial^2 g(x(t, c), t)}{\partial x^2} . \right.\right. \\
& \left.\left. \int_0^t \frac{\partial^2 g(x(s, c), s)}{\partial x^2} . \frac{\partial x}{\partial c} ds + \exp(\delta(t, c)) \frac{\partial^3 g(x(t, c), t)}{\partial x^3} . \frac{\partial x}{\partial c}\right) dt\right] \\
& + \left(\int_0^T \frac{\partial^2 g(x(t, c), t)}{\partial x^2} \exp(\delta(t, c)) dt\right) . \\
& \left(\exp(\delta(T, c)) \int_0^T \frac{\partial^2 g(x(t, c), t)}{\partial x^2} . \frac{\partial x}{\partial c} dt\right) . \\
& = \exp(\delta(T, c)) \left[\left(\int_0^T D(t, c) dt\right)^2\right. \\
& + \int_0^T (\exp(\delta(t, c)))^2 \frac{\partial^3 g(x(t, c), t)}{\partial x^3} dt \\
& \left. + \int_0^T \left(D(t, c) \int_0^t D(s, c) ds\right) dt\right].
\end{aligned}$$

The last term inside the square brackets is, by straightforward integration by parts,

$$\frac{1}{2}\left[\int_0^T D(t, c) dt\right]^2 = \frac{1}{2}(G(T, c))^2,$$

so that

$$q'''(c) = \exp(\delta(T,c)) \left[\frac{3}{2}(G(T,c))^2 + \int_0^T (\exp(\delta(t,c)))^2 \frac{\partial^3 g(x(t,c),t)}{\partial x^3} dt \right].$$

Proposition 2 ([39]) *Let us consider a general first order equation*

$$x' = g(t,x), \tag{2.8}$$

with g continuous and T-periodic in t. Fix $k \in \{1,2,3\}$. Let J be an open interval and let us assume that $g(t,x)$ has continuous partial derivatives $\frac{\partial^k}{\partial x^k} g(t,x)$ for all $(t,x) \in [0,T] \times J$. If $\frac{\partial^k}{\partial x^k} g(t,x) \geq 0$ for all $(t,x) \in [0,T] \times J$ (resp. $\frac{\partial^k}{\partial x^k} g(t,x) \leq 0$ for all $(t,x) \in [0,T] \times J$), then the equation (2.8) *has at most k limit cycles with range contained in J.*

Proof. For $k = 1$, the first derivative of the displacement function

$$q'(c) = \exp(\delta(T,c)) - 1 \geq 0.$$

Hence $q(c)$ is increasing, so it has at most one isolated zero.
For $k = 2$, the second derivative of the displacement function

$$q''(c) = \exp(\delta(T,c)) \int_0^T \exp(\delta(t,c)) g_{xx}(x(t,c),t) dt \geq 0,$$

hence $q(c)$ is concave up, so it has at most 2 zeros.
Finally, for $k = 3$, $q'''(c) \geq 0$, so $q'(c)$ is concave up, hence $q'(c)$ has at most 2 zeros and it implies that $q(c)$ has at most 3 zeros, for between two zeros of q there is always one zero of q'. Note that when $\frac{\partial^3}{\partial x^3} g(t,x) \leq 0$, we use the change of variable $y = -x(-t)$.

2.2 The equation with four monomials

As it was observed in the Introduction, it is natural in some sense to assume that two coefficients of the polynomial nonlinearity have a definite sign. A result following this idea was proved by Alwash in [9], where it is proved that if $n \geq 3$ and $a_{n-2}(t) \leq 0$, the equation

$$u' = u^n + a_{n-1}(t)u^{n-1} + a_{n-2}(t)u^{n-2}, \tag{2.9}$$

has at most one positive limit cycle.
A. Alvarez, J. L. Bravo, M. Fernández [4] study the equation of the form

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m, \tag{2.10}$$

with $m = 1$ and $n_1 > n_2 > n_3 > m := 1$. They obtain two new criteria for setting bounds to the number of limit cycles.
In [4], applying the same ideas as Gasull and Guillamon [13], the following result is obtained.

Theorem 1 ([4]) *Consider the differential equation*

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m, \tag{2.11}$$

where $n_1 > n_2 > n_3 > m = 1$. Suppose that $a_{n_1}(t)$ and $a_{n_2}(t)$, or $a_{n_2}(t)$ and $a_{n_3}(t)$ have the same definite sign, or that $a_{n_1}(t)$ and $a_{n_3}(t)$ have opposite definite sign. Then, (2.11) *has at most two positive limit cycles. If moreover $a_m(t)$ has null integral over $[0, T]$, then* (2.11) *has at most one positive limit cycle.*

The second criterion states that if

$$a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} - u'(t) \tag{2.12}$$

has definite sign, where

$$u(t) = \left(\frac{(n_1 - n_3)a_{n_3}(t)}{(n_2 - n_1)a_{n_2}(t)} \right)^{\frac{1}{n_2 - n_3}} \tag{2.13}$$

then equation (2.10) with $m = 0$ has at most two positive limit cycles.
Here we prove some related results which can be seen as a complement to the previous ones. Our main result is as follows.

Theorem 2 *Let us assume that a_{n_1} has a definite sign.*
Fix $n_1, n_2, n_3, m \in \mathbb{Z}$ entire numbers such that $n_1 > n_2 > n_3$ verify the condition

$$n_1 - 2n_2 + n_3 = 0. \tag{2.14}$$

If

$$\Delta = a_{n_2}^2 (m - n_2)^2 - 4a_{n_1} a_{n_3} (m - n_1)(m - n_3) \leq 0, \tag{2.15}$$

then (2.11) *has at most one positive limit cycle.*

Proof. By means of the change in the independent variable $\tau = -t$, we can assume that $a_{n_1} \succ 0$ without loss of generality. Let us first consider the case $m = 1$. We write the equation as

$$u' = uF(t, u),$$

where

$$F(t, u) = a_{n_1} u^{n_1 - 1} + a_{n_2} u^{n_2 - 1} + a_{n_3} u^{n_3 - 1} + a_1.$$

By using the change of variable $u = e^x$, we get

$$x' = F(t, e^x) := g(t, x). \tag{2.16}$$

Now,

$$\begin{aligned} g_x(t, x) &= e^x F_x(t, e^x) = e^{(n_3 - 1)x} [(n_1 - 1) a_{n_1} e^{(n_1 - n_3)x} \\ &+ (n_2 - 1) a_{n_2} e^{(n_1 - n_2)x} + (n_3 - 1) a_{n_3}]. \end{aligned}$$

If we call $S = e^{(n_1 - n_2)x}$, then $S^2 = e^{(n_1 - n_3)x}$ as a result of (2.14). Therefore, $g_x(t, x)$ can be written as

$$g_x(t, x) = e^{(n_3 - 1)x} [(n_1 - 1) a_{n_1} S^2 + (n_2 - 1) a_{n_2} S + (n_3 - 1) a_{n_3}].$$

The last factor is a quadratic polynomial with negative discriminant by hypothesis (2.15). Hence by Proposition 2 there exists at most one limit cycle of equation (2.16), which correspond to at most one positive limit cycle of (2.11).

For $m \neq 1$, the equation is written as

$$u' = u^m F(t, u).$$

Now the adequate change is $u = x^\alpha$, satisfying $(m-1)\alpha + 1 = 0$. This change is well defined for positive solutions and keeps the number of positive limit cycles. It leads to

$$x' = \frac{1}{\alpha} F(t, x^\alpha) := g(t, x).$$

The derivative is

$$\begin{aligned} g_x(t,x) &= x^{\alpha-1}F_x(t,x^\alpha) \\ &= \alpha x^{(n_3-m+1)\alpha-2}[a_{n_1}(n_1-m)S^2 \\ &+ a_{n_2}(n_2-m)S + a_{n_3}(n_3-m)], \end{aligned}$$

where $S = x^{(n_1-n_2)\alpha}$. The conclusion is analogous.

Once the proof is finished, it is good to state some clarifying consequences for the comparison between this result and those previously published. As it is remarked in the Introduction, the first original feature is that negative powers are possible. About condition (2.14) , it is easy to realize that it is equivalent to impose that three of the terms of the equation have powers following an arithmetic sequence, that is, there exist $r \in \mathbb{N}, \beta \in \mathbb{Z}$, such that

$$n_1 = 2r + \beta, n_2 = r + \beta, n_3 = \beta.$$

If $r = 1$ we get consecutive powers. In spite of that, m is free so the result is quite flexible and give a whole family of new criteria.

Next, we will compare with the related literature through some corollaries. The first one generalizes the result by [10] mentioned in the Introduction.
In particular, we obtain the results of Alwash.

Corollary 1 *If $n_1 > n_2 > n_3$ holds the condition (2.14) and a_{n_1}, a_{n_3} have opposite definite signs, then the equation* (2.10) *has at most two nontrivial limit cycles, at most one positive and at most one negative.*

Proof. Take $m = n_2$ and apply Theorem 2, then (2.11) has at most one positive limit cycle. For the negative one, make the change $y = -x$.

For comparison with Theorem 1, note that it does not cover the case of $a_{n_1}(t)$ and $a_{n_3}(t)$ with the same definite sign. In fact, in [4] the authors provide examples under this assumption with at least three limit cycles. Now we get the following complementary result.

Corollary 2 *Fix $n_1 > n_2 > n_3 > m = 1$ verifying (2.14) and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$a_{n_1}(t)a_{n_3}(t) \geq \frac{(n_2-1)^2}{4(n_1-1)(n_3-1)}a_{n_2}(t)^2$$

for all t, then (2.10) *has at most one positive limit cycle.*

The proof is direct. Other variant is the following one.

Corollary 3 *Fix $n_1 > n_2 > n_3$ verifying* (2.14) *and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$4a_{n_1}(t)a_{n_3}(t) > a_{n_2}^2(t),$$

for all t, then there exists $m_0 > 0$ such that if $|m| > m_0$, then (2.10) *has at most one positive limit cycle.*

The number m_0 is explicitly computable, for the proof follows easily from a pass to the limit in condition (2.15).

