Long-time stability estimates for the non-periodic Littlewood boundedness problem

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Abstract

We consider the nonlinear oscillator equation $\ddot{x} + |x|^{\alpha-1}x = p(t)$ for $\alpha \geq 3$ and non-periodic forcing p. For any solution x which is unbounded in the $(x, \dot{x})$-phase plane it is shown that, for $\varepsilon > 0$ small enough, there is a solution x^ε which is bounded in the phase plane and such that the actions of x^ε and x remain close on a time interval of length ε^{-2}; further precise information is available on the location of the zeros of x^ε. In addition, it is possible to take the bounded solution x^ε from a fixed countable family of such solutions.

Contents

1 Introduction and statement of main results

In the 1960's Littlewood [13, 14] posed the problem to investigate boundedness properties of the solutions to differential equations of the form

$$\ddot{x} + g(x) = p(t), \tag{1.1}$$

where $x = x(t)$ is scalar-valued, p is a given bounded forcing, and g has superlinear growth: $g(x)/x \to \infty$ as $|x| \to \infty$. One issue of particular interest was to decide whether all solutions would have to be bounded in the $(x, \dot{x})$-phase plane, no matter what the initial data $x(0) = x_0$ and $\dot{x}(0) = v_0$ were. As opposed to its harmless appearance this question turned out to be quite intricate, with the results obtained so far being closely tied to the regularity and periodicity hypotheses imposed on p. For instance Littlewood himself could prove: (i) in [13, Thm. 1] that given g with some additional properties there is a (non-periodic) bounded function p such that (1.1) admits at least one unbounded solution, and (ii) in [14] that there are g and a periodic function p such that (1.1) has at least one unbounded solution. See also [11] and [16] for some corrections. The result mentioned in (i) was later refined for $g(x) = x^3$ (cf. [12, Section 1.3] and the references therein) to also provide growth rates of the energy of an unbounded solution. A further related reference is [20], where for given g it was shown that there is a bounded and arbitrarily small forcing p such that 'most' initial data (in a topological sense) will lead to an unbounded motion.

Concerning bounded solutions, the first complete positive answer was obtained in [17] for $g(x) = 2x^3$ and p piecewise continuous and periodic by using KAM theory after suitably transforming the system for large energies. This result underwent many generalizations and extensions leading to a wealth of work over the years, by far too many to be reviewed here. If $p \in L^\infty(\mathbb{R})$ is not necessarily periodic, then [21] yields the existence of infinitely many bounded solutions by using variational methods.

If we summarize the preceding discussion and specialize to

$$\ddot{x} + |x|^{\alpha-1}x = p(t) \tag{1.2}$$

for $\alpha \geq 3$ and non-periodic p, then there may co-exist bounded and unbounded solutions. In view of the above it is therefore important to obtain a more refined understanding of the underlying structure and the rôle of the bounded solutions for the overall dynamics. To state our first main theorem, for a given solution $x = x(t)$ we let

$$E(t) = \frac{1}{2}\,\dot{x}(t)^2 + \frac{1}{\alpha+1}\,|x(t)|^{\alpha+1}, \tag{1.3}$$

and furthermore we write $E(t) = E(t; x)$ if the solution x is to be emphasized. The homogeneous problem $\ddot{x} + |x|^{\alpha-1}x = 0$ has a global center at the origin and the action $\varrho = \int_{\gamma_E} y\,dx$ represents the area enclosed by the periodic orbit γ_E of energy $E = \frac{1}{2}\,y^2 + \frac{1}{\alpha+1}\,|x|^{\alpha+1}$ for $y = \dot{x}$. Due to the homogeneity of the problem it is found that $\varrho = kE^{\frac{\alpha+3}{2(\alpha+1)}}$ with a certain constant $k > 0$, cf. (7.9) below. For the forcing function we need to assume that $p \in C_b^6(\mathbb{R})$, i.e., p has continuous and bounded derivatives up to the sixth order.

Theorem 1.1 *Suppose that $\alpha \geq 3$ and $p \in C^6_b(\mathbb{R})$. Then there exist a function $\varepsilon^* = \varepsilon^*(\rho) > 0$ of $\rho > 0$ (also depending upon α and $\|p\|_{C^6_b(\mathbb{R})}$) and constants $a_1, A_1, A_2, A_3, A_4 > 0$ (depending upon α and $\|p\|_{C^6_b(\mathbb{R})}$) such that the following holds. Let $x = x(t)$ be a solution of (1.2) such that $\limsup_{t\to\infty} E(t;x) = \infty$ and $\inf_{t\geq t_0} E(t;x) < \rho$ for some $t_0 \in \mathbb{R}$. Then for every $\varepsilon \in]0,\varepsilon^*]$ there are a solution $x^\varepsilon = x^\varepsilon(t)$ of (1.2) and a time $t_\varepsilon \geq t_0$ so that*

$$a_1\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \leq E(t;x^\varepsilon) \leq A_1\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \quad \text{for} \quad t \in \mathbb{R} \tag{1.4}$$

and

$$|\varrho^\varepsilon(t) - \varrho(t)| \leq A_2\,\varepsilon^{\frac{\alpha-3}{\alpha-1}} \quad \text{for} \quad 0 \leq t - t_\varepsilon \leq A_3\,\varepsilon^{-2}. \tag{1.5}$$

Furthermore, the sets of zeros of x^ε and x in $\{0 \leq t - t_\varepsilon \leq A_3\,\varepsilon^{-2}\}$ can be labelled as $\{t^\varepsilon_n : n \in \mathcal{I}\}$ and $\{t_n : n \in \mathcal{I}\}$ such that

$$|t^\varepsilon_n - t_n| \leq A_4\,\varepsilon \quad \text{for all} \quad n \in \mathcal{I}.$$

The proof is elaborated in Section 8, with an extended outline being given in Section 2; here we shall restrict ourselves to some comments. The starting point is an unbounded solution x of (1.2), in the most general sense that $\limsup_{t\to\infty} E(t;x) = \infty$; the condition $\inf_{t\geq t_0} E(t;x) < \rho$ for some $t_0 \in \mathbb{R}$ should be viewed as a quantitative statement measuring which ε are allowed. Then, for $\varepsilon > 0$ small enough, there is a solution x^ε, bounded in the phase plane with energy $\sim \varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}}$, such that the actions ϱ^ε of x^ε and ϱ of x stay close on a time interval of length ε^{-2}; further precise information is available on the location of the zeros of x^ε. In fact the proof will show that the bounded solution x^ε can be taken from a fixed countable family $\mathcal{B}$ of bounded solutions, which therefore plays a key rôle for the dynamics, since every unbounded solution has to remain close not only to the set $\mathcal{B}$, but to an individual member of this set, for a long time before it can escape. This also highlights a difference of the result as compared to standard averaging or adiabatic invariant techniques [1, 4]; see Section 10 for some more remarks.

As is to be expected the difficulties in the proof of Theorem 1.1 are due to the fact that p is non-periodic. After some transformations (1.2) can be written as a system generated by a Hamiltonian function of the form

$$H(\phi, I;\tau) = \omega_0 I^a + R(\phi, I;\tau),$$

which is 2π-periodic in the 'time variable' τ, but non-periodic in the 'angular variable' ϕ. If it were not for the latter, then KAM theory could be applied (provided that R is sufficiently regular). For R non-periodic in ϕ we have to rely on our second main result, whose proof in turn will be based on precise quantitative refinements of the theory of non-periodic twist maps [5, 6, 7] in the case of small twist at infinity; see Section 2 for more details. To state the theorem consider $\mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$ with coordinates $(\phi, I;\tau)$. Let G be an open and connected subset such that

$$\mathbb{R} \times [I_*, \infty[\times (\mathbb{R}/2\pi\mathbb{Z}) \subset G \subset \mathbb{R} \times]0, \infty[\times (\mathbb{R}/2\pi\mathbb{Z}) \tag{1.6}$$

for some $I_* > 0$. We shall consider a Hamiltonian function $H : G \to \mathbb{R}$ given by

$$H(\phi, I;\tau) = \omega_0 I^a + R(\phi, I;\tau), \tag{1.7}$$

where $\omega_0 > 0$ and $a \in]0,1[$. For the remainder term R we assume that $R \in C^{2,0}(G)$ and it satisfies

$$|R| + |\partial_\phi R| + I|\partial_I R| + |\partial^2_{\phi\phi} R| + I|\partial^2_{\phi I} R| + I^2|\partial^2_{II} R| \leq C_0 I^b, \quad I \geq I_*, \tag{1.8}$$

for some constant $C_0 > 0$ and some $b \in \mathbb{R}$ such that $b \le a-1$. (Here the class $C^{2,0}(G)$ is composed by those continuous functions having derivatives w.r. to ϕ and I up to the second order such that all these derivatives are continuous in $(\phi, I; \tau)$. Also notice that functions defined on G are 2π-periodic in τ.) The Hamiltonian system associated to H is

$$\phi' = \partial_I H = a\omega_0 I^{a-1} + \partial_I R, \quad I' = -\partial_\phi H = -\partial_\phi R. \tag{1.9}$$

Theorem 1.2 *There exist a function $\varepsilon^* = \varepsilon^*(\rho) > 0$ of $\rho > 0$ (also depending upon ω_0, a, b, C_0, I_*) and constants $a_1, A_1, A_2, A_3 > 0$ (depending upon ω_0, a, b, C_0, I_*) so that the following holds. Let (ϕ, I) be a solution of (1.9) defined on $[\tau_0, \infty[$ and satisfying $\limsup_{\tau\to\infty} I(\tau) = \infty$ as well as $\inf_{\tau\ge\tau_0} I(\tau) < \rho$. Then for every $\varepsilon \in]0, \varepsilon^*]$ there are a solution $(\phi^\varepsilon, I^\varepsilon)$ of (1.9) defined on $\mathbb{R}$ and a time $\tau_\varepsilon > \tau_0$ so that*

$$a_1\, \varepsilon^{-\frac{1}{1-a}} \le I^\varepsilon(\tau) \le A_1\, \varepsilon^{-\frac{1}{1-a}} \quad \textit{for} \quad \tau \in \mathbb{R} \tag{1.10}$$

and

$$|\phi^\varepsilon(\tau) - \phi(\tau)| + |I^\varepsilon(\tau) - I(\tau)| \le A_2\, \varepsilon \quad \textit{for} \quad 0 \le \tau - \tau_\varepsilon \le A_3\, \varepsilon^{-\frac{\sigma}{1-a}}, \tag{1.11}$$

where $\sigma = \min\{2-a, -b\} > 0$.

This result is verified in Section 4. It is reminiscent of a Nekhoroshev type statement under finite differentiability assumptions (as in [2]); again see Section 10 for some more comments.

2 Outline of the proof

The proof of Theorem 1.2 is based on a result (Theorem 3.2) on certain twist maps $(\theta, r) \mapsto (\theta', r')$ with non-periodic 'angular variable'. For large r these maps have the form

$$\theta' = \theta + \frac{c}{r^\alpha} + \mathcal{R}_1(\theta, r), \quad r' = r + \mathcal{R}_2(\theta, r),$$

where $\alpha > 0$ and $\mathcal{R}_1$, $\mathcal{R}_2$ are remainder terms which are assumed to be small in a suitable sense. Although there is little danger of confusion let us point out that here θ' and r' denote the iterates rather than derivatives. If θ were a periodic variable and $\mathcal{R}_1$, $\mathcal{R}_2$ were sufficiently regular, then all large orbits would be known to be bounded, by Moser's invariant curve theorem. For non-periodic θ, it still can be shown that there is a large number of bounded orbits, and every unbounded orbit has to stay close to a bounded orbit for many iteration steps. It should be noticed that concerning regularity we only need to assume that $\mathcal{R}_1, \mathcal{R}_2 \in C^1$. Therefore Theorem 3.2 gives something new even in the periodic case: Although then the existence of the bounded orbits follows from the Poincaré-Birkhoff theorem or from Aubry-Mather theory, a fact that is not contained in the standard theory is that every unbounded orbit has to stay close to some bounded orbit for many iterations. In the case where θ is non-periodic, the bounded orbits are obtained via a variational method using a generating function of the map. To show that unbounded orbits remain near bounded ones for many steps we use the fact that the bounded orbits that we constructed lie in horizontal strips of the form

$$\mathcal{S}_\varepsilon = \{(\theta, r) \in \mathbb{R}^2 : \varepsilon^{-1/\alpha} \le r \le \Gamma \varepsilon^{-1/\alpha}\}$$

for some fixed $\Gamma > 0$. Also $\theta' - \theta \sim \varepsilon \to 0$ as $\varepsilon \to 0$ on the bounded orbits due to the small twist at infinity, which means that the orbits only can make very small steps in their θ-variable. In addition r acts as an adiabatic invariant for the map, in that $r'-r \to 0$ as $r \to \infty$. Hence given an unbounded orbit (θ_n, r_n) it has to enter into some $\mathcal{S}_\varepsilon$ and then it gets close to the corresponding bounded orbit $(\theta_n^\varepsilon, r_n^\varepsilon)$ which is contained in $\mathcal{S}_\varepsilon$. By a kind of 'abstract' averaging method we are then able to improve to $T = \mathcal{O}(\varepsilon^{-\mu})$ for a suitable $\mu > 0$ on the mere continuous dependence, which would predict that only $T = \mathcal{O}(1)$ for the time for which

$$|\theta_n^\varepsilon - \theta_n| + |r_n^\varepsilon - r_n| \le C\varepsilon \quad \text{for} \quad n_0 \le n \le n_0 + T$$

is verified, with a certain $n_0 \in \mathbb{N}$.

Regarding the proof of Theorem 1.1, we need to check that the equation (1.2) could be fit into the abstract framework from Theorem 1.2. First (1.2) is transformed to action-angle variables (θ, ϱ) as in [17]. The next step is to change the time and angle variable in the associated Hamiltonian function $\mathcal{H}(\theta, \varrho; t)$, taking $\tau = \theta$ as the new time, $\phi = t$ as the new angle, and $I = \mathcal{H}$ as the new action. This trick leads to a new Hamiltonian function $H(\phi, I; \tau)$; it was pioneered in [10] and has found many applications since. Then H may be written as

$$H(\phi, I; \tau) = \omega_0 I^a + \tilde{R}(\phi, I; \tau), \tag{2.1}$$

where $\omega_0 > 0$ and $a \in [\frac{1}{2}, 1[$. The remainder term $\tilde{R}$ is non-periodic in ϕ, 2π-periodic in τ, and such that (1.8) holds for a certain constant $b \le 0$. Here we encounter the problem that the key assumption $b \le a-1$ from Theorem 1.2 will not be satisfied for any $\alpha \ge 3$. To remedy this problem we can perform three further canonical transformations on (2.1) using Theorem 6.7. For that the first thing to notice is that in fact $\tilde{R}$ has the special structure

$$\tilde{R}(\phi, I; \tau) = f(\phi)g(\tau)I^b + R(\phi, I; \tau),$$

where $f \in C_b^4(\mathbb{R})$, g is continuous, 2π-periodic, and has zero average, and R satisfies (1.8), with b being replaced by some smaller c. Each such transformation preserves this special structure of H and improves the power b to $b-(1-a)$. Then after three steps we arrive at $b-3(1-a) \le a-1$, so that Theorem 1.2 applies to the Hamiltonian system resulting from the last step. If we finally undo all the transformations and use Theorem 1.2, we can complete the proof of Theorem 1.1. For the transformation result (Theorem 6.7) we follow along the lines of earlier work for Hamiltonian functions that are periodic in ϕ; see [3, 9, 15] for instance. However, one always has to keep track of the differences arising through the non-periodicity of the system in ϕ.

3 Unbounded orbits close to bounded orbits

Consider the plane $\mathbb{R}^2$ with coordinates (θ, r) and a one-to-one map Φ of class C^1,

$$\Phi : D \to \mathbb{R}^2, \quad \Phi(\theta, r) = (\theta', r'),$$

where $D \subset \mathbb{R}\times]0,\infty[$ is open and contains a half-plane $\{r \ge r_*\}$ for some $r_* > 0$. By an orbit of Φ we understand a maximal solution of the difference equation

$$(\theta_{n+1}, r_{n+1}) = \Phi(\theta_n, r_n), \quad (\theta_n, r_n) \in D.$$

Typical orbits are defined for indices in some $I = \{n \in \mathbb{Z} : a < n < b\}$, where $-\infty \le a < b \le \infty$. An orbit is said to be complete, if $I = \mathbb{Z}$. We recall that throughout this paper the notion 'bounded' for an orbit will only refer to its r-component, i.e., a bounded orbit satisfies $\sup_{n\in I} r_n < \infty$, whereas it is called unbounded in the case that $\sup_{n\in I} r_n = \infty$.

Next we introduce an assumption on the existence of complete orbits.

Assumption 3.1 (EXC) *There are numbers $r^* > r_*$ and $\Gamma > 1$ such that for every $\mu \geq r^*$ there exists a complete orbit $(\hat{\theta}_n, \hat{r}_n)_{n\in\mathbb{Z}}$ of Φ satisfying*

$$\mu \leq \hat{r}_n \leq \Gamma\mu \quad \textit{for all} \quad n \in \mathbb{Z}.$$

These orbits are bounded and they can coexist with unbounded orbits. We would like to show that the orbits produced by assumption (EXC) play an important rôle in the dynamics. To this end we need to impose some additional conditions on the map Φ. Assume that it can be expressed as

$$\theta' = \theta + \mathcal{T}(\theta, r), \quad r' = r + \mathcal{R}(\theta, r),$$

for functions $\mathcal{T}, \mathcal{R} : D \to \mathbb{R}$. In addition, there should exists constants $\alpha, \sigma > 0$ and $a, A, K, M > 0$ so that

$$\frac{a}{r^\alpha} \leq \mathcal{T}(\theta, r) \leq \frac{A}{r^\alpha}, \tag{3.1}$$

$$|\mathcal{R}(\theta, r)| \leq \frac{K}{r^\alpha}, \tag{3.2}$$

$$\left|\frac{\partial\mathcal{T}}{\partial\theta}(\theta, r)\right| + \left|\frac{\partial\mathcal{T}}{\partial r}(\theta, r)\right| + \left|\frac{\partial\mathcal{R}}{\partial\theta}(\theta, r)\right| + \left|\frac{\partial\mathcal{R}}{\partial r}(\theta, r)\right| \leq \frac{M}{r^\sigma}, \tag{3.3}$$

holds for $r \geq r_*$.

Theorem 3.2 *Let the previously stated hypotheses be satisfied. Then there exists a number $\varepsilon^* > 0$ (depending upon r^*, r_*, α, σ, A, K, and M) such that the following holds. Let $(\theta_n, r_n)_{n\in I}$ be an orbit of Φ so that*

$$\inf_{n\in I} r_n < \varepsilon^{-1/\alpha} \leq \Gamma\varepsilon^{-1/\alpha} < \sup_{n\in I} r_n \tag{3.4}$$

for some $\varepsilon \in]0, \varepsilon^*[$. Then there is a complete orbit $(\theta_n^\varepsilon, r_n^\varepsilon)_{n\in\mathbb{Z}}$ of Φ satisfying

$$\frac{a}{\Gamma^\alpha}\,\varepsilon \leq \theta_{n+1}^\varepsilon - \theta_n^\varepsilon \leq A\varepsilon \quad \textit{and} \quad \varepsilon^{-1/\alpha} \leq r_n^\varepsilon \leq \Gamma\varepsilon^{-1/\alpha} \quad \textit{for} \quad n \in \mathbb{Z},$$

and furthermore there is an integer $N \in I$ such that

$$|\theta_n^\varepsilon - \theta_n| + |r_n^\varepsilon - r_n| \leq 2^{\alpha+2}(A + K)\varepsilon \quad \textit{for} \quad N \leq n \leq N + T,$$

provided that T is an integer so that

$$T \leq \frac{\ln 2}{2^\sigma M}\,\varepsilon^{-\sigma/\alpha}.$$

In particular, the set $\{N, N+1, \ldots, N+T\}$ is contained in I.

The above statement is designed for future applications and this explains its purely quantitative nature. To look at it from a more qualitative perspective, let us assume for illustration that $(\theta_n, r_n)_{n\geq 0}$ is an unbounded orbit. Then ε can be made arbitrarily small and the orbit will remain close to a bounded orbit $(\theta_n^\varepsilon, r_n^\varepsilon)_{n\in\mathbb{Z}}$ for a long period of time.

Before we go on to the proof we need three preliminary results. The first lemma shows that horizontal lines $\{r = c\}$ are mapped by Φ to explicit curves $\{r' = \psi(\theta')\}$, if c is large enough.

