

Asymptotic stability of forced oscillations emanating from a limit cycle

Oleg Makarenkov^{1*} and Rafael Ortega^{2†}

¹Department of Mathematics, Imperial College

London SW7 2AZ, London, United Kingdom

`o.makarenkov@imperial.ac.uk`

²Departamento de Matemática Aplicada, Facultad de Ciencias

Universidad de Granada, 18071-Granada, Spain.

`rortega@ugr.es`

1 Introduction

The study of forced oscillations emanating from a limit cycle is a classical problem in the theory of bifurcation. Around 1950 the basic method to deal with this problem was developed by Malkin in [11] and this study was continued by Loud in [10]. The state of the art before the contributions of Malkin and Loud can be found in the book by Lefschetz [9]. To describe the general framework we start with an autonomous system

$$\dot{x} = f(x)$$

having a closed orbit Γ associated to a periodic solution $x_0(t)$ with period $T > 0$. Notice that T is not necessarily the minimal period. The perturbation considered is

$$\dot{x} = f(x) + \varepsilon g(t, x; \varepsilon)$$

where g is periodic in t and its period is precisely T . The beginning of Malkin's method is the construction of a T -periodic function $M = M(\theta)$ depending upon $x_0(t)$ and $g(\cdot, \cdot; 0)$. The zeros of M are intimately linked to the possible bifurcations to T -periodic solutions for $\varepsilon > 0$. Assuming some non-degeneracy conditions on $x_0(t)$ one can prove that if θ_* is a non-degenerate zero of M ($M(\theta_*) = 0$, $M'(\theta_*) \neq 0$) then

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the perturbed system has a family of T -periodic solutions satisfying

$$x_\varepsilon(t) = x_0(t + \theta_*) + O(\varepsilon), \text{ as } \varepsilon \downarrow 0.$$

It is also possible to analyze the case of a zero of higher multiplicity ($M(\theta_*) = 0$, $M'(\theta_*) = 0, \dots, M^{(k-1)}(\theta_*) = 0$, $M^{(k)}(\theta_*) \neq 0$) but this requires long computations, see e.g. [10] and [6]. More recently a topological approach has been taken in [4]. A bifurcation exists as soon as θ_* is a zero where M changes sign. The next step after the existence of bifurcating branches is the study of the stability properties. This was already considered in [11], [10] and [6]. Assuming that Γ is an exponential attractor it can be proved that the bifurcating periodic solution is asymptotically stable when $M'(\theta_*) > 0$ and unstable when $M'(\theta_*) < 0$. If θ_* is a zero of a higher multiplicity, then the implicit function approach taken in [10] and [6] does not allow to detect bifurcation of stable periodic solution on the basis of the sign of $M^{(k)}(\theta_*)$ and some further computations have to be done. See in particular equations (3.5) in [6] and (4.23) in [10]. The purpose of our paper is to obtain a topological version of this stability result for increasing or decreasing zeros when the derivative of M at θ_* can vanish. In particular, we are interested in an unified answer which does not depend on the multiplicity of θ_* . We will get a positive answer in the case of analytic systems. For this class of systems we will use a variant of Lyapunov-Schmidt reduction that will allow us to prove that if M is not identically zero then the number of T -periodic solutions is finite. This is inspired by the results of Nakajima and Seifert in [12] and R.A. Smith in [15]. Once we know that T -periodic solutions are isolated we can talk about their topological index. This is just a localized version of the topological degree and the connections of this index with the stability properties of the corresponding solutions have been discussed in [7, 8, 5, 13]. The computation of the index is then obtained via a result in the line of those in [4].

The rest of the paper is organized in three sections. In Section 2 we present some preliminary results on the autonomous system. The main Theorem as well as an example illustrating its applicability can be found in Section 3. This section also shows how to prove the main result via topological degree. Finally Section 4 is devoted to the proofs of three Lemmas previously employed.

2 The autonomous system

In this section we present some elementary facts about the non-perturbed system. They will be needed later in order to state our main Theorem. Let us start with the autonomous system

$$\dot{x} = f(x) \tag{1}$$

defined on an open subset Ω of \mathbb{R}^n . The vector field $f : \Omega \rightarrow \mathbb{R}^n$ is real analytic.

Assume that $x_0(t)$ is a non-constant periodic solution of (1) with period $T > 0$. The associated variational equation is

$$\dot{y} = f'(x_0(t))y. \tag{2}$$

This is a T -periodic equation having the solution $\dot{x}_0(t)$. The Floquet multipliers are labelled as μ_1, \dots, μ_n and counted according to their multiplicity. It will be assumed that they satisfy

$$\mu_1 = 1, \quad |\mu_2| < 1, \quad \dots, \quad |\mu_n| < 1. \quad (3)$$

This condition implies that the closed orbit $\Gamma = \{x_0(t), t \in [0, T]\}$ is an attractor (see [1]). The region of attraction is an open neighborhood of Γ which will be denoted by $\mathcal{A} \subset \Omega$.

In view of the condition on the Floquet multipliers we know that the space of T -periodic solutions of (2) has dimension one. The same property must hold for the adjoint system

$$\dot{z} = -f'(x_0(t))^* z. \quad (4)$$

The next result will provide an orientation in the space of T -periodic solutions of (4).

Lemma 1 *There exists an unique T -periodic solution $z_0(t)$ of (4) satisfying*

$$\langle \dot{x}_0(t), z_0(t) \rangle = 1, \quad \text{for any } t \in \mathbb{R}.$$

Proof. It is based on Perron's lemma [14] (see also [2], Sec. III, §12). This result says that if $y(t)$ and $z(t)$ are arbitrary solutions of (2) and (4) then

$$\langle y(t), z(t) \rangle \equiv \text{constant}.$$

We will prove that if $z_1(t)$ is a non-trivial T -periodic solution of (4) then

$$\langle \dot{x}(0), z_1(0) \rangle \neq 0. \quad (5)$$

Since the space of T -periodic solutions has dimension one this will complete the proof.

