Invariant manifolds around equilibria of Newtonian equations: some pathological examples*

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Abstract

Let the equation $\ddot{x} = f(t,x)$ be periodic in time, and let the equilibrium $x_* \equiv 0$ be a periodic minimizer. If it is hyperbolic, then the set of asymptotic solutions is a smooth curve in the plane; this is stated by the Stable Manifold Theorem. The result can be extended to nonhyperbolic minimizers provided only that they are isolated and the equation is analytic [5]. In this paper we provide an example showing that one cannot say the same for $\mathcal{C}^2$ equations. Our example is pathological both in a global sense (the global stable manifold is not arcwise connected), and in a local sense (the local stable manifolds have points which are not accessible from the exterior and are not locally connected).

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1 Introduction

Consider the Newtonian equation

$$\ddot{x} = f(t,x)\,, \tag{1}$$

where the force $f = f(t,x)$ is defined on the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ and smooth. Throughout this paper, we shall always assume that f is *bounded* (so that solutions do not explode in finite time), and

$$f(t,0) = 0\,, \qquad t \in \mathbb{R}/\mathbb{Z}\,,$$

so that $x_* \equiv 0$ is an equilibrium (it will be referred to as *the trivial equilibrium*). The force f (or equation (1), or the trivial equilibrium) will be said to be *repulsive* if

$$f(t,-x) < 0 < f(t,x)\,, \qquad t \in \mathbb{R}/\mathbb{Z},\ x > 0\,. \tag{2}$$

It follows from this assumption that $\partial_x f(\cdot,0) \geq 0$ on $\mathbb{R}/\mathbb{Z}$. If $\partial_x f(\cdot,0) \not\equiv 0$, then multiplication by ξ and integration by parts in the left side of the equation shows that all eigenvalues λ of the periodic eigenvalue problem

$$\ddot{\xi} = \big(\partial_x f(t,0) - \lambda\big)\xi\,, \qquad \xi(0) = \xi(1),\ \dot{\xi}(0) = \dot{\xi}(1)\,, \tag{3}$$

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are strictly positive. Under this condition, the trivial equilibrium $x_* \equiv 0$ is hyperbolic; this is a well-known result which may be traced back to Liapounoff [2]. The Stable Manifold Theorem (see, for instance, [3]), then implies that the associated (global) stable manifold

$$W^S(x_*) := \left\{ (x(0), \dot{x}(0)) \text{ such that } x : [0, +\infty[\to \mathbb{R} \text{ solves (1) and } \lim_{t \to +\infty} (x(t), \dot{x}(t)) = (0,0) \right\},$$

is an injectively immersed curve in the plane.

However, this analysis is not valid anymore if $\partial_x f(\cdot, 0) \equiv 0$; it is, for instance the case of the Duffing equation $\ddot{x} = x^3$. In such situations, the first eigenvalue of (3) is $\lambda = 0$, so that the trivial equilibrium is *parabolic*, and the usual formulations of the Stable Manifold Theorem do not apply. Yet, our repulsive assumption (2) implies that the trivial equilibrium is an *isolated minimizer* of the periodic action functional (the potential $V(t,x) := -\int_0^x f(t,y)dy$ attains its maximum at $x = 0$ for any value of t). In this framework it has been recently shown [5] that, at least when the force f is *analytic* in x, there is a topological version of the Stable Manifold Theorem which states that the stable manifold $W^S(x_*)$ is an injectively immersed topological curve in the plane. It motivates the question of whether this analyticity assumption is actually necessary or, on the contrary, the above-mentioned result could be extended to repulsive forces, say, of class $\mathcal{C}^{0,1}$. In this paper we answer to this question by constructing an example of a repulsive force $f = f(t,x)$ of class $\mathcal{C}^{0,2}$ for which $W^S(x_*)$ fails to be a curve:

Theorem 1.1. *There exists a repulsive force $f : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^{0,2}$, such that the stable manifold $W^S(x_*)$ of the trivial equilibrium $x_* \equiv 0$ is not arcwise connected.*

Let us have a second look now to the Stable Manifold Theorem [3]. We observe that the global stable manifold $W^S(x_*)$ being an immersed curve is just a particular consequence of this result, which, in its full strength, states the existence a basis of neighborhoods of the origin such that, for any element B of that basis, the associated local stable manifold $W^S_B(x_*)$ of the trivial equilibrium $x_* \equiv 0$,

$$W^S_B(x_*) := \left\{ (x(0), \dot{x}(0)) \text{ such that } x : [0, +\infty[\to \mathbb{R} \text{ solves (1) and } (x(n), \dot{x}(n)) \in B \ \forall n \geq 0 \right\},$$

is a smooth curve in B. This holds if one assumes that the trivial equilibrium is hyperbolic, but also, -after replacing 'smooth curve' by 'topological curve'-, in the parabolic case provided that the repulsive force f is analytic; this is the main result of [5]. Of course, the stable manifold associated to the equation of Theorem 1.1 must be pathological also in a local sense, because, if $W^S_{B_0}(x_*)$ were arcwise connected for some small neighborhood B_0 of the origin, then the global stable manifold $W^S(x_*)$, which may be expressed as the union of all past iterates of $W^S_{B_0}(x_*)$ by the associated Poincaré mapping, would have the same property. However, looking only at Theorem 1.1 one might still wonder if perhaps all the nonsmoothness of the pathological stable manifold found in Theorem 1.1 is concentrated at the fixed point $(0,0)$. Thus, we shall show that our counterexample is strange also in the sense that for any bounded neighborhood B of the origin, the local stable manifold W^S_B has many points which are *not accessible* from the exterior, i.e., points $p \in W^S_B(x_*)$ for which there are not continuous curves $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma([0,1[) \subset \mathbb{R}^2 \backslash W^S_B(x_*)$ and $\gamma(1) = p$. We remark that the validity of this result does not contradict the fact that, as it will follow from the combination of Proposition 5.1 and item 4. of Lemma 3.2, all local stable manifolds $W_B^S(x_*)$ have empty interior. A topological argument will subsequently be used to conclude that these local stable manifolds W_B^S are not even locally connected:

Theorem 1.2. *There exists a repulsive force $f : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^{0,2}$ such that, for any bounded neighborhood B of the origin, the local stable manifold $W_B^S(x_*)$ is not locally connected and contains points $p \neq (0,0)$ which are not accessible from $\mathbb{R}^2 \backslash W_B^S(x_*)$.*

This paper is distributed as follows. We begin by Section 2, which is devoted to give an intuitive overview of some of the key arguments of the paper. The next Sections 3,4 and 5 are devoted to proof the Stable Manifold Theorem for analytic minimizers in the special case of repulsive equilibria. On the other hand, Sections 6 and 7 are devoted to obtain the pathologies for a certain equation of class $\mathcal{C}^{0,2}$ whose existence was announced in Section 2. The actual construction of this equation is the aim of Sections 16 and 6, while the Appendix is devoted to obtain some properties of the winding number which were crucial in Section 6.

Before closing this Introduction, I want to express my gratitude to R. Ortega. His indications have been crucial in Section 7 and Theorem 1.2, as well as with the references.

I am also pleased to thank Prof. Ch. Pommerenke, who kindly answered to my questions on planar topology.

2 Towards the proofs

In this Section we introduce the main ideas of this work. The starting point will be a planar set $W \subset \mathbb{R}^2$ where we have distinguished three different points O, p, q. The parameterized curve $\gamma : [0,1] \to \mathbb{R}^2$ is said to *pass through* z if $z \in \gamma([0,1])$.

Lemma 2.1. *Assume that:*

($\mathbf{H}_1$) *Every continuous curve $\gamma_p : [0,1] \to W$ passing though O and p also passes through q, while every continuous curve $\gamma_q : [0,1] \to W$ passing though O and q also passes through p.*

Then, no continuous curve $\gamma_p : [0,1] \to W$ passing through O and p does exist; in particular, W is not arcwise connected.

This elementary result will be the basis of our proof of Theorem 1.1, for we shall construct a force f for which the stable manifold $W^S(x_*) = W$ contains three points O, p, q verifying **($\mathbf{H}_1$)**. The precise details will be given later, but the main idea may be introduced in a few lines now. It consists in building f in such a way that equation (1) has two solutions x_1, x_2 which are asymptotic to the trivial equilibrium $x_* \equiv 0$ as $t \to +\infty$, while crossing with each other infinitely many times:

Proposition 2.2. *There exists a bounded and repulsive force $f : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^{0,2}$ such that (1) has two different positive solutions $x_1, x_2 : [0, +\infty[\to \mathbb{R}$ with*

(i) $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$,

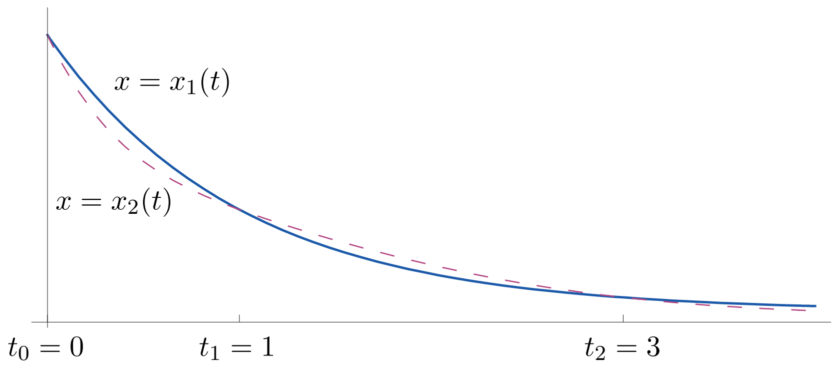


Figure 1: The graphs of x_1 and x_2.

(ii) x_1 *and* x_2 *coincide on infinitely many points* $0 = t_0 < t_1 < t_2 < \ldots \to +\infty$.

Observe that, the force f being repulsive, the positive solutions x_1, x_2 must be convex, and yet, it does not contradict the fact that they intersect infinitely many times on $[0, +\infty[$. Of course, uniqueness of solutions to initial value problems means that all these crossing points must be transversal. The reasoning will be completed in Section 6, where we shall show that assumption $(\mathbf{H}_1)$ holds for the set $W = W^S(x_*)$ and the points $O = (0,0)$, $p = (x_1(0), \dot{x}_1(0))$, $q = (x_2(0), \dot{x}_2(0))$.

We do not know whether the regularity of the function f of Proposition 2.2 may be improved. The question has some importance, as any improvement in that direction would immediately yield more regular examples for Theorems 1.1 and 1.2. On the other hand, f cannot be analytic, as, in this case, Proposition 4.2 of [5] states that two solutions cannot intersect twice as long as they remain near to the trivial equilibrium x_*.

The same force f of Proposition 2.2 will be a suitable example for Theorem 1.2. This result will follow from the preliminary version below:

Proposition 2.3. *Let* f *be as in Proposition 2.2. Then, the (global) stable manifold of the trivial equilibrium* $W^S(x_*)$*, contains a point* $p \neq (0,0)$ *which is not accessible from the exterior.*

A key role in our argumentations will be played by the so-called winding number $w_a^b(\gamma, q)$ of the continuous curve $\gamma : [a,b] \to \mathbb{R}^2$ around the base point q, which is defined (see, for instance, [1]) as follows:

$$w_a^b(\gamma, q) := \frac{\theta_\gamma(b) - \theta_\gamma(a)}{2\pi},$$

the function $\theta_\gamma : [a,b] \to \mathbb{R}$ being any continuous determination of the (multivalued) argument function along $\gamma - q$. This winding number is defined provided that $q \notin \gamma([a,b])$, and, since we are *not* assuming γ to be closed, it may take any real value and not necessarily an integer. As it is well known, it does not depend on the choice of the lifting θ_γ, and remains invariant under orientation-preserving reparametrizations of γ.

