Non-autonomous saddle-node bifurcation in a canonical electrostatic MEMS

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Abstract

We study the existence and stability of periodic solutions of a canonical mass-spring model of electrostatically actuated micro-electro-mechanical system (MEMS) by means of classical topological techniques like a-priori bounds, Leray-Schauder degree and topological index. A saddle-node bifurcation is revealed, in analogy with the autonomous case.

1 Introduction.

The purpose of this paper is to study analytically the existence and stability of periodic solutions of an idealized mass-spring model of electrostatically actuated micro-electro-mechanical system (MEMS) which has become canonical in the related literature. The system is illustrated in Figure 1.1 and consists on two parallel capacitor plates separated by a distance d, one of them is fixed and the second one is movable and attached to a linear spring with stiffness coefficient $k > 0$. When time-periodic voltage $V(t)$ is applied, the Coulomb force between the plates makes the system highly nonlinear. Oscillations are ruled by the second order differential equation

$$my'' + cy' + ky = \frac{\varepsilon_0 A}{2} \frac{V^2(t)}{(d-y)^2}, \tag{1.1}$$

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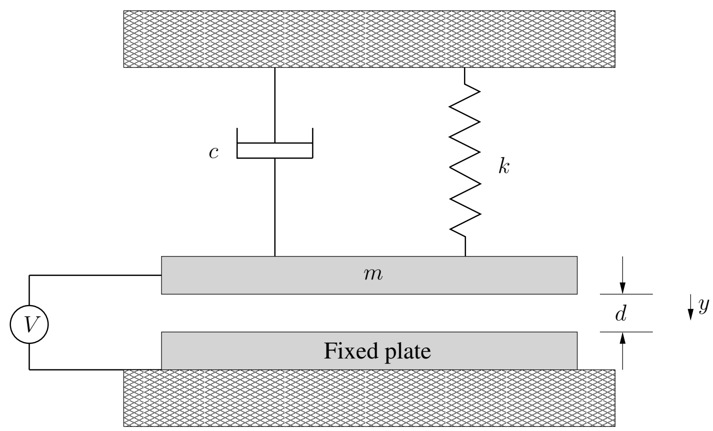

Figure 1.1: Idealized mass-spring model of electrostatically actuated MEMS.

where y is the vertical displacement of the moving plate (y is always assumed to be less than d), m is its mass, c is a viscous damping coefficient, ε_0 is the absolute dielectric constant of vacuum and A is the area of the plates.

This model has been studied for more than 40 years after its introduction by Nathanson et al [15] in 1967. In consequence, a comprehensive bibliography is beyond the scope of this work. We refer for instance to [1, 2, 3, 8, 14] and the references therein. In spite of the large number of related papers, the mathematical understanding of this system is still far from being complete.

An interesting recent work is [1], where the authors analyze in detail the so-called viscosity dominated regime, that is the case when the damping coefficient is very high and damping effects dominate over inertial effects. This leads to a reduced first-order equation, revealing a saddle-node bifurcation. When the applied voltage is constant (autonomous case), such saddle-node bifurcation is easily proved by an elementary local stability analysis of equilibria. Calling the constant voltage $V(t) \equiv V_0$, constant solutions (equilibria) of (1.1) correspond to the roots of the third-order polynomial $y(d-y)^2 - h$, where for convenience we have called $h = \dfrac{\varepsilon_0 A V_0^2}{2k} > 0$. This equation has always a root bigger than d and hence without physical meaning. A direct analysis provides a threshold value

$$d_0 = \frac{3}{2}(2h)^{1/3}, \tag{1.2}$$

such that if $d < d_0$ equation (1.1) has no equilibria and if $d > d_0$ a saddle and a node come into play, see 1.2.

