Some Applications of the Topological Degree to Stability Theory

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Abstract

These notes are devoted to show that the topological degree is an useful tool in the study of the properties of stability of periodic solutions of a scalar, time-dependent differential equation of Newton's type. Two different situations are considered depending of whether the equation has damping or not. When there is a linear friction the asymptotic stability of a periodic solution can be characterized in terms of degree. When there is no friction the equation has a hamiltonian structure and some connections between Lyapunov stability and degree are discussed. These results are applied in two different directions: to prove that some classical methods in the theory of existence lead to instability (minimization of the action functional, upper and lower solutions) and to study the stability of the solutions of a concrete class of equations (equations of pendulum-type).

The general results are presented in an abstract setting also applicable to other two-dimensional periodic systems.

Introduction

Let us consider the differential equation

$$x'' + cx' = f(t,x) \tag{1}$$

where $c \geq 0$ is a given constant and $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is periodic in time; that is,

$$f(t+T,x) = f(t,x) \text{ for each } (t,x) \in \mathbf{R}^2.$$

This class of equations has a well known interpretation in the theory of forced oscillations in Mechanics and includes some famous second order equations such as the forced

Duffing's equation or the equation of the pendulum with periodic torque. The model includes the conservative or frictionless case ($c = 0$) and the damped case ($c > 0$). The equation (1) can have different kinds of recurrent solutions: periodic solutions of period T, also called harmonic solutions; periodic solutions with minimal period nT, also called subharmonics solutions; quasi-periodic solutions; etc. The periodic solutions of period T are the simplest among all of them and play a role in the periodic equations that is similar to the role played by the equilibria in the autonomous equations.

The theory of topological degree is one of the basic tools in the study of the periodic problem for (1). This theory is normally employed to obtain result on the existence and number of T-periodic solutions. Less often it is used in the analysis of the stability properties of the solutions. However, there are connections between the theories of degree and stability when one is restricted to the periodic problem for equation (1). In fact, several authors have considered the question in a more or less explicit way. In [26], Levinson already applied degree theory to deduce some relationships between the number of stable and unstable periodic solutions of certain dissipative equations of the class (1). These results were refined by Massera in [31] and it was required in both papers that the periodic solutions were hyperbolic. In [46], Seifert considered a forced equation of pendulum-type and applied the results in [26] to deduce the existence of an unstable periodic solution from the value of a certain degree. Results on the existence of an asymptotically stable periodic solutions based on similar ideas were obtained by Cronin [11] and by Mawhin [32], who obtained applications to the Duffing's equation. In the book [22], Krasnoselskii deduced some implications of the asymptotic stability on the degree that, in contrast to all the previous results, were also proved in the non-hyperbolic case. Some other related references are included in [36] and [40].

In this notes we give a survey of some recent results on the connections between stability and degree. For equation (1) and in the case $c > 0$, it will be shown that the asymptotic stability of a periodic solution can be completely characterized in terms of the degree. In the case $c = 0$ the connections between Lyapunov stability and degree are not completely understood by the author and only partial results will be given. The same kind of ideas can be applied in the study of other equations (Lienàrd equation [2], prey-predator system [3], ...) For this reason it is convenient to present the results in a more abstract setting and we shall study the stability properties of fixed points of mappings. As it is well known, the link with the periodic problem for (1) is established via the Poincaré map.

The rest of these notes is divided in three chapters. The first chapter develops some general principles on stability of fixed points. These ideas are applied in the second chapter to the periodic problem for (1). In particular, it is proved that two of the classical methods in the theory of existence (upper and lower solutions and minimization of the action functional) normally lead to unstable solutions. The last chapter deals with a concrete equation: the forced equation of pendulum-type. The previous results are applied to this case to obtain precise information on stability. We remark that applications to other kinds of nonlinearities are also possible (see [37], [38], [41]).

Chapter 1 Stability and index of fixed points

1.1 Definition of stability

Let U be a domain of $\mathbf{R}^N$, $N \geq 1$, and let $p \in U$ be a given point. A map $F : U \to \mathbf{R}^N$ belongs to the class $\mathcal{H}(U,p)$ if it is satisfies:

(i) F is continuous and one-to-one,

(ii) p is a fixed point of F; i.e. $F(p) = p$.

A map $F \in \mathcal{H}(U,p)$ is always a homeomorphism from U onto its image $U_1 := F(U)$ and U_1 is also a domain of $\mathbf{R}^N$. This can be proved using the theorem of invariance of the domain.

Definition 1. *Assume that $F \in \mathcal{H}(U,p)$. The fixed point p is stable in the sense of Lyapunov if every neighborhood V of p contains another neighborhood W such that, for each $n \geq 0$, $F^n(W)$ is well defined and $F^n(W) \subset V$.*

A set $A \subset U$ is positively invariant if $F(A) \subset A$. The previous notion of stability can be characterized in terms of positive invariance. This is shown by the following well known result (see [48]). We leave the proof as an exercise.

Proposition 2. *Assume that $F \in \mathcal{H}(U,p)$. The following statements are equivalent:*

(i) p is stable,

(ii) There exists a basis of neighborhoods of p, $\{U_\lambda\}$, such that U_λ is positively invariant.

Definition 3. *Asume that $F \in \mathcal{H}(U,p)$. The fixed point p is asymptotically stable if it is stable and, in addition, there exists a neighborhood A of p such that*

$$\lim_{n\to\infty} F^n(q) = p \text{ for each } q \in A.$$

It is not difficult to prove that the previous convergence is uniform on compact subsets of A. In consequence, if p is asymptotically stable, one can find a disk D centered at p such that

$$\lim_{n\to\infty} \operatorname{diam}(F^n(D)) = 0, \tag{2}$$

where "diam" means the diameter of a set.

When $F \in C^1(U,\mathbf{R}^N)$ the principle of linearization can be used. Let A denote the differential of F at p and let $r(A)$ be corresponding spectral radius. If $r(A) < 1$ (resp. $r(A) > 1$) then p is asymptotically stable (resp. unstable). This is the so called principle of linearization that is valid whenever $r(A) \neq 1$.

Sometimes it will be convenient to consider iterated maps. Given $F \in \mathcal{H}(U,p)$, $k \geq 1$, F^k is defined in a recursive way by

$$F^k : U_{-k+1} \to \mathbf{R}^N,\ F^k = F \circ \overset{k)}{\cdots} \circ F$$

where $U_{-k+1} = F^{-(k-1)}(U) \cap ... \cap F^{-1}(U) \cap U$.

It is clear that U_{-k+1} is a domain in $\mathbf{R}^N$ and $F^k \in \mathcal{H}(U_{-k+1}, p)$, so that we can ask about the stability or asymptotic stability of p with respect to F^k. It is not hard to show that F^k-stability is equivalent to F-stability for each $k \geq 2$.

1.2 The topological degree

In what follows we shall only use the more classical versions of the degree theory: Brouwer's degree in finite dimensions and Leray-Schauder's degree in Banach spaces. It will be assumed that these theories are familiar to the reader. Some references on degree theory are [27], [44].

Let X be a Banach space and let $\Omega \subset X$ be a bounded and open set. A map

$$\Phi : \overline{\Omega} \to X$$

is of Leray-Schauder type if it satisfies

(i) $\Phi(x) \neq 0 \ \forall x \in \partial\Omega$.

(ii) Φ can be expressed in the form $\Phi = I - \varphi$ where I is the identity in X and $\varphi : \overline{\Omega} \to X$ is compact.

(Remark that (ii) is equivalent to the continuity of Φ when X has finite dimension.)

Given Φ of Leray-Schauder type in Ω, the degree of Φ in Ω is an integer denoted by

$$\deg[\Phi, \Omega].$$

The degree can be axiomatically characterized in terms of the properties of addition-excision, homotopy invariance and normalization (see [27], [24]). It has many other properties and perhaps the most important one is the following: if Φ is of Leray-Schauder type and $\deg[\Phi, \Omega] \neq 0$ then there exists $x \in \Omega$ such that $\Phi(x) = 0$. It is this property what makes degree theory useful in the proofs of existence theorems.

To compute the topological degree in applications one can use the properties just mentioned and also certain known criterions that can be of analytical or topological nature. The simplest analytical criterion for the computation of the degree is the linearization principle. To simplify matters we assume that $X = \mathbf{R}^N$ (see [44] for general case). Given $\Phi \in C^1(\overline{\Omega}, \mathbf{R}^N)$ such that the equation

$$\Phi(x) = 0, \ x \in \overline{\Omega}$$

has only a finite number of solutions $x_1, \dots, x_n \in \Omega$ with $\Phi'(x_i)$ non singular for each $i = 1, \dots, n$, the degree can be computed by the formula

$$\deg[\Phi, \Omega] = \sum_{i=1}^{n} \operatorname{sign}\{\det \Phi'(x_i)\}. \tag{3}$$

When 0 is not a regular value of Φ one can use sometimes more complicated analytical criterions that depend on the nonlinear terms of the Taylor expansion of Φ at x_i (see [24]).

A criterion for the computation of the degree of topological type is the following:

Assume that Ω *is convex and* $\Phi = I - \varphi$ *is of Leray-Schauder type and such that*

$$\varphi(\overline{\Omega}) \subset \overline{\Omega},$$

then $\deg[\Phi, \Omega] = 1$.