There is a further property of the winding number function, which also follows immediately from its definition, and concerns its continuous dependence with respect to the endpoints a, b of the interval where it is measured. In Lemma 2.4 below, the continuous curve $\gamma : [a, b] \to \mathbb{R}^2$ and the base point $q \in \mathbb{R}^2 \backslash \gamma([a, b])$ are assumed to be fixed:

Lemma 2.4. *For any sequences $a_n \searrow a$ and $b_n \nearrow b$, $w_{a_n}^{b_n}(\gamma, q) \to w_a^b(\gamma, q)$.*

Notice that, in some particular cases, this result may be used to prove that a given point p of the subset $W \subset \mathbb{R}^2$ is not accessible from the exterior of W. For instance, it will happen provided that there is another point $q \in W$, $q \neq p$, with the following property:

($\mathbf{H}_2$) For any third point $r_0 \in \mathbb{R}^2 \backslash W$ and any natural number N there exists some $\epsilon > 0$ such that any continuous curve $\gamma : [0, 1] \to \mathbb{R}^2 \backslash W$ with $\gamma(0) = r_0$ and $|\gamma(1) - p| < \epsilon$ verifies that $|w_0^1(\gamma, q)| \geq N$.

To illustrate this assumption we observe that it is verified, for instance, by the set W composed by some segment $[p, q]$ together with an spiral winding around it, see Fig. 2 below:

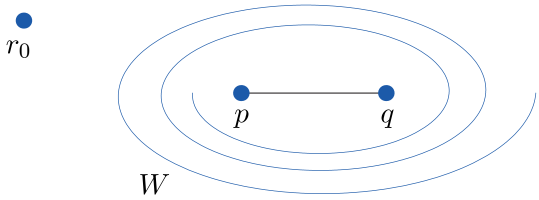


Figure 2: In order to approach p from r_0 one has to rotate many times around q.

Corollary 2.5. *Assume* **($\mathbf{H}_2$)**. *Then, p is not accessible from the exterior of W.*

The proof of this result is an easy consequence of Lemma 2.4, for if there were a continuous curve $\gamma : [0, 1] \to \mathbb{R}^2$ with $\gamma(1) = p$ and $\gamma([0, 1[) \subset \mathbb{R}^2 \backslash W$, the winding number of γ around q should be infinite. We shall see that, if f is chosen as in Proposition 2.2, the stable manifold $W^S(x_*) = W$ and the points $p = (x_1(0), \dot{x}_1(0))$, $q = (x_2(0), \dot{x}_2(0))$ verify **($\mathbf{H}_2$)**, thus leading us to Proposition 2.3. This work will be postponed to Section 6, since, at this moment, we are going to concentrate ourselves on the precise implications that analyticity has on the structure of the stable manifold associated to a repulsive equilibrium.

3 The role of analyticity

The main contribution of [5] is an stable/unstable manifold Theorem for isolated minimizers when the equation is analytic. Although elementary, the proof provided there combines several arguments and is therefore somewhat lengthy. However, in the particular case in which the minimizer is the trivial equilibrium of a repulsive potential, the upper and lower solutions argument which was employed in the general case is no longer required, and the reasoning may be considerably simplified. We devote the next three sections to this proof, on the one hand motivated by our belief that it has some interest on its own, on the other because it will shed some light on how our example has been built, and finally, also because it contains some aspects which will be needed subsequently. Thus, we shall prove the following

Proposition 3.1. *Let the repulsive force $f : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ have class $\mathcal{C}^{0,\omega}$. Then there exists a basis of neighborhoods of the origin such that, for any neighborhood B of that basis, the associated local stable manifold $W^S_B(x_*)$ of the trivial equilibrium $x_* \equiv 0$ may be described as the graph $v_0 = v(x_0)$ of a continuous function v defined on some open interval containing $x_0 = 0$.*

Our proof of Proposition 3.1 will be based on five facts, which concern to the dynamics of (not necessarily analytic) repulsive equations. From these properties, which are presented below, the first three refer to the structure of the global stable manifold $W^S(x_*)$ of the trivial equilibrium, while the last two point out to the connections between this set and the local stable manifolds $W^S_B(x_*)$ associated to different neighborhoods B of the origin.

Lemma 3.2. *Let the $\mathcal{C}^{0,1}$ force $f : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ be repulsive. Then:*

1. *For each $x_0 \in \mathbb{R}$ there exists at least one $v_0 \in \mathbb{R}$ such that $(x_0, v_0) \in W^S(x_*)$.*
2. *If $\partial_x f \geq 0$ on some band $(\mathbb{R}/\mathbb{Z}) \times]-\delta, \delta[$, then for any $x_0 \in]-\delta, \delta[$ there exists exactly one $v_0 \in \mathbb{R}$ such that $(x_0, v_0) \in W^S(x_*)$.*
3. *$W^S(x_*)$ is closed in $\mathbb{R}^2$.*
4. *$W^S_B(x_*) \subset W^S(x_*)$ for any bounded neighborhood B of the origin.*
5. *For any $\delta > 0$, the vertical band $B_\delta =]-\delta, \delta[\times \mathbb{R}$ verifies that $W^S_{B_\delta}(x_*) = W^S(x_*) \cap B_\delta$.*

Proof of Proposition 3.1 modulus Lemma 3.2: Observe, to start, that the force f being $\mathcal{C}^{0,\omega}$ and repulsive, there must exist some small band $(\mathbb{R}/\mathbb{Z}) \times]-\delta_0, \delta_0[$ where $\partial_x f \geq 0$ (this is the only moment of the proof of Proposition 3.1 where the analyticity of f will be used). Thus, item 2. of Lemma 3.2 implies the existence of some function $v :]-\delta_0, \delta_0[\to \mathbb{R}$ such that

$$W^S(x_*) \cap B_{\delta_0} = \big\{(x_0, v(x_0)) : |x_0| < \delta_0\big\},$$

and, in view of 5., we obtain

$$W^S_{B_\delta}(x_*) = \big\{(x_0, v(x_0)) : |x_0| < \delta\big\}, \qquad 0 < \delta \leq \delta_0 . \tag{4}$$

We further deduce from 3. that the graph of v is closed on the band B_{δ_0}, and thus, v is continuous. In particular, $v(x_0) \to v(0) = 0$ as $x_0 \to 0$. It means that the family $\{C_\delta\}_{0<\delta\leq\delta_0}$ of subsets of the plane defined by

$$C_\delta := \{(x_0, v_0) \in \mathbb{R}^2 : |x_0| < \delta,\ |v_0| < \max_{[-\delta,\delta]} |v| + \delta\}, \qquad 0 < \delta \leq \delta_0 ,$$

is a basis of neighborhoods of the origin. Now, since $C_\delta \subset B_\delta$, we deduce that $W^S_{C_\delta}(x_*) \subset W^S_{B_\delta}(x_*)$, while the inclusion $W^S_{B_\delta}(x_*) \subset C_\delta$, which follows from (4) and the choice of C_δ, implies that both local stable manifolds coincide. Consequently,

$$W^S_{C_\delta}(x_*) = W^S_{B_\delta}(x_*) = \{(x_0, v(x_0)) : |x_0| < \delta\},$$

for any $0 < \delta \leq \delta_0$. The proof is complete. □

The argumentation above gives some insight into the choice of the force f of Proposition 2.2 in our search of a pathological stable manifold. Because, as already observed, the only stage in the proof of Proposition 3.1 where analyticity is used consists in claiming the existence of some $\delta_0 > 0$ such that $\partial_x f \geq 0$ on $(\mathbb{R}/\mathbb{Z})\times]-\delta_0, \delta_0[$. Thus, if our pathological example is to exist, the stable manifold $W^S(x_*)$ cannot be a graph of the position, not even locally near the origin and, furthermore, $\partial_x f$ must oscillate infinitely many times around zero near the line $\{x = 0\}$.

We still have to prove Lemma 3.2; this will be done in Section 5. With this goal, it will be convenient to start with a classification of the solutions of (1) attending to their qualitative behavior.

4 Classifying solutions of repulsive equations

Our repulsive assumption (2) may be interpreted mechanically. In fact, we can think of (1) as modeling the motion, under the influence of a constant gravity force of intensity 1, of a particle of mass 1 which is subjected to glide on the slope of the pulsing mountain contoured by the graph of the potential $V(t,x) = -\int_0^x f(t,y)dy$. This mountain changes its shape as the time goes on, but repulsiveness means that, at any time, it has its only peak at $x_* = 0$. Consequently, solutions should not have negative local minima nor positive local maxima.

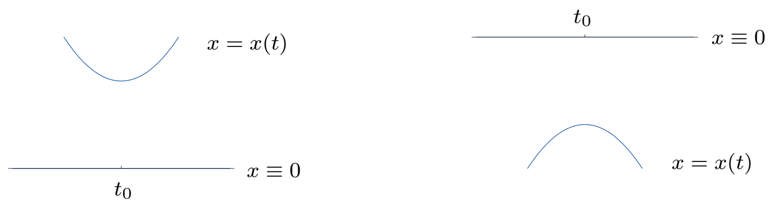


Figure 3: The behavior of a solution near a critical point t_0

Let us check now that this intuition is accurate. With this aim, let $x \not\equiv 0$ be a solution of our repulsive equation (1). Then, $\ddot{x}(t) = f(t, x(t))$ has the same sign as $x(t)$ at each time $t \in \mathbb{R}$. As a consequence, at a critical point t_0, the solution x attains either a strict local maximum or a strict local minimum depending, respectively, on whether it is negative or positive there (the nontrivial solution x cannot vanish at a critical point by uniqueness).

As a consequence, x has at most one critical point t_0 in $\mathbb{R}$. Should such a point exist, one of the two following possibilities must hold:

$(--)$ $x < 0$, $\dot{x} > 0$ on $]-\infty, t_0[$, $\dot{x} < 0$ on $]t_0, +\infty[$, and $\lim_{t\to\pm\infty} x(t) = -\infty$.

$(++)$ $x > 0$, $\dot{x} < 0$ on $]-\infty, t_0[$, $\dot{x} > 0$ on $]t_0, +\infty[$, and $\lim_{t\to\pm\infty} x(t) = +\infty$.

The only part in the classification above which does not completely follow from the previous comments is the statement concerning the limits at $\pm\infty$ of x. However, having just a critical point, x must be eventually monotonous, so that the mentioned limits do exist in the extended real line $[-\infty, +\infty]$. Should one of these limits be finite, it must be a zero of $f(t, \cdot)$ for any $t \in \mathbb{R}$, and the repulsive condition that f verifies implies that this limit must

Figure 4: Possible behaviors of solutions having some (unique) critical point t_0

be zero. But it cannot be the case if x has just a critical point t_0 where it attains a positive minimum or a negative maximum.

This argument may be repeated when the solution x has no critical points in $\mathbb{R}$ to deduce that the limits at $\pm\infty$ of x, if finite, must vanish. For instance, should x be increasing, one of the following three possibilities must hold:

$(-+)$ $\dot x > 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = -\infty$, $\lim_{t\to+\infty} x(t) = +\infty$,

(-0) $\dot x > 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = -\infty$, $\lim_{t\to+\infty} x(t) = 0$,

$(0+)$ $\dot x > 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to+\infty} x(t) = +\infty$.

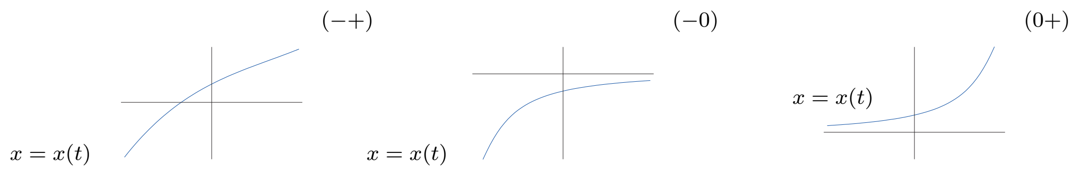


Figure 5: Possible behaviors of increasing solutions

Then, we have also the three analogous possibilities for decreasing solutions:

$(+-)$ $\dot x < 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = +\infty$, $\lim_{t\to+\infty} x(t) = -\infty$,

$(0-)$ $\dot x < 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to+\infty} x(t) = -\infty$,

$(+0)$ $\dot x > 0$ on $\mathbb{R}$, $\lim_{t\to-\infty} x(t) = +\infty$, $\lim_{t\to+\infty} x(t) = 0$.