Lemma 3.3 *Assume that $c > r_*$ is such that*

$$Mc^{-\sigma} \leq \frac{1}{2}. \tag{3.5}$$

Then there exists a function $\psi \in C^1(\mathbb{R})$ such that

$$\Phi(\{(\theta, c) : \theta \in \mathbb{R}\}) = \{(\theta', \psi(\theta')) : \theta' \in \mathbb{R}\}.$$

Proof: The function $\theta \mapsto \theta + \mathcal{T}(\theta, c)$ has derivative $1 + \partial_\theta \mathcal{T}(\theta, c) \geq 1 - Mc^{-\sigma} \geq 1/2$, as a consequence of (3.3) and (3.5). Therefore this function is an increasing diffeomorphism of the real line and we can consider its inverse, denoted by $\theta = \chi(\theta')$. Then defining

$$\psi(\theta') = c + \mathcal{R}(\chi(\theta'), c), \tag{3.6}$$

we obtain the claim. □

The second lemma will describe the behavior of Φ^{-1} near infinity. For this purpose we introduce the functions

$$f_\pm(r) = r \pm \frac{K}{r^\alpha}$$

and fix a number $s_* \geq r_*$ such that

$$f'_+(s_*) > 0 \quad \text{and} \quad f_-(s_*) > 0.$$

Then both functions are positive and increasing on $[s_*, \infty[$. Given $(\theta', r') = \Phi(\theta, r)$, we deduce from (3.2) that

$$f_-(r) \leq r' \leq f_+(r);$$

in what follows this observation will be employed several times without further mention. Also notice that, in view of (3.6), the function ψ appearing in the previous lemma satisfies

$$f_-(c) \leq \psi(\theta') \leq f_+(c) \quad \text{for} \quad \theta' \in \mathbb{R}. \tag{3.7}$$

Lemma 3.4 *Assume that for some $(\theta, r) \in D$ the point $(\theta', r') = \Phi(\theta, r)$ satisfies $r' > f_+(c)$ for some $c \geq s_*$. Moreover, let $f_-(c) > f_+(s_*)$ and (3.5) hold. Then $r \geq c$.*

Proof: The set $D_- = D \cap \{r < c\}$ is connected and hence also $\Phi(D_-)$ is connected. Since Φ is one-to-one, the set $\Phi(D_-)$ has to be contained in one of the components of the complement of the curve $\{r' = \psi(\theta')\}$. Given a point on the line $\{r = s_*\}$, its iterate $(\theta^+, r^+) = \Phi(\theta, s_*)$ satisfies

$$r^+ = s_* + \mathcal{R}(\theta, s_*) \leq f_+(s_*) < f_-(c) \leq \psi(\theta^+).$$

Here we have used (3.7). This estimate shows that $\Phi(D_-)$ has to be contained in the 'lower' component $\{(\theta', r') : r' < \psi(\theta')\}$. Finally we invoke the assumption $r' > f_+(c)$ and (3.7) to conclude that $r' > \psi(\theta')$. This shows that (θ, r) cannot belong to D_-. □

The third lemma is useful to translate certain arguments using connectedness to a discrete setting. It is based on the notion of an ε-connected set. Following [19] we consider a metric space (A, d) and $\varepsilon > 0$. We say that the space is ε-connected if given $a_*, a^* \in A$ there are finitely many points $a_0 = a_*, a_1, \ldots, a_{n-1}, a_n = a^*$ such that $d(a_j, a_{j+1}) < \varepsilon$ for $j = 0, \ldots, n-1$.

Lemma 3.5 *Let $(X, \|\cdot\|)$ be a normed space and suppose that $A \subset X$ is ε-connected for some $\varepsilon > 0$. Let $F, C_1, C_2 \subset X$ be such that $C_1 \neq C_2$ are connected components of $X \setminus F$ satisfying $A \cap C_1 \neq \emptyset$ and $A \cap C_2 \neq \emptyset$. Then there exist $x \in A$ and $y \in F$ such that $\|x - y\| \leq \varepsilon/2$.*

Proof: By assumption there are $a_0 \in A \cap C_1$, $a_{n+1} \in A \cap C_2$, and $a_1, \dots, a_n \in A$ verifying $\|a_{j+1} - a_j\| < \varepsilon$ for $j = 0, \dots, n$. Let $L \subset X$ denote the polygonal line connecting $a_0 \to a_1 \to \dots \to a_{n+1}$. Since $a_0 \in C_1$ and $a_{n+1} \in C_2$, one of the segments $a_k \to a_{k+1}$ of L must intersect F in a point y. Then $y = \lambda a_{k+1} + (1-\lambda) a_k$ for some $\lambda \in [0,1]$ yields $\|a_k - y\| \leq \lambda\varepsilon$ and $\|a_{k+1} - y\| \leq (1-\lambda)\varepsilon$. Thus depending on whether $\lambda \in [0, 1/2]$ or $\lambda \in [1/2, 1]$ one can take $x = a_k$ or $x = a_{k+1}$. □

Proof of Theorem 3.2: We fix $\varepsilon^* > 0$ so small that the following conditions hold for any $\varepsilon \in]0, \varepsilon^*[$:

$$\varepsilon^{-1/\alpha} > r^*, \quad \frac{1}{2}\,\varepsilon^{-1/\alpha} > r_*, \quad 2^{\alpha+3}(A+K)\,\varepsilon^{\frac{\alpha+1}{\alpha}} \leq 1, \tag{3.8}$$

$$\varepsilon^{-1/\alpha} > f_+\Big(\frac{1}{2}\,\varepsilon^{-1/\alpha}\Big), \quad \frac{1}{2}\,\varepsilon^{-1/\alpha} > s_*, \quad f_-\Big(\frac{1}{2}\,\varepsilon^{-1/\alpha}\Big) > f_+(s_*), \quad 2^\sigma M \varepsilon^{\sigma/\alpha} \leq \frac{1}{2}. \tag{3.9}$$

Then the first relation in (3.8) allows to use assumption (EXC) with $\mu = \varepsilon^{-1/\alpha}$. Let $(\theta_n^\varepsilon, r_n^\varepsilon)_{n\in\mathbb{Z}}$ be a complete orbit of Φ satisfying

$$\varepsilon^{-1/\alpha} \leq r_n^\varepsilon \leq \Gamma\varepsilon^{-1/\alpha} \quad \text{for} \quad n \in \mathbb{Z}. \tag{3.10}$$

From (3.1) it is easy to deduce that

$$\frac{a}{\Gamma^\alpha}\,\varepsilon \leq \theta_{n+1}^\varepsilon - \theta_n^\varepsilon \leq A\varepsilon \quad \text{for} \quad n \in \mathbb{Z}. \tag{3.11}$$

The assumption (3.4) on the given orbit $(\theta_n, r_n)_{n\in I}$ can be used to find $n_0, n_1 \in I$ such that

$$r_{n_0} < \varepsilon^{-1/\alpha} < \Gamma\varepsilon^{-1/\alpha} < r_{n_1}.$$

From now on we assume that $n_0 < n_1$, the other case being similar. Define

$$n_2 = \max\{n \in \mathbb{Z} : n_0 \leq n \leq n_1, r_n < \varepsilon^{-1/\alpha}\}.$$

Then $n_0 \leq n_2 \leq n_1$, $r_{n_2} < \varepsilon^{-1/\alpha}$, and $r_n \geq \varepsilon^{-1/\alpha}$ for $n_2 < n \leq n_1$. We are going to apply Lemma 3.4 with $(\theta, r) = (\theta_{n_2}, r_{n_2})$ and $c = \frac{1}{2}\,\varepsilon^{-1/\alpha}$. In fact, we have $r_{n_2+1} \geq \varepsilon^{-1/\alpha}$, and all of the other required conditions are satisfied due to (3.9). Hence the conclusion of Lemma 3.4 ensures that

$$r_{n_2} \geq \frac{1}{2}\,\varepsilon^{-1/\alpha}.$$

Next we define a piecewise linear function $\phi^\varepsilon : \mathbb{R} \to \mathbb{R}$ by the assignment $\phi^\varepsilon(\theta_n^\varepsilon) = r_n^\varepsilon$. Since (3.11) implies that the points $\{\theta_n^\varepsilon : n \in \mathbb{Z}\}$ form a partition of the real axis, this function ϕ^ε is well-defined and continuous. Let $\mathcal{F} \subset \mathbb{R}^2$ be the polygonal line joining the complete orbit, i.e., the graph of ϕ^ε. Then $\mathcal{F} \subset \mathcal{S} = \{(\theta, r) \in \mathbb{R}^2 : \varepsilon^{-1/\alpha} \leq r \leq \Gamma\varepsilon^{-1/\alpha}\}$ by (3.10). Furthermore define $\mathcal{C}_1 = \{(\theta, r) \in \mathbb{R}^2 : r < \phi^\varepsilon(\theta)\}$ as well as $\mathcal{C}_2 = \{(\theta, r) \in \mathbb{R}^2 : r > \phi^\varepsilon(\theta)\}$. Finally let $\mathcal{A} = \{(\theta_n, r_n) : n = n_2, \dots, n_1\}$. To apply Lemma 3.5 to $X = \mathbb{R}^2$ and the norm $\|x\| = |x_1| + |x_2|$, notice first that $(\theta_{n_2}, r_{n_2}) \in \mathcal{A} \cap \mathcal{C}_1$ and $(\theta_{n_1}, r_{n_1}) \in \mathcal{A} \cap \mathcal{C}_2$; this is due to the fact that $\mathbb{R}\times]-\infty, \varepsilon^{-1/\alpha}[\subset \mathcal{C}_1$ and

$\mathbb{R}\times]\Gamma\varepsilon^{-1/\alpha},\infty[\subset \mathcal{C}_2$. For showing that $\mathcal{A}$ is δ-connected with $\delta = 2^\alpha(A+K)\varepsilon$, we observe that $r_n \geq \frac{1}{2}\,\varepsilon^{-1/\alpha}$ for $n = n_2, \dots, n_1$. Then, from (3.1) and (3.2),

$$|\theta_{n+1}-\theta_n| + |r_{n+1}-r_n| \leq \delta \quad \text{for} \quad n = n_2, \dots, n_1 - 1,$$

which means that $\mathcal{A}$ is δ-connected. Hence we can apply Lemma 3.5 to obtain $N \in \{n_2, \dots, n_1\}$ and $(\theta, r) \in \mathcal{F}$ such that

$$|\theta_N - \theta| + |r_N - r| \leq \frac{\delta}{2}.$$

Since $N \in \{n_2, \dots, n_1\}$, in particular

$$r_N \geq \frac{1}{2}\,\varepsilon^{-1/\alpha}. \tag{3.12}$$

By definition of $\mathcal{F}$ we may write $\theta = \lambda\theta^\varepsilon_{m+1} + (1-\lambda)\theta^\varepsilon_m$ and $r = \lambda r^\varepsilon_{m+1} + (1-\lambda) r^\varepsilon_m$ for some $\lambda \in [0,1]$ and $m \in \mathbb{Z}$. From (3.11), (3.2), and (3.10) we have

$$|\theta^\varepsilon_{m+1} - \theta^\varepsilon_m| + |r^\varepsilon_{m+1} - r^\varepsilon_m| \leq (A+K)\varepsilon.$$

As a consequence,

$$|\theta - \theta^\varepsilon_m| + |r - r^\varepsilon_m| = \lambda(|\theta^\varepsilon_{m+1} - \theta^\varepsilon_m| + |r^\varepsilon_{m+1} - r^\varepsilon_m|) \leq (A+K)\varepsilon,$$

and therefore

$$\begin{aligned}
|\theta_N - \theta^\varepsilon_m| + |r_N - r^\varepsilon_m| &\leq |\theta_N - \theta| + |\theta - \theta^\varepsilon_m| + |r_N - r| + |r - r^\varepsilon_m| \\
&\leq \frac{\delta}{2} + (A+K)\varepsilon = (2^{\alpha-1}+1)(A+K)\varepsilon \\
&\leq 2^{\alpha+1}(A+K)\varepsilon
\end{aligned} \tag{3.13}$$

for those $N \in \{n_2, \dots, n_1\}$ and $m \in \mathbb{Z}$. Now consider the shifted complete orbit $(\hat{\theta}^\varepsilon_n, \hat{r}^\varepsilon_n)_{n\in\mathbb{Z}} = (\theta^\varepsilon_{n+m-N}, r^\varepsilon_{n+m-N})_{n\in\mathbb{Z}}$ of Φ. Then (3.13) says that

$$|\theta_N - \hat{\theta}^\varepsilon_N| + |r_N - \hat{r}^\varepsilon_N| \leq 2^{\alpha+1}(A+K)\varepsilon. \tag{3.14}$$

Denote $T_* = \frac{\ln 2}{2^\sigma M}\,\varepsilon^{-\sigma/\alpha}$ as well as $L = 1 + 2^\sigma M\varepsilon^{\sigma/\alpha}$. Then for each integer k such that $0 \leq k \leq T_*$,

$$|\hat{\theta}^\varepsilon_{N+k} - \theta_{N+k}| + |\hat{r}^\varepsilon_{N+k} - r_{N+k}| \leq 2^{\alpha+1}(A+K)L^k\varepsilon, \tag{3.15}$$

$$r_{N+k} \geq \frac{1}{2}\,\varepsilon^{-1/\alpha}. \tag{3.16}$$

To establish this claim, we make some auxiliary observations. Assume first that $(\theta, r), (\Theta, R) \in D$ are such that $r, R \geq \frac{1}{2}\,\varepsilon^{-1/\alpha}$. In particular, then $r, R > r_*$ holds due to the second relation in (3.8). If $(\theta', r') = \Phi(\theta, r)$ and $(\Theta', R') = \Phi(\Theta, R)$, then by the mean-value theorem and (3.3),

$$\begin{aligned}
|\Theta' - \theta'| + |R' - r'| &\leq |\Theta - \theta| + |R - r| + |\mathcal{T}(\Theta, R) - \mathcal{T}(\theta, r)| + |\mathcal{R}(\Theta, R) - \mathcal{R}(\theta, r)| \\
&\leq \Big(1 + 2^\sigma M\varepsilon^{\sigma/\alpha}\Big)\Big[|\Theta - \theta| + |R - r|\Big].
\end{aligned} \tag{3.17}$$

Next we assert that $L^{T_*} \leq 2$. In fact, from $\ln(1+x) \leq x$ for $x \geq 0$ it follows that

$$T_* \ln L \leq T_*\, 2^\sigma M\varepsilon^{\sigma/\alpha} = \ln 2.$$

For the actual proof of (3.15) and (3.16), we proceed by (finite) induction on k. For $k = 0$ this is just (3.14) and (3.12). Now suppose that (3.15) and (3.16) hold for some k with $k+1 \le T_*$. Then $r_{N+k} \ge \frac{1}{2}\,\varepsilon^{-1/\alpha}$, and from (3.10) we see that $\hat{r}^{\varepsilon}_{N+k} \ge \frac{1}{2}\,\varepsilon^{-1/\alpha}$ also. Hence (3.17) and (3.15) yield

$$\begin{aligned} |\hat{\theta}^{\varepsilon}_{N+k+1} - \theta_{N+k+1}| + |\hat{r}^{\varepsilon}_{N+k+1} - r_{N+k+1}| &\le \left(1 + 2^{\sigma} M \varepsilon^{\sigma/\alpha}\right)\Big[|\hat{\theta}^{\varepsilon}_{N+k} - \theta_{N+k}| + |\hat{r}^{\varepsilon}_{N+k} - r_{N+k}|\Big] \\ &\le 2^{\alpha+1}(A+K)L^{k+1}\varepsilon. \end{aligned}$$

Thus in particular by (3.10), due to $L^{k+1} \le L^{T_*} \le 2$, and by the third relation in (3.8),

$$r_{N+k+1} \ge \hat{r}^{\varepsilon}_{N+k+1} - |\hat{r}^{\varepsilon}_{N+k+1} - r_{N+k+1}| \ge \varepsilon^{-1/\alpha} - 2^{\alpha+2}(A+K)\varepsilon \ge \frac{1}{2}\,\varepsilon^{-1/\alpha},$$

which completes the inductive argument and hence the proof of the theorem. □

4 From the Hamiltonian to the discrete system

Throughout this section it will be assumed that the assumptions of Theorem 1.2 are satisfied. Also, all quantities that depend only upon ω_0, a, b, C_0, or I_* will be called a constant.

We are going to consider the Poincaré map Φ associated to the periodic system (1.9). For $(\phi_0, I_0) \in \mathbb{R}^2$ such that $(\phi_0, I_0; 0) \in G$ let $\tau \mapsto (\phi(\tau; \phi_0, I_0), I(\tau; \phi_0, I_0))$ denote the solution of (1.9) satisfying $\phi(0) = \phi_0$ and $I(0) = I_0$. From now on the coordinates (θ, r) used in the previous section should be identified with (ϕ_0, I_0). Consider the map

$$\Phi : D \to \mathbb{R}^2, \quad \Phi(\phi_0, I_0) = (\phi(2\pi; \phi_0, I_0), I(2\pi; \phi_0, I_0)),$$

where $D \subset \mathbb{R}^2$ is composed by those initial data (ϕ_0, I_0) leading to solutions that are defined on the whole interval $[0, 2\pi]$. Notice that D is open but not necessarily connected. In order to check that Φ fits into the general framework set up in Section 3, we first need to establish that some half-plane $\{I_0 \ge I_{**}\}$ is contained in D. This is a consequence of the following result.

Lemma 4.1 *There exists a constant $I_{**} \ge I_*$ such that for every $\phi_0 \in \mathbb{R}$ and $I_0 \ge I_{**}$ the solution*

$$(\phi(\tau), I(\tau)) = (\phi(\tau; \phi_0, I_0), I(\tau; \phi_0, I_0))$$

to (1.9) with initial data (ϕ_0, I_0) exists on $[0, 2\pi]$ and satisfies

$$\frac{I_0}{4} \le I(\tau) \le 4I_0 \quad \textit{for} \quad \tau \in [0, 2\pi]. \tag{4.1}$$

Proof: Denote $[0, \tau_0[\subset [0, 2\pi]$ the maximal half-open subinterval such that $I_0/4 < I(\tau) < 4I_0$ for $\tau \in [0, \tau_0[$. Then

$$(I^{(1-b)})' = (1-b)I^{-b}I' = -(1-b)I^{-b}(\partial_\phi R)$$

implies that $|(I^{1-b})'| \le (1-b)C_0$ on $[0, \tau_0[$. Thus for $\tau \in [0, \tau_0[$,

$$\left(\frac{I_0}{2}\right)^{1-b} \le I_0^{1-b} - 2\pi(1-b)C_0 \le I(\tau)^{1-b} \le I_0^{1-b} + 2\pi(1-b)C_0 \le (2I_0)^{1-b}, \tag{4.2}$$

provided that we take I_{**} large enough. Therefore we must have $\tau_0 = 2\pi$ and $I_0/4 \le I(\tau) \le 4I_0$ for $\tau \in [0, 2\pi]$. □

Remarks 4.2 (a) It is no loss of generality to assume that $I_{**}/4 \ge I_*$ is verified, so that (1.8) is at our disposal for $\tau \in [0, 2\pi]$.
(b) In the preceding proof it is not necessary to suppose that $b \le a - 1$, it only matters that $a \in]0,1[$ and $b < 1$. This observation will be useful in Section 7.

Given a solution $(\phi(\tau;\phi_0,I_0), I(\tau;\phi_0,I_0))$ of (1.9) defined on an interval J, the discrete Φ-orbit (ϕ_n, I_n) is well-defined for each $n \in J \cap 2\pi\mathbb{Z}$, where

$$\phi_n = \phi(2n\pi;\phi_0,I_0) \quad \text{and} \quad I_n = I(2n\pi;\phi_0,I_0).$$

Assuming that $J \cap 2\pi\mathbb{Z} \neq \emptyset$ and $\inf_{n\in J\cap 2\pi\mathbb{Z}} I_n \ge I_{**}$, the above lemma yields that

$$\frac{1}{4} \inf_{n\in J\cap 2\pi\mathbb{Z}} I_n \le \inf_{\tau\in J} I(\tau) \le \sup_{\tau\in J} I(\tau) \le 4 \sup_{n\in J\cap 2\pi\mathbb{Z}} I_n. \tag{4.3}$$

This inequality will be employed several times. It implies in particular that bounded and complete Φ-orbits correspond to bounded solutions of the Hamiltonian system defined on the whole real line.

Furthermore we need to know that if two orbits (ϕ_n, I_n) and $(\hat{\phi}_n, \hat{I}_n)$ are close to each other, then the corresponding solutions $(\phi(\tau), I(\tau))$ and $(\hat{\phi}(\tau), \hat{I}(\tau))$ are also close. This can be made precise using the following elementary stability result.