We finish the section by pointing out that Theorem 2 and its corollaries can be complemented with stability and exact multiplicity information by using the explicit behavior near the origin, as it is done for instance in [6, 13].

2.3 The complete fourth-order equation

The aim of this section is to provide some sufficient conditions for limiting the number of limit cycles of the (4,3,2,1,0)-polynomial equation

$$u' = a_4(t)u^4 + a_3(t)u^3 + a_2(t)u^2 + a_1(t)u + a_0(t). \tag{2.17}$$

In [13, Theorem 5], it is proved that (2.17) with $a_4(t) \equiv 1$ may have an arbitrary number of T-periodic solutions. On the other hand, when $a_0 \equiv 0$, the main result of [4] implies that (2.17) has at most two positive T-periodic solutions if $a_4, a_3 \succ 0$, or $a_3, a_2 \succ 0$, or $a_4 \succ 0 \succ a_2$. Our results can be seen as a partial counterpart.

Our first result is very similar to some results in [8] for the fifth-order homogeneous equation.

Theorem 3 *If $a_2, a_4 \succ 0$ and $a_3^2 - \frac{8}{3}a_4a_2 \leq 0$, equation* (2.17) *has at most two limit cycles.*

Proof. The second derivative with respect to $u$of the right-hand side of equation (2.17) is

$$12a_4(t)u^2 + 6a_3(t)u + 2a_2(t).$$

Looking this as a second-order polynomial, the discriminant is

$$36a_3^2 - 96a_4a_2.$$

By hypothesis, this is negative, then by Proposition 2 there exist at most two limit cycles.

On the other hand, next results are of a different nature.

Theorem 4 *Let us assume that* $a_0(t)a_4(t) > 0$ *for all* t. *If*

$$4\sqrt[4]{a_0 a_4^3} + a_3 \geq 0,$$

equation (2.17) *has at most two positive limit cycles.*

Proof. We can assume without loss of generality that a_0, a_4 are both strictly positive functions (if $a_0, a_4 \prec 0$, we can use the change $v = -u$). After the change $x = \frac{1}{u}$, the equation is

$$x' = -xF(t, \frac{1}{x}),$$

where

$$F(t,x) = a_4(t)x^3 + a_3(t)x^2 + a_2(t)x + a_1(t) + \frac{a_0(t)}{x}.$$

By defining $g(t,x) := -xF(t, \frac{1}{x})$, the second derivative is

$$g_{xx}(t,x) = \frac{-1}{x^3} F_{xx}(t, \frac{1}{x}).$$

Therefore, the proof is reduced to show that $F_{xx}(t,x)$ is positive for $x > 0$. It turns out that

$$F_{xx}(t,x) = 6a_4(t)x + 2a_3(t) + \frac{2a_0(t)}{x^3}.$$

Since, a_0, a_4 are strictly positive, the function $6a_4(t)x + \frac{2a_0(t)}{x^3}$ attains its global minimum at $a_0(t)^{1/4}a_4(t)^{-1/4}$. Hence, for any $x > 0$

$$F_{xx}(t,x) \geq 8a_0(t)^{1/4}a_4(t)^{3/4} + 2a_3(t) \geq 0$$

and the proof is done by a direct application of Proposition 2.

Theorem 5 *Let us assume that* $a_4(t) > 0$ *for all* t. *Then, equation* (2.17) *has at most three limit cycles verifying the property*

$$u(t) > \frac{-a_3(t)}{4a_4(t)} \text{ for all } t. \tag{2.18}$$

Analogously, equation (2.17) *has at most three limit cycles verifying the property*

$$u(t) < \frac{-a_3(t)}{4a_4(t)} \text{ for all } t. \tag{2.19}$$

Proof. Firstly, we consider the case that the function $\varphi(t) := \frac{-a_3(t)}{4a_4(t)}$ has a continuous derivative. By introducing the change $x = u - \varphi$ in equation (2.17), the resulting equation is

$$\begin{aligned} x' = a_4(t)(x+\varphi)^4 + a_3(t)(x+\varphi)^3 + a_2(t)(x+\varphi)^2 \\ +a_1(t)(x+\varphi) + a_0(t) + \varphi'(t). \end{aligned} \tag{2.20}$$

The third derivative of the right-hand side of equation is

$$g_{xxx}(t,x) = 24a_4(t)x.$$

Then $g_{xxx}(t,x) > 0$ if $x > 0$. By Proposition 2, with $J =]0, +\infty[$, there are at most three positive limit cycles of equation (2.20). Going back to the original equation, it gives at most three limit cycles of equation (2.17) verifying (2.18).

Now, let us prove the general case of a continuous function $\varphi(t)$ by a limiting argument. The set C_T^1 of T-periodic functions with continuous derivatives is dense on the set C_T of T-periodic and continuous functions. Then, it is easy to prove that there exists a sequence $\{\varphi_n(t)\} \subset C_T^1$ converging uniformly to $\varphi(t)$ and such that $\varphi_n(t) > \varphi(t)$ for all n, t. Using the previous reasoning for each $\varphi_n(t)$ and passing to the limit we get the desired result.

In the same way, it is proved that there are at most three limit cycles verifying (2.18).

Of course, in this latter result additional T-periodic solutions crossing $\frac{-a_3(t)}{4a_4(t)}$ may appear. This possibility is excluded with an additional assumption.

Corollary 4 *Let us assume that* $\frac{-a_3(t)}{4a_4(t)}$ *is strict an upper (resp. lower) solution of equation* (2.17). *Then, there are at most* 6 *limit cycles.*

Proof. If $\frac{-a_3(t)}{4a_4(t)}$ is an upper (or lower) solution, by Lemma 1 a T-periodic solution can not cross it, so there are at most 3 of them above and at most 3 below.

2.4 Applications to polynomial system in the cylinder

In this section we study the maximum number of limit cycles of some polynomial vector fields in $\mathbb{R}^2$, the so-called Hilbert number. The planar system

$$x' = -y + xP(x,y) \quad , \quad y' = x + yP(x,y) \tag{2.21}$$

where $P(x,y)$ is a polynomial, it is known in the related literature as a *rigid system* (see for instance [5, 16, 17, 44] and their references). In polar coordinates, the system is rewritten as

$$r' = rP(rcos\theta, rsin\theta), \quad \theta' = 1.$$

If r is considered as a function of θ, we get the first order differential equation

$$\frac{dr}{d\theta} = rP(rcos\theta, rsin\theta), \tag{2.22}$$

and now it is easy to give applications of the results of Section 2.3 for suitable choices of the polynomial P.
Similarly, the results contained in Section 2.3 can be applied to rigid systems when the polynomial $P(x,y)$ is a sum of homogeneous polynomials up to fourth degree or to a suitable system in the cylinder. We omit further details.

In the recent paper [7], the authors study the number of non-contractible limit cycles of a family of systems in the cylinder $\mathbb{R} \times \mathbb{R}/[0,2\pi]$ of the form

$$\left\{ \begin{array}{rcl} \frac{d\rho}{dt} & = & \tilde{\alpha}(\theta)\rho + \tilde{\beta}(\theta)\rho^{k+1} + \tilde{\gamma}(\theta)\rho^{2k+1}, \\ \frac{d\theta}{dt} & = & b(\theta) + c(\theta)\rho^k, \end{array} \right. \tag{2.23}$$

where $k \in \mathbb{Z}^+$ and all the above functions in θ are continuous and 2π-periodic. A contractible limit cycle is an isolated periodic orbit which can be deformed continuously to a point, on the contrary it is called non-contractible. This type of systems arises as the polar expression of several types of planar polynomial systems. Of course, when $b(\theta) \equiv 1$ and $c(\theta) \equiv 0$ we have a rigid system. In general, if $b(\theta)$ does not vanish, a widely used change of variables due to Cherkas [12] transforms the system into a common Abel equation.

We will consider the reciprocal case $b(\theta) \equiv 0$, $c(\theta) \equiv 1$. Let us consider the system

$$\left\{ \begin{array}{rcl} \frac{d\rho}{dt} & = & \tilde{\alpha}(\theta)\rho + \tilde{\beta}(\theta)\rho^{N_3} + \tilde{\gamma}(\theta)\rho^{N_2} + \tilde{\delta}(\theta)\rho^{N_1}, \\ \frac{d\theta}{dt} & = & \rho^k, \end{array} \right. \tag{2.24}$$

where $N_1 > N_2 > N_3 > 0$ and $k > 0$. A limit cycle of this system is always non-contractible and as a function of θ it is a limit cycle or the first order equation

$$r' = \tilde{\beta}(\theta)r^{n_1} + \tilde{\gamma}(\theta)r^{n_2} + \tilde{\delta}(\theta)r^{n_3} + \tilde{\alpha}(\theta)r^m,$$

where $n_i = N_i - k$ for $i = 1, 2, 3$ and $m = 1 - k$. Now, a direct application of Theorem 2 gives the following result.

Corollary 5 *Take N_1, N_2, N_3 such that $N_1 - 2N_2 + N_3 = 0$ and assume*

$$\tilde{\gamma}(\theta)^2(N_2 - 1)^2 - 4\tilde{\beta}(\theta)\tilde{\delta}(\theta)(N_1 - 1)(N_3 - 1) \leq 0.$$

Then, system (2.24) *has at most one limit cycle in the semiplane $\{\rho > 0\}$.*

In particular, the result holds if $\tilde{\gamma}(\theta) \equiv 0$ and $\tilde{\beta}(\theta), \tilde{\delta}(\theta)$ have the same definite signs.

As a last remark, let us comment that the study of non-contractible limit cycles on a general system on the cylinder

$$\left\{ \begin{array}{rcl} \frac{d\rho}{dt} & = & P(\theta, \rho), \\ \frac{d\theta}{dt} & = & Q(\theta, \rho) \end{array} \right.$$

where components of the field (P, Q) are periodic in θ and polynomial in ρ, leads to the study of the existence and multiplicity of periodic solutions of a first order equation with a rational (quotient of two polynomials) nonlinearity. This is a difficult problem which deserves further developments.