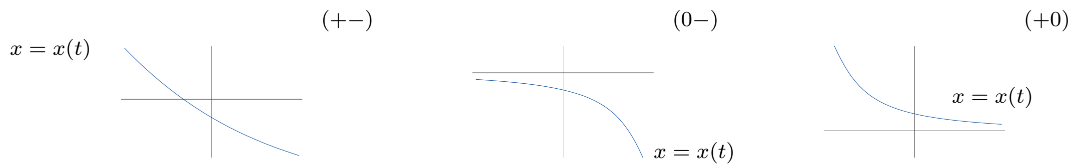


Figure 6: Possible behaviors of decreasing solutions

In this way, we have obtained a classification of the solutions of repulsive equations.

Proposition 4.1. *Let $x \not\equiv 0$ be a nontrivial solution of the repulsive equation (1). Then, there is one of the eight possibilities listed above which holds.*

We emphasize the following immediate consequence:

Corollary 4.2. *Let the solution $x = x(t)$ of the repulsive equation (1) have limit* 0 *as $t \to +\infty$. Then, x is monotonous.*

Observe that Proposition 4.1 may be seen as a particular case of Proposition 3.1 of [5], where a related result was established in the more general framework of (not necessarily repulsive) periodic minimizers. Also, the reader may find close links between Corollary 4.2 and Corollary 3.2 of [5]. Proposition 4.1 will be key through our next section to explore the stable manifold $W^S(x_*)$ of the trivial equilibrium.

5 A closer look into the stable manifold

Having at hand the classification of solutions of repulsive equations given by Proposition 4.1, we have now all the ingredients needed to establish Lemma 3.2, upon which relied our proof of Proposition 3.1:

Proof of Lemma 3.2. Observe that item 4. follows directly from Proposition 4.1, while 5. follows from 4. and Corollary 4.2. In order to prove items 1. and 3. we consider the so-called 'resolvent function' $\mathcal{X} = \mathcal{X}(t, x_0, v_0)$, defined as the value at time t of the solution x of (1) verifying the initial condition $x(0) = x_0$, $\dot{x}(0) = v_0$. Since our force f is assumed to be bounded, $\mathcal{X}$ is globally defined on $\mathbb{R}^3$, although we shall only consider it on the half space $\{t \geq 0\}$. We consider the sets $\mathcal{S}_\pm$ defined by

$$\mathcal{S}_\pm := \Big\{(x_0, v_0) : \lim_{t\to+\infty} \mathcal{X}(t, x_0, v_0) = \pm\infty\Big\} = \\ = \Big\{(x(0), \dot{x}(0)) : x \text{ solves (1) and belongs to one of the classes } (i\pm)\Big\} \quad (5)$$

It follows from Proposition 4.1 that

$$\mathcal{S}_+ = \Big\{(x_0, v_0) : \mathcal{X}(t_0, x_0, v_0) > 0,\ \partial_t \mathcal{X}(t_0, x_0, v_0) > 0 \text{ for some } t_0 > 0\Big\},$$

while $\mathcal{S}_-$ admits a similar characterization after reversing both inequalities in the right hand side above. Thus, continuous dependence on the initial conditions implies that $\mathcal{S}_\pm$ are open in $\mathbb{R}^2$, so that the stable manifold $W^S(x_*) = \mathbb{R}^2 \backslash (\mathcal{S}_- \cup \mathcal{S}_+)$ is closed, as stated by 3. On the other hand, the boundedness of the force f implies the existence, for each $x_0 \in \mathbb{R}$, of some $M = M(x_0) > 0$ such that $\{x_0\}\times]-\infty, -M[\subset \mathcal{S}_-$ and $\{x_0\}\times]M, +\infty[\subset \mathcal{S}_+$; in particular, $(\{x_0\} \times \mathbb{R}) \cap \mathcal{S}_\pm \neq \emptyset$, and these sets being open and disjoint, we deduce that $W^S(x_*) \cap (\{x_0\} \times \mathbb{R}) = (\{x_0\} \times \mathbb{R}) \backslash (\mathcal{S}_- \cup \mathcal{S}_+) \neq \emptyset$, proving 1. In order to check 2., assume, by a contradiction argument, that there were two different solutions $x_1, x_2 : [0, +\infty[\to \mathbb{R}$ of (1) with $x_1(0) = x_2(0) \in]-\delta, \delta[$ and $\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} \dot{x}_i(t) = 0$. Then, Corollary 4.2

implies that $|x_i(t)| < \delta$ for any $t \in [0, +\infty[$. We consider the function $\varphi : [0, +\infty[\times[0, 1] \to \mathbb{R}$ defined by

$$\varphi(t, \lambda) := \int_0^t L\Big(s, (1-\lambda)x_1(s) + \lambda x_2(s), (1-\lambda)\dot{x}_1(s) + \lambda\dot{x}_2(s)\Big)ds, \quad (t, \lambda) \in [0, +\infty[\times[0, 1]\,,$$

where $L(s, x, \dot{x}) = \dot{x}^2/2 + \int_0^x f(s, y)dy$ is the lagrangian associated with our equation (1). Then, integration by parts may be used to obtain

$$\partial_\lambda\varphi(t, 0) = \dot{x}_1(t)\,(x_2(t) - x_1(t))\ , \qquad \partial_\lambda\varphi(t, 1) = \dot{x}_2(t)\,(x_2(t) - x_1(t))\ ,$$

so that $\partial_\lambda\varphi(t, 1) - \partial_\lambda\varphi(t, 0) = \displaystyle\int_0^1 \partial^2_{\lambda\lambda}\varphi(t, \lambda)d\lambda \to 0$ as $t \to +\infty$. However, our assumption $\partial_x f \geq 0$ implies

$$\partial^2_{\lambda\lambda}\varphi(t, \lambda) \geq \int_0^t (\dot{x}_1(s) - \dot{x}_0(s))^2\, ds\,, \qquad (t, \lambda) \in [0, +\infty[\times[0, 1]\,,$$

a contradiction because, the positive and increasing function $t \mapsto \int_0^t (\dot{x}_1(s) - \dot{x}_0(s))^2\, ds$ cannot tend to zero as $t \to +\infty$ (remember that, by assumption, $x_1(0) = x_2(0)$ and $x_1 \not\equiv x_2$). It completes the proof. □

Let us investigate a little bit closer the sets $\mathcal{S}_\pm$ defined in (5). As we have already seen, these sets are open and disjoint, and $W^S(x_*) = \mathbb{R}^2\backslash(\mathcal{S}_- \cup \mathcal{S}_+)$. However, something more can be said:

Lemma 5.1. *$\mathcal{S}_\pm$ are simply connected, and their union $\mathcal{S}_+ \cup \mathcal{S}_- = \mathbb{R}^2\backslash W^S(x_*)$ is dense in the plane.*

Proof. Let us start by showing that $\mathcal{S}_+ \cup \mathcal{S}_-$ is dense in $\mathbb{R}^2$, or, what is the same, that $W^S(x_*)$ has empty interior. Indeed, a stronger result holds: that $W^S(x_*)$ has Lebesgue measure zero. This is a well-known argument for measure-preserving flows which we pass to sketch. Indeed, if $W^S(x_*)$ had positive measure, Egorov's Theorem would imply that, on some positive-measure subset of $W^S(x_*)$, the flow of the equation would converge uniformly to $(0, 0)$, which is not possible, since it is measure-preserving.

Let us show now that the sets $\mathcal{S}_+$ and $\mathcal{S}_-$ defined above are simply connected; we shall concentrate ourselves in the case of $\mathcal{S}_+$ since $\mathcal{S}_-$ admits a completely analogous analysis. Our argument will be based in the construction of an homeomorphism Φ between $\mathcal{S}_+$ and the three-quadrant set $\mathcal{D} := \mathbb{R}^2\backslash(] - \infty, 0]\times] - \infty, 0])$.

Given $(x_0, v_0) \in \mathcal{S}_+$ it may happen that $x_0, v_0 \geq 0$; in this case, we define $\Phi(x_0, v_0) := (x_0, v_0)$.

It may also happen that $x_0 > 0 > v_0$; in this case, there exists an unique time $t_0 = t_0(x_0, v_0) > 0$ such that $\partial_t\mathcal{X}(t_0, x_0, v_0) = 0$, and we define $\Phi(x_0, v_0) := (\mathcal{X}(t_0, x_0, v_0), -t_0)$.

Finally, if $x_0 < 0 < v_0$, then $\mathcal{X}(\cdot, x_0, v_0)$ is strictly increasing and divergent, so that there is an unique $s_0 = s_0(x_0, v_0) > 0$ such that $\mathcal{X}(t_0, x_0, v_0) = 0$. This allows us to define $\Phi(x_0, v_0) := (-s_0, \partial_t\mathcal{X}(s_0, x_0, v_0))$ in this case.

The mapping Φ constructed in this way is continuous and injective, and the domain $\mathcal{S}_+$ being open, also Φ is open. Thus, Φ it is an homeomorphism into its image, which is easily seen to be $\mathcal{D}$. In particular, $\mathcal{S}_+$ is simply connected and the proof is complete. □

Before closing this Section, a few lines to explore the set of crossing points between two solutions $x, y : [0, +\infty[\to \mathbb{R}$ of our repulsive equation (1). In general, this set might be infinite (remember the example of Proposition 2.2), but that cannot be the case anymore if x is asymptotic to the trivial equilibrium -i.e., $(x(0), \dot{x}(0)) \in W^S(x_*)$-, while y is not. In this situation, Proposition 4.1 implies that the set of crossing points is bounded, and, as these crossing points are all of them transversal (by uniqueness), then there are only finitely many of them. In Lemma 5.2 below we generalize this result to the situation where, instead of having just two solutions x, y, we deal with continuous paths $\{x_s\}_s$ and $\{y_s\}_s$. More precisely, we assume that we are given continuous curves $\alpha, \beta : [0, 1] \to \mathbb{R}^2$ such that

$$\alpha([0,1]) \subset W^S(x_*)\,, \qquad\qquad \beta([0,1]) \subset \mathbb{R}^2 \backslash W^S(x_*)\,,$$

and we define, for each $s \in [0, 1]$, $x_s := \mathcal{X}(\cdot, \alpha(s))$ and $y_s := \mathcal{X}(\cdot, \beta(s))$.

Lemma 5.2. *Assume the above. Then, there exists some positive constant $b > 0$ (not depending on s) such that every crossing point between x_s and y_s is smaller than b.*

Proof. We start by using the continuity of α and β to pick some number $R > 0$ such that

$$|x_s(0)| < R\,, \qquad |y_s(0)| < R\,, \qquad\qquad s \in [0, 1]\,.$$

Now, we recall that $\beta([0, 1]) \subset \mathbb{R}^2 \backslash W^S(x_*) = \mathcal{S}_- \cup \mathcal{S}_+$, and, since these sets are open and disjoint, the curve β must actually be contained in one of them. We assume, for instance, that $\beta([0, 1]) \subset \mathcal{S}_+$, and observe, with the help of Proposition 4.1, that for any $s \in [0, 1]$ there exists exactly one time $T_s > 0$ such that $y_s(T_s) = R$. Moreover, at this point $\dot{y}(T_s) > 0$, and the Implicit Function Theorem implies that the function $s \mapsto T_s$ defined in this way is continuous on $[0, 1]$.

We define $b := \max_{[0,1]} T_s$. Observe that if $t \geq b$ then $y_s(t) \geq R$, independently of the value of s. It implies that x_s and y_s do not cross on $[b, +\infty[$ since, by Corollary 4.2, the solutions x_s are monotonous and thus verify $|x_s(t)| < R$ for any s, t. The proof is complete. □

At this point, we have concluded our study of general repulsive equations. The next section is devoted to prove that the stable manifold $W^S(x_*)$ associated to the force f of Proposition 2.2 is pathological in the sense established by Theorem 1.1 and Proposition 2.3.