To prove (5) we find a $n \times n$ matrix S such that

$$S^{-1}Y(T)S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \boxed{} & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix},$$

where $Y(t)$ is the matrix solution of (2) with $Y(0) = I_N$ and $\det(A - I) \neq 0$. From the definition of S we have that its first column S_1 is an eigenvector of $Y(T)$ corresponding to the eigenvalue $\mu_1 = 1$. In particular S_1 is parallel to $\dot{x}(0)$. Consider the matrix $\Sigma = (S_2 | \dots | S_n)$ composed by the remaining columns of S . From the definition of S and A ,

$$Y(T)\Sigma = \Sigma A.$$

Next we apply Perron's Lemma to the solutions $Y(t)S_i$ and $z_1(t)$,

$$\langle z_1(0), S_i \rangle = \langle z_1(0), Y(T)S_i \rangle, \quad i = 2, \dots, n.$$

This implies

$$z_1(0)^* \Sigma = z_1(0)^* Y(T) \Sigma = z_1(0)^* \Sigma A.$$

Hence $z_1(0)^* \Sigma(I - A) = 0$ and so $z_1^*(0) \Sigma = 0$. Now we can conclude that (5) holds, for otherwise we should have $z_1^*(0) S = 0$ which is impossible if $z_1(t)$ is non-trivial. □

As a simple example we consider the planar system

$$\dot{x} = (1 - |x|^2)x + i|x|^2x, \quad x = x_1 + ix_2 \in \mathbb{C}.$$

It has the periodic solution $x_0(t) = e^{it}$ whose orbit $\Gamma = \mathbb{S}^1$ attracts $\mathcal{A} = \mathbb{C} - \{0\}$. The period is $T = 2N\pi$, where $N \geq 1$ is an integer arbitrarily chosen. The variational equation along $x_0(t)$ is

$$\dot{y} = (-1 + 2i)y + (-1 + i)e^{2it}\bar{y}$$

and has the Floquet solutions

$$y_1(t) = \dot{x}_0(t) = ie^{it}, \quad y_2(t) = e^{(-2+i)t}(-1 + i).$$

In consequence $\mu_1 = 1$ and $\mu_2 = e^{-2T}$. The computation of $z_0(t)$ follows from the proof of Lemma 1. We know that

$$\langle y_1(t), z_0(t) \rangle = 1, \quad \langle y_2(t), z_0(t) \rangle = \text{constant} = 0.$$

The periodicity of $e^{2t}y_2(t)$ and $z_0(t)$ implies that this last constant must vanish. From these equations one obtains that

$$z_0(t) = (1 + i)e^{it}.$$

3 Main result and an example

Let us consider the perturbed system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \tag{6}$$

where $g : \mathbb{R} \times \Omega \times [0, \varepsilon_*] \mapsto \mathbb{R}^n$ is continuous and T -periodic in t . We also assume that for each $t \in \mathbb{R}$ the function $g(t, \cdot, \cdot)$ has partial derivatives up to the second order with respect to (x, ε) and these derivatives are continuous as functions of the three variables (t, x, ε) . The most important assumption on the regularity of g will be the analyticity with respect to x . This means that for each $x_* \in \Omega$ there exists $r > 0$ such that if $\|x - x_*\| < r$ then for $j = 1, \dots, n$

$$g_j(t, x, \varepsilon) = \sum_{\alpha \in \mathbb{N}^n} g_{\alpha, j}(t, \varepsilon)(x - x_*)^\alpha, \quad t \in \mathbb{R}, \quad \varepsilon \in [0, \varepsilon_*].$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and we employ the notation for powers $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$. The coefficients $g_{\alpha, j}$ are continuous and T -periodic in t and the convergence in the above series is uniform in t and ε . As in the previous Section the vector field f is real analytic on Ω and this is enough to guarantee that the solutions of (6) depend analytically upon initial conditions once ε and t have been fixed (see [9]).

Again $x_0(t)$ is a non-constant T -periodic solution of (1) satisfying (3). We consider the function

$$M(\theta) = \int_0^T \langle g(t, x_0(t + \theta), 0), z_0(t + \theta) \rangle dt,$$

where z_0 is given by Lemma 1. This function is T -periodic and real analytic and so it will have a finite number of zeros in $[0, T[$ unless it is identically zero.

Given $\theta_* \in [0, T[$ a zero of M , $M(\theta_*) = 0$, we say that $\text{index}(M, \theta_*) = 1$ if $M(\theta) \cdot (\theta - \theta_*) > 0$ when $\theta \neq \theta_*$ is close to θ_* . When the inequality is reversed we say that $\text{index}(M, \theta_*) = -1$. In any other case we say that $\text{index}(M, \theta_*) = 0$. We shall say that a solution $x(t)$ passes through a set $S \subset \mathbb{R}^2$ if $x(t) \in S$ for some real t .

Theorem 1 *In the previous setting assume that M is not identically zero and let U be a bounded and open set satisfying*

$$\Gamma \subset U \subset \bar{U} \subset \mathcal{A}$$

(Recall that Γ is the closed orbit associated to $x_0(t)$ and \mathcal{A} is its region of attraction). Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ the system (6) has a finite number of T -periodic solutions passing through \bar{U} . Moreover, if θ_* is a zero of M with $\text{index}(M, \theta_*) \neq 0$ then there exists a T -periodic solution $x_\varepsilon(t)$ of (6) with

$$x_\varepsilon(t) - x_0(t + \theta_*) \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0,$$

uniformly in $t \in \mathbb{R}$. This solution is asymptotically stable if $\text{index}(M, \theta_*) = 1$ and unstable if $\text{index}(M, \theta_*) = -1$.