This classical result has an easy proof because Φ is linearly homotopic to the identity if $0 \in \Omega$. An interesting variant that was inspired by Browder's fixed-point-theorem [8] is the following:

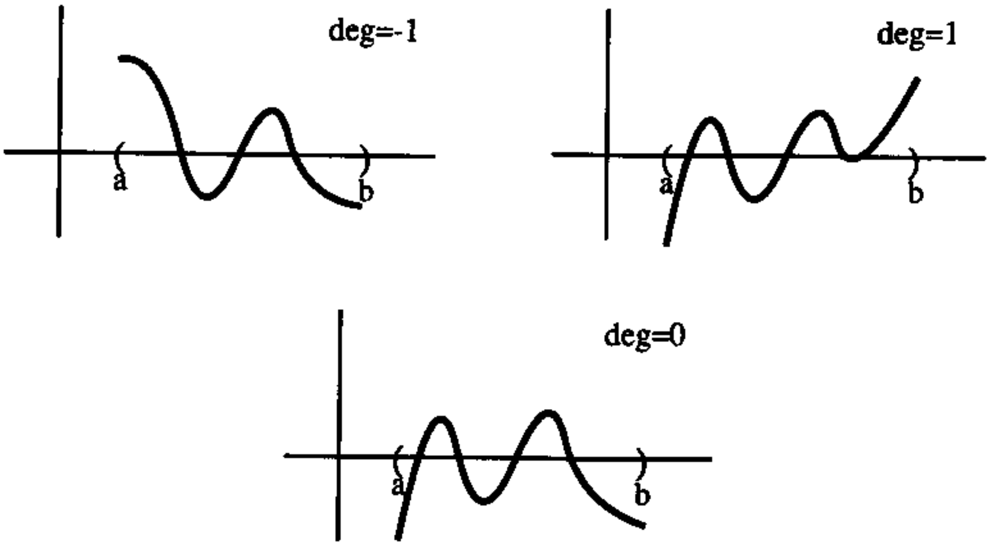
Assume that Ω *is convex and* $\Phi = I - \varphi$ *is of Leray-Schauder type and there exists* $n_0 \in \mathbf{N}$ *such that*

$$\varphi^n(\overline{\Omega}) \subseteq \overline{\Omega},\ \varphi^n(x) \neq x,\ \forall x \in \partial\Omega,\ n \geq n_0, \tag{4}$$

then $\deg[\Phi, \Omega] = 1$. (The proof is not simple, see [24]).

The computations of the degree are particularly simple in one dimension. Assuming $X = \mathbf{R},\ \Omega = (a, b),\ \Phi : (a, b) \to \mathbf{R}$,

$$\deg[\Phi, \Omega] = \begin{cases} 1 & \text{if } \Phi(a) < 0 < \Phi(b) \\ 0 & \text{if } \Phi(a)\Phi(b) > 0 \\ -1 & \text{if } \Phi(a) > 0 > \Phi(b). \end{cases}$$



1.3 The fixed point index

Let U be a domain of $\mathbf{R}^N$ and let $F : U \to \mathbf{R}^N$ be a continuous map such that $p \in U$ is an isolated fixed point of F. The fixed point index of F at p is defined as

$$i[F,p] = \deg(I - F, B_\rho)$$

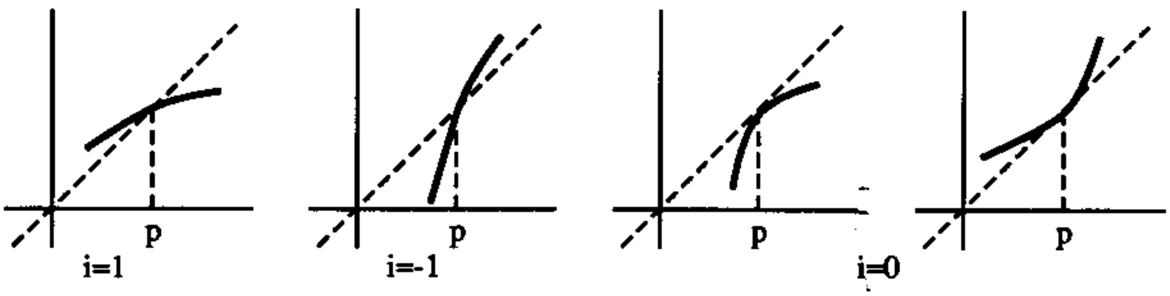
where B_ρ is a small ball centered at p that does not contain other fixed points of F. The excision property of the degree shows that this definition is independent of the radius ρ.

The linearization principle (3) for the degree shows that if $F \in C^1(U, \mathbf{R}^n)$ and $1 \notin \sigma(F'(p))$ then

$$i[F,P] = \text{sign } \{\det(I - F'(p))\}. \tag{5}$$

(From now on, $\sigma(A)$ denotes the spectrum of the linear operator A.)

In dimension $N = 1$ the index can only take the the values $1, -1$, or 0. They appear in the following situations:



In more dimensions the index can take any integer value. For instance, in two dimensions and using complex notation, the maps $F_n(z,\bar{z}) = z + z^n$, $F_{-n}(z,\bar{z}) = z + \bar{z}^n$, $n = 1, 2, \ldots$ have an isolated fixed point at the origen and $i[F_{\pm n}, 0] = \pm n$. However, for $N = 2$, it is still possible to reduce the range of the index if one imposes additional conditions.

Proposition 4. *Assume $N = 2$, $F \in C^1(U, \mathbf{R}^2)$ and p is an isolated fixed point of F.*
a) If $\det[F'(x)] = 1$ $\forall x \in U$ (area-preserving condition) then $i[F,p] \leq 1$.
b) If $0 < \det[F'(x)] < 1$ $\forall x \in U$ (area-contracting condition) then $|i[F,p]| \leq 1$.

The statement a) is proved in [50]. In [23] it is proved that if $N = 2$ and $F'(p)$ is not the identity matrix then $|i[F,p]| \leq 1$. The contracting condition implies that at least one of the eigenvalues of $F'(p)$ lies on the open unit disk, so that $F'(p) \neq I$ in case b).

A fixed point p of F is also a fixed point of the iteration F^n for each $n = 2, 3, \ldots$ Assuming that p is isolated as a fixed point of F^n one can consider the iterated index

$i[F^n, p]$. In dimension one it is easy to check that the sequence of iterated indexes $\{i[F^n, p]\}_{n\geq 1}$ is 2-periodic; that is $i[F^{n+2}, p] = i[F^n, p]$. For more dimensions, assuming that F is C^1, it can be proved that the sequence $\{i[F^n, p]\}_{n\geq 1}$ is k-periodic for some period $k \geq 1$. (See [47], [12]).

1.4 The index of an asymptotically stable fixed point

A fixed point that is asymptotically stable is isolated. In consequence the index is well defined in such case.

Theorem 5. *Assume that $F \in \mathcal{H}(U,p)$ and p is asymptotically stable. Then $i[F,p] = 1$.*

This result was presented in [22] in the contex of periodic solutions of nonautonomous differential equations. The proof given in [24] is based on Browder's principle (4). We reproduce it.

Proof. Since p is asymptotically stable there exists a small ball B, centered at p, and such that $F^n(\overline{B}) \subset B \;\forall n \geq n_0$. This follows from (2). Now (4) holds and the Browder's principle stated in 1.2 applies. Thus

$$i[F,p] = \deg(I - F, B) = 1.$$

As already mentioned in 1.1, if p is asymptotically stable with respect to F, the same property holds with respect to each iteration F^n.

Corollary 6. *Assume that $F \in \mathcal{H}(U,p)$ and p is asymptotically stable. Then $i[F^n,p] = 1$ for each $n \geq 1$.*

1.5 The index of a stable fixed point

A fixed point that is Lyapunov stable is not necessarily isolated and, in consequence, the index may be undefined. We assume that our fixed point is stable and isolated and ask about the value of the index. We shall see that the answer depends on the dimension N.

For $N = 1$ it is easy to show that a stable and isolated fixed point is always asymptotically stable. In consequence the index will be 1. For $N = 2$ we have

Theorem 7. *Assume $N = 2$, $F \in \mathcal{H}(U,p)$, F is orientation-preserving and p is Lyapunov stable and isolated. Then*

$$i[F,p] = 1.$$

This result is stated without proof in [22]. Recently a proof has been presented in [14]. This proof is based on some specific aspects of the topology of the plane. In particular, it uses a version of Brouwer's lemma on translation arcs that is inspired by [7].

For $N = 3$ the situation changes and the index can be different from 1.

Theorem 8. *Assume that $N = 3$ and γ is an integer with $\gamma \leq 1$ or $N \geq 4$ and γ is an arbitrary integer. Then there exists $F \in \mathcal{H}(B_1, 0)$ such that $p = 0$ is stable, isolated and*

$$i[F,p] = \gamma.$$

(B_1 is the unit ball of $\mathbf{R}^N$ centered at the origin)

We give a proof of this result that is based on a similar theorem for equilibria of vector fields that was presented in [24] (see also[15]). The result for vector fields is the following:

"Given γ in the conditions of the previous theorem, there exists a C^∞-vector field $f : \mathbf{R}^N \to \mathbf{R}^N$ such that $x = 0$ is the only equilibrium on $\overline{B}_1$, $\deg(-f, B_1) = \gamma$ and $x = 0$ is stable with respect to the autonomous system $x' = f(x)$"

Proof of Theorem 8. Let $\Phi_t(x)$ be the solution of the previous system $x' = f(x)$ satisfying $\Phi_0(x) = x$. The result in [53] implies that there are no closed orbits contained in $\overline{B}_1$ and having a small minimal period. In fact, the period of a closed orbit in $\overline{B}_1$ must satisfy $p \geq p_0$ where $p_0 = \frac{2\pi}{L}$, $L = \max\{\| f'(x) \| : x \in \overline{B}_1\}$ and $\| \cdot \|$ denotes the spectral norm. By continuous dependence, there exist a small ball B_δ such that $\Phi_t(\overline{B}_\delta) \subset \overline{B}_1$, $\forall t \in [0, p_0]$. The previous facts imply that

$$\Phi_t(x) \neq x \ \forall t \in (0, p_0), \ x \in \overline{B}_\delta - \{0\}.$$

A direct argument of homotopy or [24] show that, if $t \in (0, p_0)$,

$$\deg[I - \Phi_t, B_\delta] = \deg[-f, B_\delta] = \gamma.$$

In consequence an example as required in the theorem has been constructed with $F = \Phi_t$, $t \in (0, p_0)$.