Lemma 4.3 *There is a constant $C_{\rm stab} > 0$ such that if $\tau_0 \in \mathbb{R}$ and if (ϕ, I) and $(\hat{\phi}, \hat{I})$ are solutions to (1.9) so that $I(\tau_0) \ge I_{**}$ as well as $\hat{I}(\tau_0) \ge I_{**}$, then*

$$|\phi(\tau) - \hat{\phi}(\tau)| + |I(\tau) - \hat{I}(\tau)| \le C_{\rm stab}\Big(|\phi(\tau_0) - \hat{\phi}(\tau_0)| + |I(\tau_0) - \hat{I}(\tau_0)|\Big) \tag{4.4}$$

for every $\tau \in [\tau_0, \tau_0 + 2\pi]$.

Proof: Given a general ODE system $\dot{x} = X(\tau, x)$ for $x \in U \subset \mathbb{R}^d$, if $x_1 = x_1(\tau)$ and $x_2 = x_2(\tau)$ are solutions, then

$$|x_1(\tau) - x_2(\tau)| \le e^{L|\tau-\tau_0|}\,|x_1(\tau_0) - x_2(\tau_0)|$$

by Gronwall's lemma, where L is a Lipschitz constant for the vector field $X(\tau,\cdot)$. In our case $x = (\phi, I)$ and $X = (\partial_I H, -\partial_\phi H)$, which is defined on the region $U = \{I \ge I_{**}/4\}$, since $I_{**} \ge 4I_*$ by definition. Hence to estimate the Lipschitz constant on U we must obtain a bound on the second derivatives of H. From (1.8) it follows that

$$\partial^2_{\phi\phi} H = \partial^2_{\phi\phi} R = \mathcal{O}(I^b), \quad \partial^2_{\phi I} H = \partial^2_{\phi I} R = \mathcal{O}(I^{b-1}), \quad \partial^2_{II} H = \omega_0 a(a-1) I^{a-2} + \partial^2_{II} H = \mathcal{O}(I^{a-2}).$$

Therefore due to $b \le 0$ and $a - 2 < 0$ these expressions are bounded on U. It remains to observe that as a consequence of $I(\tau_0) \ge I_{**}$ and $\hat{I}(\tau_0) \ge I_{**}$ the corresponding solutions cannot leave U for $\tau \in [\tau_0, \tau_0 + 2\pi]$, in view of Lemma 4.1. $\square$

Remark 4.4 In the preceding proof it is sufficient to assume that $a \in]0,1[$ and $b \le 0$.

The theorem on the differentiability with respect to initial conditions implies that $\Phi \in C^1(D;\mathbb{R}^2)$. In addition, by uniqueness we know that Φ is one-to-one. Define

$$\mathcal{T}(\phi_0, I_0) = \phi(2\pi;\phi_0,I_0) - \phi_0 \quad \text{and} \quad \mathcal{R}(\phi_0, I_0) = I(2\pi;\phi_0,I_0) - I_0.$$

Lemma 4.5 *The Poincaré map satisfies the conditions (3.1), (3.2), and (3.3) for $I \geq I_{**}$ and appropriate constants a, A, K, M, and with*

$$\alpha = 1 - a \quad \textit{and} \quad \sigma = \max\{2 - a, -b\}.$$

Proof: Since $b < a$, we can select $\gamma_2 > \gamma_1 > 0$ such that

$$\gamma_1 I^{a-1} \leq a\omega_0 I^{a-1} - C_0 I^{b-1} \leq a\omega_0 I^{a-1} + C_0 I^{b-1} \leq \gamma_2 I^{a-1}$$

for all $I \in \mathbb{R}$ so that $I \geq I_{**}/4$. Thus from the first relation in (1.9), (1.8), and (4.1),

$$\begin{aligned} \gamma_1\, 4^{a-1}\, I_0^{a-1} &\leq \gamma_1 I(\tau)^{a-1} \leq a\omega_0 I(\tau)^{a-1} - C_0 I(\tau)^{b-1} \leq \phi'(\tau) \leq a\omega_0 I(\tau)^{a-1} + C_0 I(\tau)^{b-1} \\ &\leq \gamma_2 I(\tau)^{a-1} \leq \gamma_2\, 4^{1-a}\, I_0^{a-1}, \end{aligned}$$

whenever $I_0 \geq I_{**}$ and $\tau \in [0, 2\pi]$; also recall that $a < 1$. Thus integration over $\tau \in [0, 2\pi]$ yields

$$2\pi\gamma_1\, 4^{a-1}\, I_0^{a-1} \leq \phi_1 - \phi_0 \leq 2\pi\gamma_2\, 4^{1-a}\, I_0^{a-1}, \tag{4.5}$$

provided that $I_0 \geq I_{**}$. This yields (3.1) due to $\mathcal{T}(\phi_0, I_0) = \phi_1 - \phi_0$ and $\alpha = 1 - a > 0$.

Concerning (3.2), a similar reasoning and $b \leq a - 1 < 0$ leads to

$$|I'(\tau)| \leq C_0 I(\tau)^b \leq C_0 4^{-b} I_0^b \leq C_0 4^{-b} I_0^{a-1}.$$

Then integration over $\tau \in [0, 2\pi]$ leads to

$$|I_1 - I_0| \leq 2\pi C_0 4^{-b} I_0^{a-1},$$

which is (3.2) in view of $\mathcal{R}(\phi_0, I_0) = I_1 - I_0$.

Next we turn to (3.3). Notice that the partial derivatives appearing in this condition are just the entries of the 2×2-matrix $D\Phi(\phi_0, I_0) - E_2$, where $E_2 \in \mathbb{R}^{2\times 2}$ denotes the identity matrix. Defining

$$\Psi(\tau) = \begin{pmatrix} \frac{\partial\phi}{\partial\phi_0}(\tau) & \frac{\partial\phi}{\partial I_0}(\tau) \\ \frac{\partial I}{\partial\phi_0}(\tau) & \frac{\partial I}{\partial I_0}(\tau) \end{pmatrix} \quad \text{and} \quad \mathcal{A}(\tau) = \begin{pmatrix} \partial^2_{\phi I} R & a(a-1)\omega_0 I^{a-2} + \partial^2_{II} R \\ -\,\partial^2_{\phi\phi} R & -\,\partial^2_{\phi I} R \end{pmatrix},$$

where $\phi(\tau) = \phi(\tau; \phi_0, I_0)$, $I(\tau) = I(\tau; \phi_0, I_0)$, and $R = R(\phi(\tau), I(\tau); \tau)$, the variational equations associated to (1.9) are just $\Psi' = \mathcal{A}(\tau)\Psi$. Furthermore, we have $\Psi(0) = E_2$ and $\Psi(2\pi) = D\Phi(\phi_0, I_0)$. From (1.8) and (4.1) we obtain in some matrix norm

$$\|\mathcal{A}(\tau)\| \leq C(I_0^{b-1} + I_0^{a-2} + I_0^{b-2} + I_0^{b}) \leq C I_0^{\max\{a-2,\, b\}} = C I_0^{-\sigma} \quad \text{for} \quad \tau \in [0, 2\pi]$$

and a suitable constant $C \geq 1$ that could be computed explicitly. Since $\sigma \geq 0$ by assumption we have $I_0^{-\sigma} \leq 1$, so that in particular $\|\mathcal{A}(\tau)\| \leq C$ for $\tau \in [0, 2\pi]$. Therefore

$$\|\Psi(\tau)\| \leq \|\Psi(0)\| + \int_0^\tau \|\mathcal{A}(s)\Psi(s)\|\, ds \leq C\Big(1 + \int_0^\tau \|\Psi(s)\|\, ds\Big)$$

in conjunction with Gronwall's lemma yields $\|\Psi(\tau)\| \leq C$ for $\tau \in [0, 2\pi]$ and a new constant C. Hence

$$\|\Psi(\tau) - E_2\| \leq \int_0^\tau \|\mathcal{A}(s)\Psi(s)\|\, ds \leq C I_0^{-\sigma} \quad \text{for} \quad \tau \in [0, 2\pi]$$

and a certain C is found. Recalling that $\Psi(2\pi) = D\Phi(\phi_0, I_0)$, we obtain (3.3). $\square$

Lemma 4.6 *There exist constants $r^* > r_* \geq I_{**}$ and $\Gamma > 1$ such that the condition (EXC) is satisfied for the Poincaré map.*

The proof of this result uses a variational technique and is independent of the rest of the discussion. For this reason we postpone it to the next section (Section 5).

The application of Theorem 3.2 to the Poincaré map Φ leads to the following result.

Theorem 4.7 *Under the hypotheses of Theorem 1.2 there exist constants $\varepsilon_*, a_1, A_1, A_2, A_3 > 0$ and $\hat{\Gamma} > 1$ such that if (ϕ, I) is a solution of (1.9) defined on some interval $J \subset \mathbb{R}$ and*

$$I_{**} \leq \inf_{\tau \in J} I(\tau) < \frac{1}{4}\,\varepsilon^{-\frac{1}{1-a}} \leq 4\,\hat{\Gamma}\varepsilon^{-\frac{1}{1-a}} < \sup_{\tau \in J} I(\tau)$$

for some $\varepsilon \in]0, \varepsilon_*[$, then there are a solution $(\phi^\varepsilon, I^\varepsilon)$ of (1.9) defined on $\mathbb{R}$ and a time $\tau_\varepsilon \in J$ so that

$$a_1\,\varepsilon^{-\frac{1}{1-a}} \leq I^\varepsilon(\tau) \leq A_1\,\varepsilon^{-\frac{1}{1-a}} \quad \textit{for} \quad \tau \in \mathbb{R}$$

and

$$|\phi^\varepsilon(\tau) - \phi(\tau)| + |I^\varepsilon(\tau) - I(\tau)| \leq A_2\,\varepsilon \quad \textit{for} \quad 0 \leq \tau - \tau_\varepsilon \leq A_3\,\varepsilon^{-\frac{\sigma}{1-a}},$$

where $\sigma = \min\{2 - a, -b\} > 0$ is from Lemma 4.5.

Proof: After a time translation $\tau \mapsto \tau - \tau_0$ we can assume that $J \cap 2\pi\mathbb{Z} \neq \emptyset$. The discrete orbit $(\phi(2n\pi), I(2n\pi))_{n \in J \cap 2\pi\mathbb{Z}}$ satisfies (3.4) with $\Gamma = \hat{\Gamma}$, in view of (4.3). Using Lemmas 4.5 and 4.6 it is therefore possible to apply Theorem 3.2 to the Poincaré map. Hence the conclusion of the theorem is a consequence of Lemma 4.3. $\square$

Theorem 1.2, as stated in the introduction, can now be obtained as a corollary to the previous result.

Proof of Theorem 1.2: We take $J = [\tau_0, \infty[$ and choose the constants $\varepsilon_*, a_1, A_1, A_2, A_3 > 0$ and $\hat{\Gamma} > 1$ according to Theorem 4.7. Given $\rho > 0$ and an unbounded solution (ϕ, I), as described in Theorem 1.2, we fix $\varepsilon^* = \varepsilon^*(\rho) > 0$ such that $\varepsilon^* < \varepsilon_*$ and $\frac{1}{4}\,(\varepsilon^*)^{-1/\alpha} > \max\{\rho, I_{**}\}$. Then (ϕ, I) satisfies the assumptions of Theorem 4.7 for any $\varepsilon \in]0, \varepsilon^*[$. $\square$

5 The generating function of a sublinear Hamiltonian

In this section we complete the proof of Theorem 1.2 by showing that assumption (EXC) is verified for the Poincaré map Φ of (1.9), as was asserted in Lemma 4.6. To prove the existence of the desired complete and bounded orbits we will employ the formalism of generating functions and [7, Thm. 2.3].

Let us first recall the notion of a generating function for a C^1-map

$$\theta_1 = F(\theta, r), \quad r_1 = G(\theta, r),$$

defined on a region $\{(\theta, r) \in \mathbb{R}^2 : r \geq r_*\}$; see [1, 18]. It must be assumed that the map is area-preserving and has twist, i.e., $d\theta_1 \wedge dr_1 = d\theta \wedge dr$ and

$$\frac{\partial F}{\partial r}(\theta, r) < 0 \quad \text{for} \quad r \geq r_*.$$

The set

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : F(\theta, r_*) > \theta_1 > F(\theta, +\infty)\} \tag{5.1}$$

is open and connected, and we can solve globally the implicit function problem

$$\theta_1 = F(\theta, r), \quad (\theta, \theta_1) \in \Omega.$$

In other words, we can find a unique C^1-function $R : \Omega \to \mathbb{R}$, $r = R(\theta, \theta_1)$, with the property that $\theta_1 = F(\theta, R(\theta, \theta_1))$ for all $(\theta, \theta_1) \in \Omega$. Differentiating this relation yields

$$\partial_\theta F + (\partial_r F)(\partial_\theta R) = 0 \quad \text{and} \quad (\partial_r F)(\partial_{\theta_1} R) = 1. \tag{5.2}$$

Furthermore, we have $r_* < R(\theta, \theta_1) < +\infty$ by definition for $(\theta, \theta_1) \in \Omega$. The differential form $r_1 d\theta_1 - r d\theta$ is closed, and so we can find a function $\tilde{S} = \tilde{S}(\theta, r)$ of class C^1 and defined in $\{r \geq r_*\}$ such that $d\tilde{S} = r_1 d\theta_1 - r d\theta$; here it is essential that we are working on a simply connected set. Also notice that this differential form $r_1 d\theta_1 - r d\theta$ is only of class C^0 and the notion of closed form has to be understood in a generalized sense. The generating function is defined as

$$S : \Omega \to \mathbb{R}, \quad S(\theta, \theta_1) = -\tilde{S}(\theta, R(\theta, \theta_1)),$$

and a straightforward computation using (5.2) then shows that

$$\frac{\partial S}{\partial \theta}(\theta, \theta_1) = R(\theta, \theta_1) \quad \text{and} \quad \frac{\partial S}{\partial \theta_1}(\theta, \theta_1) = -G(\theta, R(\theta, \theta_1)).$$

Let us assume now that $(\theta_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ is such that $(\theta_n, \theta_{n+1}) \in \Omega$ and

$$\frac{\partial S}{\partial \theta}(\theta_n, \theta_{n+1}) + \frac{\partial S}{\partial \theta_1}(\theta_{n-1}, \theta_n) = 0 \quad \text{for} \quad n \in \mathbb{Z}.$$

Defining $r_n = R(\theta_n, \theta_{n+1})$, we have $r_n > r_*$ as well as $\theta_{n+1} = F(\theta_n, R(\theta_n, \theta_{n+1})) = F(\theta_n, r_n)$. In addition,

$$r_{n+1} = R(\theta_{n+1}, \theta_{n+2}) = \frac{\partial S}{\partial \theta}(\theta_{n+1}, \theta_{n+2}) = -\frac{\partial S}{\partial \theta_1}(\theta_n, \theta_{n+1}) = G(\theta_n, R(\theta_n, \theta_{n+1})) = G(\theta_n, r_n)$$

for $n \in \mathbb{Z}$. Therefore $(\theta_n, r_n)_{n \in \mathbb{Z}}$ is a complete orbit of the original map lying in $\{r > r_*\}$.

This method will be applied to the map Φ from the previous section. We continue to use the variables (ϕ_0, I_0) and notice that $\Phi : (\phi_0, I_0) \mapsto (\phi_1, I_1)$ defined on $\{I_0 \geq I_{**}\}$ is area-preserving. This is due to the Liouville's Theorem for Hamiltonian flows. The next result ensures that Φ has a twist. We continue to apply the convention on constants as introduced in the previous section.

Lemma 5.1 *There exists $\hat{I}_{**} \geq I_{**}$ such that*

$$\frac{\partial \phi}{\partial I_0}(\tau; \phi_0, I_0) \leq -\frac{1}{2}\, a(1-a)\omega_0 \tau I_0^{a-2} \tag{5.3}$$

for $\tau \in [0, 2\pi]$, $\phi_0 \in \mathbb{R}$, and $I_0 \geq \hat{I}_{**}$.

Proof: Let us start by observing that

$$\lim_{I_0\to\infty}\frac{I(\tau;\phi_0,I_0)}{I_0}=1 \tag{5.4}$$

uniformly in $\tau\in[0,2\pi]$ and $\phi_0\in\mathbb{R}$. This is a direct consequence of the inequalities in (4.2). Defining

$$\xi_1(\tau)=\frac{\partial\phi}{\partial I_0}(\tau;\phi_0,I_0)\quad\text{and}\quad \xi_2(\tau)=\frac{\partial I}{\partial I_0}(\tau;\phi_0,I_0),$$

we have $\xi_1(0)=0$, $\xi_2(0)=1$, and the variational system

$$\begin{aligned}\xi_1' &= (\partial^2_{\phi I}R)\,\xi_1+\Big(-a(1-a)\omega_0 I^{a-2}+\partial^2_{II}R\Big)\,\xi_2,\\ \xi_2' &= -(\partial^2_{\phi\phi}R)\,\xi_1-(\partial^2_{\phi I}R)\,\xi_2,\end{aligned}$$

is satisfied. In the new variables $x_1=I_0^{2-a}\xi_1$ and $x_2=\xi_2$, this reads as $x_1(0)=0$, $x_2(0)=1$, and

$$x_1' = (\partial^2_{\phi I}R)\,x_1+\Big(-a(1-a)\omega_0\Big(\frac{I}{I_0}\Big)^{a-2}+(\partial^2_{II}R)\,I_0^{2-a}\Big)\,x_2, \tag{5.5}$$

$$x_2' = -(\partial^2_{\phi\phi}R)\,I_0^{a-2}x_1-(\partial^2_{\phi I}R)\,x_2. \tag{5.6}$$

This means that the vector $x=(x_1,x_2)$ is the solution to the initial-value problem

$$x'=A(\tau;\phi_0,I_0)x,\quad x(0)=(0,1)^t, \tag{5.7}$$

where $A(\tau;\phi_0,I_0)\in\mathbb{R}^{2\times2}$ is the coefficient matrix associated to (5.5), (5.6). To study the behavior of A as $I_0\to\infty$, notice first that for this we may suppose that $I_0\geq I_{**}$. Then the bounds from Lemma 4.1 apply, which in conjunction with (1.8) yield

$$\begin{aligned}&|\partial^2_{\phi I}R(\phi(\tau),I(\tau);\tau)|+|\partial^2_{\phi\phi}R(\phi(\tau),I(\tau);\tau)|\,I_0^{a-2}+|\partial^2_{II}R(\phi(\tau),I(\tau);\tau)|\,I_0^{2-a}\\ &\leq C_0 I(\tau)^{b-1}+C_0 I(\tau)^{b}\,I_0^{a-2}+C_0 I(\tau)^{b-2}\,I_0^{2-a}\\ &\leq C\left(I_0^{b-1}+I_0^{a+b-2}+I_0^{b-a}\right)\to 0\end{aligned}$$

as $I_0\to\infty$ uniformly in $\tau\in[0,2\pi]$ and $\phi_0\in\mathbb{R}$. Hence (5.4) implies that

$$\lim_{I_0\to\infty}A(\tau;\phi_0,I_0)=A_\infty(\tau):=\begin{pmatrix}0 & -a(1-a)\omega_0\\ 0 & 0\end{pmatrix}$$

uniformly in $\tau\in[0,2\pi]$ and $\phi_0\in\mathbb{R}$. Thus the uniform limiting problem of (5.7) as $I_0\to\infty$ is

$$x'=A_\infty(\tau)x,\quad x(0)=(0,1)^t, \tag{5.8}$$

whose solution x_∞ is given by $x_{\infty,1}(\tau)=-a(1-a)\omega_0\tau$ and $x_{\infty,2}(\tau)=1$. To verify the convergence of the solutions we use (5.7) and (5.8) to write

$$\begin{aligned}x(\tau)-x_\infty(\tau) &= \int_0^\tau A(\sigma;\phi_0,I_0)\,(x(\sigma)-x_\infty(\sigma))\,d\sigma\\ &\quad+\int_0^\tau (A(\sigma;\phi_0,I_0)-A_\infty(\sigma))\,x_\infty(\sigma)\,d\sigma\end{aligned}$$

for $\tau \in [0, 2\pi]$. Taking norms and applying Gronwall's lemma, we get

$$|x(\tau) - x_\infty(\tau)| \le 2\pi\gamma\, e^{C\tau} \|A(\cdot;\phi_0, I_0) - A_\infty\|_\infty$$

for $\tau \in [0, 2\pi]$, where $\gamma = \|x_\infty\|_\infty$ and $C > 0$ is a bound on $\|A(\cdot;\phi_0, I_0)\|_\infty$. Therefore the preceding observation on the convergence of the matrices yields

$$\lim_{I_0\to\infty} \sup_{\tau\in[0,2\pi]} \sup_{\phi_0\in\mathbb{R}} |x(\tau) - x_\infty(\tau)| = 0,$$

which reads as

$$\lim_{I_0\to\infty} \sup_{\tau\in[0,2\pi]} \sup_{\phi_0\in\mathbb{R}} \left(\left| I_0^{2-a} \frac{\partial\phi}{\partial I_0}(\tau;\phi_0, I_0) + a(1-a)\omega_0\tau \right| + \left| \frac{\partial I}{\partial I_0}(\tau;\phi_0, I_0) - 1 \right| \right) = 0$$

when transformed back to the variables ξ_1, ξ_2. Thus if $\hat{I}_{**} \ge I_{**}$ is large enough, then the conclusion of the lemma holds. □