The main purpose of the present paper is to prove the existence of such saddle-node bifurcation in the full non-autonomous second order model for concrete regions of the involved parameters. To this purpose, let us remark that (1.1) is an example of ODE with singularity at the state variable and periodic dependence on time. A similar equation was introduced by Lazer and Solimini

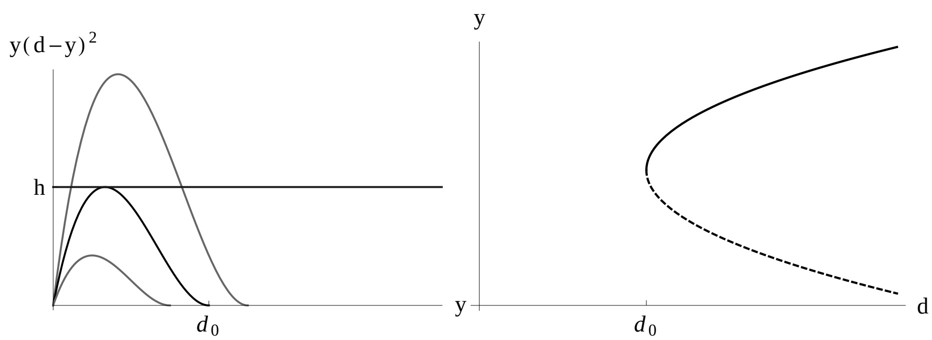

Figure 1.2: Illustration of the saddle-node bifurcation in the case where the voltage is constant.

in the seminal paper [10]. This work has become a hallmark in the area, and since its publication a wide variety of topological and variational methods have been systematically employed in the study of the existence of periodic solutions for this kind of equations (see for instance the reviews [5, 19] and the references therein). Concerning the stability of solutions, the number of papers is considerably lower [4, 12, 20, 21, 22]. It is interesting to note that, despite of the large amount of work devoted to MEMS, up to our knowledge this is the first work where this connection with a well-developed line of research like equations with singularities is reported.

The paper is structured into five section. After this Introduction, in Section 2 the main equation is written in an equivalent form by translating the singularity to the origin and the main results are stated. In Section 3, we present some a priori bounds for the solutions. Section 4 contains the proofs of the main results exposed in Section 2. The proof of the multiplicity result is adapted from [9] and relies on classical arguments from topological degree. The stability information is obtained from a well-known connection between stability and index of a periodic solution, developed by Ortega in [16, 17, 18]. Finally, in Section 5 the main results are applied to the original MEMS model (1.1) and an illustrative example is given by using concrete values of the physical constants taken from the literature. Our result is optimal in the sense that in the autonomous case we recover the exact bifurcation value given by (1.2).

In the rest of the paper, we consider the Banach space $X_T = C^1(\mathbb{R}/T\mathbb{Z})$ endowed with the usual C^1-norm. $V(t)$ is a positive, continuous and $\frac{2\pi}{w}$-periodic function. The period is denoted by $T = \frac{2\pi}{w}$. For any positive and continuous $\frac{2\pi}{w}$-periodic function $p(t)$, we denote $p_M = \max_t p(t)$, $p_m = \min_t p(t)$. If f and g are real functions defined over $[0,T]$, we shall write $f \ll g$ if $f(t) \leq g(t)$ for all $t \in [0,T]$ and the strict inequality holds on a subset of positive Lebesgue measure. We write $\|\cdot\|_p$ for the usual norm in $L^p(0,T)$.

2 An equivalent singular equation.

First, let us observe that the parameter m can be scaled out just dividing (1.1) by m and renaming c, k, A. Equivalently, we can consider $m = 1$ without loss of generality. With the change of variable $u = d - y$, (1.1) is rewritten as

$$u'' + c\,u' + ku + \frac{a^2(t)}{u^2} = s, \tag{2.1}$$

where k, c, s, are positive constants and $a(t)$ is a positive continuous T-periodic function. We are interested in the existence and stability of positive T-periodic solutions of (2.1). Let us fix $s_* := 3\left(\frac{k}{2}\,a_m\right)^{2/3}$, $s^* := 3\left(\frac{k}{2}\,a_M\right)^{2/3}$. The first main result is as follows.

Theorem 1 (Multiplicity)**.** *There exists $s_0 \in [s_*, s^*]$, such that*

1. *If $s < s_0$, problem (2.1) has no T-periodic solutions.*
2. *If $s = s_0$, problem (2.1) has at least one T-periodic solution.*
3. *If $s > s_0$, problem (2.1) has at least two T-periodic solutions.*

In the related mathematical literature, this kind of result is often known as of Ambrosetti-Prodi type. A similar result is proven in [9] for the regular equation $u'' + cu' + g(t, u) = s$ satisfying the coercive condition

$$\lim_{|u|\to\infty} g(t, u) = +\infty, \quad \text{uniformly in } t.$$

The presence of the singularity in (2.1) prevents from a direct application of this theorem.