To illustrate the result we consider the planar system

$$\dot{x} = (1 - |x|^2)x + i|x|^2x + \varepsilon(a(t) + b(t)x + c(t)\bar{x}), \quad (7)$$

where $x \in \mathbb{C}$ and $a, b, c : \mathbb{R} \rightarrow \mathbb{C}$ are continuous and 2π -periodic. The autonomous system ($\varepsilon = 0$) was already analyzed in the previous section and we can now construct the function M for $x_0(t) = e^{it}$, $z_0(t) = (1 + i)e^{it}$ and $T = 2\pi$. A direct computation leads to the formula

$$M(\theta) = \text{Re} \left[\int_0^{2\pi} (a(t) + b(t)e^{i(t+\theta)} + c(t)e^{-i(t+\theta)})(1-i)e^{-i(t+\theta)} dt \right] = 2\pi \text{Re} \left[\left(\hat{a}_1 e^{-i\theta} + \hat{b}_0 + \hat{c}_2 e^{-2i\theta} \right) (1-i) \right],$$

where \hat{a}_m, \hat{b}_m and \hat{c}_m refer to the Fourier coefficients of a, b and c , namely

$$\hat{f}_m = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-imt} dt.$$

In principle Theorem 1 would provide information on a bounded region U whose closure is contained in $\mathbb{C} - \{0\}$. However the specific properties of (7) will allow us to deduce global results. To illustrate this we first claim that for $0 \leq \varepsilon < 1$ any 2π -periodic solution $x(t)$ will satisfy

$$\max_{t \in \mathbb{R}} \|x(t)\| \leq \rho_+ := [1 + \|a\|_\infty + \|b\|_\infty + \|c\|_\infty]^{1/2}.$$

Indeed if t_* is an instant when $m := \max \|x(t)\| = \|x(t_*)\|$ then the derivative $\frac{d}{dt} \|x(t)\|^2 = 2 \langle x(t), \dot{x}(t) \rangle$ must vanish at t_* . From the equation (7) we deduce that

$$\|x(t_*)\|^4 = \|x(t_*)\|^2 + \varepsilon \left\langle a(t_*) + b(t_*)x(t_*) + c(t_*)\overline{x(t_*)}, x(t_*) \right\rangle.$$

It is not restrictive to assume that $m > 1$ and by dividing the latter equality by m^2 the claimed estimate follows. Next we observe that $x \equiv 0$ is a 2π -periodic solution for $\varepsilon = 0$. The variational equation is $\dot{y} = y$ with Floquet multipliers $\mu_1 = \mu_2 = e^{2\pi}$. A standard perturbation result guarantees the existence of some $\rho_- \in (0, 1)$ such that, for small ε , there is a unique 2π -periodic solution $z_\varepsilon(t)$ satisfying $\max \|z_\varepsilon(t)\| \leq \rho_-$. Moreover this solution is unstable since all Floquet multipliers are greater than one. Now we apply Theorem 1 in the region

$$U = \{x \in \mathbb{C} : \rho_- < \|x\| < \rho_+\}.$$

The function M can be expressed as a trigonometric polynomial of the type

$$M(\theta) = \beta + \alpha \cos(\theta + \phi) + \gamma \cos 2(\theta + \varphi),$$

with $\beta = 2\pi \operatorname{Re}[\widehat{b}_0(1-i)]$, $2\pi \widehat{a}_1(1-i) = \alpha e^{-i\phi}$, $2\pi \widehat{c}_2(1-i) = \gamma e^{-2i\varphi}$. Now it is clear that M is not identically zero if and only if

$$|\widehat{a}_1| + |\operatorname{Re}[\widehat{b}_0(1-i)]| + |\widehat{c}_2| > 0.$$

In such a case (7) has a finite number of 2π -periodic solutions passing through \overline{U} , say N . From the above discussions we conclude that also the number of 2π -periodic solutions on the whole plane is finite, namely $N + 1$. When the function M does not vanish we obtain a uniqueness result: z_ε is the unique 2π -periodic solution. When M changes sign we obtain at least two additional 2π -periodic solutions, one asymptotically stable and one unstable. Summing up, we observe that in this example the function M gives conditions for the existence and stability that are rather sharp. Notice also that the function M can have zeros of the type $M(\theta_0) = M'(\theta_0) = M''(\theta_0) = 0$, $M'''(\theta_0) \neq 0$ and they lead to an asymptotically stable solution.

Before the proof of the Theorem we will state three lemmas that will be proved in the next section. Our first preliminary result goes back to [11, page 387] and [10]. It shows that the zeros of the function M are relevant for the location of T -periodic solutions.

Lemma 2 *Assume that $\varepsilon_k \downarrow 0$ is a given sequence and let $x_k(t)$ be a T -periodic solution of (6) with $\varepsilon = \varepsilon_k$ and passing through \overline{U} . Then it is possible to extract a subsequence $\{x_k(t)\}$ and a number $\theta_* \in [0, T[$ such that $M(\theta_*) = 0$ and*

$$x_k(t) - x_0(t + \theta_*) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

uniformly in $t \in \mathbb{R}$.

For the next statements it will be convenient to employ the Poincaré map \mathcal{P}_ε associated to (6). Denoting by $x(t; \zeta, \varepsilon)$ the solution of (6) satisfying $x(0) = \zeta$, we notice that for small ε and $\zeta \in \overline{U}$ this solution is well defined in $[0, T]$. This is a consequence of the theorem on continuous dependence since \overline{U} is compact and for $\varepsilon = 0$ the solutions starting at $\overline{U} \subset \mathcal{A}$ are globally defined in the future. This observation allows us to define

$$\mathcal{P}_\varepsilon : \overline{U} \rightarrow \mathbb{R}^n, \quad \zeta \mapsto x(T; \zeta, \varepsilon).$$

This map is analytic and its fixed points are in a one-to-one correspondence with the T -periodic solutions starting at \bar{U} .

Lemma 3 *Assume that $\theta_0 \in \mathbb{R}$ is an isolated zero of M , then there exist $\varepsilon_0 > 0$ and $R > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the Poincaré map \mathcal{P}_ε of (6) has at most a finite number of fixed points in $B_R(x_0(\theta_0))$.*

The third preliminary result will establish a link between the index of the zeros of M and the fixed point index of the Poincaré map. Results of this type were already obtained in [4] but we will present later an independent proof. The Brouwer degree of a map f on a domain Ω will be denoted by $\deg(f, \Omega)$. It is assumed that Ω is open and bounded and f does not vanish on its boundary.