6 Branches of the stable manifold wrapping many times around some points

As announced in Section 2, one of the main ideas of this paper consists in estimating the winding number of certain curves in the plane. With this goal, a crucial role will be played by the result which opens the present Section. We assume that $u, v, \eta : [a, b] \to \mathbb{R}$ are $\mathcal{C}^1$ functions such that u and η cross $m \geq 0$ times, while v and η cross $n \geq 0$ times. These crossing points are not allowed to be tangential, nor to lie at the right endpoint b of the interval where the functions are defined. Then, it is possible to establish a connection between the difference $n-m$, and the winding number of the planar curve

$$\gamma(s) := (H(a,s), \partial_t H(a,s)), \qquad s \in [0,1], \tag{6}$$

around the base point $q := (\eta(a), \dot\eta(a))$. Here, the $\mathcal{C}^{1,0}$ homotopy $H : [a,b] \times [0,1] \to \mathbb{R}$, $(t,s) \mapsto H(t,s)$, is assumed to verify:

(i) $H(t,0) = u(t)$, $H(t,1) = v(t)$ for any $t \in [a,b]$.

(ii) $(H(t,s), \partial_t H(t,s)) \neq (\eta(t), \dot\eta(t))$ for any $(t,s) \in [a,b] \times [0,1]$.

(iii) $H(b,s) \neq \eta(b)$ for any $s \in [0,1]$.

We may interpret the homotopy H as being the continuous family $\{H(\cdot,s)\}_s$ of $\mathcal{C}^1[a,b]$ functions. In this way, (i) states that this family connects u and v, (ii) presupposes that any crossing point between η and the curves of this family must be transversal, and (iii) establishes that these crossing points do not pass through $t=b$.

Observe that, by assumption (ii), the crossing points between η and the curves in the homotopy should move continuously on the interval $[a,b]$ as the parameter s varies. If $n \neq m$, some extra crossing points have appeared or disappeared along the homotopy, while, by (iii), they have not passed through $t=b$. Then, the additional crossing points must have entered (or exited) the interval $[a,b]$ from $t=a$. On the other hand, transversality also implies that, for any value of s, the sign of the derivative $H_t(\cdot,s) - \dot\eta$ alternates from each zero to the next. As the zeroes go through $t=a$, it forces the curve γ defined in (6) to intersect the vertical line $\{\eta(a)\} \times \mathbb{R}$, consecutively, above and below the point $q = (\eta(a), \dot\eta(a))$, making each time half a revolution more (clockwise if the crossing point leaves the interval $[a,b]$). This leads us to conjecture the following result:

Proposition 6.1. *Assume (i),(ii) and (iii) above. Then,*

$$\left| w_0^1(\gamma, q) - \left(\frac{n-m}{2}\right) \right| < \frac{3}{2}, \tag{7}$$

where $q = (\eta(a), \dot\eta(a))$, m and n denote, respectively, the number of crossing points between u and η and v and η, and γ is given by (6).

Proposition 6.1 is elementary; however, we could not find it in the literature and it will be rigourously proved in the Section 10. We observe that the constant $3/2$ appearing in the right hand side of the inequality is not optimal, and, actually, it might be improved to $1/2$. However, we shall not need this fact in our argumentation, and consequently, we sacrifice sharpness to gain simplicity. Through the remaining of this Section we trust upon Proposition 6.1 (and Proposition 2.2) to complete our proofs of Theorem 1.1 and the weaker version of Theorem 1.2 given by Proposition 2.3:

Proof of Proposition 2.3. We consider the force f of Proposition 2.2. Remembering Corollary 2.5 it suffices to check that the associated stable manifold $W^S(x_*) = W$ and the points $p = (x_1(0), \dot x_1(0))$ and $q = (x_2(0), \dot x_2(0))$ verify assumption $(\mathbf{H}_2)$. Thus, we fix some natural number $N > 0$ and some point $r_0 \in \mathbb{R}^2 \backslash W^S(x_*)$. This point must belong to one of the sets $\mathcal{S}_\pm$ defined in (5), and, in particular, the number m of crossing points between $u := \mathcal{X}(\cdot, r_0)$ and $\eta := x_2$ must be finite. On the other hand, x_1 and x_2 intersect infinitely many times, and the resolvent function $\mathcal{X}$ being continuous, there must exist some $\epsilon > 0$ such that for any $r \in \mathbb{R}^2 \backslash W^S(u_*)$ with $|r - p| < \epsilon$ the number of crossing points between $\mathcal{X}(\cdot, r)$ and η is at least $m + 2N + 3$. To conclude the argumentation we choose some continuous curve $\gamma : [0, 1] \to \mathbb{R}^2 \backslash W^S(u_*)$ such that $\gamma(0) = r_0$ and $|\gamma(1) - p| < \epsilon$. Applying Lemma 5.2 to the curves $\alpha \equiv (x_2(0), \dot{x}_2(0))$ and $\beta = \gamma$, we find that there exists some number $b > 0$ (not depending on s) such that any crossing point between x_2 and $\mathcal{X}(\cdot, \gamma(s))$ is strictly smaller than b. We call $v := \mathcal{X}(\cdot, \gamma(1))$, which crosses with η at $n \geq m + 2N + 3$ points, and consider the homotopy $H : [0, b] \times [0, 1] \to \mathbb{R}$ defined by $H(t, s) := \mathcal{X}(t, \gamma(s))$. This homotopy verifies the assumptions of Proposition 6.1 and hence, inequality (7) holds. But, in this case, $(n - m)/2 \geq N + 3/2$ and it follows that $w_0^1(\gamma, q) \geq N$. Corollary 2.5 may be now applied to deduce that the point p is not accessible from $\mathbb{R}^2 \backslash W^S(x_*)$.

□

Proof of Theorem 1.1. We consider again the force f of Proposition 2.2, and we claim that the associated stable manifold $W^S(x_*) = W$ and the points $p = (x_1(0), \dot{x}_1(0))$, $q = (x_2(0), \dot{x}_2(0))$, $O = (0, 0)$ verify assumption $(\mathbf{H}_1)$; Lemma 2.1 will then complete the proof. Thus, let the continuous curve $\gamma : [0, 1] \to W$ with $\gamma(0) = O$ and $\gamma(1) = p$ be given. Our reasoning will consist in showing that there exists a sequence $\{q_k\}_k \to q$ with $q_k \notin W$ for any $k \in \mathbb{N}$ and such that $w_0^1(\gamma, q_k) \to +\infty$. Of course, this implies that γ passes through q, as the continuity of the winding number with respect to the base point would otherwise imply the winding number of γ with respect to q to be infinite. The symmetry on our assumptions on x_1 and x_2 means that we can exchange the points p and q, so that also any continuous curve going from O to q must pass through p.

We choose *any* sequence $\{q_k\}_k \to q$ with $q_k \notin W^S(x_*)$ for any $k \in \mathbb{N}$. The existence of such sequence is ensured by the fact that $W^S(x_*)$ has empty interior (Lemma 5.1). For any $k \in \mathbb{N}$ we denote by n_k to the number of intersection points between $\mathcal{X}(\cdot, q_k)$ and x_1 on $[0, +\infty[$. Observe that $n_k \to +\infty$ as $k \to \infty$, as the continuity of $\mathcal{X}$ implies that $\mathcal{X}(\cdot, q_k) \to \mathcal{X}(\cdot, q) = x_2$ uniformly on compact subsets of $[0, +\infty[$. We claim that

$$w_0^1(\gamma, q_k) \geq n_k - 2\,, \qquad k \in \mathbb{N}\,. \tag{8}$$

so that $w_0^1(\gamma, q_k) \to +\infty$ and the result follows. To check (8) we fix some $k \in \mathbb{N}$ and apply Lemma 5.2 to the curves $\alpha(s) := \gamma(s)$, $\beta(s) \equiv q_k$. It follows that there exists some $b > 0$ (not depending on s) such that any crossing point between $\mathcal{X}(\cdot, \gamma(s))$ and $\eta := \mathcal{X}(\cdot, q_k)$ is strictly smaller than b. We define $u \equiv 0$ and $v := x_1$ and consider the homotopy $H : [0, b] \times [0, 1] \to \mathbb{R}$ defined by $H(t, s) := \mathcal{X}(t, \gamma(s))$. This homotopy links u and v, and it verifies all other assumptions of Proposition 6.1, and hence, inequality (7) holds. But, in this particular case, m, the number of crossing points between $u \equiv 0$ and η, is either 0 or 1 (by Proposition 4.1), while n is what we called n_k. This implies (8) and concludes the proof.

□

7 Accessible points, simply connected domains and local connectedness

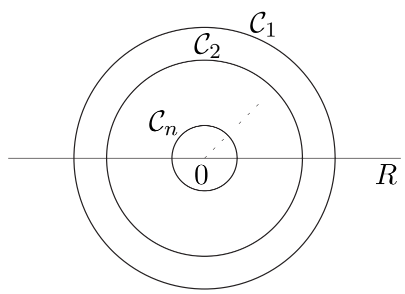
Proposition 2.3, whose proof was carried out in the previous Section, was a first step towards Theorem 1.2. In this Section we complete the proof of this result, and, with this aim, we shall first show that, under the framework of Proposition 2.2, the (global) stable manifold $W^S(x_*)$ of the trivial equilibrium cannot be locally connected:

Proposition 7.1. *Let f be as given by Proposition 2.2. Then, $W^S(x_*)$ is not locally connected.*

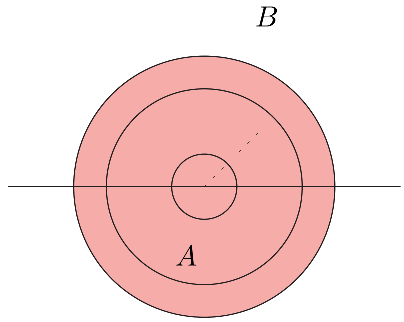
At first sight, Proposition 7.1 does not seem to be a consequence from Proposition 2.3, since there are closed subsets $W \subset \mathbb{R}^2$, even with empty interior, which have some points which are not accessible from the exterior while being locally connected; consider, for instance, the set W composed by some straight line R passing through the origin, plus a sequence of concentric circumferences $\mathcal{C}_n$ centered there and whose decreasing radii converge to zero, see Fig. 7 a). However, such examples cannot happen under a further assumption:

($\mathbf{H}_3$) W divides the plane into two (open) connected components A, B, and both of them are simply connected.

Proposition 7.2. *Let the closed set $W \subset \mathbb{R}^2$ have empty interior and verify* **($\mathbf{H}_3$)** *above. Assume further that W has a point which is not accessible from $\mathbb{R}^2 \backslash W$. Then, W is not locally connected.*



a) The set W



b) The sets A and B.

Figure 7: a) The set W is locally connected and contains the origin, which is not accessible from the exterior. b) It does not contradict the fact that the complement of W may be written as the disjoint union of the open sets A (which is disconnected), and B.

Proposition 7.2 is a planar topology result; observe that no mention to differential equations or dynamical systems is made there. We do not pretend to be original in this result, which, as most of this Section, is probably well known to the specialists. On the other hand, we do not know whether the assumption on A and B being simply connected is fully necessary; however, the result does not hold if these sets are not assumed to be at least connected.

To check this, we consider again the set W constructed above, we call B the exterior region of the circumference $\mathcal{C}_1$, and define A as the infinite sequence of annular regions which remains when the sequence $\mathcal{C}_n$ of circumferences is removed from the interior region of $\mathcal{C}_1$, see Fig. 7 b).

Having said this, we observe that the stable manifold $W = W^S(x_*)$ associated to the trivial equilibrium of any repulsive potential does *always* verify assumption $(\mathbf{H}_3)$; this was already seen in Lemma 5.1 . Thus, the combination of Propositions 2.3 and 7.2 immediately implies Proposition 7.1.

For this reason, we devote our efforts now to prove of Proposition 7.2; it will occupy us through much of this Section. It will be convenient to work on the compactified plane $\mathbb{R}^2 \cup \{\infty\}$. This set is endowed with the metric d transported from the geodesic distance on the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ by means of the stereographic projection. In this way, $\mathbb{R}^2 \cup \{\infty\}$ becomes a compact metric space; moreover, $d(x, y) \leq \pi$ for any $x, y \in \mathbb{R}^2 \cup \{\infty\}$. For any points $p, q \in \mathbb{R}^2 \cup \{\infty\}$ with $d(p, q) < \pi$ (i.e., points which are not antipodal when seen on the sphere), it makes sense to define the *segment* $[p, q]$ as the image by the stereographic projection of the shorter geodesic of the sphere connecting p and q. Observe that $d(r, s) \leq d(p, q)$ for any $r, s \in [p, q]$; in other words, the diameter of the segment $[p, q]$ is $d(p, q)$.