Chapter 3

Multiplicity of Limit Cycles of a Generalized Abel Equation

3.1 Preliminary results

There is a wealth of material on roots of polynomials. Here we review some basic properties under two categories: complex roots and real roots. We indicate some lemmas from [25, 46] which provide a classical bound for the roots of a polynomial.

Lemma 4 *Any solution of the polynomial equation*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{3.1}$$

with real or complex coefficients and $a_n \neq 0$, verifies the bound

$$|x| \leq \max\left\{1, \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|\right\}.$$

Proof. Assume that $|x| \geq 1$, hence $1 \leq |x|^k \leq |x|^{n-1}$ for $k \leq n-1$. From equation (3.1),

$$x^n = \frac{-1}{a_n}(a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0).$$

Then,

$$|x|^n \leq \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right| |x|^k \leq \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|\right) |x|^{n-1}.$$

Hence $|x| \leq \sum_{k=0}^{n-1} \left|\frac{a_k}{a_n}\right|$.

Let us consider the polynomial function

$$f(t,x) = \sum_{k=0}^{n} a_k(t)x^k, \tag{3.2}$$

where the coefficients a_i are continuous and T-periodic functions.

Two simple consequences of the previous result are stated below.

Lemma 5 *Let us assume that $a_0(t) \neq 0$ for all t. If $x \neq 0$ is a solution of $f(t,x) = 0$ for some t, then*

$$|x| \geq \frac{1}{\max\left\{1, \max_t \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right|\right\}}.$$

Proof. By using the change of variable $y = \frac{1}{x}$, equation $f(t,x) = 0$ becomes

$$a_n + a_{n-1}y + \cdots + a_1 y^{n-1} + a_0 y^n = 0,$$

Using Lemma 4,

$$|y| = \frac{1}{|x|} \leq \max\left\{1, \max_t \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right|\right\}.$$

Analogously, one can prove the next lemma.

Lemma 6 *Let us assume that $a_n(t) \neq 0$ for all t. If $x \neq 0$ is a solution of $f_x(t,x) = 0$ for some t, then*

$$|x| \leq \max\left\{1, \max_t \sum_{k=1}^{n-1} \left|\frac{ka_k}{na_n}\right|\right\}.$$

3.2 Main results

Our purpose is to contribute to the literature with some results which complement those available in the related literature. To this aim, let us write the general non-linear differential equation in more general form

$$x' = x^m \sum_{k=0}^{n} a_k(t)x^k, \tag{3.3}$$

where $m \in \mathbb{Z}$. Thus, we consider a polynomial nonlinearity with possible negative powers, a situation which has been scarcely explored in the related literature. We will justify the interest of this case in the last section with the study of a new family of planar vector fields. Along the whole chapter, we assume the standing hypothesis

(H1) $a_n(t)a_0(t) \neq 0$ for all t.

Our main result is the following one.

Theorem 6 *Let us assume that for all t,*

$$\sum_{k=1}^{n} |a_k(t)| < |a_0(t)| \tag{3.4}$$

and

$$\sum_{k=0}^{n-1} |(m+k)a_k(t)| \leq |(m+n)a_n(t)|\,. \tag{3.5}$$

Then

i) Equation (3.3) has at most one positive limit cycle and at most one negative limit cycle.

ii) If $n+m$ is odd, equation (3.3) has exactly one limit cycle. If $n+m$ is even and $a_n(t)a_0(t) < 0$ for all t, then equation (3.3) has exactly one positive limit cycle and one negative limit cycle. If $n+m$ is even and $a_n(t)a_0(t) > 0$ for all t, then equation (3.3) has no limit cycles.

In [10, Theorem 1.3 (i)], it is proved that if

$$a_n(t) \equiv 1, |a_0(t)| > \frac{2n-1}{n-1}$$

and

$$|a_k(t)| < \frac{n}{(n-1)^2} \text{ for } 1 \leq k \leq n-1,$$

then equation (3.3) with $m = 0$ has at most one positive limit cycle and at most one negative limit cycle. This result is generalized by the previous one as it is shown below.

Corollary 6 *For all t, let us assume that*

$$a_n(t) \equiv 1, |a_k(t)| \leq P \leq \frac{2}{n-1} \text{ for } 1 \leq k \leq n-1$$

and

$$|a_0(t)| > (n-1)P + 1.$$

Then equation (3.3) *with $m = 0$ has at most one positive limit cycle and at most one negative limit cycle.*

We will show that this is a direct corollary of Theorem 6. Then [10, Theorem 1.3 (i)] corresponds to the particular case $P = \frac{n}{(n-1)^2}$ (which obviously is less than $\frac{2}{n-1}$ for the nontrivial case $n \geq 2$).

Using the same technique of proof that in Theorem 6, the following result holds.

Theorem 7 *Let us assume* (3.4). *Then,*

i) If

$$\sum_{k=0}^{n-1} |(m+k)(m+k-1)a_k(t)| \leq |(m+n)(m+n-1)a_n(t)|, \tag{3.6}$$

for all t, then equation (3.3) *has at most two positive limit cycles and at most two negative limit cycles.*

ii) If $\sum_{k=0}^{n-1} |(m+k)(m+k-1)(m+k-2)a_k(t)|$

$$\leq |(n+m)(n+m-1)(n+m-2)a_n(t)|, \tag{3.7}$$

for all t, then (3.3) *has at most three positive limit cycles and at most three negative limit cycles.*

Note that condition (3.7) is weaker than (3.6) and (3.6) is weaker than (3.5).

In the next Section we prove the main results, finally, the last Section contains an application to the estimation of the number of limit cycles of a family of polynomial planar systems which generalize the well-known rigid systems by using action-angle variables instead of polar coordinates.

3.3 Proof of the main result

During this section, we will call

$$g(t,x) = x^m \sum_{k=0}^{n} a_k(t)x^k.$$

Let us define the constants

$$\lambda = \min\{|x| \neq 0 \,:\, g(t,x) = 0 \text{ for some } t\}$$

$$\mu = \max\{|x| \neq 0 \,:\, g_x(t,x) = 0 \text{ for some } t\}.$$

Note that such quantities are well-defined by hypothesis $(H1)$.

Proposition 3 *If $\mu < \lambda$, then* (3.3) *has at most one positive limit cycle and at most one negative limit cycle.*

Proof. First, we prove that a given limit cycle does not change its sign. If $m > 0$, it is trivial since $x \equiv 0$ is a solution, so other solutions never cross it. In the case $m = 0$, note that $(H1)$ implies that $\alpha \equiv 0$ is an strict upper or lower solution (depending on the sign of a_0). Finally, if $m < 0$ the equation is not defined in 0 and changes of sign are not allowed. Therefore, eventual limit cycles have constant sign. Let us prove that there exists at most one positive limit cycle, being analogous the proof for the existence of at most one negative limit cycle. This is done in three steps:

- **Step 1**: *Any eventual positive limit cycle $x(t)$ verifies*

$$x(t) \geq \lambda, \qquad \forall t.$$

  It $x(t_m) > 0$ is the global minimum of $x(t)$, then $x'(t_m) = 0$, hence $g(t_m, x(t_m)) = 0$ and the inequality holds by the own definition of λ.

- **Step 2**: $g_x(t,x) \neq 0$ *for all $x > \mu$.* This is a direct consequence of the definition of μ.

- **Step 3**: *Conclusion:* Let us call $J = (\mu, +\infty)$. By continuity, $g_x(t,x) > 0$ (if $a_n > 0$) or $g_x(t,x) < 0$ (if $a_n < 0$) for all $x \in J$. Then Proposition 2 says that there exists at most one limit cycle in J, but by Step 1 and the condition $\mu < \lambda$, all the positive limit cycles belong to J, therefore the proof is done.

Proof of Theorem 6.

i) Condition (3.4) means that

$$\max\left\{1, \sum_{k=1}^{n}\left|\frac{a_k}{a_0}\right|\right\} = 1 \text{ for every } t,$$

hence we conclude that $\lambda \geq 1$ by Lemma 5. Moreover, by Lemma 6 and condition (3.5), we get $\mu \leq 1$. Therefore $\mu \leq \lambda$. In order to apply Proposition 3, it remains to prove that $\lambda > 1$. By contradiction, if $\lambda = 1$ then $g(t, 1) = 0$ for some t, that is,

$$\sum_{k=0}^{n} a_k(t) = 0,$$

but then

$$|a_0(t)| = \left|-\sum_{k=1}^{n} a_k(t)\right| \leq \sum_{k=1}^{n} |a_k(t)|,$$

contradicting (3.4).

ii) To fix ideas, along the whole proof we can consider that $a_0(t) > 0$ for all t. In fact the remaining case can be reduced to this one by inverting the time variable $\tau = -t$.

Assume that $n + m$ is odd and $a_n(t) < 0$ for all t. Using the sign of a_0, it is easy to verify that $\alpha(t) \equiv \epsilon > 0$ is a strict lower solution for a small enough ϵ. On the other hand, $\beta(t) \equiv M > 0$ is a strict upper solution for a big enough M. This gives a positive limit cycle. Having in mind the sign of a_n, such limit cycle has negative characteristic exponent and therefore it is asymptotically stable. Let us prove that it is unique. By the first part of the theorem, if there is a second limit cycle $\varphi(t)$, it should be negative and the function $g_x(t, \varphi(t))$ does not vanish. By using that $n + m$ is odd and $a_n < 0$, the sign of this function is negative by continuity, so the characteristic exponent of $\varphi(t)$ is again negative. Therefore, we have two zeros of the displacement function $q(c)$ with negative derivative, this means that it should be a third solution in between, but this is a contradiction with part (i). The case $a_n(t) > 0$ is solved in the same way, resulting in this case a unique limit cycle which will be negative.