The second main result concerns the stability of the solutions provided by Theorem 1.

Theorem 2 (Stability)**.** *Under the conditions of Theorem 1, assume that*

$$k - 2\,(a_m)^2 \left(\frac{c}{Ts}\right)^3 \left(\frac{a_m}{a_M}\right)^6 < \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}. \tag{2.2}$$

Then

1. *If $s = s_0$, problem (2.1) has a unique T-periodic solution which is not asymptotically stable.*
2. *If $s > s_0$, problem (2.1) has exactly two T-periodic solutions, one uniformly asymptotically stable and another unstable.*

Both results together depict a canonical saddle-node bifurcation at the threshold valued s_0.

3 A priori bounds.

This section is devoted to obtain explicit bounds on the eventual T-periodic solutions of (2.1). Fix $g(t,u) := ku + \frac{a^2(t)}{u^2}$. Let us define the following constants

$$\epsilon(s) := \frac{a_m}{\sqrt{s}}, \tag{3.1}$$

$$M_1(s) := T\frac{s}{c}\left(\frac{a_M}{a_m}\right)^2, \tag{3.2}$$

$$M_2(s) := Ts\left(\frac{a_M}{a_m}\right)^2\left(2 + \frac{k}{c} + \left(\frac{a_m}{a_M}\right)^2\right). \tag{3.3}$$

Lemma 3. *If u is a T-periodic solution of* (2.1) *then $u > \epsilon(s)$.*

Proof. Let $t_1 \in]0,T[$ be such that $u(t_1) = \min_{t\in\mathbb{R}} u(t)$. Then $u''(t_1) \geq 0$, $u'(t_1) = 0$ and

$$u''(t_1) + g(t,u) = s.$$

Consequently, $\frac{(a_m)^2}{u^2(t_1)} < s$. Then

$$u(t_1) > \frac{a_m}{\sqrt{s}} = \epsilon(s).$$

□

Next result provides an upper bound for eventual T-periodic solutions of (2.1) in terms of the parameter s.

Lemma 4. *If u is a $T-$periodic solution of* (2.1) *then $u < M_1(s)$.*

Proof. Assume that u is a $T-$periodic solution of (2.1). Multiplying by u' and integrating over a period, we see that

$$c\|u'\|_2^2 = -\int_0^T \frac{a^2(t)}{u^2}u'\,dt < \sqrt{T}\frac{(a_M)^2}{\epsilon(s)^2}\|u'\|_2.$$

Thus

$$\|u'\|_2 < \sqrt{T}\frac{s}{c}\left(\frac{a_M}{a_m}\right)^2. \tag{3.4}$$

Then

$$u(t) \leq |u(t)| \leq \left|\int_{t_0}^t u'(s)ds\right| \leq \sqrt{T}\|u'\|_2 < T\frac{s}{c}\left(\frac{a_M}{a_m}\right)^2 = M_1(s).$$

□

Finally, we establish bounds on the derivative of the possible T-periodic solutions of (2.1).

Lemma 5. *If u is a $T-$periodic solution of* (2.1) *then* $\|u'\|_\infty < M_2(s)$.

Proof. First note that a direct integration of (2.1) over a whole period implies that that if there exists a $T-$periodic solution, then $s > 0$. From (2.1) we obtain

$$\begin{aligned} |u''| &= \left| -cu' - \frac{a^2(s)}{u^2(s)} - ku + s \right| \\ &\leq c|u'| + \left| \frac{a^2(s)}{u^2(s)} \right| + k|u| + s. \end{aligned}$$

Integrating over the period and using Lemma 4, Cauchy-Schwarz inequality and (3.4),

$$\begin{aligned} \|u''\|_1 &< c\sqrt{T}\|u'\|_2 + T\frac{(a_M)^2}{\epsilon(s)^2} + kM_1(s) + Ts \\ &< Ts\left(\frac{a_M}{a_m}\right)^2 \left(2 + \frac{k}{c} + \left(\frac{a_m}{a_M}\right)^2\right) = M_2(s). \end{aligned}$$