Lemma 4 *Assume that θ_0 is an isolated zero of M and \mathcal{V} is an open neighborhood of $x_0(\theta_0)$. Then there exist a number $\varepsilon_\star > 0$ and a family of open sets $V_\varepsilon \subset \mathbb{R}^n$, $\varepsilon \in (0, \varepsilon_\star)$, satisfying*

$$x_0(\theta_0) \in V_\varepsilon, \quad V_\varepsilon \subset \mathcal{V}$$

and such that

$$\deg(id - \mathcal{P}_\varepsilon, V_\varepsilon) = -\text{index}(\theta_0, M), \quad \text{whenever } \varepsilon \in (0, \varepsilon_\star).$$

We are now in the position to prove Theorem 1.

Proof of Theorem 1. If the function M does not vanish then (6) has no T -periodic solutions passing through \bar{U} when $\varepsilon > 0$ is small enough. This is a consequence of Lemma 2. From now on we assume that M vanishes somewhere. Let $T^\star > 0$ be the minimal period of $x_0(t)$, so that $T = kT^\star$ for some $k = 1, 2, \dots$. The function M has period T^\star and the sequence of zeros of M on $[0, T^\star[$ is denoted by

$$0 \leq \theta_1 < \theta_2 < \dots < \theta_m < T^\star.$$

Another consequence of Lemma 2 is that for small ε any T -periodic solution of (6) passing through \bar{U} must remain close to the orbit Γ for all time. In particular we can assume that all T -periodic solutions passing through \bar{U} have an initial condition corresponding to a fixed point of \mathcal{P}_ε .

Step 1. There exists $\varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$ then \mathcal{P}_ε has a finite number of fixed points.

Once again we apply Lemma 2 and restrict ε so that all the fixed points are contained in some of the balls $B_R(x_0(\theta_i))$, $i = 1, \dots, m$, where R is given by Lemma 3. The union of these balls contains all the fixed points of \mathcal{P}_ε and we know by Lemma 3 that they contain a finite number of fixed points.

We can also assume that R has been chosen so that these balls are pairwise disjoint. This will be employed later and it is possible since T^\star is the minimal period and so the points $x_0(\theta_i)$ and $x_0(\theta_j)$ are different whenever $i \neq j$.

After this step we can define the index of a T -periodic solution passing through \bar{U} . Assume that $x(t)$ is such a solution for some $\varepsilon \in (0, \varepsilon_1)$. We can find an open set $\mathcal{W} \subset U$ such that $x(0) \in \mathcal{W}$ is the only fixed point of \mathcal{P}_ε lying on $\bar{\mathcal{W}}$. The index of $x(t)$ is defined as

$$\gamma_T(x) = \deg(id - \mathcal{P}_\varepsilon, \mathcal{W}).$$

In principle this index could take any integer value but the condition (3) implies that

$$\gamma_T(x) \in \{-1, 0, 1\}. \quad (8)$$

This fact was already noticed by Krasnoselskii in [7]. We refer to [7] or [13] for the proof.

Step 2. If $x(t)$ is a T -periodic solution of (6) passing through \bar{U} , then $x(t)$ is asymptotically stable if $\gamma_T(x) = 1$ and unstable if $\gamma_T(x) \neq 1$.

The condition (3) and the continuity of the Floquet multipliers with respect to parameters imply the existence of a positive number $\sigma > 0$ such that if $B(t)$ is a T -periodic and continuous matrix with $\|B(t)\| \leq \sigma$ for all t then the system

$$\dot{y} = (f'(x_0(t)) + B(t))y$$

has Floquet multipliers μ_1^*, \dots, μ_n^* with μ_1^* positive and dominant and $|\mu_i^*| < 1$ for $i = 2, \dots, n$. After a time translation we conclude that the same property holds for the more general class of systems

$$\dot{y} = (f'(x_0(t + \theta)) + B(t))y, \quad \max \|B(t)\| < \sigma, \quad B(t + T) = B(t).$$

For small ε any T -periodic solution passing through \bar{U} has a variational equation in this class and so the Floquet multipliers have the structure described above. The conclusion of Step 2 is a consequence of [5] and [13].

Step 3. Assume that $\text{index}(M, \theta_i) \neq 0$. Then for any $\varepsilon \in (0, \varepsilon_2]$ the equation (6) has a T -periodic solution x with

$$x(0) \in B_R(x_0(\theta_i)) \quad \text{and} \quad \gamma_T(x) = -\text{index}(M, \theta_i).$$

This is a consequence of Lemma 4. Indeed we can find an open set $V_\varepsilon \subset B_R(x_0(\theta_i))$ with

$$\deg(id - \mathcal{P}_\varepsilon, V_\varepsilon) = -\text{index}(M, \theta_i)$$

and the additivity of the degree implies that

$$\deg(id - \mathcal{P}_\varepsilon, V_\varepsilon) = \sum_{j=1}^m \gamma_T(x_j),$$

where x_1, \dots, x_m are the T -periodic solutions of (6) with $x_j(0) \in V_\varepsilon$. The conclusion follows from (8). Notice that the convergence of this periodic solution to $x_0(t + \theta_i)$ as $\varepsilon \rightarrow 0$ is a consequence of Lemma 2 since the balls $B_R(x_0(\theta_i))$ are pairwise disjoint. This completes the proof of the Theorem.

4 Proofs of the Lemmas

Proof of Lemma 2. We present a proof for completeness. Since x_k passes through \bar{U} one can find $\tau_k \in [0, T]$ such that $x_k(\tau_k) \in \bar{U}$. After extracting subsequences we can assume that

$$\tau_k \rightarrow \tau \quad \text{and} \quad x_k(\tau_k) \rightarrow \zeta.$$

Let $\widehat{x}(t)$ denotes the solution of (1) with initial condition $\widehat{x}(\tau) = \zeta$. Since ζ is a point in the region of attraction \mathcal{A} we know that $\widehat{x}(t)$ is well defined in $[\tau, \infty[$. By continuous dependence we know that $x_k(t)$ converges to $\widehat{x}(t)$ and the convergence is uniform on every compact interval where $\widehat{x}(t)$ is well defined. In particular this applies to $[\tau, \tau + T]$ and so $\widehat{x}(\tau) = \lim x_k(\tau) = \lim x_k(\tau + T) = \widehat{x}(\tau + T)$. This implies that $\widehat{x}(t)$ is a periodic solution of (1). Since \mathcal{A} is invariant for (1) and $\widehat{x}(\tau) \in \mathcal{A}$ we deduce that the closed orbit associated to \widehat{x} must be contained in \mathcal{A} . This implies that this orbit is precisely Γ and so there exists $\theta_* \in [0, T[$ such that $\widehat{x}(t) = x_0(t + \theta_*)$. In particular $x_k(0) \rightarrow x_0(\theta_*)$. It remains to prove that $M(\theta_*) = 0$. To this end we consider the map

$$\Phi(\zeta, \varepsilon) = \mathcal{P}_\varepsilon(\zeta) - \zeta, \quad \zeta \in \overline{U}, \quad \varepsilon \in [0, \varepsilon_0].$$