A well known result says that if the closed set $E \subset \mathbb{R}^2 \cup \{\infty\} \equiv \mathbb{S}^2$ has a locally connected boundary ∂E, then E itself is locally connected. In our argumentation we shall need an auxiliary lemma which slightly generalizes this statement:

Lemma 7.3. *Let the closed sets $E, F \subset \mathbb{R}^2 \cup \{\infty\}$ verify that $\partial E \subset F$. If F is locally connected, so is $E \cup F$.*

Proof. Before going into the details we recall a well known characterization of local connectedness for compact metric spaces (see, for instance, [4], page 19): the compact set S is locally connected if and only if for each $\epsilon > 0$ there is some $\delta > 0$ such that, for any points $p, q \in S$ whose distance is smaller than δ, there is a continuum $C \subset S$ containing both p and q, and with diameter smaller than ϵ.

Now, let $0 < \epsilon < \pi$ be given, and choose $0 < \delta < \epsilon/3$ as given by the characterization above for the compact set F and the positive quantity $\epsilon/3$. Given points $p, q \in E \cup F$ with $d(p, q) < \delta$, it may happen that the segment $[p, q]$ is contained inside $E \cup F$, or not. In the first case, the segment $[p, q]$ itself provides the required continuum with diameter smaller than ϵ which joins p and q, and thus, we may assume that $[p, q] \not\subset E \cup F$. We call a to the last point $x \in [p, q]$ which verifies $[p, x] \subset E \cup F$, and we denote by b to the first point y in the same segment with $[y, q] \subset E \cup F$. The points a and b belong to F, and, since $d(a, b) \leq d(p, q) < \delta$, there must exist some continuum $C_1 \subset F$ containing both a and b, and with diameter smaller than $\epsilon/3$. Now, it suffices to take $C := C_1 \cup [p, a] \cup [b, q]$.

□

A key role in our reasoning will be played by Theorem 2.1 of [4], which characterizes those simply connected domains G of the sphere $\mathbb{R}^2 \cup \{\infty\} \equiv \mathbb{S}^2$ for which conformal mappings from the open disc $\mathbb{D}$ onto G can be continuously extended to the closure $\bar{\mathbb{D}}$. An immediate consequence of this result is the following

Lemma 7.4. *Let $G \subset \mathbb{R}^2 \cup \{\infty\}$ be open and simply connected. If ∂G (boundary relative to the sphere $\mathbb{R}^2 \cup \{\infty\}$) has a point which is not accessible from G, then $(\mathbb{R}^2 \cup \{\infty\})\backslash G$ is not locally connected.*

Proof. If ∂G has some points which are not accessible from G, then no conformal mapping $h : \mathbb{D} \to G$ can be extended to the boundary. Thus, the above-mentioned Theorem 2.1 of [4] states that $(\mathbb{R}^2 \cup \{\infty\})\backslash G$ is not locally connected. □

One easily checks that the boundary of simply connected subsets G of the plane may be disconnected if G is unbounded. Our next result shows that this is no longer true when we add the infinity point:

Lemma 7.5. *Let $G \subset \mathbb{R}^2 \cup \{\infty\}$ be simply connected. Then ∂G is connected.*

Proof. Assume that the result were not true; then, there would be open sets $U, V \subset \mathbb{R}^2 \cup \{\infty\}$ such that

$$U \cap (\partial G) \neq \emptyset \neq V \cap (\partial G), \qquad U \cup V \supset \partial G, \qquad U \cap V \cap (\partial G) = \emptyset\,. \tag{9}$$

Remark that $U \cap (\partial G)$ and $V \cap (\partial G)$ are both open and closed in ∂G. Since ∂G is compact, the distance between these sets must be positive and we may replace U, V by smaller sets so that $U \cap V = \emptyset$ and (9) still holds. We remember the Riemann mapping theorem and choose some homeomorphism $\varphi : \mathbb{D} \to G$ between the (open) unit disc in the complex plane and G. Finally, we define

$$\tilde{U} := \varphi^{-1}(U \cap G)\,, \qquad \tilde{V} := \varphi^{-1}(V \cap G)\,,$$

which are open sets in $\mathbb{D}$. We claim that there exists some $0 < r < 1$ such that the ring $\mathbb{D}\backslash(r\mathbb{D})$ is contained inside $\tilde{U} \cup \tilde{V}$. Indeed, the contrary would mean the existence of a sequence $\{\tilde{p}_n\}_n \subset \mathbb{D}$ such that $|\tilde{p}_n| \to 1$ and $\tilde{p}_n \notin \tilde{U} \cap \tilde{V}$ for any $n \in \mathbb{N}$. We let $p_n := \varphi(\tilde{p}_n)$; in this way we obtain a sequence of points in G with no adherence points in G. It means that $\mathrm{dist}(p_n, \partial G) \to 0$, but this is not possible since ∂G is a compact subset of $U \cup V$ and $p_n \notin U \cup V$ for any $n \in \mathbb{N}$. Thus, there is some $0 < r < 1$ such that, as claimed, $\mathbb{D}\backslash(r\mathbb{D}) \subset \tilde{U} \cup \tilde{V}$. But $\tilde{U}$ and $\tilde{V}$ are open and disjoint, implying that $\mathbb{D}\backslash(r\mathbb{D})$ is disconnected, a contradiction. It concludes the proof.

□

The combination of the last three lemmas immediately implies a first version of Proposition 7.2, even though it refers to subsets W of the sphere $\mathbb{R}^2 \cup \{\infty\}$ instead of the plane.

Proposition 7.6. *Let the closed set $W \subset \mathbb{R}^2 \cup \{\infty\}$ have empty interior and divide the sphere $\mathbb{R} \cup \{\infty\}$ into two connected components which are simply connected. Then:*

(i) W is connected.

(ii) If W has a point which is not accessible from $\mathbb{R}^2\backslash W$, then W is not locally connected.

Proof. *(i):* Let us denote by A, B to the connected components of $(\mathbb{R}^2 \cup \{\infty\})\backslash W$. Observe that, the set W having empty interior, $(\partial A) \cup (\partial B) = W$ and $(\partial A) \cap (\partial B) \neq \emptyset$. But ∂A and ∂B are connected; this is given by Lemma 7.5. The result follows.

(ii): Using a contradiction argument, assume instead that W were locally connected. Since $\partial(A \cup W) \subset W$, Lemma 7.3 above implies that $A \cup W = (\mathbb{R}^2 \cup \{\infty\})\backslash B$ is locally connected. But then, Lemma 7.4 states that every point in ∂B is accessible from B. Similarly, every point in ∂A should be accessible from A, and consequently, all points in $W = \partial A \cup \partial B$ are accessible from $A \cup B = (\mathbb{R}^2 \cup \{\infty\})\backslash W$, contradicting our assumptions.

□

At first glance, Proposition 7.6 does not seem to imply Proposition 7.2, since our assumption $(\mathbf{H}_3)$ implies W to be unbounded, and then, it might happen that the only point without a basis of connected neighborhoods were the infinity. The following lemma states that the last part of this reasoning was mistaken:

Lemma 7.7. *Let the metric space X be connected and locally compact. We consider the set*

$$S := \left\{ p \in X \text{ such that } p \text{ has a basis of connected neighborhoods} \right\}.$$

Then, $X\backslash S$ does not have isolated points.

Proof. Using a contradiction argument we assume, on the contrary, that $p_0 \in X\backslash S$ were isolated. Using the local compactness of X we may find some number $r > 0$ verifying the three properties below:

(a) $B_r(p)\backslash\{p_0\}$ is locally connected.

(b) Every neighborhood of p_0 contained inside $B_r(p_0)$ is disconnected.

(c) $\overline{B_r(p_0)}$ is compact.

Choose now positive numbers $0 < r_2 < r_1 < r$. We are going to find a contradiction with *(a)* by showing that the ring

$$R := \overline{B_{r_1}(p_0)}\backslash B_{r_2}(p_0)$$

contains another point $q \in X\backslash S$.

It follows from *(b)* that $B_r(p_0)$ is disconnected, and we shall call $\mathfrak{F}$ to the family of its connected components. Again using *(b)* we see that the connected component C_0 containing p_0 cannot be a neighborhood of p_0. But, in view of *(a)*, all other connected components, $C \in \mathfrak{F}\backslash\{C_0\}$ are open; moreover, also $C_0\backslash\{p_0\}$ is open. On the other hand, all elements $C \in \mathfrak{F}$ are closed relative to $B_r(p_0)$.

We claim that there are infinitely many elements $C \in \mathfrak{F}$ for which $C \cap B_{r_2}(p_0) \neq \emptyset$. Because, if there were only finitely many, say $C_0, C_1, \ldots, C_k$, then the intersections $B_{r_2}(p_0) \cap C_i$ would be closed relative to $B_{r_2}(p_0)$, but then they would be also open, and C_0 would be an open connected neighborhood of p_0, contradicting *(b)*.

We also claim that all elements $C \in \mathfrak{F}\backslash\{C_0\}$ with $C \cap B_{r_2}(p_0) \neq \emptyset$ verify that $C \cap R \neq \emptyset$. Since, otherwise, such a set C would be contained inside $B_{r_2}(p_0)$, and hence, it would be clopen in X, contradicting the connectedness of our space

Then, there are infinitely many elements $C \in \mathfrak{F}$ such that $C \cap R \neq \emptyset$, and we may choose sequences $\{C_n\}_n \subset \mathfrak{F}$, $\{p_n\}_n \subset R$ with $p_n \in C_n$ for all n and $C_n \neq C_m$ if $n \neq m$. But *(c)* implies that R is compact, and thus, $\{p_n\}$ has some adherence point q.

Observe that q belongs to the open set $B_r(p_0)\backslash\{p_0\}$. We claim that q does not have connected neighborhoods $V \subset B_r(p_0)\backslash\{p_0\}$. Indeed, if V were such a neighborhood, then it could be decomposed as a disjoint union of open sets (here, the word open is understood relative to V):

$$V = \bigcup_{C \in \mathfrak{F}} C \cap V \,,$$

implying that there exists some element $C_* \in \mathfrak{F}$ such that $C_* \supset V$. But there is at most one element of the sequence $\{p_n\}_n$ in the set C_*, contradicting the fact that q is an adherence point of this sequence. This concludes the proof.

□

At this point, the combination of Proposition 7.6 and Lemma 7.7 yields Proposition 7.2. And Propositions 2.3 and 7.2 together imply Proposition 7.1. We conclude this Section by showing how Propositions 2.3 and 7.1 imply Theorem 1.2:

Proof of Theorem 1.2. Choose some bounded neighborhood of the origin B. Remembering the boundedness of f and Proposition 4.1 we observe that the band $B_\delta :=]-\delta, \delta[\times \mathbb{R}$ verifies that $W^S_{B_\delta}(x_*) \subset B$ provided only that $\delta > 0$ is small enough. We fix such a number δ and observe that $W^S_{B_\delta}(x_*) \subset W^S_B(x_*)$; it follows from Corollary 4.2. Define next $N := B \cap B_\delta$, which is a bounded neighborhood of the origin. Items *4.* and *5.* of Lemma 3.2 imply that

$$W^S_B(x_*) \cap N = W^S(x_*) \cap N \,. \tag{10}$$

On the other hand, the combination of Propositions 2.3 and 7.1 ensure the existence of points $\tilde{q}_1, \tilde{q}_2 \in W^S(x_*)$ such that $\tilde{q}_1$ is not accessible from $\mathbb{R}^2 \backslash W^S(x_*)$ and $\tilde{q}_2$ does not have basis of connected neighborhoods in $W^S(x_*)$. We call P to the Poincaré mapping associated to our periodic equation (1),

$$P : \mathbb{R}^2 \to \mathbb{R}^2 \,, \qquad (x_0, v_0) \mapsto \big(\mathcal{X}(1, x_0, v_0), \partial_t \mathcal{X}(1, x_0, v_0)\big) \,,$$

the resolvent function $\mathcal{X}$ being defined as in Section 5. We choose some $k \in \mathbb{N}$ such that the iterates $q_1 := P^k(\tilde{q}_1)$, $q_2 := P^k(\tilde{q}_2)$ belong to the interior of N. In view of (10) one sees that

$$q_1, \; q_2 \in W^S_B(x_*) \cap N \,.$$

We define now $\tilde{N} := P^{-k}(N)$, which is a neighborhood of both $\tilde{q}_1$ and $\tilde{q}_2$. The mapping P^k establishes an homeomorphism from $\tilde{N}$ to N which sends $\tilde{N} \cap W^S(x_*)$ into $N \cap W^S(x_*) = W^S_B(x_*) \cap N$ and $\tilde{q}_1, \tilde{q}_2$ into q_1, q_2. Then q_1 is not accessible from the exterior of $W^S_B(x_*)$ and q_2 does not have basis of connected neighborhoods in $W^S_B(x_*)$. It concludes the proof. □

8 Interpolating Newtonian equations

We devote the next two sections to construct the function $f = f(t,x)$ whose existence was claimed by Proposition 2.2. Our approach will consist in building f from the solution curves $x_1, x_2 : [0,+\infty[\to \mathbb{R}$ described there. Accordingly with the just-mentioned proposition, the positive functions x_1, x_2 should verify two properties, which we rewrite here for the reader's convenience:

(i) $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$,

(ii) x_1 and x_2 coincide on infinitely many points $0 = t_0 < t_1 < t_2 < ... \to +\infty$.