Assume now that n is even and $a_n(t)a_0(t) < 0$. Then, taking $\epsilon, M > 0$ small and big enough respectively, $-M$ is a strict upper solution and $-\epsilon$ is a strict lower solution, so there exists a negative limit cycle. Similarly, ϵ is a strict lower solution and M is a strict upper solution, so there exists a positive limit cycle, and there are no more by part (i).

Finally, let us assume that $n + m$ is even and $a_n(t)a_0(t) > 0$. Then, taking $\epsilon, M > 0$ small and big enough respectively, $-M, -\epsilon, \epsilon, M$ are a strict lower solutions.
By Lemma 2, $q(-M), q(-\epsilon), q(\epsilon), q(M)$ are all positive. Assume by contradiction that there exists a limit cycle $\varphi(t)$, say a positive one (the remaining case is similar). Then it should be unique, so from $q(\epsilon), q(M) > 0$ we have that $\varphi(0)$ as a zero of the displacement function must be degenerate, that is, the characteristic exponent

$$\delta = \int_0^T g_x(t, \varphi(t))dt = 0.$$

But using the arguments contained in part (i), under our conditions $\varphi(t) > 1$ and $g_x(t, \varphi(t)) \neq 0$ for all t, which is a clear contradiction with $\delta = 0$.

Let us remark that Theorem 6 remains true if $<$ and $\leq$ are interchanged in conditions (3.4) – (3.5). The proof of Theorem 7 is analogous to the first part of Theorem 6 by using λ as defined above and modifying the definition of μ with the use of the second or third derivative of g.

3.4 Real Periodic Solutions

In [8, 9, 10] Alwash have proved related results giving more precise information about the number of limit cycles.
We consider the differential equation of the form

$$x' = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \tag{3.8}$$

In equation (3.8) if the leading coefficient, $a_n(t)$ is not 1 but does not vanish anywhere, then the transformation of the independent variable reduces the equation into a similar equation but with a leading coefficient equal 1. In this section first we give the result of Alwash [10], he gave upper bounds for the number of real periodic solutions when $|a_2(t)|, |a_1|$ or $|a_0(t)|$ is large.
We show that the jth derivative of $q(c)$ does not change sign in $c > 1$ and in $c < -1$, and the $(j-1)$th derivative of $q(c)$ does not change sign in $|c| < 1$. If the ith derivative $(i = 1, 2, 3)$ of a function does not change sign on an interval then the function has at most i zeros in this interval. This implies that q has at most $3j - 1$ zeros.

Theorem 8 ([10]) *Consider the equation* (3.8)

i) If for $0 \le t \le T$

$$|a_i(t)| < \frac{n}{(n-1)^2}, 1 \le i \le n-1,$$

and

$$|a_0| > \frac{2n-1}{n-1}$$

then there is at most one positive real periodic solution and at most one real negative periodic solution.

ii) If for $0 \le t \le T$

$$|a_i(t)| < \frac{n}{(n-2)^2}, 2 \le i \le n-1,$$

and

$$|a_1| > \frac{n(2n-3)}{n-2}$$

then there are at most five real periodic solutions; at most three of them are positive and at most three of them are negative.

iii) If for $0 \leq t \leq T$

$$|a_i(t)| < \frac{n}{(n-3)^2}, 3 \leq i \leq n-1,$$

and

$$|a_2| > \frac{n(n-1)(2n-5)}{2(n-3)}$$

then there are at most eight real periodic solutions; at most five of them are positive and at most five of them are negative.

The conditions in Theorem 8 on $|a_i|$ can be replaced by conditions on $\sum |a_i|$. The main results of the section are stated and proved below.

Theorem 9 *Consider the equation* (3.8)

i) If for $0 \leq t \leq T$

$$\sum_{i=1}^{n-1} |a_i(t)| < \frac{n}{n-1},$$

and

$$|a_0| > \frac{2n-1}{n-1}$$

then there is at most one positive real periodic solution and at most one real negative periodic solution.

ii) If for $0 \leq t \leq T$

$$\sum_{i=2}^{n-1} |a_i(t)| < \frac{n}{n-2},$$

and

$$|a_1| > \frac{n(2n-3)}{n-2}$$

then there are at most five real periodic solutions; at most three of them are positive and at most three of them are negative.

iii) If for $0 \leq t \leq T$

$$\sum_{i=3}^{n-1} |a_i(t)| < \frac{n}{n-3},$$

and

$$|a_2| > \frac{n(n-1)(2n-5)}{2(n-3)}$$

then there are at most eight real periodic solutions; at most five of them are positive and at most five of them are negative.

Proof. By letting

$$f(t,x) = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t),$$

i) If $x \geq 1$, then

$$\begin{aligned}
f_x(t,x) &= nx^{n-1} + (n-1)a_{n-1}(t)x^{n-2} + \ldots + 2a_2(t)x + a_1(t) \\
&\geq nx^{n-1} - (n-1)|a_{n-1}(t)|x^{n-2} - \ldots - 2|a_2(t)|x - |a_1(t)| \\
&> nx^{n-1} - (n-1)x^{n-2}\sum_{i=1}^{n-1}|a_i(t)| \\
&> nx^{n-1} - (n-1)x^{n-2}(\frac{n}{n-1}) \\
&> nx^{n-2}(x-1) \geq 0.
\end{aligned}$$

For $0 < x < 1$, then if $a_0 < -\frac{2n-1}{n-1}$ we have

$$\begin{aligned}
f &\leq x^n + |a_{n-1}(t)|x^{n-1} + \ldots + |a_1(t)|x + a_0(t) \\
&< 1 + |a_{n-1}(t)| + \ldots + |a_1(t)| + a_0(t) \\
&< 1 + \frac{n}{n-1} + a_0(t) \\
&< 1 + \frac{n}{n-1} - \frac{2n-1}{n-1} = 0.
\end{aligned}$$

On the other hand if $a_0 > \frac{2n-1}{n-1}$ we have

$$\begin{aligned}
f &\geq x^n - |a_{n-1}(t)|x^{n-1} - \ldots - |a_1(t)|x + a_0(t) \\
&\geq x^n - (|a_{n-1}(t)| + \ldots + |a_1(t)|) + a_0(t) \\
&\geq x^n - \frac{n}{n-1} + \frac{2n-1}{n-1} \\
&= x^n + 1 > 0.
\end{aligned}$$

Therefore $f_x > 0$ if $x \geq 1$ and $f < 0$ or $f > 0$ when $0 < x < 1$.

If $x \le -1$ and n is even, then

$$\begin{aligned} f_x &\le nx^{n-1} + (n-1)|a_{n-1}||x|^{n-2} + \\ &\quad \ldots + 2|a_2||x| + |a_1| \\ &< nx^{n-1} + (n-1)|x|^{n-2}\sum_{k=1}^{n-1}|a_k| \\ &< nx^{n-1} + nx^{n-2} \\ &= nx^{n-2}(x+1) \le 0. \end{aligned}$$

If $x \le -1$ and n is odd, then

$$\begin{aligned} f_x &\ge nx^{n-1} - (n-1)|a_{n-1}||x|^{n-2} - \ldots - 2|a_2||x| - |a_1| \\ &> nx^{n-1} - (n-1)|x|^{n-2}\sum_{k=1}^{n-1}|a_k| > nx^{n-1} - n|x|^{n-2} \\ &= n|x|^{n-1} - n|x|^{n-2} \\ &= nx^{n-1}(1 - \frac{1}{|x|}) \ge 0. \end{aligned}$$

If $-1 < x < 0$, then

$$\begin{aligned} f &\le x^n + |a_{n-1}|x^{n-1} + \ldots + |a_2|x^2 + |a_1|x + a_0 \\ &< x^{n-1}(x + |a_{n-1}| + \ldots + 2|a_2| + |a_1|) + a_0 \\ &< 1 + \sum_{i=1}^{n-1}|a_i| + a_0 \\ &< 1 + \frac{n}{n-1} - \frac{2n-1}{n-1} = 0. \end{aligned}$$

Therefore, $f_x > 0$ in $x \ge 1$, $f_x < 0$ in $x \le -1$, when n is even, but $f_x > 0$ when n is odd. Moreover, $f < 0$ in $-1 < x < 0$ if $a_0 \le -\frac{2n-1}{n-1}$.
Hence there is at most one positive real periodic solution and at most one real negative periodic solution.

ii) If $x \geq 1$ then

$$\begin{aligned} f_{xx} &\geq n(n-1)x^{n-2} - (n-1)(n-2)|a_{n-1}|x^{n-3} - \\ &\quad \ldots - 6|a_3|x - 2|a_2| \\ &> n(n-1)x^{n-2} - (n-1)(n-2)x^{n-3}\sum_{k=2}^{n-1}|a_k| \\ &> n(n-1)x^{n-2} + n(n-1)x^{n-3} \\ &= n(n-1)x^{n-3}(x+1) > 0. \end{aligned}$$

If $x \leq -1$, then

$$\begin{aligned} f_{xx} &\geq n(n-1)|x|^{n-2} - (n-1)(n-2)|a_{n-1}||x|^{n-3} - \\ &\quad \ldots - 6|a_3||x| - 2|a_2| \\ &> n(n-1)|x|^{n-2} - (n-1)(n-2)|x|^{n-3}\sum_{k=2}^{n-1}|a_k| \\ &> n(n-1)|x|^{n-3}(|x|+1) > 0. \end{aligned}$$

If $0 < x < 1$, then

$$\begin{aligned} f_x &\leq nx^{n-1} + (n-1)|a_{n-1}|x^{n-2} + \ldots + 2|a_2|x + a_1 \\ &< n + (n-1)\sum_{k=2}^{n-1}|a_k| + a_1 \\ &< n + (n-1)\frac{n}{n-2} - \frac{n(2n-3)}{n-2} = 0. \end{aligned}$$

If $-1 < x < 0$, then

$$\begin{aligned} f_x &\leq n + (n-1)|a_{n-1}| + (n-2)|a_{n-2}| + \\ &\quad \ldots + 2|a_2| + a_1 \\ &< n + (n-1)\sum_{k=2}^{n-1}|a_k| + a_1 \\ &< n + (n-1)\frac{n}{n-2} - \frac{n(2n-3)}{(n-2)} = 0. \end{aligned}$$