This is a C^1 map and the derivative $D\Phi(\zeta, \varepsilon)$ is an $n \times (n + 1)$ matrix. We claim that the rank of $D\Phi(x_0(\theta_*), 0)$ is strictly less than n . Otherwise the equation $\Phi(\zeta, \varepsilon) = 0$ should describe a curve in a small neighborhood of $(x_0(\theta_*), 0)$. However the set $\Phi = 0$ contains the curve $(x_0(\theta), 0)$ and also the set of points $(x_k(0), \varepsilon_k)$ accumulating on $(x_0(\theta_*), 0)$. Once we know that $\text{rank} D\Phi(x_0(\theta_*), 0) < n$, it remains to prove that

$$\text{rank} D\Phi(x_0(\theta), 0) = n \quad \text{if} \quad M(\theta) \neq 0.$$

The partial derivative with respect to ξ is the $n \times n$ matrix

$$\partial_\zeta \Phi(x_0(\theta), 0) = Y(T + \theta)Y(\theta)^{-1} - I_n,$$

where $Y(t)$ is the matrix solution of (2) with $Y(0) = I_n$. The Fredholm alternative for linear endomorphisms is applied to deduce that

$$\text{Im} \partial_\zeta \Phi(x_0(\theta), 0) = \left[\text{Ker} \left([Y(\theta)^*]^{-1} Y(\theta + T)^* - I_n \right) \right]^\perp.$$

The kernel in the above formula corresponds to the initial conditions at time $t = \theta$ of the T -periodic solutions of (4). Hence it is spanned by $z_0(\theta)$ and so

$$\text{Im} \partial_\zeta \Phi(x_0(\theta), 0) = \{\eta \in \mathbb{R}^n : \eta \perp z_0(\theta)\}.$$

By differentiability with respect to parameters, the function $y(t) = \partial_\varepsilon x(t, \zeta, \varepsilon)$ with $\zeta = x_0(\theta)$, $\varepsilon = 0$ solves

$$\dot{y} = f'(x_0(t + \theta))y + g(t, x_0(t + \theta), 0), \quad y(0) = 0.$$

A direct computation shows that

$$\frac{d}{dt} \langle y(t), z_0(t + \theta) \rangle = \langle g(t, x_0(t + \theta), 0), z_0(t + \theta) \rangle$$

and, integrating over the period,

$$\langle y(T), z_0(\theta) \rangle = M(\theta).$$

When $M(\theta) \neq 0$ the vector $y(T)$ is not in the range of $\partial_\zeta \Phi(x_0(\theta), 0)$ and so

$$\text{rank} (\partial_\zeta \Phi(x_0(\theta), 0) | \partial_\varepsilon \Phi(x_0(\theta), 0)) = (n - 1) + 1 = n.$$

□

Proof of Lemma 3. It is based on a variant of the Lyapunov-Schmidt reduction. We divide it in four steps.

1. The change of variables. The dominant eigenvalue of $L = (\mathcal{P}_0)'(x_0(\theta_0))$ is $\mu_1 = 1$ with eigenvector $\dot{x}_0(\theta_0)$. This eigenvalue is simple and so we can find a linear projection π in \mathbb{R}^n satisfying

$$\pi^2 = \pi, \quad \pi L = L\pi, \quad \text{Ker } \pi = \{\lambda \dot{x}_0(\theta_0); \lambda \in \mathbb{R}\}.$$

This is so-called spectral projection and the hyperplane $Y = \text{Im}(id - L) = \text{Im } \pi$ is invariant under L .

Moreover,

$$\sigma(L_Y) = \{\mu_2, \dots, \mu_n\}, \tag{9}$$

where $L_Y : Y \rightarrow Y$ is the restriction of L to Y . In the rest of the proof v denotes a generic vector lying in Y .

Consider the map

$$\Phi : (\theta, v) \in \mathbb{R} \times Y \mapsto x_0(\theta) + v \in \mathbb{R}^n.$$

This is an analytic function with partial derivatives at $(\theta_0, 0)$,

$$\partial_\theta \Phi(\theta_0, 0) = \dot{x}_0(\theta_0), \quad \partial_Y \Phi(\theta_0, 0) = id_Y.$$

The Inverse Function Theorem implies that Φ is a local diffeomorphism mapping $(\theta_0, 0)$ onto $x_0(\theta_0)$. In a neighborhood of this point we reduce the search of fixed points of \mathcal{P}_ε to the equation $\mathcal{P}_\varepsilon \circ \Phi = \Phi$. More precisely we consider the equation

$$\mathcal{P}_\varepsilon(x_0(\theta) + v) = x_0(\theta) + v, \quad |\theta - \theta_0| < \Delta, \quad \|v\| < \Delta, \tag{10}$$

for some small $\Delta > 0$. Notice that Φ is independent of ε and so Δ is uniform in $\varepsilon \geq 0$.