Uniqueness of solutions to initial value problems means that x_1 and x_2 should always intersect transversally. As a consequence, intersection points must be isolated and they all may be integrated into the sequence $\{t_n\}$. With other words, there is no loss of generality in assuming that

(iii) $x_1(t) \neq x_2(t)$ if $t \notin \{t_0, t_1, t_2, \ldots\}$.

Other properties that x_1 and x_2 must verify are

(iv) $\dot{x}_1(t_n) \neq \dot{x}_2(t_n)$ and $\ddot{x}_1(t_n) = \ddot{x}_2(t_n)$ for any $n \geq 0$.

(v) $\dot{x}_1(t), \dot{x}_2(t) < 0 < \ddot{x}_1(t), \ddot{x}_2(t)$ for any $t \in [0,+\infty[$.

(vi) $\lim_{t\to+\infty} \dot{x}_i(t) = \lim_{t\to+\infty} \ddot{x}_i(t) = 0\,, \qquad i = 1,2\,.$

In the main result of this section, we pick arbitrary $\mathcal{C}^2$ functions $x_1, x_2 : [0,+\infty[\to \mathbb{R}$ verifying all these conditions. We shall make a further assumption:

(vii) $\max\{x_1(t+1), x_2(t+1)\} < \min\{x_1(t), x_2(t)\}$ for any $t \geq 0$,

which guarantees that new crossing points do not appear when the graphs of x_1 and x_2 are projected into the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. In this framework we construct a Newtonian equation which is solved at the same time by x_1 and x_2. Indeed, this can be done with $\mathcal{C}^{0,\infty}$ regularity on the upper half cylinder $(\mathbb{R}/\mathbb{Z})\times]0,+\infty[$:

Lemma 8.1. *Let the positive, $\mathcal{C}^2$ functions $x_1, x_2 : [0,+\infty[\to \mathbb{R}$ verify conditions (i)-(vii) above. Then, there exists a 1-periodic in time, $\mathcal{C}^{0,\infty}$ and positive function $f : (\mathbb{R}/\mathbb{Z})\times]0,+\infty[\to \mathbb{R}$ such that x_1 and x_2 both solve (1).*

Proof. Consider the functions x_m, x_M defined on $[0,+\infty[$ by

$$x_m(t) := \min\{x_1(t), x_2(t)\}, \qquad x_M(t) := \max\{x_1(t), x_2(t)\}\,, \qquad t \geq 0\,. \tag{11}$$

We continuously extend them to on the negative part of the real axis by means of a same arch of parabola arriving at $t = 0$ with speed $v_0 := \dot{x}_1(0) + \dot{x}_2(0)$ and curvature $\ddot{x}_1(0) = \ddot{x}_2(0)$:

$$x_m(t) = x_M(t) := x_1(0) + v_0 t + \ddot{x}_1(0)t^2/2\,, \qquad t \in]-\infty, 0[\,.$$

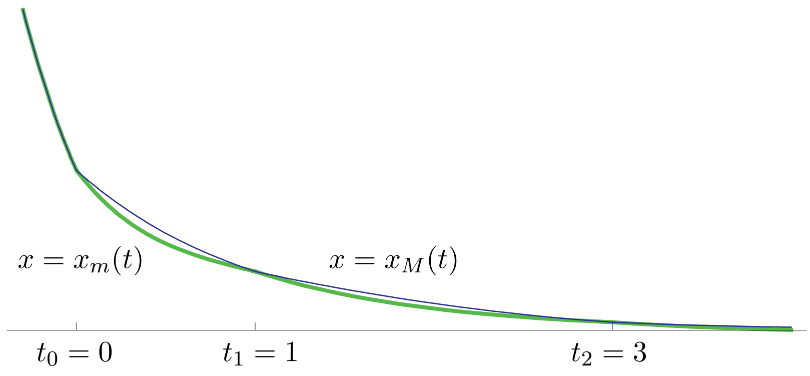


Figure 8: The graphs of x_M and x_m.

We observe now that x_1 and x_2 are solutions of (1) if and only if

$$f(t, x_1(t)) = \ddot{x}_1(t)\,, \qquad f(t, x_2(t)) = \ddot{x}_2(t)\,, \qquad\qquad t \in [0, +\infty[\,,$$

or, what is the same,

$$f(t, x_m(t)) = \ddot{x}_m(t)\,, \qquad f(t, x_M(t)) = \ddot{x}_M(t)\,, \qquad\qquad t \in [0, +\infty[\,. \tag{12}$$

Equality (12) requires some explanation. For the functions x_m and x_M are continuous, but only piecewise differentiable; they have class $\mathcal{C}^2$ on each closed interval $]-\infty, t_0]$ or $[t_n, t_{n+1}]$, but, since x_1 and x_2 intersect always transversally, the lateral derivatives are different at the nodes t_n. However, (12) is meaningful because, by the second part of our assumption *(iv)*, the left and right limits of $\ddot{x}_m$ and $\ddot{x}_M$ coincide at t_n for each $n \geq 0$. In fact, one may regard $\ddot{x}_m, \ddot{x}_M$ as continuous functions on $\mathbb{R}$. With this in mind, we extend f to the left side of the graph of x_m (which coincides with that of x_M) by letting

$$f(t, x_m(t)) := \ddot{x}_m(t) = \ddot{x}_1(0)\,, \qquad\qquad t \in]-\infty, 0[\,. \tag{13}$$

Observe now that, if f is to be 1-periodic in time, the discussions above also establish the value of f on the translations of these graphs by integer multiples of the vector $(1, 0)$. For instance, from (13) and the second part of (12), we deduce

$$f(t, x_M(t+1)) = \ddot{x}_M(t+1)\,, \qquad\qquad t \in \mathbb{R}\,. \tag{14}$$

This new definition does not contradict those already made, because, in view of assumption *(vii)*, $x_M(t+1) < x_m(t)$ for any $t \in \mathbb{R}$. It motivates us to consider the open subset Ω of the plane defined by

$$\Omega := \{(t, x) \in \mathbb{R}^2 : x_M(t+1) < x < x_M(t)\}\,.$$

Our function f will be first defined on Ω. With this goal, observe that this is a connected domain; however, it is divided into infinitely many connected components by the right half of the graph of x_m. One of them, which we shall call Ω_0, is unbounded:

$$\Omega_0 := \{(t, x) \in \mathbb{R}^2 : x_M(t+1) < x < x_m(t)\}\,.$$

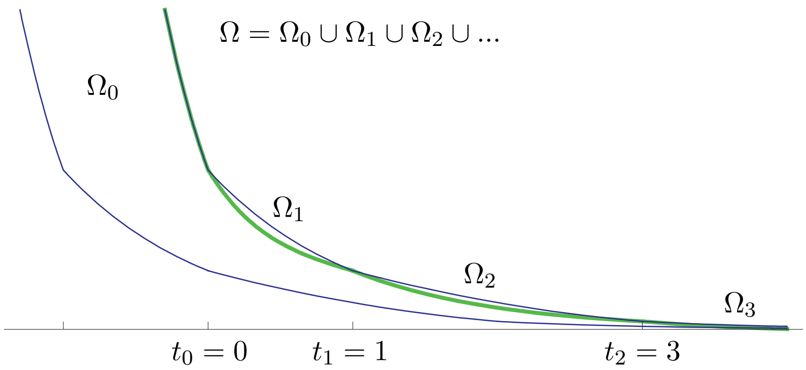


Figure 9: The set Ω.

The other connected components are bounded, and may be ordered into a sequence $\{\Omega_n\}_n$:

$$\Omega_n := \{(t, x) \in \mathbb{R}^2 : t_{n-1} < t < t_n,\ x_m(t) < x < x_M(t)\}\,, \qquad n \geq 1\,.$$

The most immediate extension of f to these sets would possibly be the piecewise linear interpolation in the vertical direction between the values of f on the respective boundaries, which have already been established in (12,13,14). However, after pasting the various definitions of f on the subdomains Ω_i, this procedure would lead to a continuous but only piecewise differentiable function f. We are interested in constructing a smooth function f, and, with this goal, we shall force the successive partial derivatives of f to vanish on the boundaries of these sets.

Thus, we choose some $\mathcal{C}^\infty$ function $h : [0, 1] \to \mathbb{R}$ with

$$h(0) = 0\,, \qquad h(1) = 1\,, \qquad h^{(k)}(0) = h^{(k)}(1) = 0 \text{ for any } k \geq 1\,,$$

which will be fixed in what follows. We define f on Ω_0 by setting

$$f(t, (1-\lambda)x_M(t+1) + \lambda x_m(t)) := (1 - h(\lambda))\ddot{x}_M(t+1) + h(\lambda)\ddot{x}_m(t)\,, \qquad t \in \mathbb{R}\,,\quad 0 < \lambda < 1\,, \tag{15}$$

and we extend f to Ω_n for any $n \geq 1$ by the rule

$$f(t, (1-\lambda)x_m(t) + \lambda x_M(t)) := (1 - h(\lambda))\ddot{x}_m(t) + h(\lambda)\ddot{x}_M(t)\,, \qquad t \in]t_{n-1}, t_n[\,,\quad 0 < \lambda < 1\,. \tag{16}$$

In this way, we have defined f on the closure of Ω so that it is a $\mathcal{C}^{0,\infty}$ function. It verifies that $\partial_x^n f(t, x) = 0$ for any $(t, x) \in \partial\Omega$ and any $n \geq 1$, and consequently, its periodic extension to the upper half plane $\mathbb{R}\times]0, +\infty[$,

$$f(t, x) := f(t - m, x)\,, \qquad (t, x) \in (m, 0) + \Omega\,, \qquad m \in \mathbb{Z}\,,$$

has also class $\mathcal{C}^{0,\infty}$, see Fig. 10 below. The construction is complete. □

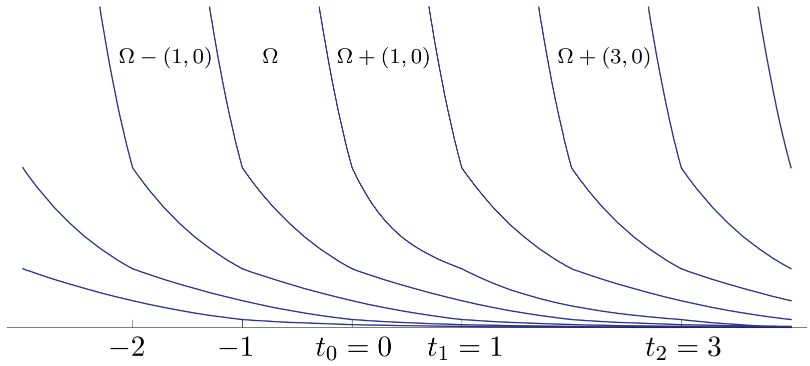


Figure 10: The translations of $\bar{\Omega}$ fill the upper half plane.