Therefore, equation (3.8) has at most five real periodic solutions; at most three of them are positive and at most three of them are negative.

iii) If $x \geq 1$ then

$$\begin{aligned} f_{xxx} \geq\ & n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)|a_{n-1}|x^{n-4} - \\ & \ldots - 24|a_4|x - 6|a_3| \\ & \geq n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)|a_{n-1}|x^{n-4} - \\ & \ldots - 24|a_4|x^{n-4} - 6|a_3|x^{n-4} \\ & > n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)x^{n-4}\sum_{k=3}^{n-1}|a_k| \\ & > n(n-1)(n-2)x^{n-3} - n(n-1)(n-2)x^{n-4} \\ & = n(n-1)(n-2)x^{n-4}(x-1) \geq 0. \end{aligned}$$

If $x \leq -1$ then

$$\begin{aligned} f_{xxx} \geq\ & n(n-1)(n-2)|x|^{n-3} - (n-1)(n-2)(n-3)|a_{n-1}||x|^{n-4} - \\ & \ldots - 24|a_4||x| - 6|a_3| \\ & > n(n-1)(n-2)|x|^{n-3} - (n-1)(n-2)(n-3)|x|^{n-4}\sum_{k=3}^{n-1}|a_k| \\ & > n(n-1)(n-2)|x|^{n-4}(|x|-1) \geq 0. \end{aligned}$$

If $-1 < x < 0$, then

$$\begin{aligned} f_{xx} \leq\ & n(n-1)|x|^{n-2} + (n-1)(n-2)|a_{n-1}||x|^{n-3} + \\ & \ldots + 6|a_3||x| + 2a_2 \\ & < n(n-1) + (n-1)(n-2)\sum_{k=3}^{n-1}|a_k| + 2a_2 \\ & < n(n-1)\frac{n(n-1)(n-2)}{n-3} - \frac{n(n-1)(2n-5)}{n-3} = 0. \end{aligned}$$

Therefore, equation (3.8) has at most eight real periodic solutions; at most five of them are positive and at most five of them are negative.

3.5 Applications to Polynomial Planar Systems

The main motivation for the study of the number of limit cycles in first-order equations with a polynomial nonlinearity is the search for information about the maximum number of limit cycles of a given autonomous polynomial planar system, the so-called Hilbert number. The most simple family of polynomial planar systems which can be reduced to a generalized Abel equation are the so-called *rigid systems*

$$\begin{cases} x' = \lambda x - y + xP(x,y) \\ y' = x + \lambda y + yP(x,y) \end{cases}$$

If

$$P(x,y) = R_1(x,y) + R_2(x,y) + \ldots + R_{n-1}(x,y),$$

where each R_k is a homogeneous polynomial of degree k , the system in polar coordinates becomes

$$\begin{cases} r' = r^n R_{n-1}(\cos\theta, \sin\theta) + r^{n-1} R_{n-2}(\cos\theta, \sin\theta) + \\ \qquad \ldots + r^2 R_1(\cos\theta, \sin\theta) + \lambda r \\ \theta' = 1. \end{cases}$$

Limit cycles of the system correspond to positive limit cycles of

$$\frac{dr}{d\theta} = r^n R_{n-1}(\cos\theta, \sin\theta) + r^{n-1} R_{n-2}(\cos\theta, \sin\theta) + \\ \ldots + r^2 R_1(\cos\theta, \sin\theta) + \lambda r.$$

Many examples contained in the literature belong to this class (see [9, 10, 8, 13, 17, 18, 19] and their references only to cite some of them). Of course, our results can be applied directly to this kind of systems, but we are going to focus our attention in a more general family which seem not to have been studied in previous works.

Let us consider the system

$$x' = -y^{2p-1} + \frac{x}{q} P(x,y) \quad , \quad y' = x^{2q-1} + \frac{y}{p} P(x,y) \tag{3.9}$$

where $P(x,y)$ is a polynomial and p, q are natural numbers. When $p = q = 1$, it is *a rigid system*.
Let us introduce action-angle variables r, θ such that

$$x = r^p C(\theta), \qquad y = r^q S(\theta), \tag{3.10}$$

where $C(\theta)$ and $S(\theta)$ are functions of θ defined implicitly by

$$\frac{C^{2q}(\theta)}{2q} + \frac{S^{2p}(\theta)}{2p} = 1. \tag{3.11}$$

Note that $(C(\theta), S(\theta))$ is a solution of the autonomous hamiltonian system

$$x' = -y^{2p-1}, \quad y' = x^{2q-1},$$

and it is uniquely determined by fixing the initial condition

$$(C(0), S(0)) = ((2q)^{\frac{1}{2q}}, 0).$$

The notion of action-angle variables is a classical tool in Celestial Mechanics and Stability Theory (a similar change was proposed by Liapunov, see [28, Pag. 43]).

In the new coordinates, the system (3.9) is rewritten as

$$r' = \frac{1}{pq} r P(r^p C(\theta), r^q S(\theta)), \qquad \theta' = r^{2pq-p-q}. \tag{3.12}$$

By taking r as a function of θ we get the single differential equation

$$\frac{dr}{d\theta} = A r^m P(r^p C(\theta), r^q S(\theta)), \tag{3.13}$$

where $A = \frac{1}{pq}$ and $m = -2pq + p + q + 1$. The coefficients will be polynomials in the generalized trigonometrical functions $C(\theta), S(\theta)$. Now we can use (3.11) in order to get

$$|C(\theta)| \leq (2q)^{\frac{1}{2q}}, |S(\theta)| \leq (2p)^{\frac{1}{2p}},$$

and such bounds can be used for practical application of our conditions. Instead of concrete examples, we will prove a somewhat general result.

Corollary 7 *Let $p, q \in \mathbb{N}$ and $Q(x, y)$ be a given polynomial of degree h. Consider the system* (3.9) *where*

$$P(x, y) = \left(\frac{x^{2q}}{2q} + \frac{y^{2p}}{2p} \right)^n + Q(x, y) + \lambda.$$

Then, there exist $n_0, \lambda_0 > 0$ (only depending on Q, p, q), such that (3.9) *has* *i) exactly one limit cycle if* $n > n_0$ *and* $\lambda < -\lambda_0$.

ii) no limit cycles if $n > n_0$ *and* $\lambda > \lambda_0$.

Proof. After the change (3.10), the resulting first order equation is

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{1}{pq} r^m P(r^p C(\theta), r^q S(\theta))) \\ &= \frac{1}{pq} r^m \left[r^{2pqn} + Q(r^p C(\theta), r^q S(\theta)) + \lambda \right].\end{aligned}$$

For this equation, the leading coefficient is

$$a_{2pqn} \equiv \frac{1}{pq}, \text{ whereas } a_0 \equiv \frac{\lambda}{pq}.$$

Finally, we can write

$$Q(r^p C(\theta), r^q S(\theta)) = \sum_{k=1}^{h \max\{p,q\}} a_k(\theta) r^k.$$

Then, condition (3.4) can be expressed as

$$|\lambda| > \lambda_0 := 1 + \max_\theta \sum_{k=1}^{h \max\{p,q\}} |a_k(\theta)|.$$

On the other hand, condition (3.5) is

$$|m+n| > \max_\theta \sum_{k=1}^{h \max\{p,q\}} |(m+k) a_k(\theta)|.$$

so it is sufficient to take

$$n > n_0 := |m| + \max_\theta \sum_{k=1}^{h \max\{p,q\}} |(m+k) a_k(\theta)|.$$

Now the result is a direct consequence of Theorem 6.

Chapter 4

The Number of Periodic Solutions of some Analytic Equations of Abel Type

4.1 The number of zeros of an analytic function

Let F be a closed rectangle with vertices $\pm a \pm ib$ and consider $h : F \to \mathbb{C}$ holomorphic in the interior of F and continuous on F. Assume also that

$$\overline{h(z)} = h(\overline{z}) \quad \forall z \in F,$$

and so h maps the interval $[-a, a]$ into $\mathbb{R}$.
Given any subset A of F we employ the notation $N(h, A)$ to indicate the number of zeros of h on A. Zeros are counted according to their multiplicity. Assume that we are given a real interval $J = [-\Gamma, \Gamma]$ with $\Gamma < a$, we would like to find an estimate on $N(h, J)$.
To state our result we need the Christoffel-Schwarz transformation f mapping the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ conformally onto F and such that $f(0) = 0$ and $f'(0) > 0$. Following [38, 27] such a map can be expressed as

$$f(z) = \frac{2b}{K(\sin\alpha)} \int_0^z \frac{d\xi}{\sqrt{1 - 2\cos(2\alpha)\xi^2 + \xi^4}}, \quad \frac{b}{a} = \frac{K(\sin\alpha)}{K(\cos\alpha)},$$

where K is the complete elliptic integral of the first kind (see Figure 4.1, where $z_+ = e^{i\alpha}$ and $z_- = e^{-i\alpha}$).

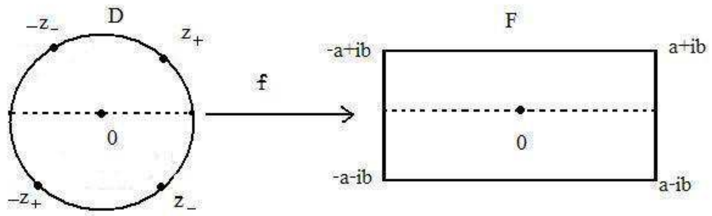

Figure 4.1:

Theorem 10 *Let $J = [-\Gamma, \Gamma]$ and $h : F \to \mathbb{C}$ be in the above conditions. In addition assume that for some positive numbers m and M,*

$$|h| \leq M \quad in \quad F, \quad \max_J |h| = m.$$

Then

$$N(h, J) \leq -\ln(m^{-1}M)/\ln(\frac{2\gamma}{1+\gamma^2}),$$

where γ is such that $f(\gamma) = \Gamma$.