Therefore

$$|u'(t)| \leq \left| \int_{t_0}^{t} u''(s)ds \right| \leq \|u''\|_1 < M_2(s).$$

□

4 Proofs of main results.

On the whole section, we denote

$$g_L(u) := ku + \frac{(a_m)^2}{u^2}, \quad g_U(u) := ku + \frac{(a_M)^2}{u^2}$$

and

$$S_j = \{s \in \mathbb{R}\colon \text{(2.1) has at least } j \ T\text{-periodic solutions}\}, \quad j \geq 1.$$

In the next results, s_*, s^* were defined on Section 2.

4.1 Multiplicity result.

The proof of Theorem 1 will be divided in a series of lemmas. In this subsection, the notion of upper and lower solutions will be used [6]. Functions $\alpha, \beta \in C^2(\mathbb{R}/T\mathbb{Z})$ are called lower and upper solutions respectively for equation $u'' + cu' + g(t,u) = s$ if, for all $t \in \mathbb{R}$,

$$\begin{aligned} \alpha'' + c\,\alpha' + g(t,\alpha) \geq s, \\ \beta'' + c\,\beta' + g(t,\beta) \leq s. \end{aligned}$$

A lower (resp. upper) solution is said to be *strict* if the above inequality is strict for all $t \in \mathbb{R}$.

Lemma 6. *If $s \in S_1$ then $s \geq s_*$.*

Proof. If u is a T-periodic solution of (2.1) *for* s, it follows that

$$\int_0^T g(t, u(t))dt = Ts.$$

Note that the function $g_L(u)$ reaches the global minimum at the point $u_1 = \left(\frac{2(a_m)^2}{k}\right)^{1/3}$ and such minimum is just s_*. Hence

$$s_* \leq \frac{1}{T}\int_0^T g_L(u(t))dt \leq \frac{1}{T}\int_0^T g(t, u(t))dt = s.$$

□

Lemma 7. $S_1 \neq \emptyset$.

Proof. We show that equation (2.1) has at least one T-periodic solution for $s > s^*$. By a known result (see [6, Theorem I-6.9]), it suffices to get strict lower and upper solutions α and β such that $\alpha(t) < \beta(t)$ for any $t \in \mathbb{R}$. First, one sees that for a given s, $\alpha_0(t) = \epsilon(s)$ is a (constant) lower solution for (2.1), in effect,

$$\alpha_0'' + c\alpha_0' + g(t, \alpha_0) \geq g_L(\alpha_0) = k\frac{a_m}{\sqrt{s}} + s > s.$$

On the other hand, it is easily checked that at the point $\beta_0 = \left(\frac{2(a_M)^2}{k}\right)^{1/3} = \frac{2}{3k}s^*$ the function $g_U(u)$ reaches its minimum value s^*. Then

$$\beta_0'' + c\beta_0' + g(t, \beta_0) \leq g_U(\beta_0) = s^* < s.$$

and β_0 is a (constant) upper solution of (2.1) for s. Note that

$$\alpha_0 < \epsilon(s^*) = \frac{2(s_*)^{3/2}}{3^{3/2}k}\frac{1}{\sqrt{s^*}},$$

and it is easy to verify that $\epsilon(s^*) < \beta_0$. Hence, $\alpha_0 < \beta_0$. □

Lemma 8. $S_1 = [s_0, +\infty)$ *with* $s_0 \geq s_*$.

Proof. By Lemma 7 we have $S_1 \neq \emptyset$. First, we show that if $\hat{s} \in S_1$, then $s \in S_1$, for every $s > \hat{s}$. Let $u_{\hat{s}}$ a T-periodic solution of (2.1) for $s = \hat{s}$. Hence $u_{\hat{s}}$ is a upper solution of (2.1) for $s > \hat{s}$, since

$$u''_{\hat{s}} + cu'_{\hat{s}} + g(t, u_{\hat{s}}) = \hat{s} < s.$$

In the other hand, $\epsilon(s)$ (defined in (3.1)) is a lower solution (2.1) for $s > \hat{s}$, moreover by Lemma 3, $\epsilon(s) < \epsilon(\hat{s}) < u_{\hat{s}}$, so the couple $(\epsilon(s), u_{\hat{s}})$ of lower and upper solutions are well-ordered and hence there is at least one T-periodic solution in between. Hence $s \in S_1$, for $s > \hat{s}$.