9 From the upper half plane to $\mathbb{R}^2$

In this Section we complete the proof of Proposition 2.2. With this aim we start by observing that the function f of Lemma 8.1 may be extended to a (continuous) repulsive function defined on the whole cylinder:

Lemma 9.1. *Let the $\mathcal{C}^2$ functions $x_1, x_2 : [0, +\infty[\to \mathbb{R}$ verify conditions (i)-(vii) from the previous section. Then, the function f of Lemma 8.1 may be continuously extended to a repulsive function on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$.*

Proof. It follows from (15,16) that

$$|f(t,x)| \leq \max\left\{\ddot{x}_M(t+1), \ddot{x}_m(t), \ddot{x}_M(t)\right\} \leq \max\left\{\ddot{x}_1(t+1), \ddot{x}_2(t+1), \ddot{x}_1(t), \ddot{x}_2(t)\right\}, \quad (t,x) \in \Omega .$$

When combined with assumption *(vi)*, these inequalities yield

$$\lim_{t\to+\infty} f(t,x) = 0 \qquad \text{uniformly with respect to } x \text{ as long as } (t,x) \in \Omega ,$$

or, what is the same, due to the geometry of the set Ω ,

$$\lim_{x\to 0} f(t,x) = 0 \qquad \text{uniformly with respect to } t \text{ as long as } (t,x) \in \Omega ,$$

and then, from the periodicity of f in its time variable, we deduce that $\lim_{x\to 0} f(t,x) = 0$ uniformly with respect to $t \in \mathbb{R}$. It follows that the antisymmetric extension of f to the whole plane $\mathbb{R}^2$,

$$f(t,0) := 0 , \qquad\qquad f(t,-x) := -f(t,x) , \quad x > 0 , \tag{17}$$

is continuous. It is also 1-periodic in time and repulsive, showing the result.

□

At this stage, we are now interested in deciding whether, at least for some particular choices of the functions x_1 and x_2, the function f built above has some additional regularity. Such a possibility seems reasonable, because our construction makes f of class $\mathcal{C}^{0,\infty}$ on the upper and lower cylinders $(\mathbb{R}/\mathbb{Z})\times]0,+\infty[$ and $(\mathbb{R}/\mathbb{Z})\times]-\infty,0[$. This motivates us to study the partial derivatives $(\partial^p f/\partial x^p)(t,x)$ as $x\to 0$. Differentiating with respect to λ in (15), we deduce that

$$\left|\frac{\partial^p f}{\partial x^p}(t,x)\right| \le \|h^{(p)}\|_\infty \frac{|\ddot{x}_m(t)-\ddot{x}_M(t+1)|}{(x_m(t)-x_M(t+1))^p}\,, \qquad (t,x)\in\Omega_0\,, \tag{18}$$

while the analogous move in (16) provides the inequality

$$\left|\frac{\partial^p f}{\partial x^p}(t,x)\right| \le \|h^{(p)}\|_\infty \frac{|\ddot{x}_M(t)-\ddot{x}_m(t)|}{(x_M(t)-x_m(t))^p}\,, \qquad (t,x)\in\bigcup_{n\ge 1}\Omega_n\,, \tag{19}$$

for any $p\ge 1$. We arrive to the following result:

Lemma 9.2. *Let the $\mathcal{C}^2$ functions $x_1,x_2:[0,+\infty[\to\mathbb{R}$ verify conditions (i)-(vii) from the previous section. We further assume that*

(viii) $$\lim_{t\to+\infty}\left[\frac{x_1(t)-x_2(t)}{x_1(t-1)-x_1(t)}\right] = \lim_{t\to+\infty}\left[\frac{x_1(t)-x_2(t)}{x_1(t)-x_1(t+1)}\right] = 0\,,$$

and, for some $p\in\mathbb{N}$,

(ix)$_p$ $$\lim_{t\to+\infty}\left[\frac{\ddot{x}_1(t)-\ddot{x}_2(t)}{\big(x_1(t)-x_2(t)\big)^p}\right] = \lim_{t\to+\infty}\left[\frac{\ddot{x}_1(t)-\ddot{x}_1(t+1)}{\big(x_1(t)-x_1(t+1)\big)^p}\right] = 0\,.$$

Then, there exists a repulsive function $f:(\mathbb{R}/\mathbb{Z})\times\mathbb{R}\to\mathbb{R}$ of class $\mathcal{C}^{0,p}$ such that x_1,x_2 both solve equation (1).

Proof. We consider the function f constructed on the upper half cylinder as in Lemma 8.1 and extended to the whole cylinder as in (17). As seen above, this makes f continuous on $(\mathbb{R}/\mathbb{Z})\times\mathbb{R}$ and $\mathcal{C}^{0,\infty}$ on $(\mathbb{R}/\mathbb{Z})\times\big(\mathbb{R}\backslash\{0\}\big)$. To complete the proof it will suffice to show how our new assumptions *(viii)* and *(ix)$_p$* imply that, for any $q\in\{1,\dots,p\}$,

$$\lim_{x\to 0}\frac{\partial^q f}{\partial x^q}(t,x)=0 \qquad \text{uniformly with respect to } t\in\mathbb{R}/\mathbb{Z}\,.$$

With this goal we recall the functions x_m,x_M defined as in (11). We may now rewrite the first part of our assumption *(ix)$_p$* as

$$\lim_{t\to+\infty}\left[\frac{\ddot{x}_M(t)-\ddot{x}_m(t)}{\big(x_M(t)-x_m(t)\big)^p}\right]=0\,, \tag{20}$$

while the combination of *(viii)* and the first part of *(ix)$_p$* gives

$$\lim_{t\to+\infty}\left[\frac{\ddot{x}_1(t)-\ddot{x}_2(t)}{\big(x_1(t-1)-x_1(t)\big)^p}\right] = \lim_{t\to+\infty}\left[\frac{\ddot{x}_1(t)-\ddot{x}_2(t)}{\big(x_1(t)-x_1(t+1)\big)^p}\right]=0\,. \tag{21}$$

On the other hand,

$$\lim_{t\to+\infty}\left[\frac{\ddot{x}_m(t)-\ddot{x}_M(t+1)}{\big(x_m(t)-x_M(t+1)\big)^p}\right]=\lim_{t\to+\infty}\left[\frac{\ddot{x}_m(t)-\ddot{x}_M(t+1)}{\big(x_1(t)-x_1(t+1)\big)^p}\right]\lim_{t\to+\infty}\left(\frac{x_1(t)-x_1(t+1)}{x_m(t)-x_M(t+1)}\right)^p .$$

Observe that the first of the limits of the right side above vanishes; this follows from (21) and the second part of assumption *(ix)*$_p$. Assumption *(viii)* implies that the last limit is one and we deduce that

$$\lim_{t\to+\infty}\left[\frac{\ddot{x}_m(t)-\ddot{x}_M(t+1)}{\big(x_m(t)-x_M(t+1)\big)^p}\right]=0\,. \tag{22}$$

But, in view of (18,19), expressions (20,22) imply

$$\lim_{t\to+\infty}\frac{\partial^q f}{\partial x^q}(t,x)=0 \qquad \text{uniformly with respect to } x \text{ as long as } (t,x)\in\Omega\,,$$

for any $1\leq q\leq p$. And having into account the geometry of the set Ω we deduce

$$\lim_{x\to 0}\frac{\partial^q f}{\partial x^q}(t,x)=0 \qquad \text{uniformly with respect to } t \text{ as long as } (t,x)\in\Omega\,.$$

The periodicity of f in its time variable now means that the above limit holds uniformly with respect to t on the upper half plane $\mathbb{R}\times]0,+\infty[$. Consequently, the antisymmetric extension (17) of f to the whole plane verifies, for any $1\leq q\leq p$

$$\lim_{x\to 0}\frac{\partial^q f}{\partial x^q}(t,x)=0 \qquad \text{uniformly with respect to } t\in\mathbb{R}\,,$$

so that $f\in\mathcal{C}^{0,p}(\mathbb{R}\times\mathbb{R})$, and further, $\partial^q f/\partial x^q\equiv 0$ on the axis $\{x=0\}$ for any $0\leq q\leq p$. This concludes the proof.

□

At a first glance, one might think that Lemma 9.2 provides a procedure to construct examples of smoother and smoother equations under the conditions of Proposition 2.2; it would suffice to start with curves x_1 and x_2 verifying assumptions *(i)-(viii)* and *(ix)*$_p$ for higher and higher choices of p. However, this argumentation fails because there are not such curves if $p\geq 3$. Let us briefly show this now: we let $\ell(t)=x_1(t)-x_1(t+1)$, which, in view of the second part of *(ix)*$_p$, must verify

$$\lim_{t\to+\infty}\frac{\ddot{\ell}(t)}{\ell(t)^p}=0\,,$$

which, if $p\geq 3$, implies

$$\lim_{t\to+\infty}\frac{\ddot{\ell}(t)}{\ell(t)^3}=0\,,$$

and consequently, by L'Hôpital rule,

$$\lim_{t\to+\infty}\frac{\dot{\ell}(t)^2}{\ell(t)^4}=0\,,$$

or, what is the same,

$$\lim_{t\to+\infty} \frac{\dot{\ell}(t)}{\ell(t)^2} = 0\,.$$

L'Hôpital rule may be applied again, to conclude

$$\lim_{t\to+\infty} \frac{1/\ell(t)}{t} = 0\,,$$

or, equivalently,

$$\lim_{t\to+\infty} t\,\ell(t) = +\infty\,.$$

In particular, there must be some $n_0 \in \mathbb{N}$ such that

$$\ell(n) \geq \frac{1}{n}\,, \qquad\qquad n \geq n_0\,,$$

implying that

$$+\infty = \sum_{n=0}^{\infty} \ell(n) = \sum_{n=0}^{\infty} \Big(x_1(n) - x_1(n+1)\Big) = x_1(0) < +\infty\,,$$

a contradiction.

Having checked that assumptions *(i)-(viii)* and *(ix)*$_p$ cannot be fulfilled if $p \geq 3$, let us close this paper by giving an example showing that these conditions are actually feasible for $p = 2$. Together with Lemma 9.2, this will complete the proof of Proposition 2.2.

We define $x_1 : [0,+\infty[\to \mathbb{R}$ by

$$x_1(t) := \frac{1}{\log(2+t)}\,, \qquad\qquad t \in [0,+\infty[\,. \tag{23}$$

Observe that this is a $\mathcal{C}^2$ function on $[0,+\infty[$. One easily verifies that

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} \dot{x}_1(t) = \lim_{t\to+\infty} \ddot{x}_1(t) = 0\,, \tag{24}$$

and also,

$$\dot{x}_1(t) < 0 < \ddot{x}_1(t)\,, \qquad\qquad t \in [0,+\infty[\,. \tag{25}$$

Moreover, straightforward computations show that, as $t \to +\infty$,

$$\dot{x}_1(t) \simeq \frac{-1}{t(\log t)^2}\,, \qquad \ddot{x}_1(t) \simeq \frac{1}{t^2(\log t)^2}\,, \qquad \dddot{x}_1(t) \simeq \frac{-2}{t^3(\log t)^2}\,, \tag{26}$$

in the sense that the limit of the quotient is 1 in every case. We combine the first and the last assertions of (26) to get

$$\lim_{t\to+\infty} \frac{\ddot{x}_1(t) - \ddot{x}_1(t+1)}{(x_1(t) - x_1(t+1))^2} = 0\,. \tag{27}$$

The construction of x_2 will be a little more sophisticated. We start from some $\mathcal{C}^4$ function $\varphi : [0,1] \to \mathbb{R}$ with

$$\varphi^{(k)}(0) = \varphi^{(k)}(1) = 0 \text{ for } k = 0,2,3,4, \qquad \varphi > 0 \text{ on }]0,1[, \qquad \dot\varphi(0) = -6\,\dot\varphi(1) > 0,$$

and we define $\Phi : [1,+\infty[\to \mathbb{R}$ by the rule

$$\Phi(t) := \frac{(-1)^n}{3^n}\varphi\left(\frac{t-2^{n-1}}{2^{n-1}}\right) \qquad \text{if } 2^{n-1} \le t < 2^n,\ n \in \mathbb{N}\,.$$

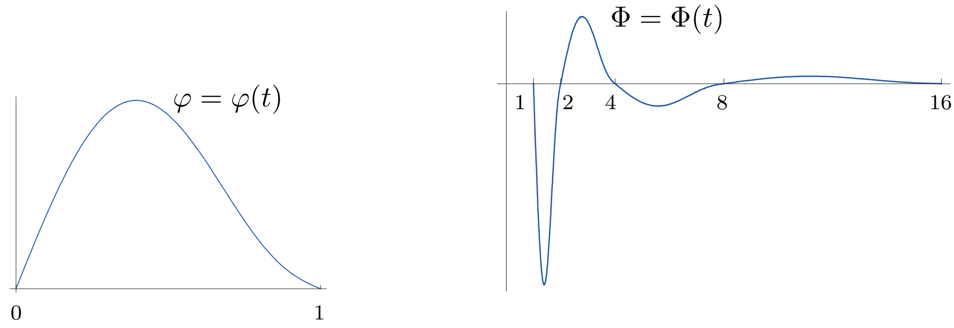


Figure 11: The graphs of φ and Φ.