Example 1 ***(Test example)*:**
Consider the function $h(z) = \sin z$, the interval $J = [\frac{-\pi}{6}, \frac{\pi}{6}]$ and the rectangle $F = [-\pi, \pi] \times [-1.5, 1.5]$. From $\frac{3}{2\pi} = \frac{K(\sin\alpha)}{K(\cos\alpha)}$ we deduce that $\alpha = 0.180762$ and $\gamma = 0.287904$. The estimate $|\sin z| \leq \cosh |\Im m z|$ implies that we can take
$M = \cosh(1.5)$. Since $m = 0.5$ we obtain

$$N(h, J) \leq -\frac{1}{\ln(\frac{2\gamma}{1+\gamma^2})} \ln(2\cosh(1.5)) \approx 2.4518.$$

An alternative approach would consist in applying Corollary 1 of [23]. In this case the rectangle has to be replaced by a domain with the shape of a stadium. In the notations of [23] we take $K = [-\frac{\pi}{6}, \frac{\pi}{6}]$ and U the ϵ- neighborhood of K with $\epsilon = 1.5$. Notice that $U \subset F$.

We notice that $|K|\epsilon^{-1} = \frac{\pi}{3}.\frac{2}{3} > \ln 2$ *and so the result in [23] is applicable. The estimate on the number of zeros now become*

$$N(h, J) \leq e^{\frac{2\pi}{9}} \ln(2\cosh(1.5)) \approx 3.11265.$$

See figure 4.2

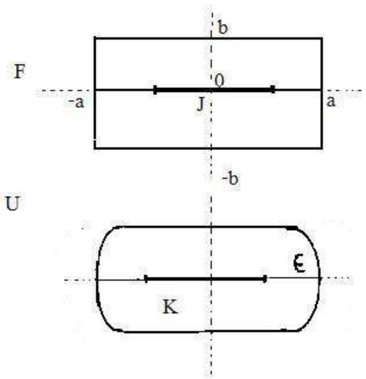

Figure 4.2:

The proof will consist in an adaptation of the classical Jensen's Lemma (usually stated on a disk) to the rectangle F. We first recall this result.(see [32] and [35]).

Lemma 7 ***(Jensen's Lemma)*** *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disk and let* $H : D \subset \mathbb{C} \to \mathbb{C}$ *be a holomorphic function. For each* $\rho \in (0, 1)$ *such that* $H(z) \neq 0$ *if* $|z| = \rho$, *we denote by* N_ρ *the number of zeros (counting multiplicities) of* H *in* $|z| < \rho$. *Then*

$$N_\rho \leq -\frac{1}{\ln \rho} \ln \frac{M}{|H(0)|},$$

where $M = \sup_{z\in D} |H(z)|$.

In the proof we will also employ the Mobius transformation

$$\mathcal{M} : w = \frac{z_0 - z}{1 - \overline{z_0} z},$$

where z_0 is a given point satisfying $|z_0| < 1$. This transformation maps the unit disk D onto itself and it is an involution; that is, the inverse $\mathcal{M}^{-1} = \mathcal{M}$. Moreover $\mathcal{M}(0) = z_0$.

Proof of theorem 10. From the Christoffel-Schwarz formula we observe that if $z \in D$, $f(-z) = -f(z)$. Moreover $f(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. Then $f([-\gamma, \gamma]) = J$ and we can pick points $\xi_0 \in J$ and $z_0 \in [-\gamma, \gamma]$ satisfying

$$|h(\xi_0)| = m, \quad f(z_0) = \xi_0.$$

(See figure 4.3, where $I_0 = [\frac{z_0-\gamma}{1-z_0\gamma}, \frac{z_0+\gamma}{1+z_0\gamma}]$ and $I_1 = [-\gamma, \gamma]$.)

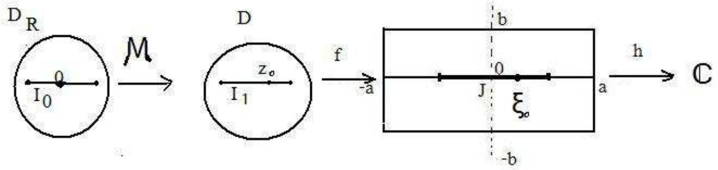

Figure 4.3:

Since z_0 is real the Mobius transformation $\mathcal{M}$ maps the real line onto itself and one immediately checks that

$$\mathcal{M}([-\gamma, \gamma]) = \mathcal{M}^{-1}([-\gamma, \gamma]) = [\frac{z_0 - \gamma}{1 - z_0\gamma}, \frac{z_0 + \gamma}{1 + z_0\gamma}].$$

The function $\varphi(x) = \frac{\gamma + x}{1+\gamma x}$, $x \in [-\gamma, \gamma]$ is increasing and so

$$\max\{|\frac{z_0 + \gamma}{1 + z_0\gamma}|, |\frac{z_0 - \gamma}{1 - z_0\gamma}|\} \leq \frac{2\gamma}{1 + \gamma^2} := R < 1.$$

These computations imply that hence $\mathcal{M}^{-1}([-\gamma, \gamma])$ is contained in the disk $|z| \leq R$.

Next we define $H = h \circ f \circ \mathcal{M}$, and observe that $|H(0)| = m$ and $|H| \leq M$ on D. Since $H(0) \neq 0$ we can apply the Identity Principle to find a sequence of numbers $\rho_n \searrow R$ and such that $H(z) \neq 0$ if $|z| = \rho_n$. From Jensen's Lemma applied to H we deduce that

$$N_{\rho_n} \leq -\frac{1}{\ln(\rho_n)} \ln(\frac{M}{m}).$$

Letting $n \to \infty$ we deduce that

$$N(h, J) \leq \lim_{n\to\infty} N_{\rho_n} \leq -\frac{1}{\ln(R)} \ln(\frac{M}{m}).$$

Assume now that $\epsilon > 0$ is a fixed number and Γ is a parameter satisfying

$$0 < \Gamma \le \overline{\Gamma} < \infty, \tag{4.1}$$

for a fixed $\overline{\Gamma}$. Consider the rectangle $F_\Gamma = [-\Gamma - \epsilon, \Gamma + \epsilon] \times [-\epsilon, \epsilon]$ and the interval $J_\Gamma = [-\Gamma, \Gamma]$.

Theorem 11 *There exists $\vartheta = \vartheta(\overline{\Gamma}, \epsilon) > 0$ such that for each Γ satisfying* (4.1) *and each continuous function $h : F_\Gamma \to \mathbb{C}$, holomorphic in the interior of F_Γ,*

$$N(h, J_\Gamma) \le \vartheta \ln(m^{-1}M),$$

where $|h| \le M$ in F_Γ and $\max_{J_\Gamma} |h| \ge m$.

Remark 1 *The above result is also applicable to a non-symmetric interval. Assuming $J = [a, b]$ with $0 < \frac{b-a}{2} \le \overline{\Gamma}$, we define $c = \frac{a+b}{2}$ and apply Theorem* 11 *to $h_1(z) = h(z + c)$ with $J_c = [-c, c]$, $F_c = [-c - \epsilon, c + \epsilon] \times [-\epsilon, \epsilon]$.*

Lemma 8 *Let I and J be open intervals and let us assume that $f : I \to J$ is strictly increasing and continuous, where $f(I) = J$. Then $f^{-1} : J \to I$ is strictly increasing and continuous.*

Proof. First we show that f^{-1} is continuous. To this end we observe that if $\alpha, \beta \in I$ with $\alpha < \beta$ then

$$f\left(]\alpha, \beta[\right) = \left]f(\alpha), f(\beta)\right[.$$

Here we are using that f is continuous and increasing. Assume now that $O \subseteq I$ is an open set. We will prove that $(f^{-1})^{-1}(O) = f(O)$ is also open. We can express the open set as

$$O = \bigcup_n]a_n, b_n[,$$

with $a_n, b_n \in I$. Then

$$f(O) = \bigcup_n f(]a_n, b_n[) = \bigcup_n]f(a_n), f(b_n)[.$$

The set $f(O)$ is open because it is an union of open intervals.
It remains to show that f^{-1} is increasing. Take $y_1, y_2 \in J$ with $y_1 \le y_2$. We want to prove that $x_1 = f^{-1}(y_1) \le x_2 = f^{-1}(y_2)$. By

contradiction assume $x_1 > x_2$, since f increasing we have $f(x_1) > f(x_2)$. But $y_1 = f(x_1) > f(x_2) = y_2$, contradiction.

Proof of theorem 11. The function $K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is strictly increasing on $k \in [0,1[$. In consequence also the function

$$\alpha \in [0,\pi/2[\rightarrowtail \frac{K(\sin\alpha)}{K(\cos\alpha)} \in [0,\infty[$$

is strictly increasing. Given $\Gamma \geq 0$ there exist a unique $\alpha = \alpha(\Gamma)$ with

$$\frac{\epsilon}{\Gamma+\epsilon} = \frac{K(\sin\alpha)}{K(\cos\alpha)}, \tag{4.2}$$

and the function $\Gamma \in [0,\infty[\mapsto \alpha(\Gamma) \in]0,\pi/4]$ is strictly decreasing and continuous. Consider the Christoffel-Schwarz transformation depending upon the parameter Γ,

$$f(z,\Gamma) = \frac{2\epsilon}{K(\sin\alpha(\Gamma))} \int_0^z \frac{d\xi}{\sqrt{1-2\cos(2\alpha(\Gamma))\xi^2+\xi^4}} \tag{4.3}$$

where we restrict z to the real axis we observe that the function $z \in [-1,1] \rightarrowtail f(z,\Gamma)$ is strictly increasing. Since $f(0,\Gamma) = 0$ and $f(1,\Gamma) = \Gamma + \epsilon$ we find a unique $\gamma = \gamma(\Gamma) \in [0,1[$ with

$$f(\gamma,\Gamma) = \Gamma. \tag{4.4}$$

Moreover $\gamma = \gamma(\Gamma)$ is a continuous function and we can define

$$\gamma_* = \max_{[0,\bar{\Gamma}]} \gamma(\Gamma). \tag{4.5}$$

We can apply Theorem 10 in F_Γ and define

$$\vartheta = -\frac{1}{\ln(\frac{2\gamma_*}{1+\gamma_*^2})}.$$

Remark 2 *ϑ can be computed numerically using* (4.2), (4.4) *and* (4.5).