Taking $\tau = 2\pi$ in (5.3), we have shown that

$$\frac{\partial\phi_1}{\partial I_0}(\phi_0, I_0) = \frac{\partial\phi}{\partial I_0}(2\pi;\phi_0, I_0) \le -\pi\, a(1-a)\omega_0 I_0^{a-2} < 0$$

for $\phi_0 \in \mathbb{R}$ and $I_0 \ge \hat{I}_{**}$, so that $I_0 \mapsto \phi_1(\phi_0, I_0)$ is invertible on $[\hat{I}_{**}, \infty[$ for every fixed $\phi_0 \in \mathbb{R}$. Since $a < 1$, (4.5) implies that $\lim_{I_0\to\infty} \phi_1(\phi_0, I_0) = \phi_0$, and hence the inverse function $\phi_1 \mapsto I_0(\phi_0, \phi_1)$ is defined on $]\phi_0, \phi_1(\phi_0, \hat{I}_{**})]$. As a consequence, the generating function $S = S(\phi_0, \phi_1)$ is defined on

$$\Omega = \{(\phi_0, \phi_1) \in \mathbb{R}^2 : \phi_1(\phi_0, \hat{I}_{**}) > \phi_1 > \phi_0\};$$

see (5.1). To compute S, notice first that

$$\tilde{S}(\phi_0, I_0) = \int_0^{2\pi} L(\tau)\, d\tau$$

satisfies $d\tilde{S} = I_1\, d\phi_1 - I_0\, d\phi_0$, where

$$L(\tau) = \phi'(\tau)I(\tau) - H(\phi(\tau), I(\tau); \tau)$$

and $(\phi(\tau), I(\tau)) = (\phi(\tau;\phi_0, I_0), I(\tau;\phi_0, I_0))$. To establish this claim, observe that $\tilde{S}$ is well-defined and of class C^1 on $\{I_0 \ge I_{**}\} \subset D$; recall Lemma 4.1. Moreover, using (1.9),

$$\begin{aligned}
\frac{\partial\tilde{S}}{\partial\phi_0} &= \int_0^{2\pi} \Big[\frac{\partial\phi'}{\partial\phi_0} I + \phi' \frac{\partial I}{\partial\phi_0} - H_\phi \frac{\partial\phi}{\partial\phi_0} - H_I \frac{\partial I}{\partial\phi_0} \Big]\, d\tau \\
&= \int_0^{2\pi} \frac{d}{d\tau}\Big(\frac{\partial\phi}{\partial\phi_0} I \Big)\, d\tau = I_1 \frac{\partial\phi_1}{\partial\phi_0} - I_0, \\
\frac{\partial\tilde{S}}{\partial I_0} &= \int_0^{2\pi} \frac{d}{d\tau}\Big(\frac{\partial\phi}{\partial I_0} I \Big)\, d\tau = I_1 \frac{\partial\phi_1}{\partial I_0},
\end{aligned}$$

which is precisely the meaning of $d\tilde{S} = I_1\, d\phi_1 - I_0\, d\phi_0$ due to $d\phi_1 = (\frac{\partial\phi_1}{\partial\phi_0})d\phi_0 + (\frac{\partial\phi_1}{\partial I_0})dI_0$. The generating function

$$S(\phi_0, \phi_1) = -\tilde{S}(\phi_0, I_0(\phi_0, \phi_1))$$

belongs to $C^1(\Omega)$. According to the discussion at the beginning of this section, we will produce sequences $(\phi_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that $(\phi_n, \phi_{n+1}) \in \Omega$ and

$$\frac{\partial S}{\partial \phi_0}(\phi_n, \phi_{n+1}) + \frac{\partial S}{\partial \phi_1}(\phi_{n-1}, \phi_n) = 0 \quad \text{for} \quad n \in \mathbb{Z}. \tag{5.9}$$

This will be achieved by an application of [7, Thm. 2.3], or more precisely by a slight improvement of the result, as contained in [8, Exercise 19]. For this, we need to check that Ω contains a strip of the type $\{0 < \phi_1 - \phi_0 < \delta_*\}$, and for each $\delta \in]0, \delta_*[$ sufficiently small the generating function satisfies

$$\underline{\alpha}(\delta)(\phi_1 - \phi_0)^{-\kappa} \leq S(\phi_0, \phi_1) \leq \overline{\alpha}(\delta)(\phi_1 - \phi_0)^{-\kappa} \tag{5.10}$$

on $\{0 < \phi_1 - \phi_0 < \delta\}$, where $\kappa = \frac{a}{1-a} > 0$ and $\overline{\alpha} > \underline{\alpha} > 0$ are such that

$$\lim_{\delta \to 0^+} \frac{\overline{\alpha}(\delta)}{\underline{\alpha}(\delta)} = 1.$$

Notice that if at this point we were to use [7, Thm. 2.3] rather than [8, Exercise 19], we would have to include the further hypothesis that $\kappa \geq 1$, which means that $a \geq 1/2$.

To begin with, we define $\delta_* = 2\pi\gamma_1 \, 4^{a-1} \, (\hat{I}_{**})^{a-1}$, where $\gamma_1 > 0$ is from (4.5). If $0 < \phi_1 - \phi_0 < \delta_*$ are fixed, then

$$\phi_1(\phi_0, \hat{I}_{**}) \geq \phi_0 + 2\pi\gamma_1 \, 4^{a-1} \, (\hat{I}_{**})^{a-1} = \phi_0 + \delta_* > \phi_1 > \phi_0$$

by (4.5), and therefore $\{0 < \phi_1 - \phi_0 < \delta_*\} \subset \Omega$. To verify (5.10) on $\{0 < \phi_1 - \phi_0 < \delta\}$, we first rewrite the Lagrange function L, using (1.9) and (1.7), as

$$L = \phi' I - H = (a-1)\omega_0 I^a + I(\partial_I R) - R.$$

Then $|I(\partial_I R) - R| \leq C I^b \leq C I_0^b$ by (1.8) and Lemma 4.1, and furthermore

$$I^a = I_0^a + a \int_0^\tau I^{a-1}(s) I'(s) \, ds$$

leads to $I^a = I_0^a + \mathcal{O}(I_0^{a+b-1})$ in a similar fashion. Therefore we get

$$L = (a-1)\omega_0 I_0^a + \mathcal{O}(I_0^b)$$

for $\tau \in [0, 2\pi]$, $\phi_0 \in \mathbb{R}$, and $I_0 \geq I_{**}$. From (5.4) and $b < a$ we thus deduce that

$$\lim_{I_0 \to \infty} \frac{L(\tau)}{I_0^a} = (a-1)\omega_0$$

uniformly in $\tau \in [0, 2\pi]$ and $\phi_0 \in \mathbb{R}$, from where it follows that also

$$\lim_{I_0 \to \infty} \frac{\tilde{S}(\phi_0, I_0)}{I_0^a} = 2\pi(a-1)\omega_0$$

uniformly in $\tau \in [0, 2\pi]$ and $\phi_0 \in \mathbb{R}$. On the other hand, if $I_0 = I_0(\phi_0, \phi_1)$, then due to (1.9), (5.4), and (1.8),

$$\frac{\phi_1 - \phi_0}{I_0^{a-1}} = \frac{1}{I_0^{a-1}} \int_0^{2\pi} \phi' \, d\tau = \frac{1}{I_0^{a-1}} \int_0^{2\pi} (a\omega_0 I^{a-1} + \partial_I R) \, d\tau \to 2\pi a\omega_0$$

as $\phi_1 - \phi_0 \to 0$; notice that $\phi_1 - \phi_0 \to 0$ enforces $I_0 \to \infty$, for instance by (4.5). Therefore

$$(\phi_1 - \phi_0)^\kappa S(\phi_0, \phi_1) = -\Big(\frac{\phi_1 - \phi_0}{I_0^{a-1}}\Big)^\kappa I_0^{-a} \tilde{S}(\phi_0, I_0) \to -(2\pi a\omega_0)^\kappa \, 2\pi(a-1)\omega_0$$

as $\phi_1 - \phi_0 \to 0$. Then the estimate (5.10) is a direct consequence, for suitable functions $\overline{\alpha}(\delta) > \underline{\alpha}(\delta) > 0$ of $\delta \in]0, \delta_*[$ sufficiently small. Now let us fix $\delta_{**} \in]0, \delta_*]$ such that (5.10) holds for $\delta \in]0, \delta_{**}]$, and furthermore

$$\overline{\alpha}(\delta) < 2^\kappa \, \underline{\alpha}(\delta)$$

is verified for $\delta \in]0, \delta_{**}]$. Then, according to [8, Exercise 19], there is a constant $\sigma_{**} \geq 1$ such that (5.9) has a solution $(\phi_n^\delta)_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that

$$\sigma_{**}^{-2}\delta \leq \phi_{n+1}^\delta - \phi_n^\delta \leq \delta \quad \text{for} \quad n \in \mathbb{Z}, \tag{5.11}$$

provided that $\delta \in]0, \delta_{**}]$. Defining $I_n^\delta = I_0(\phi_n^\delta, \phi_{n+1}^\delta)$, this generates a family $(\phi_n^\delta, I_n^\delta)_{n\in\mathbb{Z}}$ of complete orbits of Φ for $\delta \in]0, \delta_{**}]$. Here (4.5) reads as

$$2\pi\gamma_1 \, 4^{a-1} \, (I_n^\delta)^{a-1} \leq \phi_{n+1}^\delta - \phi_n^\delta \leq 2\pi\gamma_2 \, 4^{1-a} \, (I_n^\delta)^{a-1},$$

so that $\mu \leq I_n^\delta \leq \Gamma\mu$ for $n \in \mathbb{Z}$ and $\delta \in]0, \delta_{**}]$ by (5.11), where

$$\mu = \Big(\frac{2\pi\gamma_1 \, 4^{a-1}}{\delta}\Big)^{\frac{1}{1-a}} \quad \text{and} \quad \Gamma = \Big(\frac{2\gamma_2}{\gamma_1} \, 4^{2(1-a)} \, \sigma_{**}^2\Big)^{\frac{1}{1-a}};$$

notice that Γ is independent of δ, whereas $\mu = \mu(\delta) \to \infty$ for $\delta \to 0$, as it is needed. This completes the verification of assumption (EXC) for Φ. Now that we have checked that the condition (EXC) holds, it is important to notice that r^* does only depend upon ω_0, a, b, C_0, and I_*; actually we can take

$$r^* = \frac{2\pi\gamma_1 4^{a-1}}{\delta_{**}}.$$

Going back to the proof of convergence of $(\phi_1 - \phi_0)^\kappa S(\phi_0, \phi_1)$, one observes that also δ_{**} can be obtained in terms of the admissible parameters.

6 Transformation of the Hamiltonian

After introducing suitable action-angle variables and thereafter exchanging angle and time, the Hamiltonian associated to the Littlewood boundedness problem has not only the form $H = \omega_0 I^a + R$, but a special structure that can be exploited further: here

$$H(\phi, I; \tau) = \omega_0 I^a + f(\phi)g(\tau)I^b + R(\phi, I; \tau)$$

where $\omega_0 > 0$, $a < 1$, $b < 1$, $\|f\|_{C_b^4(\mathbb{R})} < \infty$, and $g : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, continuous, even or odd, and has zero mean-value.

We shall prove the transformation theorem in Section 6.2, whereas Section 6.1 introduces a convenient function class framework (inspired by [3]) and some auxiliary results needed for its proof.

6.1 The function classes $\mathcal{F}_{p,\mu}$

Fix $\mu > 0$ and consider functions in $C^{2,0}(\mathbb{R} \times [\mu, \infty[\times \mathbb{T})$, denoted by F, F_1, F_2, etc.; here we write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $p \in \mathbb{R}$ define

$$\begin{aligned}|F|_{p,\mu} \quad = \quad \sup_{(\phi,I;\tau)\in\mathbb{R}\times[\mu,\infty[\times\mathbb{T}} & I^{-p}\Big[|F(\phi,I;\tau)| + |\partial_\phi F(\phi,I;\tau)| + I|\partial_I F(\phi,I;\tau)| \\ & + |\partial^2_{\phi\phi}F(\phi,I;\tau)| + I|\partial^2_{\phi I}F(\phi,I;\tau)| + I^2|\partial^2_{II}F(\phi,I;\tau)|\Big],\end{aligned}$$

which will become a norm on the vector space

$$\mathcal{F}_{p,\mu} = \{F : |F|_{p,\mu} < \infty\}.$$

Then

$$|F_1F_2|_{p+q,\mu} \le 30\,|F_1|_{p,\mu}\,|F_2|_{q,\mu} \tag{6.1}$$

is a straightforward consequence of the definitions and the product rule, which yields 15 terms, among them two with a factor of 2. Applying (6.1) to $F_2 = I^q$, we deduce that the map $\mathcal{F}_{p,\mu} \ni F \mapsto I^qF \in \mathcal{F}_{p+q,\mu}$ is a bounded linear isomorphism.

Next we are going to discuss some properties of $\mathcal{F}_{p,\mu}$ related to the composition of maps.

Lemma 6.1 *Assume that $g \in C^2_b(\mathbb{R})$ and $F \in \mathcal{F}_{p,\mu}$ for some $p \le 0$. Define*

$$N_1(\phi,I;\tau) = g(\phi + F(\phi,I;\tau)) \quad \text{and} \quad N_2(\phi,I;\tau) = g(F(\phi,I;\tau)).$$

Then $N_i \in \mathcal{F}_{0,\mu}$ and

$$|N_i|_{0,\mu} \le C_{p,\mu}\,\|g\|_{C^2_b(\mathbb{R})}(1 + |F|^2_{p,\mu})$$

for $i = 1, 2$ and constants $C_{p,\mu} > 0$ independent of g and F.

Proof: Since $|F|_{0,\mu} \le \mu^p|F|_{p,\mu}$ for $p \le 0$, it suffices to verify the result for $p = 0$ only. In this case, however, it is a direct consequence of the chain rule. □

Remark 6.2 To deal with N_2 it is not necessary that g is defined on the whole real line. Instead it suffices to assume that $g \in C^2_b(I)$, where $I \subset \mathbb{R}$ is an interval such that $F(\mathbb{R} \times [\mu, \infty[\times \mathbb{T}) \subset I$.

For the next result notice first that in general $\Lambda \mapsto |N|_{b,\Lambda}$ is decreasing. Hence if $b < 1$ and μ are given, then $\Lambda \ge \mu + \Lambda^b|N|_{b,\Lambda}$ will be satisfied for Λ large enough.

Lemma 6.3 *Assume that $R \in \mathcal{F}_{p,\mu}$, $M \in \mathcal{F}_{0,\mu}$, and $N \in \mathcal{F}_{b,\mu}$ for some $p \in \mathbb{R}$ and $b < 1$. Select Λ so large that $\Lambda \ge \mu + \Lambda^b|N|_{b,\Lambda}$ and define*

$$R_1(\theta,\lambda;\tau) = R(\phi,I;\tau), \quad \lambda \ge \Lambda,$$

where

$$\phi = \theta + M(\theta,\lambda;\tau) \quad \text{and} \quad I = \lambda + N(\theta,\lambda;\tau).$$

Then $R_1 \in \mathcal{F}_{p,\Lambda}$ and $|R_1|_{p,\Lambda} \le C|R|_{p,\mu}$, where C only depends upon $|M|_{0,\Lambda}$, $|N|_{b,\Lambda}$, and Λ.

Proof: Since $|N(\theta,\lambda;\tau)| \le |N|_{b,\Lambda}\lambda^b$ for $\lambda \ge \Lambda$, it follows that

$$\mu \le c_1\lambda \le I \le c_2\lambda, \quad \lambda \ge \Lambda, \tag{6.2}$$

for $c_1 = 1 - \Lambda^{-(1-b)}|N|_{b,\Lambda} > 0$ and $c_2 = 1 + \Lambda^{-(1-b)}|N|_{b,\Lambda} > c_1$. To establish the actual claim, let us for instance consider $\partial^2_{\lambda\lambda}R_1$ for $\lambda \ge \Lambda$. Here pointwise

$$\begin{aligned} \partial^2_{\lambda\lambda}R_1 &= (\partial^2_{\phi\phi}R)(\partial_\lambda M)^2 + (\partial_\phi R)(\partial^2_{\lambda\lambda}M) + 2(\partial^2_{\phi I}R)(1+\partial_\lambda N)(\partial_\lambda M) \\ &\quad + (\partial^2_{II}R)(1+\partial_\lambda N)^2 + (\partial_I R)(\partial^2_{\lambda\lambda}N), \end{aligned}$$

so that by (6.2),

$$\begin{aligned} \lambda^2|\partial^2_{\lambda\lambda}R_1| &\le |M|^2_{0,\Lambda}\,|\partial^2_{\phi\phi}R| + |M|_{0,\Lambda}\,|\partial_\phi R| + 2c_1^{-1}\,|M|_{0,\Lambda}\,I\,|\partial^2_{\phi I}R|\,(1+\|\partial_\lambda N\|_\infty) \\ &\quad + c_1^{-2}\,I^2|\partial^2_{II}R|\,(1+\|\partial_\lambda N\|_\infty)^2 + c_1^{-1}I|\partial_I R|\,\|\lambda(\partial^2_{\lambda\lambda}N)\|_\infty. \end{aligned}$$

Then $\|\partial_\lambda N\|_\infty + \|\lambda(\partial^2_{\lambda\lambda}N)\|_\infty \le \Lambda^{b-1}|N|_{b,\Lambda}$ due to $\lambda \ge \Lambda$ in conjunction with $I \ge \mu$ yields

$$\sup_{(\theta,\lambda;\tau)\in\mathbb{R}\times[\Lambda,\infty[\times\mathbb{T}} \lambda^{-p}\lambda^2|\partial^2_{\lambda\lambda}R_1| \le C|R|_{p,\mu},$$

where C depends upon $|M|_{0,\Lambda}$, $|N|_{b,\Lambda}$, and Λ. Since the other terms can be handled similarly, it follows that $|R_1|_{p,\Lambda} \le C|R|_{p,\mu}$. $\square$

Lemma 6.4 *Assume that $N \in \mathcal{F}_{p,\Lambda}$ for some $p \le 1$ and*

$$|N(\theta,\lambda;\tau)| \le \frac{1}{2}\,\lambda, \quad \lambda \ge \Lambda. \tag{6.3}$$

Define $I = \lambda + N$. If $\sigma \le 1$, then

$$I^\sigma = \lambda^\sigma + \sigma\lambda^{\sigma-1}\,N(\theta,\lambda;\tau) + P(\theta,\lambda;\tau)$$

for $P \in \mathcal{F}_{2(p-1)+\sigma,\Lambda}$ such that

$$|P|_{2(p-1)+\sigma,\Lambda} \le C(1+|N|^4_{p,\Lambda}). \tag{6.4}$$

Proof: We start with the general Taylor expansion

$$(1+\varepsilon)^\sigma = 1 + \sigma\varepsilon + \sigma(\sigma-1)\varepsilon^2 Z(\varepsilon) \tag{6.5}$$

for $\varepsilon \in \mathbb{R}$, where

$$Z(\varepsilon) = \int_0^1 (1-s)(1+\varepsilon s)^{\sigma-2}\,ds.$$

Taking $\varepsilon = \lambda^{-1}N$, we obtain

$$I^\sigma = \lambda^\sigma(1+\lambda^{-1}N)^\sigma = \lambda^\sigma\Big[1 + \sigma\,\frac{N}{\lambda} + \sigma(\sigma-1)\,\frac{N^2}{\lambda^2}\,Z\Big(\frac{N}{\lambda}\Big)\Big].$$

Thus we need to verify (6.4) for

$$P(\theta,\lambda;\tau) = \sigma(\sigma-1)\,\frac{N^2(\theta,\lambda;\tau)}{\lambda^{2-\sigma}}\,Z\Big(\frac{N(\theta,\lambda;\tau)}{\lambda}\Big).$$

First we consider $M(\theta,\lambda;\tau) = Z(\frac{N(\theta,\lambda;\tau)}{\lambda})$ and apply the case $i=2$ from Lemma 6.1 in conjunction with Remark 6.2 for $g = Z$ and $F = \frac{N}{\lambda}$. Then $F(\mathbb{R}\times[\Lambda,\infty[\times\mathbb{T}) \subset [-\frac{1}{2},\frac{1}{2}] \subset]-1,1[$ by (6.3) and $Z \in C_b^2(]-1,1[)$. Therefore Lemma 6.1 for $p-1 \le 0$ yields

$$|M|_{0,\Lambda} \le C\,\|Z\|_{C_b^2(]-1,1[)}\left(1+\left|\frac{N}{\lambda}\right|^2_{p-1,\Lambda}\right) \le C(1+|N|^2_{p,\Lambda}),$$

where (6.1) was used for the last estimate. Furthermore,

$$|\lambda^{-(2-\sigma)}N^2|_{2(p-1)+\sigma,\Lambda} \le C|N^2|_{2p,\Lambda} \le C|N|^2_{p,\Lambda},$$

and hence

$$|P|_{2(p-1)+\sigma,\Lambda} \le C\,|\lambda^{-(2-\sigma)}N^2|_{2(p-1)+\sigma,\Lambda}\,|M|_{0,\Lambda} \le C|N|^2_{p,\Lambda}(1+|N|^2_{p,\Lambda}),$$

all by (6.1). □

In the previous lemmas we examined the properties of the classes $\mathcal{F}_{p,\Lambda}$ under composition of functions. To introduce suitable coordinate transforms we shall also need a result on solving an implicit equation in $\mathcal{F}_{p,\Lambda}$.