Taking $s_0 = \inf S_1$, by Lemma 6 we have $s_0 \geq s_*$, and we show that $s_0 \in S_1$. Take a sequence $\{s_n\}$ in S_1 such that $s_n \to s_0$ as $n \to +\infty$. We know already that equation (2.1) has, for each s_n, a solution u_n. Using Lemmas 3, 4 and 5, it then follows that the sequence $\{u_n\}$ is uniformly bounded and equicontinuous. By Arzela-Ascoli theorem $\{u_n\}$ contains a subsequence uniformly converging and the limit of that subsequence is a solution of (2.1) for $s = s_0$. Hence the proof is complete. □

Lemma 9. $s_0 \leq s^*$.

Proof. This follows directly from the argument employed in the proof of Lemma 7. This proof gives that $s \in S_1$ for every $s > s_*$. Since by definition $s_0 = \inf S_1$, the conclusion is evident. □

At this point it is pertinent to introduce some elementary notions of Leray-Schauder degree (see [11, 13] for a comprehensive treatment of this topic). The first step is to write our equations as an abstrat fixed point problem. Given $p(t) \in X_T$, consider the linear problem

$$u'' + cu' - u = p(t) \quad x \in X_T.$$

It follows from Fredholm alternative that it has a unique solution $x = Lp$. The linear operator $L\colon X_T \to X_T$ can be explicitly expressed by means of the Green's function as an integral operator and it is easy to verify that it is compact with the usual norm of X_T. The periodic problem for (2.1) is equivalent to

$$u = L(N_s u - u),$$

where the Nemitskii operator $N_s : X_T \to X_T$ is given by

$$N_s[u] := s - g(t, u).$$

By defining $\Phi_s := L(N_s - I)$, finding T-periodic solutions of equation (2.1) is equivalent to finding solutions of the abstract equation

$$u - \Phi_s(u) = 0. \tag{4.1}$$

Since Φ_s is a compact operator, the Leray-Schauder degree $\deg_{LS}(I - \Phi_s, \Omega)$ is well-defined whenever Φ_s has no fixed points in the boundary of Ω.

Before presenting the next result, define the set

$$\Omega_s = \{u \in X_T : \epsilon(s) < u(t) < M_1(s), \, t \in \mathbb{R}, \, \|u'\|_\infty < M_2(s)\},$$

where $s > 0$ and ϵ, M_1, M_2 are defined in (3.1), (3.2) and (3.3) respectively.

Lemma 10. *Let $s_0 < s_1 < s_2$. Then $\Omega_s \subset \Omega_{s_2}$ for any $s \in [s_1, s_2]$, and any possible solution u of* (2.1) *with $s \in [s_1, s_2]$ belongs to Ω_{s_2}.*

Proof. Note that function $\epsilon(s)$ is decreasing, the functions $M_1(s), M_2(s)$ are increasing for all $s \in [s_1, s_2]$. Then $\Omega_s \subset \Omega_{s_2}$ for all $s \in [s_1, s_2]$. By Lemma 3, 4 and 5, any possible solution u of (2.1) with $s \in [s_1, s_2]$ belongs to Ω_{s_2}. $\square$

Lemma 11. $]s_0, +\infty[\subset S_2$.

Proof. Fix $s_1 > s_0$. The set Ω_{s_1} is open, convex and bounded in X_T. By Lemma 10 any possible solution u of (2.1) with $s \in]s_0, s_1]$ belongs to Ω_{s_1} and by Lemmas 3, 4 and 5 the equation (4.1) has no solutions belonging to the boundary of Ω_{s_1}. Then $\deg_{LS}(I - \Phi_s, \Omega_{s_1})$ is well defined and independent of $s \in]s_0, s_1]$ by the homotopy property of the degree. On the other hand, the equation (4.1) has no solution for $s < s_0$ (see Lemma 8), then

$$\deg_{LS}[I - \Phi_s, \Omega_{s_1}] = 0. \tag{4.2}$$

Then the invariance of degree with respect to a homotopy that (4.2) holds for $s \in]s_0, s_1]$.