Take $\epsilon = 1.5$ and $\overline{\Gamma} = 22^{\frac{1}{3}}$.
Numerically by Mathematica6 program $\gamma(\alpha(\Gamma))$ is increasing (see Table 1) then

$$\gamma_* = \max_{[0,\overline{\Gamma}]} \gamma(\Gamma) = \gamma(22^{\frac{1}{3}}) = 0.910597..$$

Hence $\vartheta = 228.349$.

Γ	α	γ
0.0	0.785398	0.000000
0.2	0.663954	0.130990
0.4	0.559028	0.248564
0.6	0.469156	0.354350
0.8	0.392628	0.448757
1.0	0.327746	0.532101
1.2	0.272932	0.604906
1.4	0.226768	0.667919
1.6	0.187999	0.722023
1.8	0.155531	0.768162
2.0	0.128413	0.807279
2.2	0.105822	0.840271
2.4	0.087051	0.867969
2.6	0.071499	0.891127
2.8	0.058623	0.910418
2.80204	0.058504	0.910597

Table 1

4.2 Estimates for real-valued solutions

Consider the differential equation

$$x' = v(x,t) \tag{4.6}$$

where $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and 1-periodic in t. We also assume that there is uniqueness for the initial value problem. Given $\xi \in \mathbb{R}$ the solution satisfying $x(0) = \xi$ will be denoted by $x(t,\xi)$ and the corresponding maximal interval $]\alpha, \omega[$ with

$$-\infty \leq \alpha = \alpha(\xi) < 0 < \omega = \omega(\xi) \leq +\infty.$$

Let $\mathcal{D}$ be the set of initial conditions such that the solution is well defined in $[0,1]$; that is,

$$\mathcal{D} = \{\xi \in \mathbb{R} : \omega(\xi) > 1\}.$$

We recall that $\mathcal{D}$ is an open interval which sometimes can be empty. For instance, this is the case for

$$v(x,t) = \lambda(1 + x^2) \text{ with } \lambda \geq \pi.$$

The Poincare map is defined as

$$P : \mathcal{D} \to \mathbb{R}, \quad P(\xi) = x(1,\xi).$$

This is an increasing and continuous function and periodic solutions do correspond to fixed points of P.

Lemma 9 *Assume that for some $r \in \mathbb{R}$*

$$v(x,t) > 0 \quad \textit{if} \quad x \geq r.$$

In addition there exists a periodic solution and another solution that blows up not later than $t = 1$; that is, $\omega(\xi) \leq 1$ for some $\xi \in \mathbb{R}$. Then

$$P(\mathcal{D}) =]B, +\infty[\quad \textit{for some} \quad B, \quad -\infty \leq B < \infty.$$

Proof. Let A be an initial condition such that $x(t,A)$ is periodic solution of (4.6). Then $A = P(A) \in P(\mathcal{D})$. Since $P(\mathcal{D})$ is an open interval we can express it as $P(\mathcal{D}) =]B, C[$ with $-\infty \leq B < A \leq C \leq +\infty$. It remains to prove that $C = +\infty$. To this end we take any number $y \in]A, +\infty[$ and consider the solution $\phi(t)$ of (4.6) satisfying $\phi(1) = y$. We claim that $\phi(t)$ is well defined in $[0, 1]$. Otherwise there should exist some $\alpha \in [0, 1[$ such that one of the following alternatives holds,

$$(i) \lim_{t \searrow \alpha} \phi(t) = -\infty \qquad (ii) \lim_{t \searrow \alpha} \phi(t) = +\infty.$$

The first alternative is ruled out because, by uniqueness, $\phi(t) > x(t, A)$ if $t \in]\alpha, 1]$. Assume by a contradiction argument that (ii) holds. Then, for t close enough to α one should have $\phi(t) > r$. In consequence $\phi'(t) = v(\phi(t), t) > 0$ and ϕ is strictly increasing on some interval of the type $]\alpha, \alpha + \delta[$. This is not compatible with (ii).

Lemma 10 *Assume that for some $r \in \mathbb{R}$*

$$v(x, t) > 0 \quad \textit{if} \quad x \geq r.$$

In addition $\omega(r) \leq 1$ and there exists a periodic solution $\varphi(t)$ with $\varphi(0) = A$. Then, for any $R > r$ we can find $x_R > A$ with $P(x_R) = R$ such that the following statements hold,

i) $[A, x_R] \subset \mathcal{D}$.

ii) $x(t, \xi) \leq R$ *for each* $t \in [0, 1], \xi \in [A, x_R]$.

iii) $\max_{[A, x_R]} |h| \geq (R - r)$ *where* $h := id - P$.

See Figure 4.4

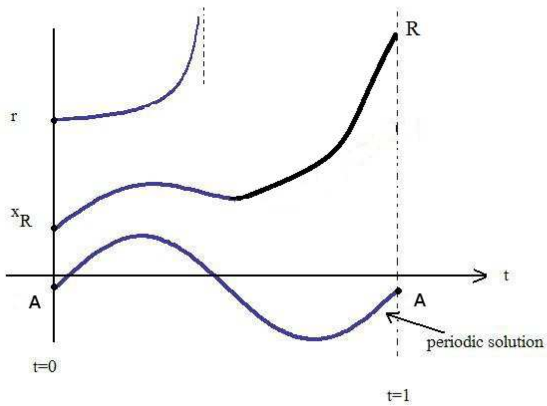

Figure 4.4:

Proof. From the previous lemma we know that $]A, \infty[\subset P(\mathcal{D})$ and so there exists $x_R \in \mathcal{D}$ such that $P(x_R) = R$. Since P is increasing and $P(A) = A < R$ we conclude that $x_R > A$. Since $\mathcal{D}$ is an interval and $A, x_R \in \mathcal{D}, (i)$ follows.

ii) First we notice that $x_R < r$ since $x_R \in \mathcal{D}$ but $[r, +\infty[\cap \mathcal{D} = \phi$. By contradiction suppose that $x(\tau, \xi) > R$ for some $\tau \in]0,1], \xi \in [A, x_R]$. Let $\psi(t)$ be the solution satisfying $\psi(1) = R$ and $\psi(0) = x_R$. Let $\tau \in]0,1]$ be the first instant such that $\psi(\tau) = r$. We claim that τ is the only number in $\psi^{-1}(r)$. Indeed, $\psi'(\widehat{\tau}) = v(\psi(\widehat{\tau}), \widehat{\tau}) = v(r, \widehat{\tau}) > 0$ for any $\widehat{\tau} \in \psi^{-1}(r)$ and this forces the uniqueness of τ. The previous discussion implies that $\psi'(t) = v(\psi(t), t) > 0$ if $t \in [\tau, 1]$. Hence ψ is strictly increasing on $[\tau, 1]$ and $\psi(t) \leq \psi(1) = R$ on this interval. The definition of τ implies that $\psi(t) \leq r$ on $[0, \tau]$ and so $\psi(t) \leq R$ on $[0, 1]$. Finally we observe that, by uniqueness, $x(t, \xi) \leq \psi(t)$ if $t \in [0, 1]$ and this is against the assumption $x(t, \xi) > R$ for some t.

iii) $x_R < r$ since $x_R \in \mathcal{D}$ but $r \notin \mathcal{D}$. Since P is increasing, then

$$\max_{[A, x_R]} |h| = \max_{[A, x_R]} |id - P| \geq |h(x_R)|,$$

but

$$|h(x_R)| = |x_R - P(x_R)| = |x_R - R| = R - x_R \geq R - r.$$

We conclude this section with an adaptation of Lemma 10 to both sides of the real line. The proof is analogous.

Lemma 11 *Assume that for some $r_+ > 0 > r_-$,*

$$v(x,t) > 0 \quad \text{if} \quad x \geq r_+, \quad v(x,t) < 0 \quad \text{if} \quad x \leq r_-.$$

In addition $\omega(r_+) \leq 1, \omega(r_-) \leq 1$, and there exists at least one periodic solution. Then, for any $R > r = \max\{r_+, |r_-|\}$, we can find an interval $J = [a, b] \subset [r_-, r_+]$ such that

i) $J \subset \mathcal{D}, P(J) = [-R, R]$ and all the fixed points of P are contained in J.

ii) $|x(t, \xi)| \leq R$ for each $t \in [0, 1], \xi \in J$.

iii) $\max_J |h| \geq (R - r)$ where $h := id - P$.

See Figure 4.5.

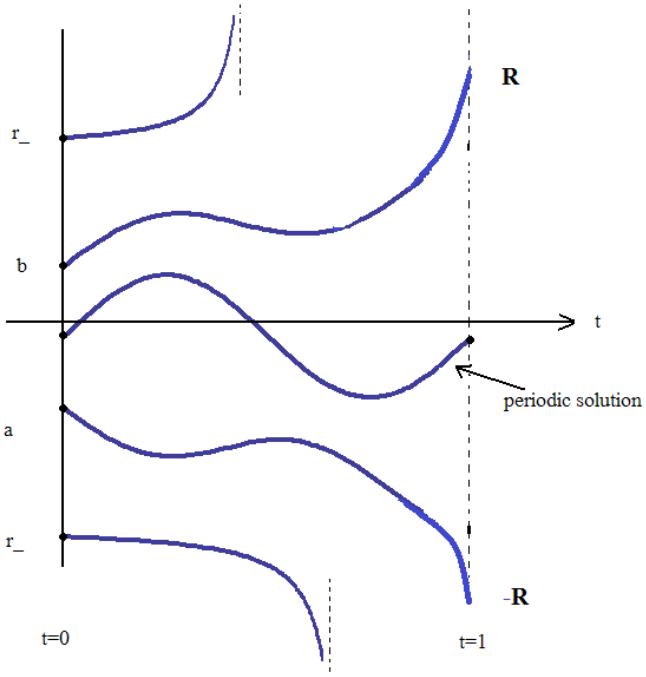

Figure 4.5:

4.3 Estimates for complex-valued solutions

In this section we consider the complex valued differential equation

$$z' = v(z,t) \tag{4.7}$$

where $v : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is continuous, holomorphic in z, 1-periodic in t and $v(z,t) \in \mathbb{R}$ if $z \in \mathbb{R}$. Since the time is real, this equation can be interpreted as a system in $\mathbb{R}^2$, namely

$$x' = \Re e v(z,t), \quad y' = \Im m v(z,t), \quad z = x + iy.$$

Since the line $y = 0$ is invariant we recover the real-valued equation on the x-axis.
We present a result similar to Lemma 2, page (1339) in [33].