Lemma 6.5 *Assume that $f \in C_b^2(\mathbb{R})$, $G \in C(\mathbb{T})$, and $b < 1$. Consider the implicit function problem*

$$\theta = \phi - b\lambda^{b-1}f(\phi)G(\tau), \tag{6.6}$$

to be solved for $\phi = \phi(\theta,\lambda;\tau)$. If $\Lambda > 0$ is such that

$$b\Lambda^{b-1}\|f\|_{C_b^2(\mathbb{R})}\|G\|_\infty < \frac{1}{2},$$

then ϕ is well-defined for $\lambda \ge \Lambda$ and can be expressed as

$$\phi = \theta + M(\theta,\lambda;\tau),$$

where $M \in \mathcal{F}_{b-1,\Lambda}$ satisfies

$$|M|_{b-1,\Lambda} \le C \tag{6.7}$$

for a constant $C > 0$ depending upon b and Λ.

Proof: For fixed $\lambda \ge \Lambda$ and $\tau \in \mathbb{R}$ the function $u(\phi) = \phi - b\lambda^{b-1}f(\phi)G(\tau)$ is such that $u'(\phi) \ge \frac{1}{2}$. Thus $u : \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, i.e., given $\theta \in \mathbb{R}$, $\lambda \ge \Lambda$, and $\tau \in \mathbb{R}$ there is a unique solution $\phi = \phi(\theta,\lambda;\tau)$ of class $C^{2,0}$ to the equation (6.6). In addition,

$$0 < \partial_\theta\phi \le 2.$$

To establish (6.7), we notice that

$$M(\theta,\lambda;\tau) = b\lambda^{b-1}f(\phi(\theta,\lambda;\tau))G(\tau).$$

First differentiate (6.6) twice w.r. to θ to obtain

$$\partial^2_{\theta\theta}\phi = \frac{b\lambda^{b-1}f''(\phi)G(\tau)}{1-b\lambda^{b-1}f'(\phi)G(\tau)}\,(\partial_\theta\phi)^2.$$

Therefore if $\lambda \geq \Lambda$, then

$$|\partial^2_{\theta\theta}\phi| \leq 8b\|f''\|_\infty \|G\|_\infty \lambda^{b-1},$$

and similarly

$$\partial_\lambda \phi = \mathcal{O}(\lambda^{b-2}), \quad \partial^2_{\theta\lambda}\phi = \mathcal{O}(\lambda^{b-2}), \quad \partial^2_{\lambda\lambda}\phi = \mathcal{O}(\lambda^{b-3}).$$

Since

$$|M|_{b-1,\Lambda} \leq C\|G\|_\infty |f \circ \phi|_{0,\Lambda} \leq C\|f\|_{C^2_b(\mathbb{R})}\|G\|_\infty (1 + |\phi|^2_{b-1,\Lambda})$$

by (6.1) and the case $i = 2$ from Lemma 6.1, we get (6.7), as $\phi = \mathcal{O}(\lambda^{b-1})$ is a consequence of $\partial_\lambda \phi = \mathcal{O}(\lambda^{b-2})$. □

6.2 The transformation theorem

Given $\mu > 0$ we define

$$\Sigma_\mu = (\mathbb{R} \times [\mu, \infty[) \times \mathbb{R}.$$

Definition 6.6 *An admissible change of variables is a map*

$$\mathcal{T} : \Sigma_\mu \to (\mathbb{R} \times]0, \infty[) \times \mathbb{R}, \quad (\phi, I; \tau) \mapsto (\theta, \lambda; \tau),$$

such that

(a) $\mathcal{T}$ is a C^1-diffeomorphism from Σ_μ onto $\mathcal{T}(\Sigma_\mu)$,

(b) $\mathcal{T}(\cdot, \cdot; \tau)$ is symplectic for all $\tau \in \mathbb{R}$, i.e., $d\theta \wedge d\lambda = d\phi \wedge dI$,

(c) $\mathcal{T}(\phi, I; \tau + 2\pi) = \mathcal{T}(\phi, I; \tau) + (0, 0; 2\pi)$,

(d) there are $\mu_2 > \mu_1 > 0$ so that $\Sigma_{\mu_2} \subset \mathcal{T}(\Sigma_\mu) \subset \Sigma_{\mu_1}$, and

(e) $I/2 \leq \lambda(\phi, I; \tau) \leq 2I$ for all $(\phi, I; \tau)$.

Let us emphasize that the independent variable τ is preserved by the transformation $\mathcal{T}$.

We are now in a position to state the main result of this section.

Theorem 6.7 *Consider a Hamiltonian of the form*

$$H(\phi, I; \tau) = \omega_0 I^a + f(\phi)g(\tau)I^b + R(\phi, I; \tau), \tag{6.8}$$

where $\omega_0 > 0$, $a < 1$, $b < 1$, $f \in C^4_b(\mathbb{R})$, and $g \in C(\mathbb{T})$ satisfies $\int_0^{2\pi} g(\tau)\, d\tau = 0$. Furthermore assume that $R \in \mathcal{F}_{c, I_0}$ for some $c \in \mathbb{R}$ and $I_0 > 0$. Then there exist $I_1 > I_0$, $\Lambda > 0$, and an admissible change of variables $\mathcal{T}$ defined on Σ_{I_1} such that $\mathcal{T}(\Sigma_{I_1}) \subset \Sigma_\Lambda$ and the system $\phi' = \partial_I H$, $I' = -\partial_\phi H$ is transformed into $\theta' = \partial_\lambda H_1$, $\lambda' = -\partial_\theta H_1$, where

$$H_1(\theta, \lambda; \tau) = \omega_0 \lambda^a + f_1(\theta)g_1(\tau)\lambda^{b-(1-a)} + R_1(\theta, \lambda; \tau). \tag{6.9}$$

The new functions appearing in H_1 satisfy

(a) $f_1(\theta) = -a\omega_0 f'(\theta)$,

(b) $g_1 \in C^1(\mathbb{T})$, $g_1'(\tau) = g(\tau)$, $\int_0^{2\pi} g_1(\tau)\,d\tau = 0$, *and*

(c) $R_1 \in \mathcal{F}_{c_1,\Lambda}$ *for* $c_1 = \max\{2b-1, c\}$.

The quantities I_1, Λ, *and* $|R_1|_{c_1,\Lambda}$ *can be estimated in terms of* ω_0, a, b, $\|f\|_{C_b^4(\mathbb{R})}$, $\|g\|_{C_b(\mathbb{R})}$, *and* $|R|_{c,I_0}$. *Furthermore, the following holds.*

(i) If $b \le 0$, *then the transformation* $\mathcal{T}$ *is bi-Lipschitzian, i.e., there is* $L > 0$ *such that*

$$L^{-1}\Big(|\hat\phi - \phi| + |\hat I - I|\Big) \le \|\mathcal{T}(\hat\phi, \hat I; \tau) - \mathcal{T}(\phi, I; \tau)\| \le L\Big(|\hat\phi - \phi| + |\hat I - I|\Big)$$

for $(\phi, I; \tau), (\hat\phi, \hat I; \tau) \in \Sigma_{I_1}$; *here* $\|\cdot\|$ *denotes any norm on* $\mathbb{R}^3$.

(ii) If g *is even, then the map* $\mathcal{T}(\cdot,\cdot;2\pi)$ *is the identity.*

Proof: The notation $C_1, C_2, \ldots, I_2, I_3, \ldots, \Lambda_1, \Lambda_2, \ldots$ will indicate quantities depending only upon ω_0, a, b, $\|f\|_{C_b^4(\mathbb{R})}$, $\|g\|_{C_b(\mathbb{R})}$, and $|R|_{c,I_0}$. The change of variables $\mathcal{T}$ is introduced by means of a generating function Ψ given by

$$\Psi(\phi, \lambda; \tau) = -\lambda^b f(\phi)\, g_1(\tau), \tag{6.10}$$

where g_1 is uniquely defined by the conditions in (b). Explicitly,

$$g_1(\tau) = \frac{1}{2\pi}\int_0^{2\pi} s g(s)\,ds + \int_0^\tau g(s)\,ds,$$

so that in particular $\|g_1\|_\infty \le (\pi/2)\|g\|_\infty$. Then

$$I = \lambda + \partial_\phi \Psi, \quad \theta = \phi + \partial_\lambda \Psi, \tag{6.11}$$

is the implicit definition of $\mathcal{T} : (\phi, I; \tau) \mapsto (\theta, \lambda; \tau)$. The rest of the proof is divided up into several steps. Step 1: $\mathcal{T}$ is well-defined. The first equation in (6.11) is

$$I = \lambda - \lambda^b f'(\phi)\, g_1(\tau).$$

The ideas from the proof of Lemma 6.5 then allow us to find a constant $I_2 > I_0$ such that the solution $\lambda = \lambda(\phi, I; \tau)$ is uniquely defined on Σ_{I_2}. In addition, λ is of class C^1 and 2π-periodic in τ. After increasing the size of I_2 we can also assume that

$$\frac{1}{2}\, I \le \lambda(\phi, I; \tau) \le 2I \quad \text{for} \quad I \ge I_2. \tag{6.12}$$

The second equation in (6.11) then says how to define the first coordinate of $\mathcal{T}$:

$$\theta(\phi, I; \tau) = \phi - b\lambda(\phi, I; \tau)^{b-1} f(\phi)\, g_1(\tau).$$

Summing up, we obtain a well-defined transformation $\mathcal{T}$ on Σ_{I_2} which is of class C^1 and 2π-periodic in τ. Step 2: $\mathcal{T}$ is admissible. The definition of $\mathcal{T}$ via a generating function implies that $\mathcal{T}(\cdot,\cdot;\tau)$ is symplectic for all $\tau \in \mathbb{R}$; this is a standard fact in Hamiltonian mechanics. In particular, $d\theta \wedge d\lambda = d\phi \wedge dI$ yields that $\mathcal{T}$ is a local diffeomorphism. We shall prove that $\mathcal{T}$ is one-to-one by constructing the inverse $\mathcal{S} = \mathcal{T}^{-1}$. By Lemma 6.5 we can find a constant $\Lambda_1 > 0$ and a solution map $\phi = \phi(\theta, \lambda; \tau)$ of (6.6) with $G = g_1$, i.e., the equation $\theta = \phi - b\lambda^{b-1} f(\phi) g_1(\tau)$, on Σ_{Λ_1}. This function ϕ is the first coordinate of $\mathcal{S}$ on Σ_{Λ_1}, the second being

$$I(\theta, \lambda; \tau) = \lambda - \lambda^b f'(\phi(\theta, \lambda; \tau))\, g_1(\tau). \tag{6.13}$$

From the existence of $\mathcal{S}$ on Σ_{Λ_1} and (6.12) we deduce that $\mathcal{T}$ is one-to-one on some region Σ_{I_3}, where $I_3 > I_2$ is chosen appropriately. To establish this claim, notice that if we take $I_3 > \max\{2\Lambda, I_2\}$, then $\mathcal{T}(\Sigma_{I_3}) \subset \Sigma_{\Lambda_1}$ and $\mathcal{S} \circ \mathcal{T} = \mathrm{id}$ on Σ_{I_3}. This completes the proof that $\mathcal{T}$ is admissible. Finally observe that $\mathcal{T}$ is admissible also in any region Σ_{I_1} for $I_1 > I_3$. For this, the only part that remains to be checked is (d). We already know that $\mathcal{T}(\Sigma_{I_1}) \subset \Sigma_{\Lambda_1}$. In view of (6.12) we can find $\Lambda_2 > \Lambda_1$ such that $I \geq I_1$, if $\lambda \geq \Lambda_2$; notice that Λ_2 depends on I_1. As a consequence, $\mathcal{S}(\Sigma_{\Lambda_2}) \subset \Sigma_{I_1}$ and $\mathcal{T} \circ \mathcal{S} = \mathrm{id}$ on Σ_{Λ_2}. Therefore $\Sigma_{\Lambda_2} = (\mathcal{T} \circ \mathcal{S})(\Sigma_{\Lambda_2}) \subset \mathcal{T}(\Sigma_{I_1})$, and thus (d) holds. Henceforth we will keep I_1 as a free parameter that will be fixed only at the end of the proof. Step 3: An expansion for $\mathcal{S}$. We assert that we can write

$$\mathcal{S}: \quad \begin{cases} \phi = \theta + M(\theta, \lambda; \tau) \\ I = \lambda + N(\theta, \lambda; \tau) \end{cases}$$

with $|M|_{b-1,\Lambda_3} + |N|_{b,\Lambda_3} \leq C_1$ for $\Lambda_3 \geq \Lambda_1$ so large that

$$b\Lambda_3^{b-1} \|f\|_{C_b^2(\mathbb{R})} \|g_1\|_\infty \leq \pi \Lambda_3^{b-1} \|f\|_{C_b^2(\mathbb{R})} \|g\|_\infty < \frac{1}{2}.$$

For this, the estimate on M is a consequence of (6.7) in Lemma 6.5. The function N can be expressed as

$$N(\theta, \lambda; \tau) = -\lambda^b f'(\theta + M(\theta, \lambda; \tau))\, g_1(\tau) \tag{6.14}$$

by (6.13). Then we may apply Lemma 6.1 for $g = f'$, $F = M \in \mathcal{F}_{b-1,\Lambda_3}$, and $i = 1$ to deduce that $N_1(\theta, \lambda; \tau) = f'(\theta + M(\theta, \lambda; \tau))$ satisfies $|N_1|_{0,\Lambda_3} \leq C_2$. Thus $|\lambda^b N_1|_{b,\Lambda_3} \leq C_3$ due to the multiplicative property (6.1), and hence the boundedness of g_1 yields $|N|_{b,\Lambda_3} \leq C_4$. Step 4: We have

$$f'(\phi) = f'(\theta) + Q(\theta, \lambda; \tau)$$

with $|Q|_{b-1,\Lambda_3} \leq C_5$; in this relation the independent variables are $(\theta, \lambda; \tau)$, and thus $\phi = \theta + M(\theta, \lambda; \tau)$ by the previous step. To establish the claim write

$$f'(\phi) - f'(\theta) = \int_0^1 \frac{d}{ds} f'(s\phi + (1-s)\theta)\, ds = D(\theta, \lambda; \tau) M(\theta, \lambda; \tau)$$

for

$$D(\theta, \lambda; \tau) = \int_0^1 f''(\theta + sM(\theta, \lambda; \tau))\, ds.$$

Since $f'' \in C_b^2(\mathbb{R})$ we can proceed as in the proof of Lemma 6.1 to deduce that $|D|_{0,\Lambda_3} \leq C_6$. Thus the bound on Q is a consequence of the multiplicative property (6.1) and the bound on M from the previous step. Step 5: We have

$$I^a = \lambda^a - a\lambda^{a+b-1} f'(\theta)\, g_1(\tau) + P_1(\theta, \lambda; \tau), \tag{6.15}$$

where $|P_1|_{2(b-1)+a,\Lambda_3} \leq C_7$. To show this we are going to apply Lemma 6.4 for $\sigma = a$. The definition of Λ_3 and (6.14) imply that (6.3) holds. Therefore

$$I^a = \lambda^a + a\lambda^{a-1} N(\theta, \lambda; \tau) + P_2(\theta, \lambda; \tau),$$

where $|P_2|_{2(b-1)+a,\,\Lambda_3} \le C_8$. Combining (6.14) with Step 4, we obtain the relation

$$P_1 = -a\lambda^{a+b-1}\, Q\, g_1(\tau) + P_2.$$

Thus using the multiplicative property (6.1) we arrive at (6.15). Step 6: We have

$$I^b = \lambda^b + P_3(\theta, \lambda; \tau),$$

where $|P_3|_{2b-1,\,\Lambda_3} \le C_9$. To establish this claim, we apply Lemma 6.4 for $\sigma = b$ and obtain

$$I^b = \lambda^b + b\lambda^{b-1} N(\theta, \lambda; \tau) + P_4(\theta, \lambda; \tau),$$

where $|P_4|_{2(b-1)+b,\,\Lambda_3} \le C_{10}$. Since $2(b-1)+b < 2b-1$, we see that $|P_4|_{2b-1,\,\Lambda_3} \le C_{11}$. The bound on $b\lambda^{b-1}N$ follows from Step 3. Step 7: Proof of (a)–(c). In view of the definition of g_1, it follows that (b) is satisfied. From the general theory of Hamiltonian systems we know that

$$H_1(\theta, \lambda; \tau) = H(\phi, I; \tau) + \partial_\tau \Psi(\phi, \lambda; \tau);$$

the presence of the term $\partial_\tau \Psi$ is due to the fact that the change of variables is time-dependent. Since

$$\partial_\tau \Psi(\phi, \lambda; \tau) = -\lambda^b f(\phi) g(\tau),$$

the previous steps allow us to write

$$H_1 = \omega_0 \lambda^a - a\omega_0 \lambda^{a+b-1} f'(\theta)\, g_1(\tau) + \omega_0 P_1 + f(\phi) g(\tau) P_3 + R \circ \mathcal{S},$$

which has the asserted form (6.9) for f_1 as in (a) and

$$R_1 = \omega_0 P_1 + f(\phi) g(\tau)\, P_3 + R \circ \mathcal{S}.$$

Observing that $2(b-1)+a < 2b-1 \le c_1$, we can derive a bound on $|P_1|_{c_1,\,\Lambda_3}$ from Step 5. From Lemma 6.1 it follows that $|f \circ \phi|_{0,\,\Lambda_3}$ is bounded, and using Step 6 we see that $|f(\phi)g(\tau)P_3|_{c_1,\,\Lambda_3}$ is bounded as well. It remains to obtain an estimate for $R \circ \mathcal{S}$, and this will be a consequence of Lemma 6.3. To use it we must decrease the size of the domain again. Let $\Lambda_4 \ge \Lambda_3$ be such that

$$\Lambda_4 \ge \mu + \Lambda_4^b\, |N|_{b,\,\Lambda_4}$$

for $\mu = \max\{I_0, \Lambda_3\}$; here we are taking into account the remark prior to Lemma 6.3. Then Lemma 6.3 applies to yield $|R \circ \mathcal{S}|_{c_1,\,\Lambda_4} \le C_{12} |R|_{c_1,\,\mu} \le C_{13} |R|_{c,\,I_0}$. To complete the argument it is therefore sufficient to adjust Λ and I_1; for this we can take $\Lambda = \Lambda_4$ and I_1 large enough so that $\mathcal{T}(\Sigma_{I_1}) \subset \Sigma_\Lambda$. Step 8: Proof of (i)–(ii). First suppose that in addition $b \le 0$. Then the functions M and N from Step 3 have bounded first order derivatives. This is due to the fact that

$$|\partial_\theta M| + |\partial_\theta N| \le |M|_{b-1,\,\Lambda} \lambda^{b-1} + |N|_{b,\,\Lambda} \lambda^b \le C_{14}$$

and

$$|\partial_\lambda M| + |\partial_\lambda N| \le |M|_{b-1,\,\Lambda} \lambda^{b-2} + |N|_{b,\,\Lambda} \lambda^{b-1} \le C_{15}$$

pointwise. This implies that $\|D\mathcal{S}\|_{L^\infty(\Sigma_\Lambda)}$ is finite, where $D\mathcal{S}$ denotes the Jacobian matrix of $\mathcal{S}$. Since $\mathcal{S}$ is symplectic, $\det D\mathcal{S} = 1$ everywhere, so that $D\mathcal{T} = (D\mathcal{S})^{-1} \circ \mathcal{T}$ is also bounded; notice that the inverse of the unity determinant matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is just $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Concerning (ii), if g is even, then g_1 is odd and hence $g_1(2\pi) = g_1(0) = 0$. From the definition of $\mathcal{T}$ it follows that $\mathcal{T}(\cdot, \cdot; 2\pi)$ is the identity. This completes the proof of the theorem. $\square$

7 Transformation of the Littlewood boundedness problem

In this section we consider the superlinear oscillator

$$\ddot{x} + |x|^{\alpha-1}x = p(t) \tag{7.1}$$

where $\alpha > 1$ and in general p will not be periodic. We are going to transform the equation by two changes of variables in order to rewrite it in a form which is appropriate for the application of the results derived previously. Each change of variables will be introduced in a separate subsection.