2. The auxiliary equation. The equation (10) can be interpreted as a system in the unknowns θ and v . As usual we apply π and solve in v . This means that we look at the implicit function problem

$$F(\theta, v; \varepsilon) := \pi \mathcal{P}_\varepsilon(x_0(\theta) + v) - \pi x_0(\theta) - v = 0.$$

This function maps $|\theta - \theta_0| < \Delta, \quad \|v\| < \Delta, \quad \varepsilon \in [0, \varepsilon_*]$ into Y and satisfies

$$\partial_v F(\theta_0, 0; 0) = L_Y - id_Y.$$

From the condition (9) we deduce that the implicit Function Theorem is applicable and so we find $r > 0$ and $\alpha : [\theta_0 - r, \theta_0 + r] \times [0, r] \rightarrow Y$ such that

$$\pi \mathcal{P}_\varepsilon(x_0(\theta) + \alpha(\theta, \varepsilon)) = \pi x_0(\theta) + \alpha(\theta, \varepsilon).$$

Moreover this is the only solution of $F(\theta, v; \varepsilon) = 0$ in some ball $\|v\| < R$. The function α is of class C^1 and analytic with respect to θ . Due to the uniqueness of α we have $\alpha(\theta, 0) = 0$ for any $\theta \in [\theta_0 - r, \theta_0 + r]$, which can be combined with the smoothness of α to find a number $\mu > 0$ such that

$$\|\alpha(\theta, \varepsilon)\| \leq \varepsilon \mu \quad \text{for any } \theta \in [\theta_0 - r, \theta_0 + r], \quad \varepsilon \in [0, r]. \tag{11}$$

In this process it can be necessary to reduce the size of r .

3. The bifurcation equation. Assume that $x(t; \Xi, \varepsilon)$ is a T -periodic solution of (6) with Ξ close to $x_0(\theta_0)$ and ε small and positive. We know from the previous steps that the initial condition can be expressed as

$$\Xi = x_0(\Theta) + \alpha(\Theta, \varepsilon)$$

for some $\Theta \in [\theta_0 - r, \theta_0 + r]$. Our next task is to show that Θ must be a zero of the function

$$M_\varepsilon(\theta) := \int_0^T \langle b_\varepsilon(t, \theta), z_0(t + \theta) \rangle dt$$

with

$$b_\varepsilon(t, \theta) := g(t, x(t, \xi, \varepsilon), \varepsilon) - \frac{1}{\varepsilon} [f(x(t, \xi, \varepsilon)) - f(x_0(t + \theta)) - f'(x_0(t + \theta)) \cdot (x(t, \xi, \varepsilon) - x_0(t + \theta))]$$

and

$$\xi = x_0(\theta) + \alpha(\theta, \varepsilon).$$

By construction $y(t) = x(t, \Xi, \varepsilon) - x_0(t + \Theta)$ has to be a T -periodic solution of the linear equation

$$\dot{y} = f'(x_0(t + \Theta))y + \varepsilon b_\varepsilon(t, \Theta).$$

The Fredholm alternative implies that Θ is a zero of M_ε .

4. Conclusion: the role of analyticity. In view of the previous steps it is enough to show that the function M_ε has a finite number of zeros in $[\theta_0 - r, \theta_0 + r]$ for small ε .

Since $\alpha(\theta, 0) = 0$ we obtain by continuous dependence that

$$b_\varepsilon(t, \theta) \rightarrow g(t, x_0(t + \theta), 0) \quad \text{as } \varepsilon \rightarrow 0$$

uniformly in $t \in [0, T]$ and $\theta \in [\theta_0 - r, \theta_0 + r]$. Indeed we also need to use that f is smooth and the estimate (11). This is required to prove that the term related to f goes to zero. Also the differentiability with respect to initial conditions and parameters plays a role here.

The function M_ε converges to M as $\varepsilon \rightarrow 0$ uniformly in $\theta \in [\theta_0 - r, \theta_0 + r]$. We are assuming that M is not identically zero and so the same must happen to M_ε for small ε . Since M_ε is analytic we conclude that it has a finite numbers of zeros in $[\theta_0 - r, \theta_0 + r]$. This is valid for $\varepsilon \in]0, \varepsilon_0[$ with $\varepsilon_0 > 0$ sufficiently small.

□

Remark The standard Lyapunov-Schmidt reduction for the equation $P_\varepsilon(\xi) = \xi$ would start with the splitting

$$\xi = \eta \dot{x}_0(\theta_0) + v, \quad \eta \in \mathbb{R}, \quad v \in Y,$$

and considering the system

$$\begin{cases} \pi \mathcal{P}_\varepsilon(\eta \dot{x}_0(\theta_0) + v) = v \\ (id - \pi) \mathcal{P}_\varepsilon(\eta \dot{x}_0(\theta_0) + v) = \eta \dot{x}_0(\theta_0). \end{cases}$$

Instead of this we are considering a sort of nonlinear splitting induced by the change of variables of Step 1. The advantage is that our bifurcation equation leads directly to $M(\theta) = 0$ as $\varepsilon \downarrow 0$. The same

approach is taken by Hale and Taboas in [3], but they prefer to work in an infinite dimensional framework.

Proof of Lemma 4. First we pick up any $n - 1$ linearly independent solutions y_1, \dots, y_{n-1} of (2) whose initial conditions at θ_0 satisfy $\langle y_i(\theta_0), z_0(\theta_0) \rangle = 0$. Next we consider the $n \times (n - 1)$ matrix $Y_1(\theta) = (y_1(\theta) | \dots | y_{n-1}(\theta))$ and notice that

$$Y_1(\theta + T) = Y_1(\theta)A_\theta \quad (12)$$

where A_θ is a $(n - 1) \times (n - 1)$ matrix with eigenvalues μ_2, \dots, μ_n . To verify this it is enough to observe that the hyperplane V_θ spanned by $y_1(\theta), \dots, y_{n-1}(\theta)$ is invariant under the monodromy operator $M_\theta : y(\theta) \mapsto y(\theta + T)$. This is a consequence of Perron's Lemma. The eigenvector of M_θ associated to $\mu_1 = 1$ is $\dot{x}_0(\theta)$ and does not belong to V_θ . In consequence the restriction of M_θ to V_θ has eigenvalues μ_2, \dots, μ_n . The matrix A_θ is precisely the representation of this restriction with respect to the basis $y_1(\theta), \dots, y_{n-1}(\theta)$. This property of the matrix $Y_1(\theta)$ will be employed several times. First we will employ it to evaluate the topological degree of the auxiliary map

$$\Phi_\varepsilon(\theta, \zeta) = -\varepsilon M(\theta)\dot{x}_0(\theta) + (Y_1(\theta) - Y_1(\theta + T))\zeta$$