Given $s \in]s_0, s_1]$, let u_0 be a T-periodic solution of (2.1) known to exist for $s = s_0$. Note that $\epsilon(s_1) < \epsilon(s) < \epsilon(s_0) < u_0$ for all $s \in]s_0, s_1[$. The degree will now be computed with respect to a different set, namely

$$\Omega := \{u \in X_T : \epsilon(s_1) < u < u_0,\, t \in \mathbb{R},\, \|u'\|_\infty < M_2(s_1)\}.$$

Now, for all $s \in]s_0, s_1]$, $\epsilon(s_1)$ is a strict lower solution for (2.1),

$$\epsilon(s_1)'' + c\epsilon(s_1)' + g(t, \epsilon(s_1)) \geq g_L(\epsilon(s_1)) = k\frac{a_m}{\sqrt{s_1}} + s_1 > s,$$

whereas, u_0 is a strict upper solution, in effect,

$$u_0'' + cu_0' + g(t, u_0) = s_0 < s.$$

Therefore, by [18, Theorem 5], if follows that for all $s \in]s_0, s_1]$

$$\deg_{LS}(I - \Phi_s, \Omega) = 1. \tag{4.3}$$

Using (4.2), it the follows from the additivity property of the degree that, for $s \in]s_0, s_1]$,

$$\deg_{LS}(I - \Phi_s, \Omega_{s_1} \setminus \Omega) = -1. \tag{4.4}$$

The relations (4.3), (4.4) imply that equation (2.1) has at least one solution in Ω and a second one in $\Omega_{s_1} \setminus \Omega$. Since s_1 is arbitrary in $]s_0, \infty[$ then $]s_0, \infty[\subset S_2$. $\square$

Proof of Theorem 1. The conclusion of Theorem 1 follows from Lemmas 8, 9 and 11. $\square$

4.2 Stability result.

Given $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^+ \times \mathbb{R}$, let $u(t;\zeta)$ be the unique solution of (2.1) satisfying the initial conditions

$$u(0) = \zeta_1, \quad u'(0) = \zeta_2.$$

The Poincaré map is defined as the mapping

$$P_T : D_T \subset \mathbb{R}^2 \to \mathbb{R}^2 \qquad\qquad P_T(\zeta) = (u(T;\zeta), u'(T;\zeta)),$$

where $D_T = \{\zeta \in \mathbb{R}^+ \times \mathbb{R} : u(t,\zeta) \text{ is defined in } [0,T]\}$. Let u be a T-periodic solution of (2.1) and $\zeta_0 = (u(0), u'(0))$. The solution u is said to be isolated (period T) if ζ_0 is an isolated fixed point of P_T. In such case the topological index of u is defined in terms of the following formula

$$\gamma_T(u) := \deg(I - P_T, B),$$

where deg is the Brouwer degree and B is a small ball centered at ζ_0 that does not contain others fixed points of P_T. Alternatively, the topological index of u can be written in terms of the Leray-Schauder degree as follows

$$\gamma_T(u) = -\deg_{LS}(I - \Phi_s, B),$$

where B is a small ball centered at u. A basic property is that $|\gamma_T(u)| \leq 1$. See [18, 16, 17] for this and other properties.

The following definition will be needed.

Definition 1. *A T-periodic solution u_0 of* (2.1) *is called* nondegenerate *if the linearized equation*

$$y'' + cy' + \alpha(t)y = 0,$$

with $\alpha(t) = \dfrac{\partial g}{\partial u}(t, u_0(t))$ does not have non-trivial T-periodic solutions. On the contrary, it is called degenerate.

An isolated T-periodic solution is nondegenerate if and only if its topological index is not zero (see again [18]).

The aim of this subsection is to characterize the stability of a T-periodic solution u to (2.1). The main tool is the following result.

Theorem 12. *[16, Theorem 1.1]Assume that u is an isolated T-periodic solution of* (2.1) *such that the condition*

$$\frac{\partial g}{\partial u}(t, u(t)) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4},$$

holds for all $t \in \mathbb{R}$. Then u is asymptotically stable (resp. unstable) if and only if $\gamma_T(u) = 1$(resp. $\gamma_T(u) = -1$).