7.1 Action-angle variables

This is well-known and repeated for completeness only and to fix the notation. For every $\lambda > 0$ denote by x_λ the solution to $\ddot{x} + |x|^{\alpha-1}x = 0$ such that $x_\lambda(0) = \lambda$ and $\dot{x}_\lambda(0) = 0$. Notice that the orbits of $\ddot{x} + |x|^{\alpha-1}x = 0$ are the closed curves $\frac{1}{2}\,y^2 + \frac{1}{\alpha+1}\,|x|^{\alpha+1} = \text{const}$. By the homogeneity of the nonlinearity, $x_\lambda(t) = \lambda x_1(\lambda^{\frac{\alpha-1}{2}}t)$. Each x_λ is periodic with minimal period $T(\lambda) = \lambda^{\frac{1-\alpha}{2}}T(1)$. Let $\Lambda > 0$ be the unique λ such that $T(\Lambda) = 2\pi$ and write

$$c(t) = x_\Lambda(t), \quad s(t) = \dot{c}(t).$$

These functions have many similarities with the trigonometric functions. For instance, c is even, s is odd, and both are 2π-periodic. Also they are anti-periodic:

$$c(t+\pi) = -c(t), \quad s(t+\pi) = -s(t),$$

as follows from the uniqueness for the initial value problem. From this anti-periodicity furthermore

$$\int_0^{2\pi} c(t)\,dt = \int_0^{2\pi} s(t)\,dt = 0 \tag{7.2}$$

is obtained. Finally, by the conservation of energy,

$$\frac{1}{2}\,s(t)^2 + \frac{1}{\alpha+1}\,|c(t)|^{\alpha+1} = \frac{1}{\alpha+1}\,\Lambda^{\alpha+1}, \quad t \in \mathbb{R}. \tag{7.3}$$

Using the functions c and s, we change variables $(x,\dot{x}) \mapsto (\theta,\varrho)$ in (7.1) by

$$x = \gamma\,\varrho^{\frac{2}{\alpha+3}}\,c(\theta), \quad \dot{x} = \gamma^{\frac{\alpha+1}{2}}\,\varrho^{\frac{\alpha+1}{\alpha+3}}\,s(\theta), \tag{7.4}$$

where $\gamma > 0$ satisfies

$$\gamma^{\frac{\alpha+3}{2}}\Big(\frac{2}{\alpha+3}\Big)\Lambda^{\alpha+1} = 1.$$

It is straightforward to check that this transformation is a symplectic diffeomorphism from $\mathbb{R}^2\setminus\{0\}$ onto $\mathbb{T}\times]0,\infty[$. Furthermore, for every $(x,\dot{x}) \in \mathbb{R}^2\setminus\{0\}$,

$$\frac{1}{2}\,\dot{x}^2 + \frac{1}{\alpha+1}\,|x|^{\alpha+1} = \omega_1\varrho^{\frac{2(\alpha+1)}{\alpha+3}}, \quad \text{where} \quad \omega_1 = \frac{1}{\alpha+1}\,(\gamma\Lambda)^{\alpha+1}. \tag{7.5}$$

This is a consequence of (7.3). The Hamiltonian

$$\mathcal{H}(x,\dot{x};t) = \frac{1}{2}\,\dot{x}^2 + \frac{1}{\alpha+1}\,|x|^{\alpha+1} - p(t)x$$

is transformed into

$$\mathcal{H}(\theta,\varrho;t)=\omega_1\varrho^{\frac{2(\alpha+1)}{\alpha+3}}-\gamma\,\varrho^{\frac{2}{\alpha+3}}\,p(t)\,c(\theta), \tag{7.6}$$

and the corresponding equations of motion are

$$\dot\theta \;=\; \partial_\varrho\mathcal{H}=\frac{2(\alpha+1)}{\alpha+3}\,\omega_1\varrho^{\frac{\alpha-1}{\alpha+3}}-\frac{2\gamma}{\alpha+3}\,\varrho^{-\frac{\alpha+1}{\alpha+3}}\,p(t)\,c(\theta), \tag{7.7}$$

$$\dot\varrho \;=\; -\partial_\theta\mathcal{H}=\gamma\,\varrho^{\frac{2}{\alpha+3}}\,p(t)\,s(\theta). \tag{7.8}$$

The solutions to (7.1) are unique and do exist globally, the latter since

$$E(t;x)=\frac{1}{2}\,\dot x(t)^2+\frac{1}{\alpha+1}\,|x(t)|^{\alpha+1}=\omega_1\varrho(t)^{\frac{2(\alpha+1)}{\alpha+3}} \tag{7.9}$$

from (1.3) satisfies $|\dot E|=|p(t)\dot x|\le|p(t)|\sqrt{2E}$. However, the change to action-angle variables has introduced a singularity at the origin $x=\dot x=0$ corresponding to $\varrho=0$. Therefore we have to make sure that in the proofs below we do only consider solutions of (7.7) and (7.8) being defined on intervals $J\subset\mathbb{R}$ such that $\varrho(t)>0$ for $t\in J$. In this case we can undo the change of variables and obtain a solution of the original problem (7.1) defined on J.

7.2 Changing time and angle

The results from the previous sections concern systems that are periodic in the independent variable. Since we do not assume p to be periodic, the Hamiltonian $\mathcal{H}$ from (7.6) does not fit into this framework. To remedy this defect, we change the rôles of time and angle. We first describe the method in a general context. Consider a Hamiltonian system

$$\dot\theta=\partial_\varrho\mathcal{H},\quad \dot\varrho=-\partial_\theta\mathcal{H}, \tag{7.10}$$

where $\mathcal{H}=\mathcal{H}(\theta,\varrho;t)$ is a function of class C^k for some $k\ge 2$ which is defined on $\mathbb{T}\times]0,\infty[\times\mathbb{R}$. In addition, we assume that for some $\varrho_0>0$ and $\mu>0$,

$$\partial_\varrho\mathcal{H}(\theta,\varrho;t)\ge\mu\quad\text{for}\quad\varrho\ge\varrho_0. \tag{7.11}$$

Let $(\theta(t),\varrho(t))$ be a solution to (7.10) defined for $t\in J$ and such that $\varrho(t)>\varrho_0$ for all $t\in J$. Then the first equation in (7.10) implies that

$$\dot\theta(t)=\partial_\varrho\mathcal{H}(\theta(t),\varrho(t);t)\ge\mu\quad\text{for}\quad t\in J,$$

and hence the inverse of $t\mapsto\theta(t)$ is well-defined. Traditionally, this inverse function would be written as $\theta\mapsto t(\theta)$, but we will introduce new variables ϕ and τ to denote it by $\phi=\phi(\tau)$; so $\tau=\theta(t)$ and $t=\phi(\tau)$. This function ϕ is a C^k-diffeomorphism from $\tilde J=\theta(J)$ onto J. Next we consider the C^k-function

$$I(\tau)=\mathcal{H}(\tau,\varrho(\phi(\tau));\phi(\tau))\quad\text{for}\quad\tau\in\tilde J.$$

We intend to construct a Hamiltonian function $H=H(\phi,I;\tau)$ so that $(\phi(\tau),I(\tau))$ becomes a solution of

$$\phi'=\partial_I H,\quad I'=-\partial_\phi H, \tag{7.12}$$

where the prime $'$ indicates differentiation w.r. to τ. Less formally, the angle θ, now denoted by τ, has become the new independent variable, whereas the old time variable $t = \phi$ plays the rôle of the new angle.

In order to determine H we differentiate w.r. to τ and obtain

$$\begin{aligned} \phi' &= \dot{\theta}^{-1} = (\partial_\varrho \mathcal{H})^{-1}, \\ I' &= \partial_\theta \mathcal{H} + (\partial_\varrho \mathcal{H})\, \dot{\varrho}\, \phi' + (\partial_t \mathcal{H})\, \phi', \end{aligned} \tag{7.13}$$

where all partial derivatives are to be evaluated at $(\tau, \varrho(\phi(\tau)); \phi(\tau))$. From (7.13) and the second relation in (7.10) it follows that $\phi = \phi(\tau)$ and $I = I(\tau)$ satisfy

$$\phi' = (\partial_\varrho \mathcal{H})^{-1}, \quad I' = (\partial_\varrho \mathcal{H})^{-1}(\partial_t \mathcal{H}). \tag{7.14}$$

Now notice that the equation

$$\mathcal{H}(\theta, \varrho; t) = I$$

defines implicitly a solution function $\varrho = H(t, I; \theta)$ of class C^k. This function is well-defined on the open set

$$G = \{(t, I; \theta) \in \mathbb{R}^2 \times \mathbb{T} : I > \mathcal{H}(\theta, \varrho_0; t)\}$$

with ϱ_0 from (7.11). Then differentiating the relation $\mathcal{H}(\theta, H(t, I; \theta); t) = I$ we see that indeed (7.12) is verified for this Hamiltonian H.

To discuss more quantitative aspects we return to the particular Hamiltonian defined by (7.6). For $\varrho \to \infty$ the first term $\varrho^{\frac{2(\alpha+1)}{\alpha+3}}$ is dominant, and hence:

$$\text{(7.11) holds for appropriate } \varrho_0 > 0 \text{ and } \mu > 0, \tag{7.15}$$

depending only upon ω_1, α, γ, $\|p\|_{L^\infty(\mathbb{R})}$, and $\|c\|_{L^\infty(\mathbb{R})}$. Thus following the general recipe outline above to change time and angle variables we can define a new Hamiltonian $H = H(\phi, I; \tau)$ implicitly by solving

$$I = \omega_1 H^{\frac{2(\alpha+1)}{\alpha+3}} - \gamma\, H^{\frac{2}{\alpha+3}}\, p(\phi)\, c(\theta). \tag{7.16}$$

We will be assuming that p is of class C^6, and since c is of class C^3 at least, it follows that H is of class C^3 at least on G. For $\mathcal{H}$ from (7.6) the set G contains the region

$$\{(\phi, I; \theta) \in \mathbb{R}^2 \times \mathbb{T} : I \geq I_*\},$$

provided that I_* is fixed sufficiently large (depending upon the same parameters as ϱ_0, μ). At the same time (and with the same dependencies) we can suppose that I_* is so large that the solution $\varrho = H$ to (7.16) satisfies

$$\alpha_0\, I^{\frac{\alpha+3}{2(\alpha+1)}} \leq \varrho \leq \beta_0\, I^{\frac{\alpha+3}{2(\alpha+1)}}, \quad I \geq I_*, \tag{7.17}$$

for suitable constants $\alpha_0, \beta_0 > 0$. Let

$$\omega_0 = \omega_1^{-\frac{\alpha+3}{2(\alpha+1)}} = \Big(\frac{2(\alpha+1)}{\alpha+3}\, \gamma^{\frac{1-\alpha}{2}}\Big)^{\frac{\alpha+3}{2(\alpha+1)}} \tag{7.18}$$

and introduce R through the relation

$$H(\phi, I; \tau) = \omega_0\, I^{\frac{\alpha+3}{2(\alpha+1)}} + \frac{(\alpha+3)}{2(\alpha+1)}\, \gamma\, \omega_0^{\frac{\alpha+5}{\alpha+3}}\, I^{\frac{3-\alpha}{2(\alpha+1)}}\, p(\phi) c(\tau) + R(\phi, I; \tau). \tag{7.19}$$

The associated equations of motion are

$$\begin{aligned}\phi' &= \partial_I H = \omega_0 \frac{\alpha+3}{2(\alpha+1)} I^{\frac{1-\alpha}{2(\alpha+1)}} + \frac{9-\alpha^2}{4(\alpha+1)^2}\gamma\,\omega_0^{\frac{\alpha+5}{\alpha+3}} I^{\frac{1-3\alpha}{2(\alpha+1)}} p(\phi)c(\tau) + \partial_I R, &(7.20)\\ I' &= -\partial_\phi H = -\frac{(\alpha+3)}{2(\alpha+1)}\gamma\,\omega_0^{\frac{\alpha+5}{\alpha+3}} I^{\frac{3-\alpha}{2(\alpha+1)}} \dot p(\phi)c(\tau) - \partial_\phi R. &(7.21)\end{aligned}$$

The following lemma contains bounds on the remainder term R and on $\partial_\tau H$; its proof is deferred to an appendix, see Section 9.

Lemma 7.1 *There are constants $C_0 > 0$ and $I_0 \geq I_* > 0$ (depending upon $\|p\|_{C_b^2(\mathbb{R})}$) such that*

$$|R| + |\partial_\phi R| + I|\partial_I R| + |\partial^2_{\phi\phi} R| + I|\partial^2_{\phi I} R| + I^2|\partial^2_{II} R| \leq C_0 I^{\frac{3(1-\alpha)}{2(\alpha+1)}} \tag{7.22}$$

and

$$|\partial_\tau H| \leq C_0 I^{\frac{3-\alpha}{2(\alpha+1)}} \tag{7.23}$$

is verified for $I \geq I_0$.

Our strategy will be to obtain results for the system (7.20), (7.21) and then pass back the conclusions to the original system. Next we shall present two auxiliary results needed for this. The first one says, roughly, that bounded solutions to (7.20), (7.21) produce bounded solutions to (7.10) or (7.1). The second result tells us that if two solutions to (7.20), (7.21) are close and have large action, then they remain close when transported back to (7.10).

Lemma 7.2 *There exists $I_{**} > I_*$ (depending upon the same parameters) such that if $(\phi(\tau), I(\tau))$ is a solution of (7.20), (7.21) defined on $\mathbb{R}$ so that if*

$$I_{**} \leq \underline{I} \leq I(2\pi n) \leq \overline{I} \quad \textit{for} \quad n \in \mathbb{Z}, \tag{7.24}$$

then $\tau \mapsto \phi(\tau)$ has a global inverse $\phi^{-1} : \mathbb{R} \to \mathbb{R}$ and

$$\theta(t) = \phi^{-1}(t), \quad \varrho(t) = H(t, I(\phi^{-1}(t)); \phi^{-1}(t)), \tag{7.25}$$

is a solution of (7.10) defined on $\mathbb{R}$ satisfying

$$\alpha_1\, \underline{I}^{\,\frac{\alpha+3}{2(\alpha+1)}} \leq \varrho(t) \leq \beta_1\, \overline{I}^{\,\frac{\alpha+3}{2(\alpha+1)}} \quad \textit{for} \quad t \in \mathbb{R}, \tag{7.26}$$

where $\alpha_1, \beta_1 > 0$ depend upon the usual parameters.

Remark 7.3 Since ϱ is positive everywhere by (7.26), the relation $x(t) = \gamma\varrho(t)^{\frac{2}{\alpha+3}} c(\theta(t))$ from (7.4) will define a global solution to (7.1) of bounded energy; recall (7.9) and (7.5). In the application of Lemma 7.2 below we will have additional information on the solution $(\phi(\tau), I(\tau))$, namely

$$0 < \delta \leq \phi(2\pi(n+1)) - \phi(2\pi n) \leq \Delta \quad \text{for} \quad n \in \mathbb{Z},$$

for certain $\Delta > \delta > 0$. This translates into information concerning the oscillatory behavior of x. If we define $T_n = \phi(2\pi n)$, then $\delta \leq T_{n+1} - T_n \leq \Delta$ for $n \in \mathbb{Z}$, and the T_n are exactly the instants (in increasing order) where

$$x(T_n) > 0 \quad \text{and} \quad \dot x(T_n) = 0.$$

To justify the last assertion, notice that $s(\theta)$ vanishes if and only if $\theta = n\pi$ for some $n \in \mathbb{Z}$. Moreover, $c(n\pi) = (-1)^n$. Then the relations

$$x(t) = \gamma \varrho(t)^{\frac{2}{\alpha+3}}\, c(\theta(t)), \quad \dot{x}(t) = \gamma^{\frac{\alpha+1}{2}}\, \varrho(t)^{\frac{\alpha+1}{\alpha+3}}\, s(\theta(t)),$$

establish the claim. Later we will be interested in the zeros of $x(t)$. It is not hard to prove that the function $c(\theta)$ vanishes if and only if $\theta = \pi/2 + n\pi$ for some $n \in \mathbb{Z}$. Hence the zeros of $x(t)$ can be labelled in an increasing order as $t_n = \phi(\pi/2 + n\pi)$, $n \in \mathbb{Z}$. They satisfy

$$x(t_n) = 0, \quad (-1)^n \dot{x}(t_n) < 0, \quad \text{and} \quad T_n < t_{2n} < t_{2n+1} < T_{n+1}.$$

Proof of Lemma 7.2: If we rewrite H from (7.19) as $H(\phi, I; \tau) = \omega_0 I^a + R_1(\phi, I; \tau)$ for $a = \frac{\alpha+3}{2(\alpha+1)} \in]\frac{1}{2}, 1[$, then, due to Lemma 7.1, H satisfies (1.8) for $b = \frac{3-\alpha}{2(\alpha+1)}$. We notice that $a > b > a-1$, and hence Theorem 1.2 cannot be applied directly. However, we may invoke Lemma 4.1 to find $I_{**} > 4I_*$ such that (7.24) implies that

$$\frac{1}{4}\, \underline{I} \le I(\tau) \le 4\, \overline{I}, \quad \tau \in [2\pi n, 2\pi(n+1)]. \tag{7.27}$$

Once we know that $I(\tau)$ lies between $\frac{1}{4}\, \underline{I}$ and $4\, \overline{I}$, it follows that $I(\tau) \ge I_*$. Thus in view of the discussion preceding the lemma we can globally undo the change of variables to obtain a solution of (7.10) from (7.25). Finally by (7.17) and (7.27) we may take $\alpha_1 = 4^{-a}\alpha_0$ and $\beta_1 = 4^a \beta_0$. $\square$

Lemma 7.4 *Suppose that $\alpha \ge 3$. Then there exists $\hat{I}_{**} > I_*$ (depending upon the same parameters) such that if $(\phi_i(\tau), I_i(\tau))$ for $i = 1, 2$ are solutions of (7.20), (7.21) defined on a common interval $\tilde{J}$ of length $|\tilde{J}| > 2\pi$ which satisfy*

$$\hat{I}_{**} \le I_m \le I_1(2\pi n) \le I_M, \tag{7.28}$$

$$|\phi_1(2\pi n) - \phi_2(2\pi n)| + |I_1(2\pi n) - I_2(2\pi n)| \le \eta < 1, \tag{7.29}$$

for all $n \in \tilde{J} \cap 2\pi\mathbb{Z}$, then the associated solutions of (7.10) are defined on a common interval J so that

$$|J| \ge c_1 I_M^{\frac{1-\alpha}{2(\alpha+1)}} |\tilde{J}| - c_2 \eta \tag{7.30}$$

and

$$|\varrho_1(t) - \varrho_2(t)| \le c_3 \left(I_m^{\frac{1-\alpha}{2(\alpha+1)}} + I_m^{\frac{3-\alpha}{2(\alpha+1)}} I_M^{\frac{\alpha-1}{2(\alpha+1)}}\right) \eta \tag{7.31}$$

for $t \in J$, where $c_1, c_2, c_3 > 0$ depend upon the usual parameters.

Proof: If we take $\hat{I}_{**} > 2$, then $\eta < 1 \le I_m/2$, so that (7.28), (7.29) leads to

$$\frac{1}{2}\, I_m \le I_1(2\pi n) - \eta \le I_2(2\pi n) \le I_1(2\pi n) + \eta \le \frac{3}{2}\, I_M$$

for $n \in \tilde{J} \cap 2\pi\mathbb{Z}$. Then the same argument as in the proof of the previous lemma yields

$$\frac{1}{4}\, I_m \le I_1(\tau) \le 4 I_M \quad \text{and} \quad \frac{1}{8}\, I_m \le I_2(\tau) \le 6 I_M \tag{7.32}$$

for $\tau \in \tilde{J}$, where it may be necessary to increase $\hat{I}_{**}$ further. If we take $a = \frac{\alpha+3}{2(\alpha+1)} \in]\frac{1}{2}, 1[$ and $b = \frac{3-\alpha}{2(\alpha+1)}$ as in the proof of Lemma 7.2, then $b \le 0$ by assumption, and hence Lemma 4.3 applies. Thus replacing $\hat{I}_{**}$ by a larger quantity if needed, it follows from (4.4) and (7.29) that

$$|\phi_1(\tau) - \phi_2(\tau)| + |I_1(\tau) - I_2(\tau)| \le C_{\mathrm{stab}}\, \eta \tag{7.33}$$

for $\tau \in \tilde{J}$. From (7.20) and (7.32) we obtain for certain constants $c_4, c_5 > 0$ the estimate

$$c_4 I_M^{a-1} \le \phi_i'(\tau) \le c_5 I_m^{a-1}, \quad i = 1, 2, \tag{7.34}$$

for $\tau \in \tilde{J}$; recall that $a < 1$.