Remark. I thank Prof. E.N. Dancer for suggesting me the use of the results of [53] in this proof.

1.6 Instability criterions in two dimensions

Theorem 7 can be employed to obtain results of instability of fixed points in two dimensions; it is enough to verify in each case that the index is different from 1. There are many results on the computation of the index that can be used (see [23]) and we shall present two examples that are of historical interest. They are taken from [14] .

1.6.1 A criterion of Levi-Civita

We consider fixed points in two dimensions such that $\sigma(f'(p)) \subset \{-1, 1\}$. Poincaré pointed out that, in a generic sense, one should expect instability for these points.

Motivated by this statement, Levi-Civita obtained in 1901 several results of instability in [25]. We now show how to obtain one of these results using degree theory.

Let $F = F(x,y)$ be a C^2-map defined in a neighborhood of the origin with the Taylor expansion

$$F(x,y) = F_0(x,y) + R(x,y),$$
$$F_0(x,y) = (x + a_1x^2 + a_2xy + a_3y^2, y + b_1x^2 + b_2xy + b_3y^2)$$
$$R(x,y) = o(x^2+y^2) \quad \text{as} \quad (x,y) \to (0,0).$$

The inverse function Theorem implies that $F \in \mathcal{H}(U,0)$ for some U. A scaling argument proves that if the origin is an isolated fixed point of F_0 then it is also isolated as a fixed point of F and

$$i[F,0] = i[F_0,0].$$

To compute $i[F_0,0]$ define

$$D_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \ D_2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \ D_3 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \ D = 4D_1D_2 - D_3^2.$$

In ([24], page 38) and [23] it is proved that if $D \neq 0$ then 0 is isolated and the value of the index is given by the following table

$D < 0$	$i[F_0,0] = 0$
$D > 0,\ D_1 > 0$	$i[F_0,0] = 2$
$D > 0,\ D_1 < 0$	$i[F_0,0] = -2$

In consequence, if $D \neq 0$, the origin is unstable.

1.6.2 Instability at the third root of unity

We now consider a C^2-map $F = F(x,y)$ with

$$F(0,0) = (0,0), \quad F'(0,0) \text{ a rotation of angle } \frac{2\pi}{3}$$

and prove that in most cases instability ocurrs. This result is relevant in Celestial Mechanics in the understanding of strong resonances (See [48]). A proof using Lyapunov functions is given in ([48], page 222). We now give a proof based on degree theory. Using complex notation the map $F = F(z,\overline{z})$ has a Taylor expansion of the form

$$F(z,\overline{z}) = \omega z + az^2 + bz\overline{z} + c\overline{z}^2 + \dots$$

where ω is a primitive third root of unity ($\omega^2+\omega+1=0$), $a,b,c \in \mathbf{C}$. The linearization principle shows that

$$i[F,0] = i[F^2,0] = 1.$$

However, the third iteration has the expansion

$$F^3(z,\overline{z}) = z + 3c\overline{\omega}\overline{z}^2 + \dots$$

If $c \neq 0$ the origin is isolated with respect to F^3 and

$$i[F^3, 0] = -2.$$

In consequence $x = 0$ is unstable with respect to F.

1.7 A sufficient condition for asymptotic stability in terms of the index

In this section we ask about the possibility of proving the asymptotic stability of a fixed point using information on the corresponding sequence of indexes. In general the answer to this question is negative and the reason is that the converse of Corollary 6. is not valid. This is shown by the following example.

Example I
$N = 2,\ F(x) = \lambda x \qquad (\lambda > 1).$

The origin repels the rest of the orbits and it is easily seen, using the principle of linealization (5), that $x = 0$ is unstable and $i[F^n, 0] = 1$ for each $n \geq 1$.

The previous example assumes $N = 2$, and one can create similar examples when $N > 2$. The case $N = 1$ is different and is discussed in the next examples.

Example II
$N = 1,\ F \in \mathcal{H}(U, p),\ F$ increasing.

An argument based on the monotonicity of F, proves that p is asymptotically stable if and only if there exists $\delta > 0$ such that

$$(x - f(x))(x - p) > 0 \quad \text{if } 0 < |x - p| < \delta.$$

This is equivalent to $i[F, p] = 1$. In this case the index $i[F, p]$ characterizes the asymptotic stability.

Example III
$N = 1,\ F \in \mathcal{H}(U, p),\ F$ decreasing.

Assuming that p is isolated with respect to F, the index $i[F, p]$ is always one (see the figure in Section 1.3). Therefore this index does not discriminate between stable and unstable fixed points. However the second iteration F^2 is in the conditions of example II. Now, it is the second index $i[F^2, p]$ the right one to characterize asymptotic stability.

The next result extends the previous remarks to a more general setting. First we introduce some notation. Let A be a square matrix of dimension N having r different eigenvalues

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_r\},\ r \leq N.$$

It is assumed that they are arranged so that

$$|\lambda_1| \geq \ldots \geq |\lambda_r|.$$

The corresponding algebraic multiplicities are denoted by $\mu_1, \ldots, \mu_r$. It is said that the matrix A satisfies the condition (C_1) if at least one of the following alternatives holds:

$$|\lambda_1| < 1, \text{ or } |\lambda_2| < 1, \mu_1 = 1$$

Theorem 9. *Assume that N is arbitrary and $F \in C^1(U, \mathbf{R}^N) \cap \mathcal{H}(U,p)$. In addition it is assumed that*

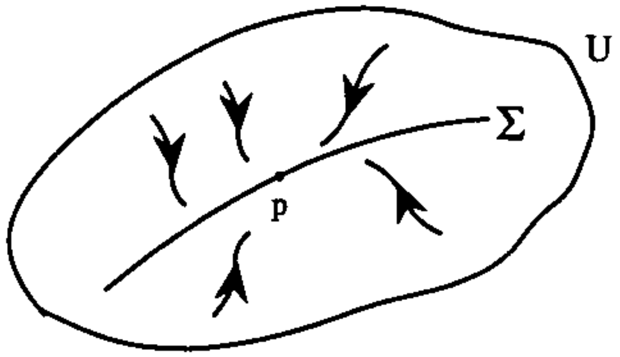
p is isolated with respect to F^2 and $F'(p)$ satisfies (C_1).

Then the following statements are equivalent:

i) p is asymptotically stable
ii) p is stable
iii) $i[F^2, p] = 1$.

This result is stated and proved in [40]. A similar result can be found in [13]. In [13] the assumption on the spectrum of $F'(p)$ is replaced by a condition of monotonicity of F that in particular implies that $F'(p)$ satisfies (C_1). This monotonicity condition is applicable in the case of parabolic equations but it does not work in the applications to Newton's equations.

A way of proving the theorem is based on the use of the Center Manifold Theorem ([30], [19], [18]). The details are given in [40] but we now give a sketch. The difficult case is $\lambda_1 = \pm 1$ because otherwise one can use linearization principles. Since $\mu_1 = 1$ the Center Manifold is in this case a curve Σ that is locally invariant and locally attracting.



The map F can be restricted to a neighborhood of p in Σ, say Σ' in such a way that $F(\Sigma') \subset \Sigma$. The map $F_\Sigma : \Sigma' \to \Sigma$ is such that the asymptotic stability of p with respect to F and with respect to F_Σ are equivalent (see [18] page 262). Now F_Σ is a one-dimensional homeomorfism and one uses the ideas of examples II and III to conclude. The delicate step is the connection between the index of F and F_Σ. The previous theorem allows to characterize asymptotic stability in terms of index when the map is planar and area-contracting.

Corollary 10. *Assume $N = 2$ and $F \in C^1(U, \mathbf{R}^2) \cap \mathcal{H}(U,p)$. In addition,*

$$0 < \det F'(x) < 1 \quad \forall x \in U.$$

Then,
p is asymptotically stable $\Longleftrightarrow$ p is isolated with respect to F^2 and $i[F^2,p] = 1$.

Proof.
$F'(p)$ satisfies (C_1) since we are in two dimensions and $\det F'(p) \in (0,1)$.

1.8 Remarks on the area-preserving case

In this section we assume that $N = 2$ and $F \in C^1(U, \mathbf{R}^N) \cap \mathcal{H}(U,p)$ satisfies

$$\det[F'(x)] = 1 \quad \forall x \in U.$$

The linearization principle shows that if p is stable and λ_1, λ_2 are the eigenvalues of $F'(p)$ then one of the following alternatives hold:
i) $\lambda_1 = \overline{\lambda}_2$, $\mid \lambda_1 \mid = \mid \lambda_2 \mid = 1$, $\lambda_i \notin \mathbf{R}$ (elliptic case)
ii)$\lambda_1 = \lambda_2 = \pm 1$, (parabolic case)
(of course, the possibility of satisfying condition (C_1) of the previous section is now excluded).