with respect to $\Omega_\delta := (\theta_0 - \delta, \theta_0 + \delta) \times B_\delta(0)$, where $\delta > 0$ is a small number and $B_\delta(0)$ is an open ball in \mathbb{R}^{n-1} . We will impose several restrictions on the size of δ , the first being that M has no zeros on $[\theta_0 - \delta, \theta_0 + \delta]$ other than θ_0 . Notice that by linear independence the equation $\Phi_\varepsilon(\theta, \zeta) = 0$ in $\bar{\Omega}_\delta$ splits as $M(\theta) = 0$ and $(Y_1(\theta) - Y_1(\theta + T))\zeta = 0$. Then $\theta = \theta_0$ and from the identity (12) we deduce that $\zeta = 0$. Thus the degree we want to compute is well defined and does not change if we replace Ω_δ by any sub-domain containing $(\theta_0, 0)$. For the effective computation we diminish $\delta > 0$ in such a way that Φ_ε is linearly homotopic to the vector field

$$\widehat{\Phi}(\theta, \zeta) = -M(\theta)\dot{x}_0(\theta_0) + (Y_1(\theta_0) - Y_1(\theta_0 + T))\zeta$$

for $\varepsilon > 0$ sufficiently small so that $\deg(\Phi_\varepsilon, \Omega_\delta) = \deg(\widehat{\Phi}, \Omega_\delta)$. The matrix $S = (\dot{x}_0(\theta_0) | Y_1(\theta_0) - Y_1(\theta_0 + T))$ is non-singular and the map $\widehat{\Phi}$ can be expressed as $S \circ [(-M) \times id]$. By the theorems on the evaluation of the topological index of a composition of vector fields (see e.g. [8], Theorem 7.1), of a product of vector fields (see e.g. [8], Theorem 7.4) and of a linear vector field (see [8], Theorem 6.1) we have that

$$\deg(\widehat{\Phi}, \Omega_\delta) = \text{index}((\theta_0, 0), \widehat{\Phi}) = \text{index}(0, S) \cdot \text{index}((\theta_0, 0), (-M) \times id) = -\text{sign det } S \cdot \text{index}(\theta_0, M). \quad (13)$$

Another restriction on δ that will be useful later is related to the map $\psi(\theta, \zeta) = x_0(\theta) + Y_1(\theta)\zeta$. This map must be a diffeomorphism from Ω_δ onto its image and $\psi(\bar{\Omega}_\delta) \subset \mathcal{V}$. Notice that this is possible since $\det \psi'(\theta_0, 0) = \det(\dot{x}_0(\theta_0) | Y_1(\theta_0)) \neq 0$.

Our next step is to show that the vector fields

$$\mathcal{F}_\varepsilon(\theta, \zeta) = (id - \mathcal{P}_\varepsilon)(x_0(\theta) + Y_1(\theta)\zeta)$$

and Φ_ε are homotopic on a sub-domain of Ω_δ for $\varepsilon > 0$ sufficiently small. Let $x(t; \theta, \zeta, \varepsilon)$ be the solution of (6) satisfying $x(0) = x_0(\theta) + Y_1(\theta)\zeta$. The Taylor expansion leads to

$$x(t, \theta, \zeta, \varepsilon) = x_0(t + \theta) + Y_1(t + \theta)\zeta + \varepsilon \int_0^t Y(t + \theta)Y(s + \theta)^{-1}g(s, x_0(s + \theta), 0)ds + O(\varepsilon^2 + \|\zeta\|^2),$$

where we recall that the matrix $Y(t)$ was defined in Section 2. This expansion is obtained by computing the derivatives with respect to ζ and ε and applying the formula of variation of constants. The matrix $Y^*(t + \theta)^{-1}Y^*(\theta)$ is fundamental at $t = \theta$ for the adjoint system and so

$$z_0(t + \theta) = Y^*(t + \theta)^{-1}Y^*(\theta)z_0(\theta).$$

From the periodicity of z_0 we deduce that

$$Y^*(T + \theta)z_0(\theta) = Y^*(\theta)z_0(\theta).$$

Thus,

$$\begin{aligned} \left\langle \int_0^T Y(T + \theta)Y^{-1}(s + \theta)g(s, x_0(s + \theta), 0)ds, z_0(\theta) \right\rangle &= \\ &= \int_0^T \langle g(s, x_0(s + \theta), 0), z_0(s + \theta) \rangle ds = M(\theta). \end{aligned}$$

In consequence,

$$\langle \mathcal{F}_\varepsilon(\theta, \zeta), z_0(\theta) \rangle = -\varepsilon M(\theta) + O(\varepsilon^2 + \|\zeta\|^2), \quad \mathcal{F}_\varepsilon(\theta, \zeta) = (Y_1(\theta) - Y_1(\theta + T))\zeta + \varepsilon\gamma(\theta) + O(\varepsilon^2 + \|\zeta\|^2),$$

where γ is defined by an integral. Perhaps after a new reduction of the size of δ we can find a positive constant Λ such that

$$\max_{k=1, \dots, n-1} |\langle (Y_1(\theta) - Y_1(\theta + T))\zeta, y_k(\theta) \rangle| \geq \Lambda \|\zeta\|, \quad \text{for every } \zeta \in \mathbb{R}^{n-1} \text{ and } |\theta - \theta_0| \leq \delta.$$

To justify this assertion we notice that, by continuity, it is enough to check it for $\theta = \theta_0$ and in this case it follows from (12) since $(Y_1(\theta_0) - Y_1(\theta_0 + T)) = Y_1(\theta_0)(I - A_{\theta_0})$ and $(I - A_{\theta_0})$ is non-singular. From now on the number δ will be kept fixed. We are going to compute the degree of \mathcal{F}_ε on the set $W_\varepsilon = \{(\theta, \zeta) : |\theta - \theta_0| < \delta, \|\zeta\| < \varepsilon^{2/3}\}$. The boundary of W_ε is composed by $\Delta_1 : \theta = \theta_0 \pm \delta, \|\zeta\| \leq \varepsilon^{2/3}$ and $\Delta_2 : |\theta - \theta_0| \leq \delta, \|\zeta\| = \varepsilon^{2/3}$. On Δ_1 we observe that for ε small enough

$$\text{sign} \langle \mathcal{F}_\varepsilon(\theta, \zeta), z_0(\theta) \rangle = -\text{sign} M(\theta), \quad \text{with } \theta = \theta_0 \pm \delta.$$