To estimate the length of the transformed interval J we write $\tilde{J} = [\tau_*, \tau^*]$, assuming without loss of generality that this interval is compact. The solutions $(\theta_i(t), \varrho_i(t))$ of (7.10) are defined for

$$t \in J_i = \phi_i(\tilde{J}) = [\phi_i(\tau_*), \phi_i(\tau^*)].$$

From (7.34) we deduce that

$$|J_i| = \phi_i(\tau^*) - \phi_i(\tau_*) \ge (\min_{\tilde{J}} \phi_i')(\tau^* - \tau_*) \ge c_4 I_M^{a-1}\, |\tilde{J}|, \quad i = 1, 2.$$

Also, by (7.33), the symmetric difference of J_1 and J_2 satisfies

$$|J_1 \Delta J_2| \le |\phi_1(\tau_*) - \phi_2(\tau_*)| + |\phi_1(\tau^*) - \phi_2(\tau^*)| \le 2C_{\mathrm{stab}}\, \eta.$$

As a consequence, the length of $J = J_1 \cap J_2$ can be bounded below by

$$|J| = |J_1 \cup J_2| - |J_1 \Delta J_2| \ge c_4 I_M^{a-1}\, |\tilde{J}| - 2C_{\mathrm{stab}}\, \eta,$$

which completes the proof of (7.30). Concerning (7.31), fix $t \in J$ and $\tau_1, \tau_2 \in \tilde{J}$ such that $t = \phi_1(\tau_1) = \phi_2(\tau_2)$. Then, by (7.33) and (7.34),

$$C_{\mathrm{stab}}\, \eta \ge |\phi_1(\tau_1) - \phi_2(\tau_1)| = |\phi_2(\tau_2) - \phi_2(\tau_1)| \ge c_4 I_M^{a-1} |\tau_2 - \tau_1|,$$

so that

$$|\tau_2 - \tau_1| \le c_5 I_M^{1-a}\, \eta \tag{7.35}$$

for $c_5 = c_4^{-1} C_{\mathrm{stab}}$. From (7.19), (7.22), and (7.23),

$$|\partial_I H| \le c_6 I^{a-1}, \quad |\partial_\phi H| + |\partial_\tau H| \le c_6 I^b. \tag{7.36}$$

Therefore by (7.21), (7.32), due to $b \le 0$, and (7.35),

$$|I_1(\tau_1) - I_1(\tau_2)| \le c_6 4^{-b} I_m^b\, |\tau_1 - \tau_2| \le c_7\, I_m^b\, I_M^{1-a}\, \eta$$

for $c_7 = 4^{-b}\, c_5\, c_6$. Hence (7.33) yields

$$|I_1(\tau_1) - I_2(\tau_2)| \le |I_1(\tau_1) - I_1(\tau_2)| + |I_1(\tau_2) - I_2(\tau_2)| \le (c_7\, I_m^b\, I_M^{1-a} + C_{\mathrm{stab}})\, \eta \le c_8 (I_m^b\, I_M^{1-a} + 1)\, \eta$$

for $c_8 = \max\{c_7, C_{\mathrm{stab}}\}$. Finally, using (7.25), (7.36), (7.32), and (7.35), this results in

$$\begin{aligned} |\varrho_1(t) - \varrho_2(t)| &= |H(t, I_1(\tau_1); \tau_1) - H(t, I_2(\tau_2); \tau_2)| \\ &\le c_6\, 8^{1-a} I_m^{a-1} |I_1(\tau_1) - I_2(\tau_2)| + c_6\, 8^{-b} I_m^b |\tau_2 - \tau_1| \\ &\le c_6\, 8^{1-a} c_8 (I_m^{a+b-1}\, I_M^{1-a} + I_m^{a-1})\, \eta + c_6\, 8^{-b} c_5\, I_m^b I_M^{1-a}\, \eta \\ &\le c_9 (I_m^{a+b-1}\, I_M^{1-a} + I_m^{a-1} + I_m^b I_M^{1-a})\, \eta. \end{aligned}$$

Since $a + b - 1 < b$, we obtain (7.31). $\square$

8 More transformations and proof of Theorem 1.1

Our starting point will be the Hamiltonian

$$H(\phi, I; \tau) = \omega_0 I^a + \omega_2\, p(\phi) c(\tau) I^b + R(\phi, I; \tau) \tag{8.1}$$

from (7.19), where

$$a = \frac{\alpha+3}{2(\alpha+1)}, \quad b = \frac{3-\alpha}{2(\alpha+1)}, \quad \text{and} \quad \omega_2 = \frac{(\alpha+3)}{2(\alpha+1)}\gamma\, \omega_0^{\frac{\alpha+5}{\alpha+3}}. \tag{8.2}$$

According to (7.22) we have $R \in \mathcal{F}_{c,\, I_0}$, where its growth rate is determined by the exponent

$$c = \frac{3(1-\alpha)}{2(\alpha+1)}.$$

The function $c(\tau)$ in (8.1) is 2π-periodic and has zero average; recall (7.2). This implies that it has a unique primitive $c_1(\tau)$ which is 2π-periodic and has zero average. More generally, let $c_n(\tau)$ be the unique solution to

$$\frac{d^n c_n}{d\tau^n}(\tau) = c(\tau), \quad c_n \text{ is } 2\pi-\text{periodic}, \quad \int_0^{2\pi} c_n(\tau)\, d\tau = 0.$$

Now we apply Theorem 6.7 three times to obtain a chain of symplectic changes of variables

$$(\phi, I; \tau) = (\theta_0, \lambda_0; \tau) \mapsto (\theta_1, \lambda_1; \tau) \mapsto (\theta_2, \lambda_2; \tau) \mapsto (\theta_3, \lambda_3; \tau) \tag{8.3}$$

such that at each stage and for large enough λ_k the Hamiltonian can be expressed as

$$H_k(\theta_k, \lambda_k; \tau) = \omega_0 \lambda_k^a + \omega_{k+2}\, p^{(k)}(\theta) c_k(\tau) \lambda_k^{b_k} + R_k(\theta_k, \lambda_k; \tau)$$

for certain constants ω_{k+2} and

$$b_0 = b, \quad b_k = b_{k-1} - (1-a).$$

Furthermore,

$$R_k \in \mathcal{F}_{c_k,\, \tilde{\lambda}_k} \tag{8.4}$$

for suitable (large enough) constants $C_k > 0$, $\tilde{\lambda}_k > 0$, and with $c_k = \max\{2b_{k-1} - 1, c_{k-1}\}$. Notice that the successive applications of Theorem 6.7 are admissible, since $p, \dot{p}, \ddot{p} \in C_b^4(\mathbb{R})$ and the functions $c_0(\tau) = c(\tau)$, $c_1(\tau)$, and $c_2(\tau)$ are 2π-periodic and have zero average. Moreover, $b_k \le b < 1$, with $b < 1$ being equivalent to $\alpha > 1$. The changes of variables $\mathcal{T}_k : (\theta_k, \lambda_k; \tau) \mapsto (\theta_{k+1}, \lambda_{k+1}; \tau)$ for $k = 0, 1, 2$ are admissible. Hence we have for the individual stages

$$\frac{\lambda_1}{2} \le I \le 2\lambda_1, \quad \frac{\lambda_2}{2} \le \lambda_1 \le 2\lambda_2, \quad \frac{\lambda_3}{2} \le \lambda_2 \le 2\lambda_3, \tag{8.5}$$

if $\lambda_1 \ge \tilde{\lambda}_1$, $\lambda_2 \ge \tilde{\lambda}_2$, $\lambda_3 \ge \tilde{\lambda}_3$, where $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 > 0$ may have to be increased further, if necessary. In particular,

$$\frac{1}{8} I \le \lambda_3 \le 8I \quad \text{for} \quad I \ge \tilde{I} \text{ or } \lambda_3 \ge \tilde{\lambda}_3, \tag{8.6}$$

provided that $\tilde{I} > 0$ and $\tilde{\lambda}_3 > 0$ are fixed sufficiently large. From Theorem 6.7(i) we also deduce that the $\mathcal{T}_k$ are bi-Lipschitzian, i.e., there are $L_k \geq 1$ and $\lambda_k^* > 0$ large such that

$$L_k^{-1}\Big(|\hat{\theta}_k - \theta_k| + |\hat{\lambda}_k - \lambda_k|\Big) \leq \|\mathcal{T}_k(\hat{\theta}_k, \hat{\lambda}_k; \tau) - \mathcal{T}_k(\theta_k, \lambda_k; \tau)\| \leq L_k\Big(|\hat{\theta}_k - \theta_k| + |\hat{\lambda}_k - \lambda_k|\Big) \tag{8.7}$$

for $\hat{\lambda}_k, \lambda_k \geq \lambda_k^*$; this is a consequence of $b_2 < 0$, $b_1 < 0$, and $b_0 \leq 0$ (notice that $b_0 = 0$ iff $\alpha = 3$).

If we continue to denote $E = \omega_1 \varrho^{\frac{2(\alpha+1)}{\alpha+3}}$, see (7.9), then also

$$\alpha_2 E \leq I \leq \beta_2 E \quad \text{for} \quad E \geq \tilde{E} \tag{8.8}$$

and with $\alpha_2 = \omega_1^{-1}\beta_0^{-\frac{2(\alpha+1)}{\alpha+3}}$ and $\beta_2 = \omega_1^{-1}\alpha_0^{-\frac{2(\alpha+1)}{\alpha+3}}$, if $\tilde{E}$ is sufficiently large; this is a consequence of (7.17).

Let us now point out why we have imposed the condition $p \in C_b^6(\mathbb{R})$. Theorem 6.7 could be applied for any b and c less than 1, but it is only of interest in the case where $b > c$, since otherwise the remainder R will dominate $f(\phi)g(\tau)I^b$ and an additional transformation step will yield no improvement. Thus if we were to assume $p \in C_b^7(\mathbb{R})$ for instance, then we can realize a further change of variables

$$(\theta_3, \lambda_3; \tau) \mapsto (\theta_4, \lambda_4; \tau),$$

but it would be of no use: we have

$$b_1 = \frac{2-\alpha}{\alpha+1}, \quad b_2 = \frac{5-3\alpha}{2(\alpha+1)}, \quad b_3 = \frac{3-2\alpha}{\alpha+1}, \quad c_1 = c_2 = c_3 = \frac{3(1-\alpha)}{2(\alpha+1)}, \tag{8.9}$$

and consequently $b_1 > c_1$ and $b_2 > c_2$, whereas $b_3 \leq c_3$ (and $b_3 = c_3$ iff $\alpha = 3$).

Proof of Theorem 1.1: Due to $b_3 \leq c_3 \leq a - 1$ it is possible to apply Theorem 4.7 for the system in the variables (θ_3, λ_3) and with the Hamiltonian H_3 written as $H_3 = \omega_0 \lambda_3^a + \tilde{R}_3$ defined on $G = \{\lambda_3 \geq \tilde{\lambda}_3\}$. Notice that $\tilde{R}_3$ satisfies the condition (1.8) with $b = c_3$; this is a consequence of (8.4) and $b_3 \leq c_3 \leq a - 1$. Let ε_*, a_1, A_1, A_2, A_3, and $\hat{\Gamma}$ be the constants given by Theorem 4.7 for this Hamiltonian. The rôle of I_{**} in Theorem 4.7 is played by a constant obtained from the application of Lemma 4.1 and denoted by λ_{3**}. Next we are going to define the function $\varepsilon^* = \varepsilon^*(\rho)$, whose existence is asserted in Theorem 1.1. First let

$$\bar{\rho} = \max\Big\{2\rho,\, \omega_1 \varrho_0^{\frac{2(\alpha+1)}{\alpha+3}},\, \alpha_2^{-1}\tilde{I},\, \tilde{E},\, 8\alpha_2^{-1}\max\{\lambda_{3**}, \tilde{\lambda}_3, \lambda_3^*, 2\tilde{\lambda}_2, 2\lambda_2^*, 4\lambda_1^*\}\Big\}$$

with ϱ_0 from (7.15), $\tilde{I}$ from (8.6), $\alpha_2, \beta_2, \tilde{E}$ from (8.8), $\tilde{\lambda}_2, \tilde{\lambda}_3$ from (8.6), (8.5), and $\lambda_1^*, \lambda_2^*, \lambda_3^*$ from (8.7). Let $\varepsilon^*(\rho)$ be a positive number satisfying

$$\varepsilon^*(\rho) < \varepsilon_*, \quad 8\beta_2\bar{\rho} < \frac{1}{4}\,\varepsilon^*(\rho)^{-\frac{1}{1-a}},$$

and furthermore

$$\varepsilon^*(\rho) \leq \min\Big\{\tilde{\lambda}_3^{-1}a_1,\, (\lambda_3^*)^{-1}a_1,\, (2\tilde{\lambda}_2)^{-1}a_1,\, (2\lambda_2^*)^{-1}a_1,\, (4\lambda_1^*)^{-1}a_1,\, (8I_{**})^{-1}a_1,\, (8\hat{I}_{**})^{-1}a_1\Big\}^{\frac{\alpha-1}{2(\alpha+1)}} \tag{8.10}$$

and

$$\varepsilon^*(\rho) \leq \frac{1}{2L_0L_1L_2},$$

with a_1 from Theorem 4.7, I_{**} from Lemma 7.2, $\hat{I}_{**}$ from Lemma 7.4, and L_0, L_1, L_2 from (8.7). Let $x = x(t)$ be a solution of (1.2) such that

$$\limsup_{t\to\infty} E(t;x) = \infty \quad \text{and} \quad \inf_{t\geq t_0} E(t;x) < \rho$$

holds for some $t_0 \in \mathbb{R}$. Fix $\varepsilon \in]0, \varepsilon^*(\rho)]$. Next select $t_1 \geq t_0$ such that $E(t_1;x) < \rho$ and take $r = r(\varepsilon)$ satisfying

$$\frac{1}{4}\,\alpha_2 r > 4\hat{\Gamma}\varepsilon^{-\frac{1}{1-a}}.$$

Then, by assumption, we find $t_r > t_1$ so that $E(t_r;x) > 2r$. Since $\rho < \bar{\rho} < 2r$ there is $t_r^* \in [t_1, t_r[$ with the property that

$$E(t_r^*;x) = \bar{\rho} \quad \text{and} \quad E(t;x) > \bar{\rho} \text{ for } t \in]t_r^*, t_r]. \tag{8.11}$$

We start the transformations by changing to action-angle variables $(x, \dot{x}) \mapsto (\theta, \varrho)$ as described in Section 7.1, writing the Hamiltonian in the form $\mathcal{H}(\theta, \varrho; t)$ given in (7.6). Since $E(t;x) \geq \bar{\rho} > 0$ for $t \in [t_r^*, t_r]$, (7.9) implies that ϱ stays away from zero during this time interval, i.e., the change of variables does not introduce a singularity at the origin. Now notice that (8.11) and $E = \omega_1 \varrho^{\frac{2(\alpha+1)}{\alpha+3}}$ yields $\varrho(t) > \varrho_0$ for $t \in [t_r^*, t_r]$. Therefore (7.11) is verified along the solution and we can change time and angle as described in Section 7.2. This leads to the Hamiltonian $H(\phi, I; \tau)$ from (7.19) and a solution $(\phi(\tau), I(\tau))$ of (7.20), (7.21) defined on the τ-interval $\tilde{J} = \phi([t_r^*, t_r]) =: [\tau_r^*, \tau_r]$. Thereafter we introduce the three further changes of variables $\mathcal{T}_k$ from (8.3), as outlined above. Then by (8.8),

$$I(\tau) \geq \alpha_2 E(\phi^{-1}(\tau); x) \geq \alpha_2 \bar{\rho} > \tilde{I} \quad \text{for} \quad \tau \in \tilde{J}, \tag{8.12}$$

and hence

$$\frac{1}{8}\,I(\tau) \leq \lambda_3(\tau) \leq 8I(\tau) \quad \text{for} \quad \tau \in \tilde{J} \tag{8.13}$$

by (8.6). Similarly,

$$I(\tau_r^*) \leq \beta_2 E(t_r^*;x) = \beta_2 \bar{\rho} \quad \text{and} \quad I(\tau_r) \geq \alpha_2 E(t_r;x) \geq 2\alpha_2 r,$$

which due to (8.13) yields

$$\lambda_{3**} \leq \inf_{\tau\in\tilde{J}} \lambda_3(\tau) \leq \lambda_3(\tau_r^*) \leq 8\beta_2\bar{\rho} < \frac{1}{4}\,\varepsilon^{-\frac{1}{1-a}}$$

and

$$\sup_{\tau\in\tilde{J}} \lambda_3(\tau) \geq \lambda_3(\tau_r) \geq \frac{1}{4}\,\alpha_2 r > 4\hat{\Gamma}\,\varepsilon^{-\frac{1}{1-a}}.$$

Hence it follows from Theorem 4.7 that there are a (globally in τ defined) solution $(\theta_3^\varepsilon(\tau), \lambda_3^\varepsilon(\tau))$ of the system associated to H_3 and a time $\tau_\varepsilon \in \tilde{J}$ so that $\{0 \leq \tau - \tau_\varepsilon \leq A_3\,\varepsilon^{-3}\} \subset \tilde{J}$,

$$a_1\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \leq \lambda_3^\varepsilon(\tau) \leq A_1\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \quad \text{for} \quad \tau \in \mathbb{R}, \tag{8.14}$$

and

$$|\theta_3^\varepsilon(\tau) - \theta_3(\tau)| + |\lambda_3^\varepsilon(\tau) - \lambda_3(\tau)| \leq A_2\,\varepsilon \quad \text{for} \quad 0 \leq \tau - \tau_\varepsilon \leq A_3\,\varepsilon^{-3}; \tag{8.15}$$

notice that $-\frac{1}{1-a} = \frac{2(\alpha+1)}{1-\alpha}$, $\sigma = \min\{2-a, -c_3\} = -c_3 = \frac{3(\alpha-1)}{2(\alpha+1)}$, and $-\frac{\sigma}{1-a} = -3$ by (8.2) and (8.9). In order to transfer (8.14) and (8.15) back to obtain the desired solution x^ε of (1.2), we first undo the three transformations involved in (8.3) to get a solution $(\phi^\varepsilon(\tau), I^\varepsilon(\tau))$ of (7.20), (7.21) which is defined for all $\tau \in \mathbb{R}$. Then (8.14) in conjunction with $\varepsilon \le \varepsilon^*(\rho)$ and (8.10) implies that $\lambda_3^\varepsilon(\tau) \ge \max\{\tilde{\lambda}_3, \lambda_3^*, 2\tilde{\lambda}_2, 2\lambda_2^*, 4\lambda_1^*\} \ge \tilde{\lambda}_3$ for $\tau \in \mathbb{R}$. This together with (8.6) and (8.14) yields the bound

$$\frac{a_1}{8}\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \le I^\varepsilon(\tau) \le 8A_1\,\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}} \quad \text{for} \quad \tau \in \mathbb{R}. \tag{8.16}$$