Observe that Φ is a $\mathcal{C}^2$ function on $[1,+\infty[$. It verifies

$$\Phi(t) = 0 \text{ if and only if } t = 2^n \text{ for some } n \ge 0\,, \tag{28}$$

and also

$$\dot\Phi(2^n) \neq 0 = \ddot\Phi(2^n)\,, \qquad n \ge 0\,. \tag{29}$$

We may estimate the rate at which Φ tends to zero as $t \to +\infty$:

$$|\Phi(t)| \le \left(\frac{1}{3^n}\right)\max_{[0,1]}\varphi < \left(\frac{1}{t}\right)^{\log(3)/\log(2)}\left(\max_{[0,1]}\varphi\right), \qquad t \in [2^{n-1},2^n[\,,\quad n \in \mathbb{N}\,. \tag{30}$$

Actually, analogous analysis, when applied to $\dot\Phi$ and $\ddot\Phi$, give

$$\left|\dot\Phi(t)\right| = \frac{2}{6^n}\left|\dot\varphi\left(\frac{t-2^{n-1}}{2^{n-1}}\right)\right| < \left(2\max_{[0,1]}|\dot\varphi|\right)\left(\frac{1}{t}\right)^{1+\log(3)/\log(2)}, \qquad t \in [2^{n-1},2^n[\,,\quad n \in \mathbb{N}\,, \tag{31}$$

$$\left|\ddot\Phi(t)\right| = \frac{4}{12^n}\left|\ddot\varphi\left(\frac{t-2^{n-1}}{2^{n-1}}\right)\right| < \left(4\max_{[0,1]}|\ddot\varphi|\right)\left(\frac{1}{t}\right)^{2+\log(3)/\log(2)}, \qquad t \in [2^{n-1},2^n[\,,\quad n \in \mathbb{N}\,. \tag{32}$$

We combine (30) with the first assertion of (26) to deduce

$$\lim_{t\to+\infty}\left(\frac{\Phi(t)}{x_1(t-1)-x_1(t)}\right) = \lim_{t\to+\infty}\left(\frac{\Phi(t)}{x_1(t)-x_1(t+1)}\right) = 0\,, \tag{33}$$

while, comparing (31,32) with the first and middle statements of (26), we get

$$\lim_{t\to+\infty} \frac{\dot{\Phi}(t)}{\dot{x}_1(t)} = \lim_{t\to+\infty} \frac{\ddot{\Phi}(t)}{\ddot{x}_1(t)} = 0\,. \tag{34}$$

Finally, we have the inequality

$$\left|\frac{\ddot{\Phi}(t)}{\Phi(t)^2}\right| = 4\left(\frac{3}{4}\right)^n \left|\left(\frac{\ddot{\varphi}}{\varphi^2}\right)\left(\frac{t-2^{n-1}}{2^{n-1}}\right)\right| \le 4\left(\frac{3}{4}\right)^n \max_{]0,1[}\left|\frac{\ddot{\varphi}}{\varphi^2}\right|\,, \qquad t\in[2^{n-1},2^n[\,,\ n\in\mathbb{N}\,,$$

(observe that $\ddot{\varphi}(t)/\varphi(t)^2 \to 0$ as $t\to 0,1$). We deduce that

$$\lim_{t\to+\infty} \frac{\ddot{\Phi}(t)}{\Phi(t)^2} = 0\,. \tag{35}$$

Proof of Proposition 2.2. We define x_1 as in (23), and x_2 by

$$x_2(t) := x_1(t) + \epsilon\,\Phi(t+1)\,, \qquad t\in[0,+\infty[\,,$$

for some positive number $\epsilon > 0$. We claim that, if ϵ is small enough, these functions verify assumptions *(i)-(viii)* and *(ix)*$_2$ for the sequence $t_n = 2^n - 1$. Indeed, *(i)* follows from (24,30), while *(ii),(iii)* come from (28). In the other hand, (29) implies *(iv)*, while *(v)* follows, if $\epsilon > 0$ is small enough, from the combination of (25) and (34). Assumption *(vi)* follows at once from (24,34), while (33) implies *(vii)* for small ϵ. Also *(viii)* was shown in (33), while the first part of *(ix)*$_2$ was obtained in (35) and the second in (27). Thus, Lemma 9.2 may be applied for $p = 2$ and the proof is complete. □

10 Appendix: On the winding number of planar curves

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a (not necessarily closed) continuous path in the plane, and let the point $q \in \mathbb{R}^2\backslash\gamma([a,b])$ be given. As we already mentioned in Section 2, the winding number of γ with respect to the base point q is defined by

$$w_a^b(\gamma,q) := \frac{\theta_\gamma(b)-\theta_\gamma(a)}{2\pi}\,,$$

for any continuous determination $\theta_\gamma : [a,b] \to \mathbb{R}$ of the argument function on $\gamma - q$. Some well known properties of the winding number function are the following:

(i) *The winding number is additive.* For any continuous path $\gamma : [a,b] \to \mathbb{R}^2$, any point $q \in \mathbb{R}^2\backslash\gamma([a,b])$ and any $t \in]a,b[$, $w_a^b(\gamma,q) = w_a^t(\gamma,q) + w_t^b(\gamma,q)\,.$

(ii) *It vanishes on constant paths and changes sign when the orientation of the path is reversed.* Given $p \in \mathbb{R}^2\backslash\{q\}$, the constant path $\gamma_* \equiv p$ (defined on any closed interval $[a,b]$) has winding number with respect to q equal to zero. On the other hand, given some continuous $\gamma : [a,b] \to \mathbb{R}^2\backslash\{q\}$, the orientation-reversed path

$$\gamma^{-1} : [a,b] \to \mathbb{R}^2\backslash\{q\}\,, \qquad t \mapsto \gamma(a+b-t)\,,$$

has winding number $w_a^b(\gamma^{-1},q) = -w_a^b(\gamma,q)$

(iii) It remains invariant under fixed endpoints homotopies. Let $M : [a,b] \times [0,1] \to \mathbb{R}^2 \backslash \{q\}$ be continuous and such that $M(a,\cdot)$ and $M(b,\cdot)$ are constant mappings. Then,

$$w_a^b(M(\cdot,0),q) = w_a^b(M(\cdot,1),q)\,. \tag{36}$$

If the condition on the homotopy -which we shall now call J- to have fixed endpoints is removed, then (36) may not hold; however, there is still a 'commutative square' equality which is the aim of Lemma 10.1 below:

Lemma 10.1. *Let the continuous homotopy $J : [a,b] \times [0,1] \to \mathbb{R}^2 \backslash \{q\}$ be given. Then,*

$$w_a^b(J(\cdot,0),q) + w_0^1(J(b,\cdot),q) = w_0^1(J(a,\cdot),q) + w_a^b(J(\cdot,1),q)\,.$$

Proof. We consider the fixed endpoints homotopy $M : [a-1,b+1] \times [0,1] \to \mathbb{R}^2 \backslash \{q\}$ defined by

$$M(t,s) := \begin{cases} J(a,(-a+1+t)s) & \text{if } a-1 \le t \le a\,, \\ J(t,s) & \text{if } a \le t \le b\,, \\ J(b,(b+1-t)s) & \text{if } b \le t \le b+1\,. \end{cases}$$

Using by the invariance property *(iii)* of the winding number, we deduce

$$w_{a-1}^{b+1}(M(\cdot,0),q) = w_{a-1}^{b+1}(M(\cdot,1),q)\,,$$

and the additive property *(i)* gives

$$\begin{aligned} w_{a-1}^{a}(M(\cdot,0),q) + w_a^b(M(\cdot,0),q) + w_b^{b+1}(M(\cdot,0),q) = \\ = w_{a-1}^{a}(M(\cdot,1),q) + w_a^b(M(\cdot,1),q) + w_b^{b+1}(M(\cdot,1),q)\,. \end{aligned}$$

Now, the curve $M(\cdot,0)$ is constant on $[a-1,a]$ and also on $[b,b+1]$, and, by *(ii)*, $w_{a-1}^{a}(M(\cdot,0),q) = w_b^{b+1}(M(\cdot,0),q) = 0$. Also by *(ii)*, $w_b^{b+1}(M(\cdot,1),q) = -w_0^1(J(1,\cdot),q)$. Since M and J coincide on $[a,b] \times [0,1]$, the result follows.

□

We shall be particularly interested in the case which occurs when the path γ has the special form $\gamma(t) = \alpha_x(t) := (x(t),\dot{x}(t))$ for some $\mathcal{C}^1$ function x with only nondegenerate zeroes. In this case, the winding number of α_x around $O = (0,0)$ may be estimated from the number k_x of zeroes of x on $[a,b]$:

Lemma 10.2. *Let $x : [a,b] \to \mathbb{R}$ be a $\mathcal{C}^1$ function with no degenerate zeroes. Then,*

$$\left| w_a^b(\alpha_x,O) + k_x/2 \right| \le 1/2\,.$$

Proof. Observe that, the curve x having only nondegenerate zeroes, $\alpha_x(t) \neq O$ for any $t \in [a, b]$, and we may assign a sign to each point where α_x intersects the vertical axis $\{0\} \times \mathbb{R}$ according to the side of the punctured axis which the point belongs to. It follows that such positive and negative intersection points alternate on the time, and, moreover, between a positive and its consecutive negative one, α_x stays to the right of the vertical axis, while, if it is a negative intersection point which comes first, then α_x stays on the left half plane until the next (positive) intersection point. It means that α_x spins clockwise and the result follows.

□

Proof of Proposition 6.1. As before, we let $O := (0, 0)$. We consider the homotopy $J : [a, b] \times [0, 1] \to \mathbb{R}^2 \backslash \{O\}$ defined by

$$J(t, s) := (H(t, s) - \eta(t), H_t(t, s) - \dot{\eta}(t)) \,, \qquad (t, s) \in [a, b] \times [0, 1] \,,$$

and, according to Lemma 10.1, one has

$$w_0^1(J(a, \cdot), O) = w_a^b(J(\cdot, 0), O) - w_a^b(J(\cdot, 1), O) + w_0^1(J(b, \cdot), O) \,. \tag{37}$$

Remembering the definition of γ in (6), $J(a, s) = \gamma(s) - q$ for any $s \in [0, 1]$; in particular, $w_0^1(J(a, \cdot), O) = w_0^1(\gamma, q)$. Concerning the right hand side of (37), we have

- $J(\cdot, 0) = \alpha_{u-\eta}$ and, in view that $m = k_{u-\eta}$, Lemma 10.2 states that

$$|w_a^b(J(\cdot, 0), O) + m/2| < 1/2 \,,$$

- $J(\cdot, 1) = \alpha_{v-\eta}$ and, in view that $n = k_{v-\eta}$, Lemma 10.2 states that

$$|w_a^b(J(\cdot, 1), O) + n/2| < 1/2 \,,$$

- $J(b, [0, 1])$ does not intersect the vertical section $\{0\} \times \mathbb{R}$, so that

$$|w_0^1(J(b, \cdot), O)| < 1/2 \,,$$

implying the result.

□

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