A characterization of stability in the line of Corollary 10. is not possible. In fact, it is not difficult to construct an area preserving analytic map with the Taylor expansion

$$F(z, \overline{z}) = \omega z + \overline{z}^2 \ldots$$

where $\omega^2 + \omega + 1 = 0$. For this construction one can use the technique of generating functions (see [4]). According to the results of section 1.6 the fixed point $z = 0$ is unstable and $i[F, 0] = i[F^2, 0] = 1$. In general, given $\mu \in \mathbf{N}$, $\mu \geq 3$, it is possible to find a mapping F that is area-preserving and analytic and satisfies

$$F(0) = 0,\ \sigma(F'(0)) = \{e^{\frac{2\pi}{\mu}}, e^{-\frac{2\pi}{\mu}}\},$$
$$i[F^n, 0] = 1,\ n = 1, \ldots, \mu - 1, i[F^\mu, 0] \neq 1.$$

(See [48], [42].)

This examples show that, if a characterization of stability in terms of the degree is going to be possible for area-preserving mapping, it should use the whole sequence

$\{i[F^n,p]\}_{n\geq 1}$. There are some partial answers to this question when F is analytic. They depend on the spectrum $\sigma(F'(p)) = \{\lambda, \overline{\lambda}\}$ with $|\lambda| = 1$. When λ is not a root of unity, (5) implies that $i[F^n,p] = 1$ for each $n \geq 1$. On the other hand in [45] it is shown that if

$$|\lambda^n - 1| > \frac{c}{n^\nu},\ n = 1,2,\dots\ (c,\nu > 0)$$

then p is always stable.

In particular this condition says that λ is not a root of unity and is in fact "far away from roots of unity" in an arithmetic sense. In the other extreme we have the case of when λ is a root of the unity. This case can be reduced to the parabolic case using iterations. There are interesting ideas in [49] on parabolic fixed points that perhaps could be useful to obtain a characterization of stability in terms of index when λ is a root of unity.

Chapter 2 The index of a periodic solution

From now on we consider the differential equation (1) where $c \geq 0$ and $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous and T-periodic in time. In most cases it will be assumed that the initial value problem associated to (1) has a unique solution.

To each isolated T-periodic solution, $\varphi(t)$, we can associate an integer called the index of φ of period T. This index can be defined in different (but essentially equivalent) forms depending on the way the periodic problem for (1) has been reduced to a fixed point equation. We shall review two of such reductions.

2.1 Definition of the index via the Poincaré map

In this section we assume uniqueness for the initial value problem associated to (1). Given $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, let $x(t;\xi)$ be the solution satisfying

$$x(0) = \xi_1,\ x'(0) = \xi_2.$$

The Poincaré map is defined as the mapping

$$P_T : D_T \subset \mathbf{R}^2 \to \mathbf{R}^2,\ P_T(\xi) = (x(T;\xi), x'(T;\xi)),$$

where $D_T = \{\xi \in \mathbf{R}^2 :\ x(t;\xi) \text{ is defined in } [0,T]\}$.

The standard theory of the Cauchy problem says that D_T is open in $\mathbf{R}^2$ and P_T is a homeomorphism between D_T and $P_T(D_T)$. In addition, the fixed points of P_T correspond to the initial conditions of the T-periodic solutions and the search of T-periodic solutions is reduced to the study of the equation in $\mathbf{R}^2$,

$$\xi = P_T(\xi). \qquad (6)$$

Let $\varphi(t)$ be a T-periodic solution of (1) and $\xi_0 = (\varphi(0), \varphi'(0))$. The solution φ is said to be isolated (period T) if ξ_0 is an isolated fixed point of P_T. In such case the index of φ is defined in terms of the following formula

$$\gamma_T(\varphi) := i[P_T, \xi_0].$$

This is the definition employed in [22].

2.2 The index in the space of periodic functions

Let $X_T = C(\mathbf{R}/T\mathbf{Z})$ be the space of functions $p : \mathbf{R} \to \mathbf{R}$ that are continuous and T-periodic with the uniform norm. We wish to formulate the periodic problem for (1) as a fixed point equation in X_T. Given $\lambda > 0$ and $p \in X_T$, consider the linear problem

$$x'' + cx' - \lambda x = p(t), \ x \in X_T. \tag{7}$$

It follows from the Fredholm alternative that it has a unique solution $x = L_\lambda p$. In addition, the linear operator $L_\lambda : X_T \to X_T$, $p \to x$, can be expressed by means of the Green's function as an integral operator and it is easy to verify that it is compact. The periodic problem for (1) is equivalent to

$$x = L_\lambda(Nx - \lambda x), \ x \in X_T, \tag{8}$$

where $N : X_T \to X_T$, $Nx = f(\cdot, x(\cdot))$.

Since the operator $\Phi_\lambda = L_\lambda(N - \lambda I)$ is compact, the Leray-Schauder degree can be applied to this case.

Given a T-periodic solution $\varphi(t)$ of (1) and $\lambda > 0$, assuming that φ is an isolated fixed point of Φ_λ, we define the new index,

$$\tilde{\gamma}_T(\varphi) := -i[\Phi_\lambda, \varphi].$$

It is easy to prove that, when there is uniqueness for the initial value problem, φ is an isolated fixed point of Φ_λ for some $\lambda > 0$ if and only if φ is isolated (period T) in the sense of section 2.1. Also, a homotopy argument shows that the previous definition is independent of the value of λ ($\lambda > 0$). The condition of positivity on λ in the definition has been imposed to make sure that (7) has a unique solution. Using the coincidence degree [17] and the principles of relatedness [24], it is possible to construct equivalent definitions of $\tilde{\gamma}_T(\varphi)$ that employ (7) with $\lambda \geq 0$.

The two indexes γ_T and $\tilde{\gamma}_T$ have been defined in different ways but it follows from ([24], Chapter 3) that they coincide whenever P_T is well defined. The negative sign in the definition of $\tilde{\gamma}_T$ had the intention of getting this coincidence.

It is interesting to discuss a little the advantages and disadvantages of each of the previous approaches. The following table sums up the most obvious differences

Index	$\gamma_T(\varphi)$	$\tilde{\gamma}_T(\varphi)$
Space	$\mathbf{R}^2$, dim $= 2$	X_T, dim $= \infty$
Operator	P_T (defined implicitly from the equation)	Φ_λ (explicitly known)
Assumption on (1)	Uniqueness of the initial value problem	——

There are other differences between both approaches. Since P_T is intimately connected with the dynamics of (1), the first approach is more direct in dynamical questions such as stability. On the other hand Φ_λ has certain properties of monotonicity that may be very useful in monotone methods. Also, the second approach can be employed to analyze questions related to the convergence of Picard iterations.

2.3 The iterated index

The differential equation (1) is periodic in time with period T. In consequence it can also be seen as a periodic equation with period nT for each $n \geq 2$. The new Poincaré map corresponding to this period is

$$P_{nT} : D_{nT} \subset \mathbf{R}^2 \to \mathbf{R}^2,\ P_{nT}(\xi) = (x(nT;\xi), x'(nT;\xi)).$$

It can also be expressed as the composition $P_{nT} = P_T \circ \overset{(n}{\cdots} \circ P_T$.

The equation

$$\xi = P_{nT}(\xi) \quad (n \geq 2) \tag{9}$$

can have more solutions than (6). The solutions of the differential equation corresponding to these initial conditions are periodic solutions of period nT. A periodic solution of (1) with minimal period nT ($n \geq 2$) will be called a subharmonic solution of order n.

Let $\varphi(t)$ be a T-periodic solution of (1). This solution is isolated (period nT) if the initial condition $\xi_0 = (\varphi_0(0), \varphi_0'(0))$ is an isolated solution of (9). In such case, the index of φ of period nT is defined as

$$\gamma_{nT}(\varphi) := i[P_{nT}, \xi_0].$$

The previous definition requires the uniqueness of the initial value problem for (1). An alternative definition (valid in the general case) can be given using the Banach space X_{nT} and the operators

$$L_{\lambda,n} : X_{nT} \to X_{nT},\ N_n : X_{nT} \to X_{nT},\ \Phi_{\lambda,n} = L_{\lambda,n} \circ (N_n - \lambda I)$$

defined in an obvious way according to section 2.2

The iterated index is now defined as

$$\tilde{\gamma}_{nT}(\varphi) := -i[\Phi_{\lambda,n}, \xi_0].$$

2.4 The computation of the index

The computation of the index of a periodic solution may be a difficult task in general, however there are some simple cases. Probably the most direct way to compute the index is the linearization. Assume that $f \in C^{0,1}(\mathbf{R}\times\mathbf{R})$ and φ is a T-periodic solution. The linearization of (1) at φ is the Hill's equation

$$y'' + cy' = f_x(t,\varphi(t))y. \tag{10}$$

Let μ_1, μ_2 be the characteristic multipliers of this equation. They are the eigenvalues of any monodromy matrix of (10). In addition, $P_T \in C^1(D_T, \mathbf{R}^2)$ and the derivative of P_T at $\xi_0 = (\varphi(0), \varphi'(0))$ is precisely a monodromy matrix of (10). (See for instance [22]). Assuming that $\mu_i \neq 1$, $i = 1, 2$, we can use the the linearization principle for the index to deduce that φ is isolated (period T) and

$$\begin{aligned}\gamma_T(\varphi) = i[P_T, \xi_0] = \operatorname{sign}\det[I - P_T'(\xi_0)] = \\ \operatorname{sign}\{(1-\mu_1)(1-\mu_2)\}.\end{aligned}$$

When $\mu_i = 1$ for some i, the previous technique does not work and the index of φ depends not only on (10) but also on the nonlinear terms of the Taylor expansion of f. The following example will illustrate this fact.