Lemma 12 *Assume that we are given a compact interval* $[a, b]$ *and a rectangle* $F = [a - \epsilon, b + \epsilon] \times [-\epsilon, \epsilon]$ *for some* $\epsilon > 0$.
In addition there are positive number ρ *and* K *such that*

- $\forall\ \widehat{\xi} \in [a, b], z(t, \widehat{\xi})$ *is well defined in* $[0, 1]$ *and* $|z(t, \widehat{\xi}) - c| \leq \rho$ $\forall\ t \in [0, 1]$, *where* $c = \frac{a+b}{2}$ *is the mid-point of the interval,*
- F *is contained in the disk of radius* ρ *centered at* c *and* $|v(z, t)| \leq K$ *if* $|z - c| \leq \rho + 2$
- $\sqrt{2}\epsilon \exp(K) < 1$.

Then, for each $\xi \in F$ *the solution* $z(t, \xi)$ *is well defined in* $[0, 1]$ *and satisfies*

$$|z(t, \xi) - c| \leq \rho + 1, |z(t, \xi) - \xi| \leq \rho + \frac{b - a}{2} + \sqrt{2}\epsilon(1 + \exp(K)), t \in [0, 1].$$

See Figure 4.6.

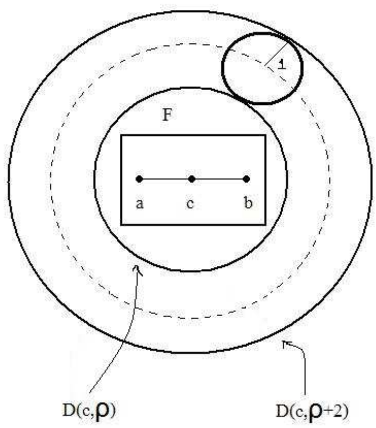

Figure 4.6:

Proof. First we notice that from the assumptions $\frac{b-a}{2} \leq \rho$. Given $\xi \in F$ we denote by $\widetilde{\omega} = \widetilde{\omega}(\xi)$ the largest number in $[0, 1]$ such that

$$|z(t, \xi) - c| \leq \rho + 1 \quad \text{if} \quad t \in [0, \widetilde{\omega}].$$

Given any ξ in F we can find $\widehat{\xi} \in [a,b]$ such that

$$|\xi - \widehat{\xi}| \leq \sqrt{2}\epsilon.$$

Then, from the differential equation when $t \in [0, \widetilde{\omega}]$,

$$|z(t,\xi) - z(t,\widehat{\xi})| \leq |\xi - \widehat{\xi}| + \int_0^t |v(z(s,\xi),s) - v(z(s,\widehat{\xi}),s)| ds.$$

The function $v(.,t)$ is holomorphic and Cauchy's inequality implies that for
$|z - c| \leq \rho + 1$,

$$|\frac{\partial v}{\partial z}(z,t)| \leq \max_{|z-c|\leq\rho+2} |v(z,t)| \leq K.$$

From the complex version of the Mean Value Theorem we deduce that,

$$|z(t,\xi) - z(t,\widehat{\xi})| \leq |\xi - \widehat{\xi}| + K\int_0^t |z(s,\xi) - z(s,\widehat{\xi})| ds \quad \text{if} \quad t \in [0, \widetilde{\omega}].$$

From Gronwall's lemma [21],

$$|z(t,\xi) - z(t,\widehat{\xi})| \leq |\xi - \widehat{\xi}| e^{Kt}, t \in [0, \widetilde{\omega}].$$

Next we claim that $\widetilde{\omega} \geq 1$. Assuming by contradiction that $\widetilde{\omega} < 1$ we notice that

$$\begin{aligned} \rho + 1 = |z(\widetilde{\omega},\xi) - c| &\leq |z(\widetilde{\omega},\xi) - z(\widetilde{\omega},\widehat{\xi})| + |z(\widetilde{\omega},\widehat{\xi}) - c| \\ &\leq |\xi - \widehat{\xi}| e^K + \rho \leq \sqrt{2}\epsilon e^K + \rho, \end{aligned}$$

and this is not compatible with the assumptions. In particular the previous reasoning implies that $z(t,\xi)$ is well defined in $[0,1]$ and

$$|z(t,\xi) - c| \leq \rho + 1 \quad if \quad t \in [0,1].$$

To obtain the other estimate we observe that

$$\begin{aligned} |z(t,\xi) - \xi| &\leq |z(t,\xi) - z(t,\widehat{\xi})| + |z(t,\widehat{\xi}) - \widehat{\xi}| + |\widehat{\xi} - \xi| \\ &\leq \sqrt{2}\epsilon(e^K + 1) + \rho + \frac{b-a}{2}, t \in [a,b]. \end{aligned}$$

4.4 An example

Next we show how to employ the previous results in order to estimate the number of periodic solutions of the equation

$$x' = v(x,t) := x^3 + 10\sin x + p(t) \tag{4.8}$$

where $p : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, 1-periodic and $\|p\|_\infty \leq C$. The numbers $\pm(C+10)^{\frac{1}{3}}$ are constant lower and upper solutions of this equation and this shows that there exists always at least one periodic solution.(see Theorem 2.1, page (383) in [34])
Next we define the number

$$r = 2^{\frac{1}{3}}(C+10)^{\frac{1}{3}}.$$

The function

$$\psi(t) = \frac{r}{\sqrt{1-r^2t}}, t \in [0, \frac{1}{r^2}[$$

is a solution of $x' = \frac{1}{2}x^3$ and satisfies $\psi(t) \geq \psi(0) = r$. From these properties it is easy to deduce that $\psi(t)$ is a sub-solution of (4.8). Actually,

$$\psi'(t) - \psi(t)^3 - 10\sin\psi(t) - p(t) \leq -\frac{1}{2}\psi(t)^3 + 10 + C \leq -\frac{1}{2}r^3 + 10 + C = 0.$$

In consequence the solution of (4.8) satisfying $x(0) = r$ must blow up before the time $t = \frac{1}{r^2}$. This fact allows up to apply Lemma 11 to (4.8) with $r_+ = r, r_- = -r$ and $R = 2r$. We find an interval $J = [a,b]$ in the conditions of the Lemma. In particular the number of periodic solutions of the differential equation coincides with the number of zeros of $h = id - P$ in the interval $[a,b]$.
Also

$$|x(t,\xi)| \leq 2r \quad \text{if} \quad t \in [0,1] \quad \text{and} \quad \xi \in J \tag{4.9}$$

and

$$\max_J |h| \geq r. \tag{4.10}$$

Next we apply the results of section 4.3. Here it is important to notice that $\sin z$ is an entire function and so

$$v(z,t) = z^3 + 10\sin z + p(t)$$

is entire as a function of z. Let us apply Lemma 12 with

$$\rho = 3r, \quad K = (4r+2)^3 + C + 10\cosh(\rho+2), \quad \epsilon = \frac{1}{2}e^{-K}.$$

Let us show that this choice is admissible. Using (4.9) we notice that if $\widehat{\xi} \in [a,b]$ and $t \in [0,1]$ then

$$|z(t,\widehat{\xi}) - c| \leq |z(t,\widehat{\xi})| + |\frac{a+b}{2}| \leq 2r + r = 3r.$$

Also, since $\epsilon < \frac{1}{2}$, we notice that the rectangle F is contained in the disc centered at c and of radius $3r$. Moreover, if $|z-c| \leq \rho + 2$,

$$\begin{aligned} |v(z,t)| &\leq (|z-c| + |c|)^3 + \cosh(\rho+2) + C \\ &\leq (\rho + 2 + r)^3 + \cosh(\rho+2) + C := K. \end{aligned}$$

From the conclusion of Lemma 12 we know that h is well defined in F and

$$|h(\xi)| \leq \rho + \frac{b-a}{2} + \sqrt{2}\epsilon(1+e^K) \leq 3r + r + \frac{\sqrt{2}}{2}(e^{-K}+1).$$

We are ready to apply Theorem 11 to the function $h = id - P$ on the interval $[a,b]$. Notice that this interval is not symmetric around the origin and we must take into account the Remark after that Theorem. Notice that the theorem on differentiability with respect to initial conditions implies that h is holomorphic. Moreover, since $\Im m z = 0$ is invariant for the equation, h is real-valued on the real line. Since $\Gamma = \frac{b-a}{2} \leq r =: \overline{\Gamma}$, $M = 4r + \frac{\sqrt{2}}{2}(e^{-K}+1)$ and $m = r$,

$$N(h,[a,b]) \leq \vartheta \ln(4 + \frac{\sqrt{2}}{2r}(e^{-K}+1)) \tag{4.11}$$

We can sum up the previous discussions in the following statement: For each $C > 0$ there exists $N(C)$ such that the equation (4.8) has at most N periodic solutions for any function $p : \mathbb{R} \to \mathbb{R}$ continuous and 1-periodic with

$$\|p\|_\infty \leq C.$$

Moreover, the number N can be computed numerically.

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