As observed before, all three changes of variables $\mathcal{T}_k$ are bi-Lipschitzian; see (8.7). Then, since $\lambda_3^\varepsilon(\tau) \ge \lambda_3^*$ for $\tau \in \mathbb{R}$ and

$$\lambda_3(\tau) \ge I(\tau)/8 \ge (\alpha_2/8)\bar{\rho} \ge \max\{\tilde{\lambda}_3, \lambda_3^*, 2\tilde{\lambda}_2, 2\lambda_2^*, 4\lambda_1^*\} \ge \lambda_3^*$$

for $\tau \in \tilde{J}$ by (8.13) and (8.12), we see that (8.7) applies for $k = 2$ and on $\{0 \le \tau - \tau_\varepsilon \le A_3\,\varepsilon^{-3}\}$. Next, we already derived that $\lambda_3^\varepsilon(\tau) \ge \tilde{\lambda}_3$ for $\tau \in \mathbb{R}$ and $\lambda_3(\tau) \ge \tilde{\lambda}_3$ for $\tau \in \tilde{J}$. Thus (8.5) yields $\lambda_2^\varepsilon(\tau) \ge \lambda_3^\varepsilon(\tau)/2 \ge \lambda_2^*$ for $\tau \in \mathbb{R}$ as well as $\lambda_2(\tau) \ge \lambda_3(\tau)/2 \ge \lambda_2^*$ for $\tau \in \tilde{J}$. Therefore (8.7) can also be used for $k = 1$ on $\{0 \le \tau - \tau_\varepsilon \le A_3\,\varepsilon^{-3}\}$. Last and final, $\lambda_2^\varepsilon(\tau) \ge \lambda_3^\varepsilon(\tau)/2 \ge \max\{\tilde{\lambda}_2, 2\lambda_1^*\}$ for $\tau \in \mathbb{R}$ and $\lambda_2(\tau) \ge \lambda_3(\tau)/2 \ge \max\{\tilde{\lambda}_2, 2\lambda_1^*\}$ for $\tau \in \tilde{J}$ implies that $\lambda_1^\varepsilon(\tau) \ge \lambda_2^\varepsilon(\tau)/2 \ge \lambda_1^*$ for $\tau \in \mathbb{R}$ and $\lambda_1(\tau) \ge \lambda_2(\tau)/2 \ge \lambda_1^*$ for $\tau \in \tilde{J}$, by (8.5). Hence (8.7) can be applied for $k = 0$ as well, on $\{0 \le \tau - \tau_\varepsilon \le A_3\,\varepsilon^{-3}\}$. In summary, we obtain

$$|\phi^\varepsilon(\tau) - \phi(\tau)| + |I^\varepsilon(\tau) - I(\tau)| \le L_0L_1L_2A_2\,\varepsilon \quad \text{for} \quad 0 \le \tau - \tau_\varepsilon \le A_3\,\varepsilon^{-3}$$

from (8.15) and (8.7). In particular, we arrive at

$$|\phi^\varepsilon(2\pi n) - \phi(2\pi n)| + |I^\varepsilon(2\pi n) - I(2\pi n)| \le \tilde{A}_2\,\varepsilon \tag{8.17}$$

for those $n \in \mathbb{Z}$ such that $0 \le 2\pi n - \tau_\varepsilon \le A_3\,\varepsilon^{-3}$, with the constant $\tilde{A}_2 = L_0L_1L_2$. Next we are going to apply Lemmas 7.2 and 7.4 for $(\phi^\varepsilon(\tau), I^\varepsilon(\tau))$, $\underline{I} = I_m = (a_1/8)\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}}$, and $\overline{I} = I_M = 8A_1\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}}$. In view of (8.10) we have $\underline{I} \ge I_{**}$ and $I_m \ge \hat{I}_{**}$, i.e., from (8.16) we see that (7.24) and (7.28) are verified. Due to Lemma 7.2 we find a solution $(\theta^\varepsilon(t), \varrho^\varepsilon(t))$ of (7.10) defined on $\mathbb{R}$ and satisfying

$$\tilde{\alpha}_1\,\varepsilon^{-\frac{\alpha+3}{\alpha-1}} \le \varrho^\varepsilon(t) \le \tilde{\beta}_1\,\varepsilon^{-\frac{\alpha+3}{\alpha-1}} \quad \text{for} \quad t \in \mathbb{R}, \tag{8.18}$$

where $\tilde{\alpha}_1 = \alpha_1(a_1/8)^{\frac{\alpha+3}{2(\alpha+1)}}$ and $\tilde{\beta}_1 = \beta_1(8A_1)^{\frac{\alpha+3}{2(\alpha+1)}}$. Since $\varrho^\varepsilon(t)$ stays away from zero, we can now define $x^\varepsilon(t) = \gamma\varrho^\varepsilon(t)^{\frac{2}{\alpha+3}}\,c(\theta^\varepsilon(t))$ according to (7.4) to obtain a solution of (1.2). Recalling $E(t; x^\varepsilon) = \omega_1\varrho^\varepsilon(t)^{\frac{2(\alpha+1)}{\alpha+3}}$, it follows from (8.18) that (1.4) holds. Concerning (1.5), we take $\eta = \tilde{A}_2\varepsilon < 1$ in (7.29). Then, by (7.30) and (7.31), on a time interval J of length

$$|J| \ge c_1 I_M^{\frac{1-\alpha}{2(\alpha+1)}} \cdot 2A_3\varepsilon^{-3} - c_2\eta = c_1(8A_1\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}})^{\frac{1-\alpha}{2(\alpha+1)}} \cdot 2A_3\varepsilon^{-3} - c_2\tilde{A}_2\varepsilon \ge \tilde{c}_1\varepsilon^{-2}$$

we get

$$\begin{aligned}
|\varrho^\varepsilon(t) - \varrho(t)| &\le c_3(I_m^{\frac{1-\alpha}{2(\alpha+1)}} + I_m^{\frac{3-\alpha}{2(\alpha+1)}} I_M^{\frac{\alpha-1}{2(\alpha+1)}})\,\eta \\
&= c_3\Big(((a_1/8)\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}})^{\frac{1-\alpha}{2(\alpha+1)}} + ((a_1/8)\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}})^{\frac{3-\alpha}{2(\alpha+1)}}(8A_1\varepsilon^{-\frac{2(\alpha+1)}{\alpha-1}})^{\frac{\alpha-1}{2(\alpha+1)}}\Big)\tilde{A}_2\varepsilon \\
&\le \tilde{c}_3(\varepsilon + \varepsilon^{-\frac{2}{\alpha-1}})\,\varepsilon \le \hat{c}_3\,\varepsilon^{\frac{\alpha-3}{\alpha-1}}
\end{aligned}$$

for $t \in J$. Thus if we let t_ε denote the midpoint of J, then we obtain (1.5). Concerning the last assertion, we notice that the sequence of increasing zeros of $x(t)$ and $x^\varepsilon(t)$ lying in J are the $t_n = \phi(\pi/2 + n\pi)$ and $t_n^\varepsilon = \phi^\varepsilon(\pi/2 + n\pi)$ such that $\pi/2 + n\pi \in J$; see Remark 7.3. Then write

$$\begin{aligned} t_n^\varepsilon - t_n &= \phi^\varepsilon(\pi/2 + n\pi) - \phi(\pi/2 + n\pi) \\ &= \phi^\varepsilon(\pi/2 + n\pi) - \phi^\varepsilon(2N\pi) + \phi^\varepsilon(2N\pi) - \phi(2N\pi) + \phi(2N\pi) - \phi(\pi/2 + n\pi) \end{aligned}$$

for N chosen such that T_N is closest. Using (8.17), $\phi^\varepsilon(\pi/2 + n\pi) - \phi^\varepsilon(2N\pi) \sim (\phi^\varepsilon)' \sim \varepsilon$, and $\phi(2N\pi) - \phi(\pi/2 + n\pi) \sim \phi' \sim \varepsilon$, we finally arrive at $|t_n^\varepsilon - t_n| \le A_4\,\varepsilon$ for $\pi/2 + n\pi \in J$, provided that $A_4 > 0$ is fixed large enough. □

9 Appendix I: Proof of Lemma 7.1

To verify (7.22) it will be understood that henceforth all assertions and constants $C = C(K)$ are uniform in p such that $\|p\|_{C_b^2(\mathbb{R})} \le K$. First the defining relation (7.16) is rewritten as

$$I = \omega_0^{-\frac{2(\alpha+1)}{\alpha+3}}\, \varrho^{\frac{2(\alpha+1)}{\alpha+3}} \Big(1 - \omega_0^{\frac{2(\alpha+1)}{\alpha+3}}\,\gamma\,\varrho^{-\frac{2\alpha}{\alpha+3}}\,p(\phi)\,c(\tau)\Big) \tag{9.1}$$

for $\varrho = \varrho(\phi, I; \tau) = H$. Our first task will be to prove that the function ϱ belongs to $\mathcal{F}_{p,\,I_0}$ for $p = \frac{\alpha+3}{2(\alpha+1)}$ and I_0 large enough. From (7.17) we deduce that, for large I,

$$\alpha_0 I^p \le \varrho \le \beta_0 I^p, \tag{9.2}$$

and so $I^{-p}|\varrho(\phi, I; \tau)|$ is bounded by β_0. Concerning the bound on $\partial_I\varrho$, note first that differentiation of (9.1) w.r. to I yields

$$\partial_I\varrho = \Big(\frac{2(\alpha+1)}{\alpha+3}\,\omega_0^{-\frac{2(\alpha+1)}{\alpha+3}} - \gamma\,\frac{2}{\alpha+3}\,\varrho^{-\frac{2\alpha}{\alpha+3}}\,p(\phi)\,c(\tau)\Big)^{-1} \varrho^{\frac{1-\alpha}{\alpha+3}}, \tag{9.3}$$

so that $I|\partial_I\varrho| \le CI^p$. Here we are using (9.2), and also that I, and hence ϱ, is very large. For the bound on $\partial_\phi\varrho$ the argument is similar. Taking the derivative of (9.1) w.r. to ϕ leads to

$$\partial_\phi\varrho = \gamma\Big[\frac{2(\alpha+1)}{\alpha+3}\,\omega_0^{-\frac{2(\alpha+1)}{\alpha+3}} - \gamma\,\frac{2}{\alpha+3}\,\varrho^{-\frac{2\alpha}{\alpha+3}}\,p(\phi)\,c(\tau)\Big]^{-1} p'(\phi)\,c(\tau)\,\varrho^{\frac{3-\alpha}{\alpha+3}},$$

and hence $|\partial_\phi\varrho| \le CI^{\frac{3-\alpha}{2(\alpha+1)}} \le CI^p$. Next we turn to the second derivatives of ϱ. By (9.3),

$$\begin{aligned} \partial_{II}^2\varrho &= \Big(\frac{1-\alpha}{\alpha+3}\Big)\Big(\frac{2(\alpha+1)}{\alpha+3}\,\omega_0^{-\frac{2(\alpha+1)}{\alpha+3}} - \gamma\,\frac{2}{\alpha+3}\,\varrho^{-\frac{2\alpha}{\alpha+3}}\,p(\phi)\,c(\tau)\Big)^{-1} \varrho^{-\frac{2(\alpha+1)}{\alpha+3}}(\partial_I\varrho) \\ &\quad - \gamma\,\frac{4\alpha}{(\alpha+3)^2}\Big(\frac{2(\alpha+1)}{\alpha+3}\,\omega_0^{-\frac{2(\alpha+1)}{\alpha+3}} - \gamma\,\frac{2}{\alpha+3}\,\varrho^{-\frac{2\alpha}{\alpha+3}}\,p(\phi)\,c(\tau)\Big)^{-2} p(\phi)\,c(\tau)\,\varrho^{-\frac{2(1+2\alpha)}{\alpha+3}}(\partial_I\varrho). \end{aligned}$$

Therefore $\varrho = \mathcal{O}(I^{\frac{\alpha+3}{2(\alpha+1)}})$ and $|\partial_I\varrho| \le CI^{\frac{1-\alpha}{2(\alpha+1)}}$ leads to

$$|\partial_{II}^2\varrho| \le C\varrho^{-\frac{2(\alpha+1)}{\alpha+3}}\,|\partial_I\varrho| + C\varrho^{-\frac{2(1+2\alpha)}{\alpha+3}}\,|\partial_I\varrho| \le CI^{-\frac{(1+3\alpha)}{2(\alpha+1)}},$$

so that $I^2|\partial_{II}^2\varrho| \le CI^p$. The bounds on $\partial_{\phi I}^2\varrho$ and $\partial_{\phi\phi}^2\varrho$ can be derived in a similar manner. This establishes the claim that $\varrho \in \mathcal{F}_{p,\,I_0}$ for I_0 sufficiently large.

Next, using (7.19), (9.1), and the general Taylor expansion (6.5) for

$$\sigma_1 = \frac{\alpha+3}{2(\alpha+1)}, \quad \sigma_2 = \frac{3-\alpha}{2(\alpha+1)}, \quad \varepsilon = -\omega_0^{\frac{2(\alpha+1)}{\alpha+3}} \gamma\, \varrho^{-\frac{2\alpha}{\alpha+3}}\, p(\phi)\, c(\tau),$$

it follows that

$$\begin{aligned} R(\phi, I; \tau) &= \varrho - \omega_0\, I^{\frac{\alpha+3}{2(\alpha+1)}} - \frac{(\alpha+3)}{2(\alpha+1)} \gamma\, \omega_0^{\frac{\alpha+5}{\alpha+3}}\, I^{\frac{3-\alpha}{2(\alpha+1)}}\, p(\phi) c(\tau) \\ &= \varrho - \varrho\Big(1 - \omega_0^{\frac{2(\alpha+1)}{\alpha+3}} \gamma\, \varrho^{-\frac{2\alpha}{\alpha+3}}\, p(\phi)\, c(\tau)\Big)^{\frac{\alpha+3}{2(\alpha+1)}} \\ &\quad - \gamma^{\frac{3-\alpha}{2}}\, \varrho^{\frac{3-\alpha}{(\alpha+3)}} \Big(1 - \omega_0^{\frac{2(\alpha+1)}{(\alpha+3)}} \gamma\, \varrho^{-\frac{2\alpha}{(\alpha+3)}}\, p(\phi)\, c(\tau)\Big)^{\frac{3-\alpha}{2(\alpha+1)}} p(\phi) c(\tau) \\ &= \varrho - \varrho\Big(1 + \sigma_1\varepsilon + \sigma_1(\sigma_1 - 1)\varepsilon^2 Z_1(\varepsilon)\Big) \\ &\quad - \gamma^{\frac{3-\alpha}{2}}\, \varrho^{\frac{3-\alpha}{(\alpha+3)}} \Big(1 + \sigma_2\varepsilon + \sigma_2(\sigma_2 - 1)\varepsilon^2 Z_2(\varepsilon)\Big) p(\phi) c(\tau) \end{aligned}$$

for

$$Z_j(\varepsilon) = \int_0^1 (1-s)(1+\varepsilon s)^{\sigma_j - 2}\, ds, \quad j = 1, 2.$$

Here, recalling the identity (7.18), some cancellations occur, and we obtain

$$\begin{aligned} R(\phi, I; \tau) &= -\sigma_1(\sigma_1 - 1)\varrho\, \varepsilon^2 Z_1(\varepsilon) \\ &\quad - \gamma^{\frac{3-\alpha}{2}}\, \varrho^{\frac{3-\alpha}{(\alpha+3)}} \Big(\sigma_2\varepsilon + \sigma_2(\sigma_2 - 1)\varepsilon^2 Z_2(\varepsilon)\Big) p(\phi) c(\tau). \end{aligned} \tag{9.4}$$

Next observe that $\varepsilon = \varepsilon(\phi, I; \tau) \in \mathcal{F}_{q, I_0}$ for $q = -\frac{\alpha}{\alpha+1}$; this is a consequence of (9.2) and $\varrho \in \mathcal{F}_{p, I_0}$. From Lemma 6.1 we thus deduce that $Z_1 \circ \varepsilon \in \mathcal{F}_{0, I_0}$, and similarly we get $Z_2 \circ \varepsilon \in \mathcal{F}_{0, I_0}$. Now it is easy to prove (7.22): From $\varrho \in \mathcal{F}_{p, I_0}$, $\varepsilon^2 \in \mathcal{F}_{2q, I_0}$, and $Z_1 \circ \varepsilon \in \mathcal{F}_{0, I_0}$ it follows that

$$-\sigma_1(\sigma_1 - 1)\varrho\, \varepsilon^2 Z_1(\varepsilon) \in \mathcal{F}_{p+2q, I_0},$$

and furthermore $p + 2q = \frac{3(1-\alpha)}{2(\alpha+1)}$ is precisely the exponent which shows up on the right-hand side of (7.22). The remaining terms in (9.4) can be treated in an analogous way, so that we have verified (7.22).

To establish (7.23), we differentiate (7.16), i.e.,

$$I = \omega_1 H^{\frac{2(\alpha+1)}{\alpha+3}} - \gamma\, H^{\frac{2}{\alpha+3}}\, p(\phi)\, c(\tau),$$

w.r.t. to τ and obtain

$$\Big[\omega_1 \frac{2(\alpha+1)}{\alpha+3} H^{\frac{\alpha-1}{\alpha+3}} - \gamma \frac{2}{\alpha+3} H^{-\frac{\alpha+1}{\alpha+3}}\, p(\phi)\, c(\tau)\Big] (\partial_\tau H) = \gamma\, H^{\frac{2}{\alpha+3}}\, p(\phi)\, \dot{c}(\tau).$$

Therefore (7.23) follows from $H = \varrho = \mathcal{O}(I^{\frac{\alpha+3}{2(\alpha+1)}})$, and the proof to Lemma 7.1 is complete. □

10 Appendix II: Some more comments on the proof

1. Theorem 4 in [12] can be applied to equation (1.2) with $\alpha = 3$. This result implies in particular the existence of a forcing $p \in C_b^6(\mathbb{R})$ such that

$$\ddot{x} + x^3 = p(t) \tag{10.5}$$

has an unbounded solution $x(t)$ satisfying

$$E(t;x) \sim t^{4/15} \quad \text{as} \quad t \to \infty.$$

As far as we know it is an open problem to decide whether the exponent 4/15 is sharp. Our results seem unrelated to this question, since the numbers t_ε could go to infinity very slowly.

2. The estimate (1.5) says that the solutions $x^\varepsilon(t)$ and $x(t)$ are close; here 'closeness' is measured in terms of the action, defined by the autonomous equation. Perhaps more refined estimates could be obtained assuming more regularity for $p(t)$ and using successive corrections of the adiabatic invariant. This kind of approach can be found in [4].

3. The averaging principle as well as the theory of adiabatic invariants can be applied in several ways to systems (1.2) and (1.9). In any case, these methods alone do not seem to be sufficient to yield a complete proof of our results. In essence, there are two steps in the proof:

(i) unbounded solutions are close to bounded solutions at some instant of time, and

(ii) solutions with close initial data remain close for long periods of time.

Certainly step (ii) is in the scope of perturbation theory, but step (i) seems to be of more topological nature; notice the crucial rôle played by Lemma 3.5 in our proof. To elaborate somewhat further on this point, consider $\alpha = 3$ and hence (10.5). Defining $y(t) = \varepsilon x(\varepsilon t)$ for a small parameter $\varepsilon > 0$, equation (10.5) is transformed into

$$\ddot{y} + y^3 = \varepsilon^3 p(\varepsilon t).$$

Adopting the framework developed in [1], we introduce the change of variables

$$s = \varepsilon t, \quad y = \gamma R^{1/3} c(\theta), \quad \dot{y} = \gamma^2 R^{2/3} s(\theta).$$

Then the equations becomes

$$\begin{aligned}
\frac{d\theta}{ds} &= \beta_1 R^{-1/3} - \beta_2\, \varepsilon^3 R^{-2/3} p(\lambda) c(\theta), \\
\frac{dR}{ds} &= \beta_3\, \varepsilon^3 R^{1/3} p(\lambda) s(\theta), \\
\frac{d\lambda}{ds} &= \varepsilon,
\end{aligned}$$

where β_1, β_2, and β_3 can be specified in terms of (7.7), (7.8). Assuming that $R(0) = 1$, the second equation leads to the estimate

$$|R(s) - R(s_0)| \leq C\varepsilon^3 |s - s_0|,$$

valid on any s-interval where for instance $\frac{1}{2} \leq R(s) \leq 2$. Undoing the change of variables and using $R = \varepsilon^3 \varrho$, we obtain

$$|\varrho(t) - \varrho(t_0)| \leq C\varepsilon t, \tag{10.6}$$

where $\varrho(t)$ denotes the action associated to $x(t)$. Let us assume that step (i) was already accomplished and that we know in addition that there is a bounded solution $x^\varepsilon(t)$ with $\varrho^\varepsilon(t) \sim \varepsilon^{-3}$ and such that $\varrho^\varepsilon(t_\varepsilon)$ and $\varrho(t_\varepsilon)$ are close. Then the estimate (10.6) can be applied to both solutions, which leads to

$$\varrho(t) - \varrho^\varepsilon(t) = \varrho(t) - \varrho(t_\varepsilon) + \varrho(t_\varepsilon) - \varrho^\varepsilon(t_\varepsilon) + \varrho^\varepsilon(t_\varepsilon) - \varrho^\varepsilon(t) = \mathcal{O}(1)$$

on an interval of the type $|t - t_\varepsilon| \le C\varepsilon^{-1}$. Most likely the previous argument can be refined using the averaging principle, since $s(\theta)$ has zero average. This could lead to the improved bound

$$\varrho(t) - \varrho^\varepsilon(t) = \mathcal{O}(1) \quad \text{for} \quad |t - t_\varepsilon| \le C\varepsilon^{-2},$$

given by our result. We have proved the existence of $x^\varepsilon(t)$ such that $\varrho^\varepsilon(t) \sim \varepsilon^{-3}$ and $\varrho^\varepsilon(t_\varepsilon) \sim \varrho(t_\varepsilon)$ by a combination of variational and topological techniques.

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