Example

$$x'' + cx' = a(t)x^n, \quad n \geq 2.$$

It is assumed that $a \in C(\mathbf{R}/T\mathbf{Z})$ does not change the sign and does not vanish identically; that is

$$a(t) \geq 0 \quad \forall t \in \mathbf{R} \qquad \text{or} \qquad a(t) \leq 0 \quad \forall t \in \mathbf{R}$$

and

$$\int_0^T |\, a \,| > 0.$$

The equilibrium $x = 0$ is a T-periodic solution with linearized equation

$$y'' + cy' = 0.$$

The characteristic multipliers are in this case $\mu_1 = 1$, $\mu_2 = e^{-cT}$ and therefore the method of linearization does not apply. Next we prove that $x = 0$ is isolated (period T) and

$$\gamma_T(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\operatorname{sign}(a) & \text{if } n \text{ is odd.} \end{cases}$$

To prove that $x = 0$ is isolated we show that any other T-periodic solution of the equation must satisfy

$$\|x\|_{L^\infty}^{n-1} \geq \frac{1}{T \mid \int_0^T a \mid}.$$

(Notice that this inequality will prove the isolation in the sense of section 2.2).

Let $x(t)$ be such solution and integrate the equation over one period to get

$$\int_0^T a(t)x(t)^n dt = 0.$$

Since a does not change sign, $x(t)$ must vanish at some $t_0 \in \mathbf{R}$. Thus, using Cauchy-Schwarz inequality,

$$|\, x(t) \,| = |\int_{t_0}^t x'(s)ds \,| \leq \sqrt{t-t_0}(\int_{t_0}^t x'(s)^2 ds)^{1/2} \text{ if } t \in [t_0, t_0+T]$$

and

$$\|x\|_{L^\infty} \leq \sqrt{T}\|x'\|_{L^2}.$$

Multiplying the equation by x and integrating again,

$$\|x'\|_{L^2}^2 = -\int_0^T a x^{n+1} dt \leq |\int_0^T a \,|\, \|x\|_{L^\infty}^{n+1}.$$

The estimate is obtained by a combination of both inequalities.

To compute the index when n is even we argue by contradiction. If $\gamma_T(0) \neq 0$, the properties of the degree would imply that the equation

$$x'' + cx' = a(t)x^n + \epsilon$$

has a T-periodic solution when ϵ is a small constant. However this equation has not T-periodic solutions if ϵ has the same sign of a because a T-periodic solution $x(t)$ must satisfy

$$\int_0^T a(t)x(t)^n dt + \epsilon T = 0.$$

To compute the index when n is odd one can proceed as follows. Let $\epsilon \in \mathbf{R}$ be of same sign of a and such that $|\,\epsilon\,| < |\int_0^T a\,|$ and consider the homotopy

$$x'' + cx' = \lambda a(t)x^n + (1-\lambda)\epsilon x,\ \lambda \in [0,1].$$

As before one shows that any nontrivial T-periodic solution of this parametrized equation satisfies $\|x\|_{L^\infty}^{n-1} \geq \frac{1}{T|\int_0^T a|}$. Now the homotopy invariance of the Leray-Schauder degree proves that it is sufficient to compute the index when $\lambda = 0$. This can be done by linearization to conclude that $\gamma_T(0) = -\text{sign}\epsilon$.

In the previous examples the index was always 1, -1 or 0. The next result determines the possible values of the index when f is smooth.

Proposition 1. *Assume that $f \in C^{0,1}(\mathbf{R} \times \mathbf{R})$ and let φ be a T-periodic solution that is isolated (period T).*

Then

$$\gamma_T(\varphi) \leq 1 \text{ if } c = 0 \quad \text{and} \quad |\,\gamma_T(\varphi)\,| \leq 1 \text{ if } c > 0.$$

This is a consequence of Proposition 4. of the first chapter because the Liouville formula shows that the Poincaré map is area-preserving when $c = 0$ and area-contracting when $c > 0$.

2.5 Index and stability of a periodic solution

Let $\varphi(t)$ be a T-periodic solution of (1) and denote by $\xi_0 = (\varphi(0), \varphi'(0))$ the corresponding initial condition. Assuming uniqueness for the initial value problem and using the continuous dependence of the solutions it is easy to prove that φ is stable or asymptotically stable with respect to (1) if and only if ξ_0 has the same property as a fixed point of P_T. In consequence we can combine ideas of the previous sections with chapter 1 and obtain the following result

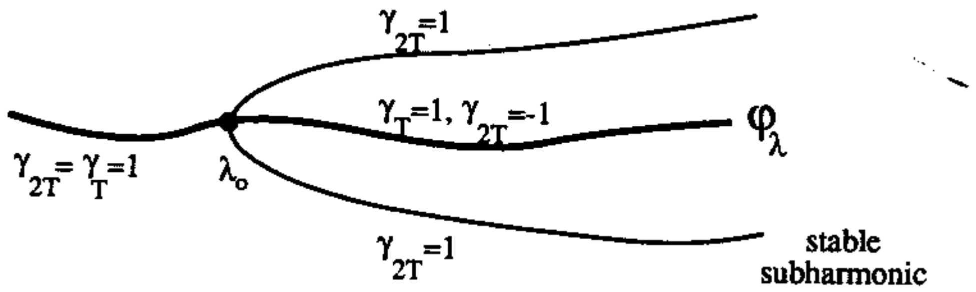
Theorem 2. *Assume that there is uniqueness for the initial value problem associated to (1) and let $\varphi(t)$ be a T-periodic solution that is stable and isolated (period T). Then $\gamma_T(\varphi) = 1$.*

This is a consequence of theorem 7. of chapter 1 since P_T is an orientation-preserving homeomorphism.

Theorem 3. *Assume $f \in C^{0,1}(\mathbf{R} \times \mathbf{R})$ and $c > 0$. Let $\varphi(t)$ be a T -periodic solution. Then*

$$\varphi(t) \text{ is asymptotically stable } \Longleftrightarrow \varphi \text{ is isolated (period } 2T) \text{ and } \gamma_{2T}(\varphi) = 1$$

This result follows from Corollary 10. of chapter 1. It was first obtained in [38] for the hyperbolic case and in the general case in [40]. The need of the index of double period in this characterization can be explained in terms of bifurcations. Consider a T-periodic solution $\varphi_\lambda(t)$ that depends of a parameter (of course, (1) will depend also of λ) and assume that φ_λ experiences a bifurcation to second order subharmonic solutions as in the diagram



The exchange of stability at λ_0 can not be detected by γ_T since the new stable solutions are not of period T.

We conclude with a simple application of this results.

Example (continuation)

Again we consider the example of section 2.4.

$$x'' + cx' = a(t)x^n, \; n \geq 2$$

with

$$a \in C(\mathbf{R}/T\mathbf{Z}) \quad \textit{and} \quad a \geq 0 \;\; \textit{or} \;\; a \leq 0, \; a \neq 0.$$

If n is even or n is odd and $a \geq 0$, $\gamma_T(0) \neq 1$ and $x \equiv 0$ is unstable.
If $c > 0$, n is odd and $a \leq 0$ we have $\gamma_T(0) = \gamma_{2T}(0) = 1$ and $x \equiv 0$ is asymptotically stable.

The previous results do not apply to the case $c = 0$, $a \leq 0$. In such case one could use techniques in the line of [42].

2.6 The metod of lower and upper solutions for first and second order equations

The method of lower and upper solutions is well known as a method of proving the existence of periodic solutions of a first or second order differential equation. We now ask a different question about this method: what can be said about the stability properties of the solution obtained with it?. For equations of first order this method usually captures an asymptotically stable solution. This fact is not only true for ordinary equations but also for parabolic equations [21] or even for some abstract periodic equations such that the Poincaré operator preserves the order [13]. However, for second order equations the solution lying between the lower and upper solution is usually unstable. We notice that in this case the solutions of the initial value problem are not ordered.

Although our interest is in second order equations, it is useful to review some known facts for first order equations.

2.6.1 The first order equation

In this paragraph we consider the equation

$$x' = f(t,x) \tag{11}$$

with $f \in C(\mathbf{R}/T\mathbf{Z} \times \mathbf{R})$. It is also assumed that there is uniqueness for the initial value problem.

Let $\alpha, \beta : [0,T] \longrightarrow \mathbf{R}$ be C^1 functions satisfying :

$$\alpha'(t) \preceq f(t,\alpha(t)), \quad \beta'(t) \succeq f(t,\beta(t)), \forall t \in [0,T] \tag{12}$$

$$a := \alpha(0) \leq \alpha(T) < \beta(T) \leq \beta(0) := b \tag{13}$$

(the simbol $\preceq$ means that the inequality is strict on some subinterval of $[0,T]$).

We say that α and β are a couple of strict lower and upper solutions that are ordered.

Let $x(t,\xi)$ be the solution of (11) with $x(0)=\xi$. The theory of diferential inequalities implies that for each ξ with $a \leq \xi \leq b$ one has $\alpha(T) < x(t,\xi) < \beta(T)$. The Poincaré map P_T is in this case a strictly increasing function, $P_T : \mathbf{R} \longrightarrow \mathbf{R}$ such that $P_T[a,b] \subset (a,b)$. In consequence if P_T has a finite number of fixed point lying in (a,b), at least one of them must be asymptotically stable. This can be proved using example II of section 1.7. We have prove the following result.