The next comparison lemma will be useful as well.

Lemma 13. *[13, Lemma 7.2] Consider the differential operator $\mathcal{L}_\alpha$ by*

$$\mathcal{L}_\alpha[u] := u'' + cu' + \alpha(\cdot)u,$$

acting on T-periodic functions, where $\alpha \in L^1(\mathbb{R}/T\,\mathbb{Z})$ satisfies condition

$$\alpha \ll \frac{\pi^2}{T^2} + \frac{c^2}{4}. \tag{4.5}$$

Then the following conclusions are true.

1. *For each real number μ, any possible T-periodic solution u of the equation $\mathcal{L}_\alpha[u] = \mu$ is either trivial or different from zero for every $t \in]0, T[$.*
2. *Let α_1 and α_2 be functions in $L^1(\mathbb{R}/T\mathbb{Z})$ satisfying (4.5) and such that $\alpha_1 \ll \alpha_2$. Then the equations*

$$\mathcal{L}_{\alpha_i}[u] = 0, \qquad i = 1, 2,$$

cannot admit nontrivial $T-$periodic solutions simultaneously.

We also need the following preliminary result about the order of T-periodic solutions of (2.1).

Lemma 14. *Assume that the inequality (2.2) is satisfied and u_1, u_2 are T-periodic solutions of (2.1) for $s = s_1$, $s = s_2$ respectively with $u_1(t) \neq u_2(t)$ for some t. Then either $u_1 > u_2$ or $u_2 < u_1$ on $[0, T]$.*

Proof. Let $s = \max\{s_1, s_2\}$. Note that g is strictly convex in $[\epsilon(s), M_1(s)]$ and s holds (2.2). As a consequence, it is easy to verify that $\dfrac{\partial g}{\partial x}(t, x) \ll \dfrac{\pi^2}{T^2} + \dfrac{c^2}{4}$ for all $x \in [\epsilon(s), M_1(s)]$. The difference $v = u_1 - u_2$ satisfies an equation of the form

$$\mathcal{L}_\alpha[v] = s_1 - s_2,$$

where $\alpha = \frac{g(t,u_1(t))-g(t,u_2(t))}{u_1(t)-u_2(t)}$ is a T-periodic function such that $\alpha(t) \leq \dfrac{\partial g}{\partial x}(t, M_1(s))$. The conclusion follows from the first part of Lemma 13. □

Proof of Theorem 2 . First, let us show that (2.1) has exactly two periodic solution for $s > s_0$. By Theorem 1, if $s > s_0$ then problem (2.1) has most two T-periodic solutions. Let u_1, u_2, u_3 be solutions of (2.1) for $s = s_1 > s_0$. Then by Lemma 14, they can be ordered, say $u_1 < u_2 < u_3$. Setting $v_1 = u_2 - u_1$, $v_2 = u_3 - u_2$, we see that v_i satisfies the equation $\mathcal{L}_{\alpha_i}[v_i] = 0$ with $\alpha_i = \dfrac{g(t, u_{i+1}) - g(t, u_i)}{v_i}$, $i = 1, 2$. The strict convexity of g implies that $\alpha_1 < \alpha_2$ on $[0, T]$. Using the second part of Lemma 13, we obtain that either v_1 or v_2 must be zero. In conclusion, (2.1) has exactly two T-periodic solutions for any $s > s_0$.

Let u_0 be a T-periodic solution of (2.1) for $s = s_0$ We are going to prove that it is unique. It follows from the continuity of the index that $\gamma_T(u_0) = 0$, because, on the contrary, equation (2.1) would have a T-periodic solution for all $s \in]s - \epsilon, s_0[$ for some $\epsilon > 0$. Therefore u_0 is degenerate. Note now that if u_1 is another T-periodic solution of (2.1) for some $s_1 \geq s_0$, then, by Lemma 14, either

$$u_1 < u_0 \text{ and hence } \frac{\partial g}{\partial u}(t, u_1) < \frac{\partial g}{\partial u}(t, u_0) \text{ on } [0, T],$$

or

$$u_1 > u_0 \text{ and hence } \frac{\partial g}{\partial u}(t, u_1) > \frac{\partial g}{\partial u}(t, u_0) \text{ on } [0, T].$$