On Δ_2 we claim that for some $k = 1, \dots, n - 1$ (depending on ζ),

$$\text{sign} \langle \mathcal{F}_\varepsilon(\theta, \zeta), y_k(\theta) \rangle = \text{sign} \langle (Y_1(\theta) - Y_1(\theta + T))\zeta, y_k(\theta) \rangle.$$

Indeed, from the expansion of \mathcal{F}_ε we find that for each k

$$\langle \mathcal{F}_\varepsilon(\theta, \zeta), y_k(\theta) \rangle = \langle (Y_1(\theta) - Y_1(\theta + T))\zeta, y_k(\theta) \rangle + O(\varepsilon).$$

For some k , $|\langle (Y_1(\theta) - Y_1(\theta + T))\zeta, y_k(\theta) \rangle| \geq \Lambda \varepsilon^{2/3}$ and this term is dominant, leading to the coincidence of the signs. Summarizing, for $(\theta, \zeta) \in \partial W_\varepsilon$ the vectors $\Phi_\varepsilon(\theta, \zeta)$ and $\mathcal{F}_\varepsilon(\theta, \zeta)$ do not point in opposite directions and, therefore, the vector fields Φ_ε and \mathcal{F}_ε are linearly homotopic on W_ε (see [8, theorem 2.1]).

By excision,

$$\text{deg}(\mathcal{F}_\varepsilon, W_\varepsilon) = \text{deg}(\widehat{\Phi}, \Omega_\delta) = -\text{sign} \det S \cdot \text{ind}(\theta_0, M). \quad (14)$$

To finish the proof we define $V_\varepsilon = \psi(W_\varepsilon)$ and observe that $(id - \mathcal{P}_\varepsilon) \circ \psi = \mathcal{F}_\varepsilon$ on W_ε . The theorem on the degree of the composition implies that

$$\deg(id - \mathcal{P}_\varepsilon, V_\varepsilon) \cdot \deg(\psi - x_0(\theta_0), W_\varepsilon) = \deg(\mathcal{F}_\varepsilon, W_\varepsilon).$$

For instance, Theorem 7.2, Formula 7.6 in [8] is applicable since $\partial V_\varepsilon = \psi(\partial W_\varepsilon)$, V_ε is connected and $x_0(\theta_0) \in V_\varepsilon$. By the linearization theorem for topological degree (see e.g. [8], Theorem 6.3) we have that

$$\deg(\psi - x_0(\theta_0), W_\varepsilon) = \text{sign det } \psi'(\theta_0, 0) = \text{sign det}(\dot{x}_0(\theta_0)|Y_1(\theta_0)). \quad (15)$$

The conclusion of the Lemma follows from these last identities and (14) because

$$\text{sign det}(\dot{x}_0(\theta_0)|Y_1(\theta_0)) = \text{sign det } S. \quad (16)$$

To prove this claim we consider the family of matrices

$$Y_1(\theta_0) - \lambda Y_1(\theta_0 + T) = Y_1(\theta_0)(I - \lambda A_{\theta_0}), \quad \lambda \in [0, 1],$$

where once again we have used (12). For $\lambda = 0$ and $\lambda = 1$ we obtain the second blocks of the matrices appearing in the identity (16). The eigenvalues of A_{θ_0} are μ_2, \dots, μ_n , all of them with modulus less than one. Hence

$$\det(\dot{x}_0(\theta_0)|Y_1(\theta_0) - \lambda Y_1(\theta_0 + T)) \neq 0$$

for all $\lambda \in [0, 1]$ and so the sign of this determinant is independent of λ . The identity (16) expresses this fact for the extreme values of λ . \square

References

- [1] E. A. Coddington, N. Levinson, Theory of ordinary differential equations. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [2] B. P. Demidovich, "Lectures on the mathematical theory of stability," Izdat. Nauka, Moscow, 1967. (In Russian)
- [3] J. K. Hale, P. Taboas, Bifurcation from families of periodic solutions. Classical and celestial mechanics (Recife, 1993/1999), 351–382, Princeton Univ. Press, Princeton, NJ, 2002.
- [4] M. Kamenskii, O. Makarenkov, P. Nistri, A continuation principle for a class of periodically perturbed autonomous systems, *Math. Nachr.* **281** (2008) 42–61.
- [5] J. S. Kolesov, Study of stability of solutions of second-order parabolic equations in the critical case, *Izv. Akad. Nauk SSSR* **33** (1969) 1277–1291.
- [6] Ju. M. Kopnin, Periodic oscillations of nonlinear nonautonomous systems with many degrees of freedom, (Russian) *Inz. Z.* **5** (1965) 217–226.

- [7] M. A. Krasnoselskii, The operator of translation along the trajectories of differential equations. Translations of Mathematical Monographs, Vol. 19. Translated from the Russian by Scripta Technica. American Mathematical Society, Providence, R.I. 1968.
- [8] M. A. Krasnosel'skii, P. P. Zabreiko, "Geometrical Methods of Nonlinear Analysis", Springer-Verlag, Berlin, 1984.
- [9] S. Lefschetz, Differential equations: geometric theory. Pure and Applied Mathematics. Vol. VI Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London 1957.
- [10] W. S. Loud, Periodic solutions of a perturbed autonomous system, *Ann. of Math.* **70** (1959) 490–529.
- [11] I. G. Malkin, Some problems of the theory of nonlinear oscillations, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. (In Russian).
- [12] F. Nakajima, G. Seifert, The number of periodic solutions of 2-dimensional periodic systems. *J. Differential Equations* **49** (1983) 430–440.
- [13] R. Ortega, A criterion for asymptotic stability based on topological degree, in Proceedings of the First World Congress of Nonlinear Analysts (Tampa, 1992). Walter de Gruyter, 1996, pp 383-394
- [14] O. Perron, Die Ordnungszahlen der Differentialgleichungssysteme, *Math. Zeitschr.* **31** (1930) 748–766.
- [15] R. A. Smith, Some applications of Hausdorff dimension inequalities for ordinary differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986) 235–259.