Proposition 4. *Asume that there exist functions $\alpha, \beta \in C^1[0,T]$ satisfying (12) and (13). In addition assume that the number of T-periodic solutions of (11) with initial condition $x(0) \in (a,b)$ is finite. Then there exists an asymptotically stable T-periodic solution φ verifying*

$$\alpha(t) < \varphi(t) < \beta(t) \quad \forall t \in [0,T].$$

An analogous result can be obtained reversing the order in (12) and keeping inequality (13). For proving it, it is sufficient to look toward the past and consider the operator P_{-T}. However the conclusion will now be the existence of an unstable solution.

2.6.2 The second order equation

We now go back to our equation (1). A couple of ordered strict lower and upper solutions of (1) is given by two functions $\alpha, \beta \in C^2(\mathbf{R}/T\mathbf{Z})$ satisfying

$$\alpha''(t) + c\alpha'(t) \succeq f(t,\alpha(t)), \quad \beta''(t) + \beta'(t) \preceq f(t,\beta(t)) \;\; \forall t \in \mathbf{R} \tag{14}$$

$$\alpha(t) < \beta(t) \; \forall t \in \mathbf{R} \tag{15}$$

It is well known that under these assumptions there exist at least one T-periodic solution between α and β (see [6], [33]). The next result sharpens the conclusion assuming that the number of T-periodic solutions is finite.

Theorem 5. *Assume that f is locally Lipschitz continuous with respect to x and let α, β be T-periodic functions of class C^2 satisfying (14), (15). In addition assume that the number of T-periodic solutions of (1) satisfying*

$$\alpha(t) < \varphi(t) < \beta(t) \quad \forall t \in \mathbf{R}$$

is finite. Then at least one of them is unstable.

Remarks. 1. A very similar result was proved in [39] in the case $c > 0$ and in [14] for $c = 0$. An extension to Lienard's equation was presented in [2] .
2. By analogy with the first order equation one could think that a result of stability could be obtained by reversing inequality (14). However in such case it is easy to construct examples of linear equations without periodic solutions or having only unstable solutions.

The proof in [39], [14] used the definition of index based on the Poincaré map. We now give a proof that is similar to the proof in [33] and is based on the second definition of the index and the following maximum principle for the periodic problem (see [43]) :

" *Let* $x \in C^2(\mathbf{R}/T\mathbf{Z})$ *be a solution of the differential inequality*

$$x'' + cx' - \lambda x \preceq 0$$

for some $\lambda \geq 0$. *Then* $x(t) > 0 \quad \forall t \in \mathbf{R}$."

Proof of the theorem. We choose $\lambda > 0$ large enough in order to guarantee that the function $f^*(t,x) = f(t,x) - \lambda x$ is strictly decreasing with respect to x in the set $\{(t,x): \ t \in \mathbf{R}, \ \alpha(t) \leq x \leq \beta(t)\}$. Following section 2.2 we rewrite the periodic problem for 1 as the fixed point equation

$$x = \Phi_\lambda(x), \ x \in X.$$

The set $\Omega = \{x \in X : \ \alpha(t) < x(t) < \beta(t) \ \forall t \in \mathbf{R}\}$ is convex and open in X, we prove $\Phi_\lambda(\overline{\Omega}) \subset \Omega$. Given $x \in \overline{\Omega}$, $\alpha \leq x \leq \beta$ we check that $y = \Phi_\lambda(x)$ satisfies $y < \beta$ (the inequality $y > \alpha$ is verified in an analogous way). In fact y is a solution of

$$y'' + cy' - \lambda y = f^*(t, x(t))$$

and, since $f^*(t,x) \geq f^*(t,\beta)$, the function $z = \beta - y$ satisfies

$$z'' + cz' - \lambda z \preceq 0,$$

and the maximum principle implies that $z > 0$.

Now the degree of $I - \Phi_\lambda$ in the sense of Leray-Schauder is well defined in Ω and

$$\deg[I - \Phi_\lambda, \Omega] = 1.$$

By assumption Φ_λ has a finite number of fixed points in Ω, say $\varphi_1, \ldots, \varphi_n$. The properties of the degree imply that

$$-1 = \sum_{i=1}^{n} \gamma_T(\varphi_i)$$

and in consequence some φ_i must have negative index and thus it is unstable.

Remark. The previous proof gives some information on the index of a solution (it must be negative). This can be combined with Proposition 1. to obtain multiplicity results.

Example

The equation

$$x'' + cx' + \frac{1}{2}x - 3\sin x = \sin t \quad (c \geq 0)$$

has at least three 2π-periodic solutions.

To prove this we can assume that it has only a finite number of 2π-periodic solutions (otherwise it is already proved). The constants $\alpha \equiv -\frac{\pi}{2}$, $\beta \equiv \frac{\pi}{2}$ are strict lower and upper solutions so that there exist a 2π-periodic solution $\varphi_1$with $\gamma_{2\pi}(\varphi_1) < 0$.

It is easy to show that the 2π-periodic solutions of

$$x'' + cx' + \frac{1}{2}x - 3\lambda \sin x = \lambda \sin t, \quad \lambda \in [0,1]$$

have a bound independent of λ. The homotopy invariance of the degree implies that

$$\deg[I - P, B] = \sum_{i=1}^{n} \gamma_{2\pi}(\varphi_i) = 1.$$

We know that $\gamma_{2\pi}(\varphi_1) < 0$ and, by Proposition 1., $\gamma_{2\pi}(\varphi_i) \leq 1 \; \forall i$. Therefore, there exist at least two more solutions with index 1.

2.7 The action functional: instability of minimizers

In this section we consider the equation (1) when $c = 0$, that is

(16) $$x'' = f(t, x)$$

where $f \in C(\mathbf{R}/T\mathbf{Z} \times \mathbf{R})$ and there is uniqueness for the initial value problem. In this case the equation has a variational structure and can be seen as the Euler equation of the action functional. The periodic problem for (16) is reduced to the search of critical points of the functional

$$\mathcal{A} \, : \, H \to \mathbf{R}, \; \mathcal{A}(x) = \int_0^T \{\frac{1}{2}x'(t)^2 - V(t, x(t))\}dt$$

where H is the Sobolev space

$$H = \{ \, x : \mathbf{R} \to \mathbf{R} \, ; \; x \text{ is absolutely continuous, } T\text{-periodic and } x' \in L^2(\mathbf{R}/T\mathbf{Z})\}$$

with inner product

$$(x, y)_H = \int_0^T x'(t)y'(t)dt + \int_0^T x(t)y(t)dt,$$

and V is the potential defined by $V(t, x) = -\int_0^x f(t, y)dy$.

The simplest critical points of $\mathcal{A}$ appear with the minima. We now prove that the corresponding T-periodic solutions are unstable.

Theorem 6. *Let φ be a T-periodic solution of (16) that is isolated (period T) and such that $\mathcal{A}$ reaches a local strict minimum at φ. Then φ is an unstable solution of (16).*

Proof. We reproduce the proof from [14]. To start we compute the gradient of $\mathcal{A}$ with respect to the inner product $(.,.)_H$. Define the operator $T: H \to H$, $Tx = z$ where z the is the periodic solution of

$$z'' - z = f(t, x(t)) - x(t).$$

It is easy to prove that T is a compact nonlinear operator satisfying

$$(Tx, y)_H = \int_0^T z'y' + zy = -\int_0^T z''y + \int_0^T zy = -\int_0^T [f(t,x) - x]y \quad \forall x, y \in H.$$

Computing the differential of $\mathcal{A}$ we obtain

$$(\nabla\mathcal{A}[x], y)_H = d\mathcal{A}[x]y = \int_0^T x'y' + f(t,x)y = (x,y)_H + \int_0^T \{f(t,x) - x\}y = (x - Tx, y)_H.$$

In consequence $\nabla\mathcal{A} = I - T$ and the theory of degree can be applied to this operator. It is known that the degree of a gradient field in the neighborhood of a minimum is 1 (see [1]). In our case

$$\deg[I - T, B] = 1$$

where B is a small neighborhood of φ in H.

Going back to the definition of the index in the space of functions we notice that T is essentially the same as Φ_{-1}, only the domain of definition changes. In fact, $\mathrm{dom}(\Phi_{-1}) = X$, dom $T = H$, $H \subset X$ and $\Phi_{-1}x = Tx$, $\forall x \in H$. Let $\tilde{B}$ a small neighborhood of φ in X. Since φ is an isolated fixed point of Φ_{-1} and T, B and $\tilde{B}$ have a common core in the sense of [24]. The principle of relatedness shows that

$$\deg[I - T, B] = \deg[I - \Phi_{-1}, \tilde{B}].$$

Thus $\gamma_T(\varphi) = -i(\Phi_{-1}, \varphi) = -1$ and Theorem 2 implies that φ is unstable.

Chapter 3 Stability of periodic solutions of pendulum-type equations

In this chapter we consider the equation

$$x'' + cx' = g(t,x) + p(t) - \mu \tag{17}$$

where μ is a real parameter, $p: \mathbf{R} \to \mathbf{R}$ is continuous, T-periodic and with mean value zero ($\int_0^T p = 0$), $c \geq 0$ and $g \in C^{0,1}(\mathbf{R} \times \mathbf{R})$ is T-periodic in t and such that

$$g(t, x + 2\pi) = g(t,x) \quad \forall (t,x) \in \mathbf{R} \times \mathbf{R}.$$

The forced pendulum equation is the model of this class of equations and corresponds to the case $g(t,x) = -A \sin x$. There are many other related equations such as the equation of the swing or pendulum of variable length ($g(t,x) = -\lambda(t)\sin x,\ p = 0,\ \mu = 0$).