By the second part of Lemma 13, we conclude that u_1 is nondegenerate, and hence $s_1 > s_0$. So we have proved that, for $s = s_0$ equation (2.1) has a unique T-periodic solution u_0 which satisfies $\gamma_T(u_0) = 0$, and hence it can not be asymptotically stable by Theorem 12.
A consequence of the above reasoning and Lemma 11 is the existence, for $s_1 > s_0$, of exactly two nondegenerate T-periodic solutions $u_1 < u_1^*$ of (2.1). From Lemma 11,

$$\deg_{LS}[I - \Phi_{s_1}, \Omega_{s_1}] = \deg_{LS}[I - \Phi_{s_0}, \Omega_{s_1}] = -\gamma_T(u_0) = 0,$$

so that necessarily

$$\gamma_T(u_1^*) = -\gamma_T(u_1) = 1.$$

The conclusion follows from a direct application of Theorem 12. $\square$

5 Application to the original MEMS model.

The initial motivation for our study has been the analysis of the mass-spring model of electrostatically actuated MEMS presented in the Introduction. In this section the main results are applied to this model. As we noted before, the mass parameter m can be scaled out just dividing (1.1) by m and renaming c, k, A. $V(t)$ is a continuous, positive, T-periodic function. We will write $T = \frac{2\pi}{w}$, where w is the frequency.

Theorem 15 (Multiplicity)**.** *There exists $d_0 > 0$ such that*

1. *If $d < d_0$, (1.1) has no T-periodic solutions.*
2. *If $d = d_0$, (1.1) has at least one T-periodic solution.*
3. *If $d > d_0$, (1.1) has at least two T-periodic solutions.*

Besides, d_0 admits the following quantitative estimate

$$\frac{3}{2}\left(\frac{\varepsilon_0 A V_m^2}{k}\right)^{1/3} \leq d_0 \leq \frac{3}{2}\left(\frac{\varepsilon_0 A V_M^2}{k}\right)^{1/3}. \tag{5.1}$$

Theorem 16 (Stability)**.** *On the conditions of Theorem 15, assume that*

$$4k < \frac{\varepsilon_0 A V_m^2}{2}\left(\frac{wcV_m^2}{\pi k d V_M^2}\right)^3 + w^2 + \frac{c^2}{m}. \tag{5.2}$$

Then,

1. *if $d = d_0$, (1.1) has a unique T-periodic solution which is not asymptotically stable,*

2. *if $d > d_0$, (1.1) has exactly two T-periodic solution, one uniformly asymptotically stable and another unstable.*

Remark 1. *As we remarked in the Introduction, the inequality (5.1) is optimal because of $V(t)$ is chosen constant (autonomous case), then the inequalities are infact equalities and d_0 is exactly the value $d_0 = \frac{3}{2}(2h)^{1/3}$ obtained in the Introduction.*

Remark 2. *It is worth to analyze in detail the physical meaning of condition (5.2). Note that if $4k \leq \dfrac{c^2}{m}$ then (5.2) holds for any frequency w. This case can be related with the "viscosity dominated regime" studied in [1]. On the other hand, if $4k > \dfrac{c^2}{m}$, elementary calculations provide a minimum frequency w_0 such that (5.2) holds for every $w > w_0$. w_0 is the unique positive root of the cubic equation. Another possibility is just to take $w^2 > 4k$. This resembles the paradigmatic phenomenon of "stabilization by high frequencies", which appear in a wide number of physical systems like the inverted pendulum with vibrating support (see for instance [7]).*

Example. *For illustrative purposes, we have taken from [14] the following values of the physical parameters: $m = 3.5 \times 10^{-11}$ Kg, $k = 0.17$ N/m, $c = 1.78 \times 10^{-6}$ Kg/s, $A = 1.6 \times 10^{-9}$ m^2, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m. If $V(t) = 10 + 2\cos(wt)$ V, then using Theorem 15 the bifurcation value is bounded by $2.62033\,\mu m < d_0 < 3.4336\,\mu m$. By Theorem 16, if $d > d_0$ and $w \geq 0.76772$ then there are exactly two periodic solutions of (1.1), one asymptotically stable and the other unstable.*

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