The configuration space of this equation can be the real line or the circle. In what follows it is always assumed that $x \in \mathbf{R}$. This is important because the class of periodic solutions of this equation on the line and on the circle may not coincide. Given a solution $x(t)$ of (17), the periodicity in x of the equation implies that $x(t)+2n\pi$ is also a solution for each integer n. In what follows we identify these solutions and consider them as one single solution.

The existence of periodic solutions of equations of pendulum-type has been studied by many authors (See [35] and the references there). The stability properties of the periodic solutions have been treated in [46], [37], [39].

We wish to combine the results of the previous chapter with the information on the degree derived from the known existence theory to obtain new results on stability.

3.1 Existence of periodic solutions

The existence of T-periodic solutions of (17) imposes some restrictions on the parameter μ. An obvious necessary condition for existence is derived as follows.

Define

$$g_M(t) = \sup_{x\in\mathbf{R}} g(t,x),\ g_L(t) = \inf_{x\in\mathbf{R}} g(t,x),$$

and

$$\underline{g} = \frac{1}{T}\int_0^T g_L(t)dt,\ \overline{g} = \frac{1}{T}\int_0^T g_M(t)dt.$$

Let $x(t)$ be a T-periodic solution of (17). By integrating the equation over one period one gets

$$\int_0^T g(t,x(t))dt = \mu T$$

and this implies

$$\underline{g} \le \mu \le \overline{g}.$$

This condition is not sufficient in general. The next result shows the form of the necessary and sufficient condition for existence. The proof can be seen in [34] or [20]. See also [9].

Proposition 1. *There exist constants μ_-, μ_+ satisfying*

$$\underline{g} \le \mu_- \le \mu_+ \le \overline{g}$$

such that (17) has a T-periodic solution if and only if

$$\mu_- \le \mu \le \mu_+.$$

Remarks 1. When μ is not in the interval $[\mu_-, \mu_+]$ then every solution of the equation is unbounded. This is a consequence of the second theorem of Massera.
2. Sometimes it may happen that the existence interval degenerates and $\mu_- = \mu_+$. In that case it can be proved that there exists always a continuum of T-periodic solutions for $\mu = \mu_\pm$. This phenomenon appears for the linear equation ($g \equiv 0$) and also for some nonlinear equations (See the example in [5]). However it can be proved that this case is in some sense excepcional ([29]) and we shall analize the situation $\mu_- < \mu_+$.

3.2 Index of periodic solutions

Proposition 2. *Assume that the set of T-periodic solutions of (17) is finite and given by $\varphi_1, \ldots, \varphi_n$. Then*

$$\sum_{i=1}^{n} \gamma_T(\varphi_i) = 0.$$

Before the proof we need a preliminary result on the computation of a degree in two dimensions. We give an elementary proof taken from [37].

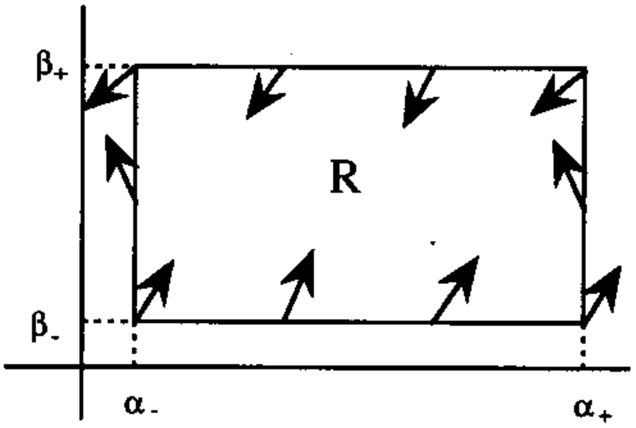
Lemma 3. *Consider the rectangle*

$$R = \{\sigma = (\sigma_1, \sigma_2) \in \mathbf{R}^2 \ : \alpha_- < \sigma_1 < \alpha_+, \ \ \beta_- < \sigma_2 < \beta_+\}, \ \ \alpha_- < \alpha_+, \ \beta_- < \beta_+,$$

and let $F = (F_1, F_2) : R \to \mathbf{R}^2$ be a continuous function satisfying

$$F(\alpha_-, \sigma_2) = F(\alpha_+, \sigma_2) \neq 0, \ \sigma \in [\beta_-, \beta_+], \ \ F_1(\sigma_1, \beta_+) < 0, \ F_1(\sigma_1, \beta_-) > 0, \ \sigma_1 \in [\alpha_-, \alpha_+].$$

Then $\deg[F, R] = 0$.



Proof. It is not restrictive to assume $\alpha_- = \beta_- = -1, \quad \alpha_+ = \beta_+ = 1$. Define $F^*(\sigma_1, \sigma_2) = F(-\sigma_1, \sigma_2)$. From the previous assumptions $\lambda F^* + (1-\lambda)F, \ \lambda \in [0,1]$ is a homotopy. Hence $\deg[F, R] = \deg[F^*, R]$. On the other hand the definition of degree implies that $\deg[F, R] = -\deg[F^*, R]$.

Proof of proposition 2. Let $P(x_0, y_0) = (x(T; x_0, y_0), x'(T; x_0, y_0))$ be the Poincaré operator of (17). Since g is bounded, it follows from the equation that there exists $\rho > 0$ such that if $|y_0| > \rho$ then

$$\mathrm{sign}(y_0)\,[x(T; x_0, y_0) - x_0] > 0.$$

Consider the rectangle

$$R = \{(x_0, y_0) \in \mathbf{R}^2 : \ \alpha < x_0 < \alpha + 2\pi \ , |y_0| < \rho\}$$

where $\alpha \neq \varphi_i(0), \ i = 1, ..., n$. It follows from Lemma 3. that

$$\deg[I - P, R] = 0.$$

Since R contains all the fixed points of P (after the identification $x(t) \equiv x(t) + 2n\pi$) the result follows.

The previous result allows us to obtain precise information on the values of the indexes.

Corollary 4. *In the conditions of the previous proposition,*

(i) If $\mu = \mu_+$ *or* μ_-*,*

$$\gamma_T(\varphi_i) = 0 \text{ for each } i = 1, ..., n.$$

(ii) If $\mu_- < \mu < \mu_+$*, there exist* $i_1, i_2 \in \{1, ..., n\}$ *such that*

$$\gamma_T(\varphi_{i_1}) < 0, \ \gamma_T(\varphi_{i_2}) = 1.$$

Proof. (i) follows form the continuity of the degree. If some index were not zero, there should exist periodic solutions for $\mu_+ + \epsilon$ or $\mu_- - \epsilon$ and ϵ small.
(ii) The periodic solutions corresponding to the value of the parameter for μ_+ and μ_- can be seen respectively as strict upper and lower solutions of (17). By adding or substracting $2n\pi$ it can be assumed that they are ordered. In consequence, the proof of Theorem 5. of Chapter 2 implies that there exists i_1 such that $\gamma_T(\varphi_{i_1}) < 0$. Now proposition 1. of Chapter 2 and proposition 2. imply the existence of i_2.

Remark. The previous corollary implies the existence of at least two T-periodic solutions when $\mu \in (\mu_-, \mu_+)$. This result was first obtained in [16] using Leray-Schauder degree.

3.3 The number of asymptotically stable T-periodic solutions

In what follows we assume $c > 0$ and obtain general estimates and bounds on the number of asymptotically stable T-periodic solutions of (17). The presentation of the results is inspired by [31]. Let $N(1)$ denote the number of T-periodic solutions of (17) and $N_s(1)$ (resp. $N_u(1)$) denote the number of asymptotically stable (resp. unstable) T-periodic solutions.

Proposition 5. *Assume that* $N(1) < \infty$, *then*

$$N_s(1) \leq N_u(1).$$

Proof. Let $\varphi_1, ..., \varphi_n$ be the T-periodic solutions of (17), $n = N(1)$. Since the index can only take the values $1, -1$ or 0 (Proposition 1. of chapter 2) the set $\{1, ..., n\}$ is decomposed in the three subsets

$$I_\lambda = \{i : \gamma_T(\varphi_i) = \lambda\}, \ \lambda = 1, -1, 0.$$

It follows from Theorem 2 of Chapter 2, that $N_s(1) \leq |I_1|$ and $N_u(1) \geq |I_{-1}| + |I_0|$. On the other hand $|I_1| = |I_{-1}|$ because the sum of all indexes must be zero (Proposition 2.). This completes the proof.

A periodic solution $x(t)$ with minimal period kT, for some $k \geq 2$, is called a subharmonic solution of order k. It is clear that $x(t + jT)$, $j = 1, ..., k-1$, are also subharmonic solutions. Let $N(k)$ be the number of subharmonic solutions of order k. The number $N(k)$ is divided by k.

Proposition 6. *Asume that* $N(1) < \infty$,

(i) If $\mu = \mu_\pm$, $N_s(1) = 0$.
(ii) If $\mu \in (\mu_-, \mu_+)$,

$$N(2) = 0 \Longrightarrow N_s(1) \geq 1$$

$$\sum_{k \geq 1} N(2^k) < \infty \Longrightarrow \exists k_0 \geq 1 : \ N_s(2^{k_0}) \geq 2^{k_0}.$$

Proof. (i) follows from Corollary 4. and Theorem 2. of chapter 2.
(ii) We apply corollary 4. with period $2T$ to obtain the existence of a $2T$-periodic solution φ such that $\gamma_{2T}(\varphi) = 1$. Since $N(2) = 0$, φ is not a subharmonic and it must be T-periodic. We apply Theorem 3. of chapter 2, to deduce that φ is asymptotically stable. To prove the second implication let k^* be such that $N(2^{k^*+1}) = 0$. Then we apply the previous result with initial period $2^{k^*}T$ to deduce the existence of a $2^{k^*}T$-periodic solution that is asymptotically stable.

Remark. It would be interesting to determine the exact restrictions for the numbers $N(k)$, $N_s(k)$ and $N_u(k)$ in the class of equations of pendulum-type. This is meant in the sense of [31].

3.4 The region of asymptotic stability

In this section we assume that $c > 0$ and g is a real entire function with respect to x; that is

$$g(t,x) = \sum_{n=0}^{\infty} \alpha_n(t)x^n, \quad |x| < \infty$$

where $\alpha_n \in C(\mathbf{R}/T\mathbf{Z})$. In addition

$$g_x(t,x) \geq -[(\frac{\pi}{T})^2 + \frac{c^2}{4}] \quad \forall (t,x) \in \mathbf{R} \times \mathbf{R}. \tag{18}$$

Theorem 7. *In the previous assumptions, if $\mu \in (\mu_-, \mu_+)$, the set of T-periodic solutions is finite and at least one of them is asymptotically stable and another unstable.*

Remarks. 1.This result was obtained in [39] under the assumption (more restrictive than (18))

$$g_x(t,x) \geq -\frac{c^2}{4} \quad \forall (t,x) \in \mathbf{R} \times \mathbf{R}. \tag{19}$$

2. A related result was obtained in [46] assuming that p was fixed and c large enough (depending on p). This kind of assumptions allows the use of local methods. In contrast, in the previous theorem the function p was arbitrary.

The previous result will be a consequence of Proposition 6. if we prove

$$N(2) = 0, \quad N(1) < \infty.$$

The non-existence of second order subharmonic solutions is a consequence of (18) while the analyticity conditions on g imply that the set of T-periodic solutions is finite. This facts are collected in the next two lemmas.

Lemma 8. *In the assumptions of the theorem,*

$$N(2) = 0.$$

Proof. Let $x(t)$ be a $2T$-periodic solution of (17), we shall prove that $x(t)$ is T-periodic. Define $y(t) = x(t+T) - x(t)$, then $y(t)$ is anti-periodic of period T $(y(t+T) = -y(t))$ and it satisfies

$$y'' + cy' = \alpha(t)y$$

where α is a T-periodic function such that

$$\alpha(t)y(t) = g(t, x(t+T)) - g(t, x(t)).$$

It follows from (18) that α satisfies

$$\alpha \geq -[(\frac{\pi}{T})^2 + \frac{c^2}{4}].$$

Using the Sturm comparison theory we deduce that if y is nontrivial the distance between two consecutive zeros is strictly greater than T. Since y is anti-periodic this is not possible unless $y = 0$. This proves that x is T-periodic.

Lemma 9. *In the assumptions of the theorem and assuming that $\mu_- \neq \mu_+$,*

$$N(1) < \infty.$$

The proof of this lemma is lengthy. It combines the alternative method and the real analytic version of the implicit function theorem. The details can be seen in [39]. See also [52].

The condition (18) was imposed to guarantee the non-existence of second order subharmonic solutions. The next example shows that in a certain sense the constant $(\frac{\pi}{T})^2$ is optimal in the theorem.

Example. A pendulum of variable length

Given a positive and T-periodic function $\alpha(t)$, we consider the equation

$$x'' + cx' + [\alpha(t) + \frac{c^2}{4}]\sin x = 0. \tag{20}$$

The equilibria $x \equiv 0$ and $x \equiv \pi$ are always T-periodic solutions of this equation. We shall prove that for each $\gamma > (\frac{\pi}{T})^2$ there exists $\alpha \in C^\infty(\mathbf{R}/T\mathbf{Z})$, $0 < \alpha \leq \gamma$ and $c_0 > 0$ such that if $0 \leq c < c_0$ the equation (20) has exactly two T-periodic solutions, $x \equiv 0$ and $x \equiv \pi$, and both are unstable with indexes

$$\gamma_T(0) = 1,\ \gamma_{2T}(0) = -1,\ \gamma_T(\pi) = \gamma_{2T}(\pi) = -1.$$

Notice that if we consider for such α the parametric equation

$$x'' + cx' + [\alpha(t) + \frac{c^2}{4}]\sin x = \mu$$

then $\mu_- < 0 < \mu_+$ because there are solutions with nonzero index for $\mu = 0$. This example shows that Theorem 7. is not valid if $(\frac{\pi}{T})^2$ is replaced by γ in (18).

We construct the example after several steps.

Step 1. There exist ϵ, $\delta > 0$ such that if

$$\| \alpha - (\frac{\pi}{T})^2 \|_{L^\infty} < \delta,\ \ 0 < c < \delta$$

then the only T-periodic solution of (20) satisfying $\| x \|_{C^2} < \epsilon$ [resp. $\| x - \pi \|_{C^2} < \epsilon$] is $x \equiv 0$ [resp. $x \equiv \pi$].

It is easy to check that the only T-periodic solutions of $x'' + (\frac{\pi}{T})^2 \sin x = 0$ are the equilibria $x \equiv 0$ and $x \equiv \pi$, moreover they are nondegenerate. (This means that the only T-periodic solution of the corresponding linearized equation is the trivial solution.) This fact allows us to apply the implicit function theorem to the equation $F = 0$ with

$$F : C^2(\mathbf{R}/T\mathbf{Z}) \times C(\mathbf{R}/T\mathbf{Z}) \times \mathbf{R} \to C(\mathbf{R}/T\mathbf{Z}), \ \ (x, \alpha, c) \to x'' + cx' + [\alpha(t) + \frac{c^2}{4}] \sin x$$

and find a unique solution $x = x(\alpha, c)$ in a neighborhood of the points $(0, (\frac{\pi}{T})^2, 0)$, $(\pi, (\frac{\pi}{T})^2, 0)$.

Step 2. If $\| \alpha - (\frac{\pi}{T})^2 \|_{L^\infty}$ and c are sufficiently small, the only T-periodic solutions of (20) are $x \equiv 0$, $x \equiv \pi$.

By contradiction assume the existence of sequences x_n, α_n, c_n such that $\alpha_n \to (\frac{\pi}{T})^2$ uniformly, $c_n \downarrow 0$ and x_n is a nonconstant solution of the corresponding equation of the kind (20). It is easy to obtain uniform bounds on $\| x_n \|_{C^2}$. Using the equation and a compactness argument one extracts a subsequence $x_n \to x$ in C^2, where x is a T-periodic solution of $x'' + (\frac{\pi}{T})^2 x = 0$. Thus $x \equiv 0$ or $x \equiv \pi$ and this fact is not consistent with Step 1.

Step 3. For each $\alpha \in C^\infty(\mathbf{R}/T\mathbf{Z})$, $\alpha > 0$,

$$\gamma_T(\pi) = \gamma_{2T}(\pi) = -1.$$

The linearization of (20) at π is

$$y'' + cy' - [\alpha(t) + \frac{c^2}{4}]y = 0.$$

It is well known (see [10] for the case $c = 0$) that if $\alpha + \frac{c^2}{4} > 0$ then the Floquet multipliers satisfy

$$\mu_1 > 1 > \mu_2 > 0.$$

The conclusion follows from the results of Section 2.4.

Step 4. There exists $\alpha \in C^\infty(\mathbf{R}/T\mathbf{Z})$ with $\| \alpha - (\frac{\pi}{T})^2 \|_{L^\infty}$ arbitrarily small such that if c is close to zero then

$$\gamma_T(0) = 1, \ \gamma_{2T}(0) = -1.$$

Since the equation $z'' + (\frac{\pi}{T})^2 z = 0$ has the double Floquet multiplier -1 with respect to period T, it is possible to find a Hill's equation $z'' + \alpha(t) z = 0$ with α close to $(\frac{\pi}{T})^2$ and multipliers satisfying

$$\mu_1^* < -1 < \mu_2^* < 0$$

(see [28]). For such α the linearization at $x \equiv 0$ of the original equation (20) is

$$y'' + cy' + [\alpha(t) + \frac{c^2}{4}]y = 0.$$

This equation is reduced to the previous Hill's equation by the change of variables $y = e^{-\frac{c}{2}t}z$. In consequence the multipliers are $\mu_1 = e^{-cT/2}\mu_1^*$, $\mu_2 = e^{-cT/2}\mu_2^*$. If c is small we still have

$$\mu_1 < -1 < \mu_2 < 0$$

and applying again Section 2.4.,

$$\gamma_T(0) = \text{sign}\{(1-\mu_1)(1-\mu_2)\} = 1, \ \gamma_{2T}(0) = \text{sign}\{(1-\mu_1^2)(1-\mu_2^2)\} = -1.$$

3.5 Some remarks

1. The condition (18) is optimal for the non-existence of subharmonic solutions of the second order. When it holds Lemma 8. shows that $N(2) = 0$. On the other hand, in the example of the pendulum of variable length one always has $N(2) \geq 2$. This follows from the values of the indexes of period $2T$ of 0 and π together with Proposition 2.

2. The extension of Massera's convergence theorem by R.A. Smith [51] implies that every bounded solution of (17) converges to a T-periodic solution when (19) holds. In this case the dynamics of the equation is simple and, in particular, $N(k) = 0$, for each $k \geq 2$. It would be interesting to decide if (19) is optimal with respect to the appearance of complicated dynamical behavior in the equation.

3. Theorem 7. is false when $c = 0$. However it seems possible to obtain in such case a generic result in the line of [37].

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